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PSEUDO-CONFORMAL ACTIONS OF THE MÖBIUS GROUP

M. BELRAOUTI, M. DEFFAF, Y. RAFFED, AND A. ZEGHIB

Abstract. We study compact connected pseudo-Riemannian manifolds (M, g) on which the conformal group $\text{Conf}(M, g)$ acts essentially and transitively. We prove, in particular, that if the non-compact semi-simple part of $\text{Conf}(M, g)$ is the Möbius group, then (M, g) is conformally flat.

CONTENTS

1. Introduction	1
2. Preliminaries	2
2.1. Notations	2
2.2. General facts	3
3. Lie algebra formulation	4
3.1. Enlargement of the isotropy group	4
3.2. Distortion	6
3.3. The isotropy group is big	7
3.4. A suitable modification of \mathfrak{g}	7
4. The Möbius conformal group: a classification theorem	8
4.1. The Frances-Melnick case	8
4.2. The non-compact semi-simple case	8
4.3. The general case	11
References	11

1. INTRODUCTION

A pseudo-Riemannian manifold is a differentiable manifold M endowed with a pseudo-Riemannian metric g of signature (p, q) . Two metrics g_1 and g_2 on M are said to be conformally equivalent if and only if $g_1 = \exp(f)g_2$ where f is C^∞ function. A conformal structure is then an equivalence class $[g]$ of a pseudo-Riemannian metric g and a conformal manifold is a manifold endowed with a pseudo-Riemannian conformal structure. A remarkable family of conformal manifolds is given by the conformally flat ones. These are pseudo-Riemannian conformal manifolds that are locally conformally diffeomorphic (i.e preserving the conformal structures) to the Minkowski space $\mathbb{R}^{p,q}$ i.e the vector space \mathbb{R}^{p+q} endowed with the pseudo-Riemannian metric $-dx_0^2 - \dots - dx_{p-1}^2 + dy_0^2 + \dots + dy_{q-1}^2$.

The conformal group $\text{Conf}(M, g)$ is the group of transformations that preserve the conformal structure $[g]$. It is said to be essential if there is no metric in the conformal class of g for which it acts isometrically. In the Riemannian case, the sphere \mathbb{S}^n is a compact conformally flat manifold with an essential conformal group.

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The Einstein universe $\text{Ein}^{p,q}$ is the equivalent model of the standard conformal sphere in the pseudo-Riemannian setting. It admits a two-fold covering conformally equivalent to the product $\mathbb{S}^p \times \mathbb{S}^q$ endowed with the conformal class of $-g_{\mathbb{S}^p} \oplus g_{\mathbb{S}^q}$. It is conformally flat and its conformal group, which is in fact the pseudo-Riemannian Möbius group $O(p+1, q+1)$, is essential. Actually the Einstein universe is the flat model of conformal pseudo-Riemannian geometry. This is essentially due to the fact that the Minkowski space embeds conformally as a dense open subset of the Einstein universe $\text{Ein}^{p,q}$ and in addition to the Liouville theorem asserting that conformal local diffeomorphisms on $\text{Ein}^{p,q}$ are unique restrictions of elements of $O(p+1, q+1)$. Hence a manifold is conformally flat if and only if it admits a $(O(p+1, q+1), \text{Ein}^{p,q})$ -structure.

In the sixties A. Lichnérowicz conjectured that among compact Riemannian manifolds, the sphere is the only essential conformal structure. This was generalised and proved independently by Obatta and Ferrand (see [20], [17]). In the pseudo-Riemannian case, a similar question, called the pseudo-Riemannian Lichnérowicz conjecture, was raised by D'Ambra and Gromov [1]. Namely, if a compact pseudo-Riemannian conformal manifold is essential then it is conformally flat. This was disproved by Frances see [8], [10].

The present article is the first of a series on the pseudo-Riemannian Lichnérowicz conjecture in a homogeneous setting [5, 4]. The general non homogeneous case, but with signature restrictions, was amply studied by Zimmer, Bader, Nevo, Frances, Zeghib, Melnick and Pecastaing (see [25], [2], [11], [22], [23], [21], [18]). Let us also quote [16] as a recent work in the Lorentz case.

We are investigating in this first part the case where the non-compact semi-simple part of the conformal group is locally isomorphic to the Möbius group $SO(1, n+1)$. More exactly, we prove the following classification result. This Möbius situation will actually play a central role towards the general case treated in [5].

Theorem 1.1. *Let $(M, [g])$ be a conformal connected compact pseudo-Riemannian manifold. We suppose that there exists G a subgroup of the conformal group $\text{Conf}(M, g)$ acting essentially and transitively on $(M, [g])$. We suppose moreover that the non-compact semi-simple part of G is locally isomorphic to the Möbius group $SO(1, n+1)$. Then $(M, [g])$ is conformally flat. More precisely $(M, [g])$ is conformally equivalent to*

- *The conformal Riemannian n -sphere or;*
- *Up to a cover, the Einstein universe $\text{Ein}^{1,1}$ or;*
- *Up to a finite cover, the Einstein universe $\text{Ein}^{3,3}$.*

Remark 1.2. *It turns out that, in the first and third cases, the acting group G is locally isomorphic to the Möbius group, that is, G is simple. In the second case, the universal cover \tilde{G} is a subgroup of $\widetilde{SL(2, \mathbb{R})} \times \widetilde{SL(2, \mathbb{R})}$. It can in particular be $\widetilde{SL(2, \mathbb{R})} \times \widetilde{SO(2)}$.*

2. PRELIMINARIES

2.1. Notations. Throughout this paper (M, g) will be a compact connected pseudo-Riemannian manifold of dimension n endowed with a transitive and essential action of the conformal group $G = \text{Conf}(M, g)$. We suppose without loss of generality that G is connected.

Fix a point x in M and denote by $H = \text{Stab}(x)$ its stabilizer in G . Denote respectively by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H . Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition of \mathfrak{g} , where \mathfrak{s} is semi-simple and \mathfrak{r} is the solvable radical of \mathfrak{g} . Denote by \mathfrak{s}_{nc} the non-compact semi-simple factor of \mathfrak{s} , by \mathfrak{s}_c the compact one and let \mathfrak{n} be the nilpotent radical of \mathfrak{g} . Note that \mathfrak{n} is an ideal of \mathfrak{g} . Let us denote respectively by S, S_{nc}, S_c, R and N the connected Lie sub-groups of G associated to $\mathfrak{s}, \mathfrak{s}_{nc}, \mathfrak{s}_c, \mathfrak{r}$ and \mathfrak{n} .

