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**PERSISTENCE, EXTINCTION AND SPREADING PROPERTIES
OF NON-COOPERATIVE FISHER–KPP SYSTEMS IN
SPACE-TIME PERIODIC MEDIA**

LÉO GIRARDIN

ABSTRACT. This paper is concerned with asymptotic persistence, extinction and spreading properties for non-cooperative Fisher–KPP systems with space-time periodic coefficients. In a preceding paper, a family of generalized principal eigenvalues associated with an appropriate linear problem was studied. Here, a relation with semilinear systems is established. When the maximal generalized principal eigenvalue is negative, all solutions to the Cauchy problem become locally uniformly positive in long-time. In contrast with the scalar case, multiple space-time periodic uniformly positive entire solutions might coexist. When another, possibly smaller, generalized principal eigenvalue is nonnegative, then on the contrary all solutions to the Cauchy problem vanish uniformly and the zero solution is the unique space-time periodic nonnegative entire solution. When the two generalized principal eigenvalues differ and zero is in between, the long-time behavior depends on the decay at infinity of the initial data. Finally, with similar arguments, a Freidlin–Gärtner-type formula for the asymptotic spreading speed of solutions with compactly supported initial data is established.

1. INTRODUCTION

This paper is concerned with reaction–diffusion systems of the form

$$(KPP) \quad \text{diag}(\mathcal{P})\mathbf{u} = \mathbf{L}\mathbf{u} - \mathbf{C}\mathbf{u} \circ \mathbf{u},$$

where $\mathbf{u} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a vector-valued function of size $N \in \mathbb{N}^*$, with a time variable $t \in \mathbb{R}$ and a space variable $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ being the dimension of the underlying space, where each operator of the family $\mathcal{P} = (\mathcal{P}_i)_{i \in [N]}$, where $[N] = \mathbb{N} \cap [1, N]$, has the form

$$\mathcal{P}_i : u \mapsto \partial_t u - \nabla \cdot (A_i \nabla u) + q_i \cdot \nabla u,$$

with $A_i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ and $q_i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ periodic functions of (t, x) , respectively square matrix-valued and vector-valued, where $\mathbf{L}, \mathbf{C} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{N \times N}$ are square matrix-valued periodic functions of (t, x) , and where \circ denotes the Hadamard product between two vectors in \mathbb{R}^N .

We will study both nonnegative entire solutions of KPP and solutions of the Cauchy problem supplemented with bounded nonnegative initial conditions,

$$(IC) \quad \mathbf{u}(0, x) = \mathbf{u}_{\text{ini}}(x) \quad \text{for all } x \in \mathbb{R}^n, \quad \mathbf{u}_{\text{ini}} \geq \mathbf{0}, \quad \max_{i \in [N]} \sup_{x \in \mathbb{R}^n} u_{\text{ini}, i}(x) < +\infty.$$

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The standing assumptions on \mathcal{P} , \mathbf{L} and \mathbf{C} are the following.

(A₁) The family $(A_i)_{i \in [N]}$ is *uniformly elliptic*:

$$0 < \min_{i \in [N]} \min_{y \in \mathbb{S}^{n-1}} \min_{(t,x) \in \mathbb{R} \times \mathbb{R}^n} (y \cdot A_i(t,x)y).$$

(A₂) The matrix $\underline{\mathbf{L}} \in \mathbb{R}^{N \times N}$, whose entries are

$$\underline{l}_{i,j} = \min_{(t,x) \in \mathbb{R} \times \mathbb{R}^n} l_{i,j}(t,x) \quad \text{for all } (i,j) \in [N]^2,$$

is *essentially nonnegative*: its off-diagonal entries are nonnegative.

(A₃) The matrix $\bar{\mathbf{L}} \in \mathbb{R}^{N \times N}$, whose entries are

$$\bar{l}_{i,j} = \max_{(t,x) \in \mathbb{R} \times \mathbb{R}^n} l_{i,j}(t,x) \quad \text{for all } (i,j) \in [N]^2,$$

is *irreducible*: it does not have a stable subspace of the form $\text{span}(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k})$, where $k \in [N-1]$, $i_1, \dots, i_k \in [N]$ and $\mathbf{e}_i = (\delta_{ij})_{j \in [N]}$. By convention, $[0] = \emptyset$ and 1×1 matrices are irreducible, even if zero.

(A₄) The matrix $\underline{\mathbf{C}} \in \mathbb{R}^{N \times N}$, whose entries are

$$\underline{c}_{i,j} = \min_{(t,x) \in \mathbb{R} \times \mathbb{R}^n} c_{i,j}(t,x) \quad \text{for all } (i,j) \in [N]^2,$$

is *positive*: its entries are positive.

(A₅) There exists $\delta \in (0,1)$ such that $\mathbf{L}, \mathbf{C} \in \mathcal{C}_{\text{per}}^{\delta/2, \delta}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{N \times N})$ and, for each $i \in [N]$, $A_i \in \mathcal{C}_{\text{per}}^{\delta/2, 1+\delta}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{n \times n})$ and $q_i \in \mathcal{C}_{\text{per}}^{\delta/2, \delta}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Moreover, $A_i = A_i^T$ for each $i \in [N]$.