Let \mathfrak{a} be a Cartan subalgebra of \mathfrak{s} associated with a Cartan involution Θ . Consider $\mathfrak{s} = \mathfrak{s}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{s}_\alpha = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{s}_\alpha$ the root space decomposition of \mathfrak{s} , where Δ is the set of roots of $(\mathfrak{s}, \mathfrak{a})$. Denote respectively by Δ^+, Δ^- the set of positive and negative roots of \mathfrak{s} for some chosen notion of positivity on \mathfrak{a}^* . Then $\mathfrak{s} = \mathfrak{s}_- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+$, where $\mathfrak{s}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{s}_\alpha$ and $\mathfrak{s}_- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{s}_\alpha$.

For every $\alpha \in \mathfrak{a}^*$, consider

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}, \forall H \in \mathfrak{a} : ad_H(X) = \alpha(H)X\}.$$

We say that α is a weight if $\mathfrak{g}_\alpha \neq 0$. In this case \mathfrak{g}_α is its associated weight space. As $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ (see [14, Theorem 13]) then, for every $\alpha \neq 0$, $\mathfrak{g}_\alpha = \mathfrak{s}_\alpha \oplus \mathfrak{n}_\alpha$, where

$$\mathfrak{n}_\alpha = \{X \in \mathfrak{n}, \forall H \in \mathfrak{a} : ad_H(X) = \alpha(H)X\}.$$

Moreover, the commutativity of \mathfrak{a} together with the fact that finite dimensional representations of a semi-simple Lie algebra preserve the Jordan decomposition implies that elements of \mathfrak{a} are simultaneously diagonalisable in some basis of \mathfrak{g} . Thus $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$.

Finally we will denote respectively by A, S_+ the connected Lie subgroups of G corresponding to \mathfrak{a} and \mathfrak{s}_+ .

2.2. General facts. We will prove some general results about the conformal group G . We start with the following general fact:

Proposition 2.1. *We have that $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$. In particular the sub-algebra $\mathfrak{s} \ltimes \mathfrak{n}$ is an ideal of \mathfrak{g} .*

Proof. For this, let us consider the semi-simple S -representation in $GL(\mathfrak{r})$. It preserves \mathfrak{n} and thus has a supplementary invariant subspace. But $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ so automorphisms of \mathfrak{r} act trivially on $\mathfrak{r}/\mathfrak{n}$ and hence $[\mathfrak{s}, \mathfrak{g}] \subset \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{n}] \subset \mathfrak{s} \ltimes \mathfrak{n}$. We deduce that $\mathfrak{s} \ltimes \mathfrak{n}$ is an ideal of \mathfrak{g} . \square

Next we will prove:

Proposition 2.2. *The non-compact semi-simple factor S_{nc} of S is non trivial.*

Let us first start with the following simple observation:

Proposition 2.3. *If a conformal diffeomorphism f of (M, g) preserves a volume form ω on M , then it preserves a metric in the conformal class of g .*

Proof. Let f be a diffeomorphism preserving the conformal class $[g]$ and a volume form ω on M . Denote by ω_g the volume form defined on M by the metric g . On the one hand, there exists a C^∞ real function ϕ such that $\omega = e^\phi \omega_g$. Hence ω is the volume form defined by the metric $e^{\frac{2\phi}{n}} g$. On the other hand, we have $f^* e^{\frac{2\phi}{n}} g = e^\psi e^{\frac{2\phi}{n}} g$, for some C^∞ function ψ . Thus $f^* \omega = e^{\frac{\psi}{2}} \omega$. But, f preserves the volume form ω , so $\psi = 0$ which means that f preserves the metric $e^{\frac{2\phi}{n}} g$. \square

As a consequence we get:

Corollary 2.4. *The conformal group G preserves no volume form on M .*

Assume that the non-compact semi-simple factor S_{nc} is trivial. Then by [24, Corollary 4.1.7] the group G is amenable. So it preserves a regular Borel measure μ on the compact manifold M . It is in particular a quasi-invariant measure with associated rho-function $\rho_1 = 1$ (in the sense of [3]). Let now ω_g be the volume form corresponding to the metric g . As the group G acts conformally and the action is C^∞ , the measure ω_g is also quasi-invariant with C^∞ rho-function ρ_2 (see [3, Theorem B.1.4]). Again by [3, Theorem B.1.4], the measures μ and ω_g are equivalent and $\frac{d\mu}{d\omega_g} = \frac{1}{\rho_2}$. This shows that μ is a volume form. Then one use Corollary 2.4 to get the Proposition 2.2.

In the general case the essentiality of the action ensure the non discreteness of the stabilizer H .

Proposition 2.5. *The stabilizer H is not discrete.*

Proof. If it was not the case then H would be a uniform lattice in G . But as the action is essential, there is an element $h \in H$ that does not preserve the metric on $\mathfrak{g}/\mathfrak{h}$. So $|\det(\text{Ad}_h)| \neq 1$ contradicting the unimodularity of G . \square

To finish this part let us prove the two following important Lemmas that will be used later in the paper:

Lemma 2.6. *Let $\pi : S_{nc} \rightarrow \text{GL}(V)$ be a linear representation of S_{nc} into a linear space V . Then, the compact orbits of S_{nc} are trivial.*

Proof. Assume that S_{nc} has a compact orbit $\mathcal{C} \subset V$. Then the convex envelope $\text{Conv}(\mathcal{C} \cup -\mathcal{C})$ is an S_{nc} -invariant compact convex symmetric set with non empty interior. Thus the action of S_{nc} preserves the Minkowski gauge $\|\cdot\|$ (which is in fact a norm) of $\text{Conv}(\mathcal{C} \cup -\mathcal{C})$. But $\text{Isom}(\text{Conv}(\mathcal{C} \cup -\mathcal{C}), \|\cdot\|)$ is compact. So the restriction of the representation π to $\text{Conv}(\mathcal{C} \cup -\mathcal{C})$ gives rise to an homomorphism from a semi-simple group with no compact factor to a compact group and hence is trivial. \square

Lemma 2.7. *A linear representation $\pi : \mathfrak{s}_{nc} \rightarrow \text{gl}(V)$ of \mathfrak{s}_{nc} into a linear space V is completely determined by its restriction to $\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+$. More precisely, $\pi_{\mathfrak{s}_{nc}}(V) = \text{Vect}(\pi_{\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+}(V))$.*

Proof. It is in fact sufficient to show that $\pi_{\mathfrak{s}_-}(V) \subset \text{Vect}(\pi_{\mathfrak{a} \oplus \mathfrak{s}_+}(V))$. For that, fix $x \in \mathfrak{s}_{-\alpha} \subset \mathfrak{s}_-$ and let $a \in \mathfrak{a}$ such that $\mathbb{R}x \oplus \mathbb{R}a \oplus \mathbb{R}\Theta(x) \cong \mathfrak{sl}(2, \mathbb{R})$ (see for example [15, Proposition 6.52]). Thus the restriction of π to $\mathbb{R}x \oplus \mathbb{R}a \oplus \mathbb{R}\Theta(x)$ is isomorphic to a linear representation of $\mathfrak{sl}(2, \mathbb{R})$ into V . Using Weyl Theorem we can assume without loss of generality that this last is irreducible. But irreducible linear representations of $\mathfrak{sl}(2, \mathbb{R})$ into V are unique up to isomorphism (see for instance [13, Theorem 4.32]). It is then easy to check that they verify $\pi(x)(V) \subset \text{Vect}(\pi_{\mathbb{R}a \oplus \mathbb{R}\Theta(x)}(V))$ (see [13, Examples 4.2]). This finishes the proof. \square

3. LIE ALGEBRA FORMULATION

3.1. Enlargement of the isotropy group. As the manifold G/H is compact, the isotropy subgroup H is a uniform subgroup of G . If H was discrete then it is a uniform lattice and in this case G would be unimodular. In the non discrete case,

this imposes strong restrictions on the group H . When H and G are both complex algebraic it is equivalent to being parabolic i.e contains maximal solvable connected subgroup of H . In the real case, Borel and Tits [6] proved that an algebraic group H of a real linear algebraic group G is uniform if it contains a maximal connected triangular subgroup of G . Recall that a subgroup of G (respectively a sub-algebra of \mathfrak{g}) is said to be triangular if, in some real basis of \mathfrak{g} , its image under the adjoint representation is triangular.

Let $H^* = \text{Ad}^{-1} \left(\overline{\text{Ad}(H)}^{\text{Zariski}} \right)$ be the smallest algebraic Lie subgroup of G containing H . By [12, Corollary 5.1.1], the Lie algebra \mathfrak{h}^* of H^* contains a maximal triangular sub-algebra of \mathfrak{g} . The sub-algebra $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$ being triangular, we get the following fact:

Fact 3.1. *Up to conjugacy, the sub-algebra \mathfrak{h}^* contains $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.*

Consider the vector space $\text{Sym}(\mathfrak{g})$ of bilinear symmetric forms on \mathfrak{g} . The group G acts naturally on $\text{Sym}(\mathfrak{g})$ by $g.\Phi(X, Y) = \Phi(\text{Ad}_{g^{-1}}X, \text{Ad}_{g^{-1}}Y)$. Let $\langle \cdot, \cdot \rangle$ be the bilinear symmetric form on \mathfrak{g} defined by

$$\langle X, Y \rangle = g(X^*(x), Y^*(x)),$$

where g is the pseudo-Riemannian metric, X^* , Y^* are the fundamental vector fields associated to X and Y and x is the point fixed previously. It is a degenerate symmetric form with kernel equal to \mathfrak{h} .

Let P be the subgroup of G preserving the conformal class of $\langle \cdot, \cdot \rangle$. It is an algebraic group containing H and normalizing the sub-algebra \mathfrak{h} . In particular, it contains H^* : the smallest algebraic group containing H . Using Fact 3.1 we get that up to conjugacy, the Lie algebra \mathfrak{p} of P contains $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.

Proposition 3.2. *The Cartan sub-group A does not preserve the metric $\langle \cdot, \cdot \rangle$.*

Proof. First as \mathfrak{h} is an ideal of \mathfrak{p} then by taking quotient of both P and H by H° , we can suppose that H is a uniform lattice of P and in particular that P is unimodular.

Assume that A preserves the metric $\langle \cdot, \cdot \rangle$. On the one hand, the groups S_+ and N preserve the conformal class of $\langle \cdot, \cdot \rangle$. On the other hand, they act on $\text{Sym}(\mathfrak{g})$ by unipotent elements. So the groups A , S_+ , and N preserve the metric $\langle \cdot, \cdot \rangle$. But by Iwasawa decomposition $(A \ltimes S_+)$ is co-compact in S . Thus the S -orbit of $\langle \cdot, \cdot \rangle$ is compact in $\text{Sym}(\mathfrak{g})$ and hence trivial by Lemma 2.6. Therefore S and N are subgroups of P . This implies that for any $p \in P$, $\left| \det(\text{Ad}_p)|_{\mathfrak{g}/\mathfrak{p}} \right| = 1$. Indeed, the action of G on $(\mathfrak{s}_c + \mathfrak{r})/\mathfrak{n}$ factors through the product of the action of S_c on \mathfrak{s}_c by the trivial action on $\mathfrak{r}/\mathfrak{n}$. As P contains S and N , its action on $\mathfrak{g}/\mathfrak{p}$ is a quotient of the action of S_c on \mathfrak{s}_c . But S_c is compact, thus it preserves some positive definite scalar product and hence the determinant $\left| \det(\text{Ad}_p)|_{\mathfrak{g}/\mathfrak{p}} \right| = 1$.

Now let $h \in H$ such that Ad_h does not preserve $\langle \cdot, \cdot \rangle$. We have that

$$1 \neq \left| \det(\text{Ad}_h)|_{\mathfrak{g}/\mathfrak{h}} \right| = \left| \det(\text{Ad}_h)|_{\mathfrak{g}/\mathfrak{p}} \right| \left| \det(\text{Ad}_h)|_{\mathfrak{p}/\mathfrak{h}} \right|$$

Finally we get $\left| \det(\text{Ad}_h)|_{\mathfrak{p}/\mathfrak{h}} \right| \neq 1$ which contradicts the unimodularity of P . \square

3.2. Distortion. The group P preserves the conformal class of $\langle \cdot, \cdot \rangle$. There exists thus an homomorphism $\delta : P \rightarrow \mathbb{R}$ such that: for every $p \in P$ and every $u, v \in \mathfrak{g}$,

$$(1) \quad \langle \text{Ad}_p(u), \text{Ad}_p(v) \rangle = e^{\delta(p)} \langle u, v \rangle = \left| \det (\text{Ad}_p)|_{\mathfrak{g}/\mathfrak{h}} \right|^{\frac{2}{n}} \langle u, v \rangle$$