The precise definition of the functional spaces appearing in (A₅) will be clarified below, if not clear already. We point out that, as usual in such a smooth and generic framework, the symmetry of the diffusion matrices is actually given for free. We also point out that the competition term $\mathbf{C}\mathbf{u}$ might be changed into a more general term, as in [9]. With appropriate assumptions, this generalization would only require minor adaptations. However, from an application point of view, we mostly have in mind the case described above, and therefore, for the sake of readability, the setting is deliberately restricted to this case.

For brevity, we will denote from now on

$$\mathcal{Q} = \text{diag}(\mathcal{P}) - \mathbf{L}$$

the linear operator coming from the linearization of (KPP) at $\mathbf{u} = \mathbf{0}$. This linear operator is cooperative, and this will be a key property in the forthcoming analysis of the non-cooperative semilinear system (KPP).

1.1. Organization of the paper. The remainder of Section 1 is devoted to a detailed introduction. Section 2 contains the proofs.

1.2. Notations. Generally speaking, notations are chosen consistently with our previous paper on space-time homogeneous coefficients [9] and with our paper with I. Mazari on the principal spectral analysis of \mathcal{Q} [13].

In the whole paper, \mathbb{N} is the set of nonnegative integers, and therefore contains 0.

We fix once and for all $n+1$ positive numbers $T, L_1, \dots, L_n \in \mathbb{R}_+^*$. For the sake of brevity, we use the notations $L = (L_1, \dots, L_n)$, $(0, L) = (0, L_1) \times \dots \times (0, L_n)$ and $[[0, L]] = \prod_{\alpha=1}^n L_\alpha$. Unless specified otherwise, time and space periodicities refer to, respectively, T -periodicity with respect to t and L_α -periodicity with respect to

x_α for each $\alpha \in [n]$ (or L -periodicity with respect to x for short). The space-time periodicity cell $(0, T) \times (0, L)$ is denoted Ω_{per} and its volume is $T|[0, L]|$.

Vectors in \mathbb{R}^N and matrices in $\mathbb{R}^{N \times N}$ are denoted in bold font. Functional operators are denoted in calligraphic typeface (bold if they act on functions valued in \mathbb{R}^N). Functional spaces, *e.g.* $\mathcal{W}^{1, \infty}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$, are also denoted in calligraphic typeface. A functional space \mathcal{X} denoted with a subscript \mathcal{X}_{per} , $\mathcal{X}_{t\text{-per}}$ or $\mathcal{X}_{x\text{-per}}$ is restricted to functions that are space-time periodic, time periodic or space periodic respectively.

For clarity, Hölder spaces of functions with $k \in \mathbb{N} \cup \{0\}$ derivatives that are all Hölder-continuous with exponent $\alpha \in (0, 1)$ are denoted $\mathcal{C}^{k+\alpha}$; when the domain is $\mathbb{R} \times \mathbb{R}^n$, it should be unambiguously understood that $\mathcal{C}^{k+\alpha, k'+\alpha'}$ denotes the set of functions that have k α -Hölder-continuous derivatives in time and k' α' -Hölder-continuous derivatives in space.

For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, $\mathbf{u} \leq \mathbf{v}$ means $u_i \leq v_i$ for all $i \in [N]$, $\mathbf{u} < \mathbf{v}$ means $\mathbf{u} \leq \mathbf{v}$ together with $\mathbf{u} \neq \mathbf{v}$ and $\mathbf{u} \ll \mathbf{v}$ means $u_i < v_i$ for all $i \in [N]$. If $\mathbf{u} \geq \mathbf{0}$, we refer to \mathbf{u} as *nonnegative*; if $\mathbf{u} > \mathbf{0}$, as *nonnegative nonzero*; if $\mathbf{u} \gg \mathbf{0}$, as *positive*. The sets of all nonnegative, nonnegative nonzero, positive vectors are respectively denoted $[\mathbf{0}, \infty)$, $[\mathbf{0}, \infty) \setminus \{\mathbf{0}\}$ and $(\mathbf{0}, \infty)$. The vector whose entries are all equal to 1 is denoted $\mathbf{1}$ and this never refers to an indicator function. Similar notations and terminologies might be used in other dimensions and for matrices. The identity matrix is denoted \mathbf{I} .

Similarly, a function can be nonnegative, nonnegative nonzero, positive. For clarity, a positive function is a function with only positive values.

To avoid confusion between operations in the state space \mathbb{R}^N and operations in the spatial domain \mathbb{R}^n , Latin indexes i, j, k are assigned to vectors and matrices of size N whereas Greek indexes α, β, γ are assigned to vectors and matrices of size n . We use mostly subscripts to avoid confusion with algebraic powers, but when both Latin and Greek indexes are involved, we move the Latin ones to a superscript position, *e.g.* $A_{\alpha, \beta}^i(t, x)$. We denote scalar products in \mathbb{R}^N with the transpose operator, $\mathbf{u}^T \mathbf{v} = \sum_{i=1}^N u_i v_i$, and scalar products in \mathbb{R}^n with a dot, $x \cdot y = \sum_{\alpha=1}^n x_\alpha y_\alpha$.