In particular if $p \in P$ preserves the metric then $\delta(p) = 0$ and

$$(2) \quad \langle \text{Ad}_p(u), \text{Ad}_p(v) \rangle = \langle u, v \rangle$$

Or equivalently

$$(3) \quad \langle \text{ad}_p(u), v \rangle + \langle u, \text{ad}_p(v) \rangle = 0$$

It follows that if the action of $p \in P$ on \mathfrak{g} is unipotent then $\delta(p) = 0$. Therefore, the homomorphism δ is trivial on S_- and N but not on A by Proposition 3.2. We continue to denote by δ the restriction of δ to A . We can see it alternatively as a linear form $\delta : \mathfrak{a} \rightarrow \mathbb{R}$, called **distortion**, verifying: for every $a \in \mathfrak{a}$ and every $u, v \in \mathfrak{g}$,

$$(4) \quad \langle \text{ad}_a(u), v \rangle + \langle u, \text{ad}_a(v) \rangle = \delta(a) \langle u, v \rangle$$

Definition 3.1. Two weights spaces \mathfrak{g}_α and \mathfrak{g}_β are said to be paired if they are not $\langle \cdot, \cdot \rangle$ -orthogonal.

Definition 3.2. A weight α is a non-degenerate weight if \mathfrak{g}_α is not contained in \mathfrak{h} .

Definition 3.3. We say that a subalgebra \mathfrak{g}' is a modification of \mathfrak{g} if \mathfrak{g}' projects surjectively on $\mathfrak{g}/\mathfrak{h}$. In this case $\mathfrak{g}'/(\mathfrak{g}' \cap \mathfrak{h}) = \mathfrak{g}/\mathfrak{h}$.

Proposition 3.3. If the weight space \mathfrak{g}_0 is degenerate then up to modification, \mathfrak{g} is semi-simple and $M = G/H$ is conformally flat.

Proof. On the one hand, $\mathfrak{g}_0 \subset \mathfrak{h}$ implies that $\mathfrak{a} \subset \mathfrak{h}$. As \mathfrak{h} is an ideal of \mathfrak{p} , we get that $\mathfrak{s}_+ = [\mathfrak{s}_+, \mathfrak{a}] \subset \mathfrak{h}$. On the other hand, $\mathfrak{r} \subset \mathfrak{g}_0 + \mathfrak{n} \subset \mathfrak{g}_0 + [\mathfrak{n}, \mathfrak{a}] \subset \mathfrak{h}$. Thus, up to modification, we can assume that \mathfrak{g} is semi-simple and that \mathfrak{h} contains $\mathfrak{a} + \mathfrak{s}_+$.

Now let α_{max} be the highest positive root and let $X \in \mathfrak{g}_{\alpha_{max}}$. Then $d_1 e^X : \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}/\mathfrak{h}$ is trivial. Yet e^X is not trivial. We conclude using [9, Theorem 1.4]. \square

A direct consequence of Equation 4, is that if \mathfrak{g}_α and \mathfrak{g}_β are paired then $\alpha + \beta = \delta$. This shows that if α is a non-degenerate weight then $\delta - \alpha$ is also a non-degenerate weight. In particular if 0 is a non-degenerate weight, then \mathfrak{g}_0 and \mathfrak{g}_δ are paired and hence δ is a non-degenerate weight. In fact:

Proposition 3.4. If 0 is a non-degenerate weight then δ is a root. Moreover $\mathfrak{s}_\delta \not\subset \mathfrak{h}$.

Proof. First we will prove that the subalgebras \mathfrak{a} and \mathfrak{n}_δ are $\langle \cdot, \cdot \rangle$ -orthogonal. Let $a \in \mathfrak{a}$ such that $\delta(a) \neq 0$. Using Equation 4 for a , $u = a$ and $v \in \mathfrak{n}_\delta$, we get, $\langle a, \text{ad}_a(v) \rangle = \delta(a) \langle a, v \rangle$. But v preserves $\langle \cdot, \cdot \rangle$, thus by Equation 3, $\delta(a) \langle a, v \rangle = 0$. Hence $\langle a, v \rangle = 0$, for every $v \in \mathfrak{n}_\delta$. We conclude by continuity.

Now if δ was not a root then $\mathfrak{s}_\delta = 0$ and $\mathfrak{g}_\delta = \mathfrak{n}_\delta$. Thus \mathfrak{a} and \mathfrak{g}_δ are orthogonal. Which implies that $\mathfrak{a} \subset \mathfrak{h}$. But \mathfrak{h} is an ideal of \mathfrak{p} , so $\mathfrak{g}_\delta = [\mathfrak{g}_\delta, \mathfrak{a}] \subset \mathfrak{h}$. This contradicts the fact that \mathfrak{g}_δ is paired with \mathfrak{g}_0 .

To finish we need to prove that $\mathfrak{s}_\delta \not\subset \mathfrak{h}$. If this was not the case then \mathfrak{a} would be orthogonal to \mathfrak{g}_δ . Hence $\mathfrak{g}_\delta \subset \mathfrak{h}$ which contradicts again the fact that \mathfrak{g}_δ is paired with \mathfrak{g}_0 . \square

3.3. The isotropy group is big. From now and until the end we will suppose that the non-compact semi-simple part S_{nc} of G is locally isomorphic to the Möbius group $SO(1, n + 1)$. In this case the Cartan Lie algebra \mathfrak{a} is one dimensional and we have $\mathfrak{s}_{nc} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha$, where α is a positive root, $\mathfrak{a} \cong \mathbb{R}$, $\mathfrak{m} \cong \mathfrak{so}(n)$, and $\mathfrak{s}_{-\alpha} \cong \mathfrak{s}_\alpha \cong \mathbb{R}^n$. Moreover, $\mathfrak{g}_{\pm\alpha} = \mathfrak{s}_{\pm\alpha} \oplus \mathfrak{n}_{\pm\alpha}$, $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c \oplus \mathfrak{r}_0$, $\mathfrak{g}_\beta = \mathfrak{n}_\beta$ for every $\beta \neq 0, \pm\alpha$ and $\mathfrak{r} = \mathfrak{r}_0 \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta$.

In section 3.1 we saw that the isotropy group H is contained in the algebraic group P which turn out to be big i.e to contain the connected Lie groups A , S_α and N . Our next result shows that the group H itself is big:

Proposition 3.5. *The Lie algebra \mathfrak{h} contains $\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta$.*

Proof. We have that $\mathfrak{a} \subset \mathfrak{h}$. Indeed, if 0 is a degenerate weight then we are done. If not, then δ is a root and $\mathfrak{a} \subset \mathfrak{g}_0$ is orthogonal to every \mathfrak{g}_β with $\beta \neq \delta$. From the proof of Proposition 3.4 we know that \mathfrak{a} and \mathfrak{n}_δ are orthogonal. Thus it remains to show that \mathfrak{a} and \mathfrak{s}_δ are orthogonal. For that, let $x \in \mathfrak{s}_\delta$ then $\Theta(x) \in \mathfrak{s}_{-\delta}$ and $[x, \Theta(x)] \neq 0$ in \mathfrak{a} . Now using Equation 3 and the fact that one of x or $\Theta(x)$ preserve $\langle \cdot, \cdot \rangle$, we get $\langle \text{ad}_x(\Theta(x)), x \rangle = 0$. But \mathfrak{a} is one dimensional so it is orthogonal to \mathfrak{s}_δ . To end this proof, we have that \mathfrak{h} is an ideal of \mathfrak{p} and so

$$\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta = \mathfrak{a} \oplus \left[\mathfrak{a}, \mathfrak{s}_\alpha \oplus \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta \right] \subset \mathfrak{h} \oplus [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{h}.$$

\square

As a consequence we get:

Corollary 3.6. *If 0 is a non-degenerate weight, then $\delta = -\alpha$.*

3.4. A suitable modification of \mathfrak{g} . We will show that \mathfrak{g} admits a suitable modification \mathfrak{g}' . This allows us to considerably simplify the proofs in the next section. More precisely, we have:

Proposition 3.7. *The solvable radical decomposes as a direct sum $\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{r}_2$, where \mathfrak{r}_1 is a subalgebra commuting with the semi-simple factor \mathfrak{s} and \mathfrak{r}_2 is an \mathfrak{s} -invariant linear subspace contained in \mathfrak{h} . In particular $\mathfrak{g}' = \mathfrak{s} \oplus \mathfrak{r}_1$ is a modification of \mathfrak{g} .*

To prove Proposition 3.7, we need the following lemma:

Lemma 3.8. *We have $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{h}$.*

Proof of Lemma 3.8. First we prove that $[\mathfrak{n}, \mathfrak{g}_0] \subset \mathfrak{h}$. For this, note that by the Jacobi identity and the fact that \mathfrak{n} is an ideal of \mathfrak{g} , we have $\left[\bigoplus_{\beta \neq 0} \mathfrak{n}_\beta, \mathfrak{g}_0 \right] = \bigoplus_{\beta \neq 0} \mathfrak{n}_\beta$ which in turn is a subset of \mathfrak{h} by Proposition 3.5. Thus one need to prove that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{h}$. We know that \mathfrak{n} preserve the metric $\langle \cdot, \cdot \rangle$. So using Equation 3 for $p \in \mathfrak{n}_0$, $u \in \mathfrak{g}_0$ and $v \in \mathfrak{g}_\delta$ gives us: $\langle \text{ad}_p(u), v \rangle + \langle u, \text{ad}_p(v) \rangle = 0$. But once again by Jacobi identity, the fact that \mathfrak{n} is an ideal of \mathfrak{g} and Proposition 3.5 we have $\text{ad}_p(v) \in \mathfrak{g}_\delta \cap \mathfrak{n} = \mathfrak{n}_\delta \subset \mathfrak{h}$. So $\langle \text{ad}_p(u), v \rangle = 0$, which means that $[\mathfrak{n}_0, \mathfrak{g}_0]$ is orthogonal to \mathfrak{g}_δ . Using the fact that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ and that \mathfrak{g}_0 is orthogonal to every \mathfrak{g}_β

for $\beta \neq \delta$ we get that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{h}$.

Next we have that $\mathfrak{s}_c \subset \mathfrak{g}_0$ thus $[\mathfrak{s}_c, \mathfrak{n}] \subset [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{h}$.

Finally we finish by proving that $[\mathfrak{s}_{nc}, \mathfrak{n}] \subset \mathfrak{h}$. On the one hand we have,

$$[\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{m}, \mathfrak{n}] \subset [\mathfrak{a} \oplus \mathfrak{s}_\alpha, \mathfrak{n}] + [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{h} + \mathfrak{h} \subset \mathfrak{h}.$$

On the other hand, as \mathfrak{s}_{nc} is semi-simple we have by Lemma 2.7 that $[\mathfrak{s}_{nc}, \mathfrak{n}] \subset \text{Vect}([\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha, \mathfrak{n}]) \subset \mathfrak{h}$.

Proof of Proposition 3.7. The subalgebra $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$ is \mathfrak{s} -invariant, so it admits an \mathfrak{s} -invariant supplementary subspace \mathfrak{r}'_1 in \mathfrak{r} . But \mathfrak{s} acts trivially on $\mathfrak{r}/[\mathfrak{s}, \mathfrak{n}]$ and thus it acts trivially on \mathfrak{r}'_1 . We take \mathfrak{r}_1 to be the \mathfrak{s} -invariant subalgebra generated by \mathfrak{r}'_1 (in fact the action of \mathfrak{s} on \mathfrak{r}_1 is trivial).

It is clear that \mathfrak{r}_1 is a direct sum of \mathfrak{r}'_1 and \mathfrak{r}''_1 : an \mathfrak{s} -invariant subspace of $[\mathfrak{s}, \mathfrak{n}]$. Consider \mathfrak{r}_2 to be the supplementary of \mathfrak{r}''_1 in $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$. It is \mathfrak{s} -invariant and by Lemma 3.8 we have $\mathfrak{r}_2 \subset \mathfrak{h}$.

4. THE MÖBIUS CONFORMAL GROUP: A CLASSIFICATION THEOREM

This section is devoted to prove Theorem 1.1. We distinguish two situations: when \mathfrak{m} is contained in \mathfrak{h} and when it is not. In this last one, we first consider the case where only the non-compact semi-simple part S_{nc} is non trivial. Then deduce from it the general case. From now and until the end we will assume, up to modification, that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}_1$.

4.1. The Frances-Melnick case. We suppose that the sub-algebra \mathfrak{m} is contained in \mathfrak{h} . Then we have the following proposition:

Proposition 4.1. *M is conformally equivalent to the standard sphere \mathbb{S}^n or the Einstein universe $\text{Ein}^{1,1}$.*

Proof. Assume first that \mathfrak{g}_0 is contained in \mathfrak{h} . Then by Proposition 3.3, M is conformally flat and after modification, $\mathfrak{r} = 0$. Moreover, $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{s}_{-\alpha}$. This is because $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c$ and $[\mathfrak{m}, X] = \mathfrak{s}_{-\alpha}$ for every $X \neq 0$ in $\mathfrak{s}_{-\alpha}$. Thus M is conformally equivalent to $\text{SO}(1, n+1)/\text{CO}(n) \times \mathbb{R}^n \cong \mathbb{S}^n$.

Now suppose that \mathfrak{g}_0 is not in \mathfrak{h} . In this case $\mathfrak{g}_{-\alpha} = \mathfrak{g}_\delta$ is paired with \mathfrak{g}_0 . But \mathfrak{a} , \mathfrak{m} and \mathfrak{n}_δ are contained in \mathfrak{h} so $\mathfrak{s}_{-\alpha}$ is paired with $\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1)$. Note that \mathfrak{m} acts on $\mathfrak{s}_{-\alpha} \oplus (\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1))$ by preserving the pairing (in fact the action of \mathfrak{m} preserves the metric $\langle \cdot, \cdot \rangle$). On the contrary for $n \geq 2$, $\mathfrak{m} \cong \mathfrak{so}(n)$ acts trivially on $\mathfrak{r}_0 \cap \mathfrak{r}_1$ and transitively on $\mathfrak{s}_{-\alpha} - \{0\}$, so $n = 1$. As the metric is of type (p, q) , we conclude that the projection of $\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1)$ on $\mathfrak{g}/\mathfrak{h}$ is $\cong \mathbb{R}$. Thus, after modification $\mathfrak{g} = \mathfrak{so}(1, 2) \oplus \mathbb{R} = \mathfrak{u}(1, 1)$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{s}_\alpha = \mathbb{R} \oplus \mathbb{R}$ and hence M is, up to cover, conformally equivalent to $\text{Ein}^{1,1}$. □

4.2. The non-compact semi-simple case. Here we suppose that \mathfrak{m} is not contained in \mathfrak{h} , the compact semi-simple part \mathfrak{s}_c and the radical solvable part \mathfrak{r}_1 are both trivial. We will show:

Proposition 4.2. *The pseudo-Riemannian manifold M is conformally equivalent to $\text{Ein}^{3,3}$*

By corollary 3.6, δ is a negative root. In particular $\delta = -\alpha$ and $\mathfrak{g}_{-\alpha}$ is paired with \mathfrak{g}_0 . In addition $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha$ and $\mathfrak{a} \oplus \mathfrak{s}_\alpha \subset \mathfrak{h}$. We have:

Proposition 4.3. *The root space \mathfrak{s}_δ does not intersect \mathfrak{h} . In particular the metric is of type (n, n) .*

Proof. If it was the case then let $0 \neq X \in \mathfrak{s}_\delta \cap \mathfrak{h}$. We have $[[X, \mathfrak{s}_{-\delta}], X] = \mathfrak{s}_\delta$ so $\mathfrak{s}_\delta \subset \mathfrak{h}$. This contradicts the fact that \mathfrak{g}_δ is paired with \mathfrak{g}_0 . \square

Consider the bracket $[\cdot, \cdot] : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{a} \oplus \mathfrak{m}$. Denote by $\cdot \wedge \cdot : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{m}$ and $\cdot \vee \cdot : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{a}$ its projections on \mathfrak{m} and \mathfrak{a} respectively. Direct computations give us:

Lemma 4.4.

- (1) $\forall X \in \mathfrak{s}_\alpha, \forall x \in \mathfrak{s}_{-\alpha} : X \vee x = -\Theta(x) \vee \Theta(X)$.
- (2) $\forall X \in \mathfrak{s}_\alpha, \forall x \in \mathfrak{s}_{-\alpha}, \forall y \in \mathfrak{s}_{-\alpha} : [X \wedge x, y] = [X \wedge y, x] - [\Theta(x) \wedge y, \Theta(X)]$

The Cartan involution identifies \mathfrak{s}_α and $\mathfrak{s}_{-\alpha}$, which when identified with \mathbb{R}^n , \mathfrak{m} acts on them as $\mathfrak{so}(n)$. In this case, the map $\cdot \vee \cdot$ can be seen as a bilinear symmetric map from $\mathbb{R}^n \times \mathbb{R}^n$ to \mathbb{R}^n , and when composed with α gives rise to an \mathfrak{m} -invariant scalar product $\langle \cdot, \cdot \rangle_0$ on \mathbb{R}^n . Moreover, by Lemma 4.4, for every $x, X \in \mathbb{R}^n$, $X \wedge x$ is the antisymmetric endomorphism of \mathbb{R}^n defined by $X \wedge x(y) = \langle X, y \rangle_0 x - \langle x, y \rangle_0 X$.

Let $x, X \in \mathbb{R}^n$ and consider P the plane generated by x, X . Then $X \wedge x$ when seen as element of $\mathfrak{m} \cong \mathfrak{so}(n)$ is the infinitesimal generator of a one parameter group acting trivially on the orthogonal P^\perp of P with respect to the scalar product $\langle \cdot, \cdot \rangle_0$. Hence $X \wedge x \in \mathfrak{so}(P)$. More generally:

Proposition 4.5. *Let E be a linear subspace of \mathbb{R}^n and let $x \in E$. Consider \mathfrak{c} the Lie subalgebra of $\mathfrak{so}(n)$ generated by $\{X \wedge x / X \in E\}$. Then \mathfrak{c} equals the Lie algebra linearly generated by $\{X \wedge X' / X, X' \in E\}$, which in turn equals $\mathfrak{so}(E)$, the Lie algebra of orthogonal transformations preserving E and acting trivially on its orthogonal (with respect to $\langle \cdot, \cdot \rangle_0$).*

Proof. First we have $\mathfrak{c}(E) \subset E$ and hence $\mathfrak{c} \subset \mathfrak{so}(E)$. It is then sufficient to prove that \mathfrak{c} and $\mathfrak{so}(E)$ have the same dimensions. For that let $\{x, X_2, \dots, X_k\}$ be a basis of E . Note that $\{X_2 \wedge x, \dots, X_k \wedge x, [X_i \wedge x, X_j \wedge x], \text{ for } 2 \leq i < j \leq k\}$ are linearly independent. Thus $\mathfrak{c} = \mathfrak{so}(E)$. \square

For every $x \neq 0 \in \mathfrak{s}_{-\alpha}$ consider:

$$Z_x = \{X \neq 0 \in \mathfrak{s}_\alpha, \text{ such that } [X, x] \in \mathfrak{h}\}.$$

Denote $\overline{\Theta(Z_x)}$ the projection of $\Theta(Z_x)$ in $\mathfrak{g}/\mathfrak{h}$. Then:

Proposition 4.6. *The family $\left\{ \overline{\Theta(Z_x)} \setminus \{0\} \right\}_{x \in \mathfrak{s}_{-\alpha}}$ form a partition of $\mathfrak{g}/\mathfrak{h}$. More precisely:*

$$\overline{\Theta(Z_x)} \cap \overline{\Theta(Z_y)} = \{0\} \iff x \notin \Theta(Z_y) \iff y \notin \Theta(Z_x).$$

Proof. By Proposition 4.5 we have,

$$[Z_x, \Theta(Z_x)] = \mathfrak{a} \oplus \text{alg}(\{X \wedge x / X \in Z_x\}) \subset \mathfrak{h}.$$

This implies that

$$\Theta(Z_x) = \Theta(Z_y) \iff x \in \Theta(Z_y) \iff y \in \Theta(Z_x).$$

Hence, the projections $\left\{ \overline{\Theta(Z_x)} \setminus \{0\} \right\}_{x \in \mathfrak{s}_{-\alpha}}$ form a partition of $\mathfrak{g}/\mathfrak{h}$. \square

Next we prove:

Proposition 4.7. *The pseudo-Riemannian manifold M is conformally flat.*

Proof. We need to prove that the Weyl tensor W (or the Cotton tensor C if the dimension of M is 3) vanishes. Actually we will just make use of their conformal invariance property. Namely: if f is a conformal transformation of M then,

$$(5) \quad d_x f W(X, Y, Z) = W(d_x f(X), d_x f(Y), d_x f(Z))$$

We denote by \bar{x} the projection in $\mathfrak{g}/\mathfrak{h}$ of an element $x \in \mathfrak{g}$. A direct application of Equation 5 gives us:

- (1) $W(\bar{x}, \bar{y}, \bar{z}) = 0$ for every $x, y, z \in \mathfrak{s}_{-\alpha}$;
- (2) $W(\bar{x}, \bar{y}, \bar{m}) = 0$ for every $x, y \in \mathfrak{s}_{-\alpha}$ and every $m \in \mathfrak{m}$;
- (3) $[X, W(\bar{x}, \bar{m}_1, \bar{m}_2)] = W([X, \bar{x}], \bar{m}_1, \bar{m}_2)$ for every $X \in \mathfrak{s}_\alpha$, $x \in \mathfrak{s}_{-\alpha}$ and every $m_1, m_2 \in \mathfrak{m}$.

Let $x \in \mathfrak{s}_{-\alpha}$, $m_1, m_2 \in \mathfrak{m}$. Then, from Equation 5 we obtain:

$$[\Theta(x), W(\bar{x}, \bar{m}_1, \bar{m}_2)] = W([\Theta(x), \bar{x}], \bar{m}_1, \bar{m}_2) = 0.$$

In other words

$$W(\bar{x}, \bar{m}_1, \bar{m}_2) \in \overline{\Theta(Z_x)}.$$

Now let $x, y \in \mathfrak{s}_{-\alpha}$, $X \in \mathfrak{s}_\alpha$ and $m \in \mathfrak{m}$. Then again Equation 5 gives us:

$$W(\bar{x}, [X, \bar{y}], \bar{m}) + W([X, \bar{x}], \bar{y}, \bar{m}) = 0.$$

But $W(\bar{x}, [X, \bar{y}], \bar{m}) \in \overline{\Theta(Z_x)}$ and $W([X, \bar{x}], \bar{y}, \bar{m}) \in \overline{\Theta(Z_y)}$. Thus, Proposition 4.6 gives us:

- (1) If $y \notin \Theta(Z_x)$ then $W(\bar{x}, [X, \bar{y}], \bar{m}) = 0$;
- (2) In the case $y \in \Theta(Z_x)$ and $X \in Z_x$, we have $W(\bar{x}, [X, \bar{y}], \bar{m}) = 0$
- (3) If $y \in \Theta(Z_x)$ and $X \notin Z_x$. Then because $\Theta(X) \notin \Theta(Z_x)$ we have:

$$W(\bar{x}, [X, \bar{y}], \bar{m}) = W(\bar{x}, [\Theta(y), \overline{\Theta(X)}], \bar{m}) = 0.$$

So as a conclusion we get $W = 0$. □

We finish this section by proving Proposition 4.2:

Proof of Proposition 4.2. First note that if $n = 1$ then $\mathfrak{m} = 0$. Thus we assume $n \geq 2$. So far we have seen that $M = \text{SO}(1, n+1)/H$ is a conformally flat pseudo-Riemannian manifold of signature (n, n) . Since the Lie algebra \mathfrak{h} contains $\mathfrak{a} + \mathfrak{s}_\alpha$, the group H° is cocompact in $\text{SO}(1, n+1)$. Therefore $\text{SO}^\circ(1, n+1)/H^\circ$ is connected and compact, with a connected isotropy and hence simply connected. As M is connected, it covers $\text{SO}^\circ(1, n+1)/H^\circ$ and thus equals it.

On the one hand, the Einstein universe $\text{Ein}^{n,n}$ is simply connected. Thus M is identified to $\text{Ein}^{n,n}$. So $\text{SO}(1, n+1)$ acts transitively on $\text{Ein}^{n,n}$ with isotropy H . By Montgomery Theorem [19, Theorem A] any maximal compact subgroup in $\text{SO}(1, n+1)$, e.g. $K_2 = \text{SO}(n+1)$, acts transitively on $\mathbb{S}^n \times \mathbb{S}^n$ the two fold cover of $\text{Ein}^{n,n}$.

On the other hand, the conformal group of $\text{Ein}^{n,n}$ is $\text{SO}(n+1, n+1)$. A maximal compact subgroup of it is $K_1 = \text{SO}(n+1) \times \text{SO}(n+1)$. Up to conjugacy, we can assume $K_2 \subset K_1$. Therefore, $K_2 = \text{SO}(n+1)$ acts via a homomorphism $\rho = (\rho_1, \rho_2) : \text{SO}(n+1) \rightarrow \text{SO}(n+1) \times \text{SO}(n+1)$.

If $\text{SO}(n+1)$ is simple, then:

- either ρ_1 or ρ_2 is trivial and the other one is bijective, in which case $\rho(\mathrm{SO}(n+1))$ does not act transitively on $\mathbb{S}^n \times \mathbb{S}^n$,
- or both are bijective, and $\rho(\mathrm{SO}(n+1))$ is up to conjugacy in $\mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$ the diagonal $\{(g, g)/g \in \mathrm{SO}(n+1)\}$. The latter, too, does not act transitively on $\mathbb{S}^n \times \mathbb{S}^n$.

Hence $\mathrm{SO}(n+1)$ must be non-simple which implies $n = 1$ or $n = 3$. but $n = 1$ was excluded, and then remains exactly the case $n = 3$, for which M is conformally equivalent to $\mathrm{Ein}^{3,3}$.

4.3. The general case. In this section we will show Theorem 1.1 in the general case. We suppose that $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_c \oplus \mathfrak{t}_1$. Let us denote by $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{h}$ so that $\mathfrak{so}(1, n+1) \cap \mathfrak{h} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{m}_0$. A priori the subalgebra \mathfrak{m}_0 could be of any dimension in \mathfrak{m} . Nevertheless the hypothesis $\mathfrak{m} \not\subset \mathfrak{h}$ restricts drastically the possibilities. So we have:

Proposition 4.8. *The subalgebra \mathfrak{m}_0 has codimension n in \mathfrak{m} .*

Proof. If $n = 2$ then $\mathfrak{m} = \mathfrak{so}(2)$. Hence $[p, \mathfrak{s}_\alpha] = \mathfrak{a} \oplus \mathfrak{m}$ for any non null $p \in \mathfrak{s}_{-\alpha}$. Recall that $\mathfrak{s}_{-\alpha}$ preserves the metric so by applying Equation 3 for $p = v \in \mathfrak{s}_{-\alpha}$, $u \in \mathfrak{s}_\alpha$ we get $\langle \mathfrak{s}_{-\alpha}, \mathfrak{m} \rangle = 0$. Thus $\mathfrak{m} \subset \mathfrak{h}$ which contradicts our hypothesis.

Assume that $n \geq 3$ and suppose that \mathfrak{m}_0 has codimension less than $n-1$. Denote by M_0 the connected subgroup of $\mathrm{SO}(n)$ corresponding to \mathfrak{m}_0 .

If the action of M_0 on $\mathfrak{s}_{-\alpha} \cong \mathbb{R}^n$ is reducible then M_0 preserves the splitting $\mathbb{R}^d \times \mathbb{R}^{n-d}$ and hence is contained in $\mathrm{SO}(d) \times \mathrm{SO}(n-d)$. Thus M_0 has codimension bigger than the codimension of $\mathrm{SO}(d) \times \mathrm{SO}(n-d)$ which in turn achieves its minimum if $d = 1$ or $n-d = 1$ and hence $M_0 = \mathrm{SO}(n-1)$. One can identify \mathfrak{m}_0 with $\mathfrak{so}(E)$ for some $n-1$ dimensional linear subspace E of $\mathfrak{s}_{-\alpha}$. Let then $e \in \mathfrak{s}_{-\alpha}$ such that $\mathfrak{s}_{-\alpha} = \mathbb{R}e \oplus E$. Fix a non zero element $x \in \Theta(E)$, we have $\langle \mathrm{ad}_x(e), X \rangle + \langle e, \mathrm{ad}_x X \rangle = 0$ for every $X \in \mathfrak{s}_{-\alpha}$ and so in particular $\langle e, \mathrm{ad}_x e \rangle = 0$. In addition by Proposition 4.5, $[E, \Theta(E)] = \mathfrak{a} \oplus \mathfrak{m}_0 \subset \mathfrak{h}$ thus $\langle \mathrm{ad}_x e, X \rangle = 0$ for every $X \in E$ and hence $\mathrm{ad}_x e$ is orthogonal to $\mathfrak{s}_{-\alpha}$. This implies that $x \wedge e \in \mathfrak{h} \cap \mathfrak{m} = \mathfrak{m}_0 = \mathfrak{so}(E)$ which contradicts the fact that $x \wedge e$ is the infinitesimal rotation of the plane $\mathbb{R}e \oplus \mathbb{R}x$.

The last case to consider is when M_0 acts irreducibly. Let $m \in \mathfrak{m}_0$, $X \in \mathfrak{s}_{-\alpha}$ and $y \in \mathfrak{s}_c \oplus \mathfrak{t}_1$ then $\langle \mathrm{ad}_m(X), y \rangle + \langle X, \mathrm{ad}_m y \rangle = 0$. But $\mathrm{ad}_m y = 0$ and hence $\mathfrak{s}_c \oplus \mathfrak{t}_1$ is orthogonal to $[\mathfrak{m}_0, \mathfrak{s}_{-\alpha}]$ which is equal to $\mathfrak{s}_{-\alpha}$ by irreducibility. Thus $\mathfrak{s}_c \oplus \mathfrak{t}_1 \subset \mathfrak{h}$ and we are in the non-compact semi-simple case. Therefore $n = 3$ and $\mathfrak{m} \cong \mathfrak{so}(3)$. Non trivial Sub-algebras of $\mathfrak{so}(3)$ have dimension one and are reducible. So the only left possibility is $\mathfrak{m}_0 = \mathfrak{m} \cong \mathfrak{so}(3)$ which show that $\mathfrak{m} \subset \mathfrak{h}$ and this is a contradiction. \square

End of Proof of Theorem 1.1. By Proposition 4.8, \mathfrak{m}_0 is of codimension n in \mathfrak{m} . But $\mathfrak{s}_{-\alpha}$ is paired with $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c \oplus (\mathfrak{t}_1 \cap \mathfrak{t}_0)$. Thus $\mathfrak{s}_c \oplus (\mathfrak{t}_1 \cap \mathfrak{t}_0) \subset \mathfrak{h}$ and we are also in the non-compact semi-simple case. Therefore $n = 3$ and M is conformally equivalent to $\mathrm{Ein}^{3,3}$.
 \square

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