For any vector $\mathbf{u} \in \mathbb{R}^N$, $\text{diag}(\mathbf{u})$, $\text{diag}(u_i)_{i \in [N]}$ or $\text{diag}(u_i)$ for short refer to the diagonal matrix in $\mathbb{R}^{N \times N}$ whose i -th diagonal entry is u_i . These notations can also be used if \mathbf{u} is a function valued in \mathbb{R}^N .

Finite dimensional Euclidean norms are denoted $|\cdot|$ whereas the notation $\|\cdot\|$ is reserved for norms in functional spaces.

The notation \circ is reserved in the paper for the Hadamard product (component-wise product of vectors or matrices) and never refers to the composition of functions.

1.3. Motivations. Extensive discussions on population dynamics models leading to systems of the form (KPP) can be found in [9, 13]. Therefore we only suggest briefly one example of application.

In [6], the following system was studied:

$$(1) \quad \begin{cases} \partial_t n_e = D_e \partial_{xx} n_e + r_e n_e (1 - m_{ee} n_e - m_{ed} n_d) + \mu_d n_d - \mu_e n_e, \\ \partial_t n_d = D_d \partial_{xx} n_d + r_d n_d (1 - m_{de} n_e - m_{dd} n_d) + \mu_e n_e - \mu_d n_d. \end{cases}$$

This system was conceived as an eco-evolutionary model for spatio-temporal dynamics of a population with two phenotypes, or morphs. Each morph $i \in \{e, d\}$

has a dispersal rate D_i , a growth rate r_i , is subjected to Lotka–Volterra dynamics with intermorph and intramorph competition rates m_{ij} , and mutates into the other morph at rate μ_i . The establisher morph e is specialized in growth, *i.e.* $r_e > r_d$, whereas the disperser morph d is specialized in dispersal, *i.e.* $D_d > D_e$. The role of each morph during a population invasion was then investigated, heuristically in [6] and subsequently rigorously in [15].

The form of the system above is clearly compatible with (KPP). In [6, 15], the environment was assumed to be homogeneous, and coefficients were therefore spatiotemporal constants. Advection terms can be added to model, *e.g.*, directional wind. Temporal periodicity of the coefficients can be added to model, depending on the timescale, seasonality or nycthemeral rhythms. Spatial periodicity of the coefficients can be added to model biological invasions in periodic landscapes, *e.g.*, vineyards. More generally, spatiotemporal periodicity is a way of introducing environmental heterogeneity while keeping strong mathematical tractability.

1.4. Results. Before stating the results, we need to introduce a family of generalized principal eigenvalues that was previously studied in [13]. The family $(\lambda_{1,z})_{z \in \mathbb{R}^n}$ is defined by:

$$(2) \quad \lambda_{1,z} = \lambda_{1,\text{per}}(e_{-z} \mathcal{Q} e_z),$$

where $\lambda_{1,\text{per}}$ denotes the periodic principal eigenvalue classically given by the Krein–Rutman theorem and where $e_{\pm z} : x \in \mathbb{R}^n \mapsto e^{\pm z \cdot x}$. The operator $e_{-z} \mathcal{Q} e_z$ can be alternatively written as:

$$(3) \quad e_{-z} \mathcal{Q} e_z = \mathcal{Q} - \text{diag}((A_i + A_i^T)z \cdot \nabla + z \cdot A_i z + \nabla \cdot (A_i z) - q_i \cdot z).$$

For any $z \in \mathbb{R}^n$, there exists a unique, up to multiplication by a positive constant, positive periodic principal eigenfunction $\mathbf{u}_z \in \mathcal{C}_{\text{per}}^{1,2}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \infty))$ satisfying $\mathcal{Q}(e_z \mathbf{u}_z) = \lambda_{1,z} e_z \mathbf{u}_z$.

Recall that $z \in \mathbb{R}^n \mapsto \lambda_{1,z}$ is strictly concave, coercive, with one global maximum. We denote:

$$(4) \quad \lambda_1 = \max_{z \in \mathbb{R}^n} \lambda_{1,z} \quad \text{and} \quad \lambda'_1 = \lambda_{1,0}.$$

The equality $\lambda_1 = \lambda'_1$ can be true or false. As was proved in [13], λ_1 and λ'_1 can be alternatively defined as:

$$(5) \quad \lambda_1 = \sup \left\{ \lambda \in \mathbb{R} \mid \exists \mathbf{u} \in \mathcal{C}_{t\text{-per}}^{1,2}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \infty)) \quad \mathcal{Q} \mathbf{u} \geq \lambda \mathbf{u} \right\},$$

$$(6) \quad \lambda'_1 = \inf \left\{ \lambda \in \mathbb{R} \mid \exists \mathbf{u} \in \mathcal{W}^{1,\infty} \cap \mathcal{C}_{t\text{-per}}^{1,2}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \infty)) \quad \mathcal{Q} \mathbf{u} \leq \lambda \mathbf{u} \right\},$$

We are now in a position to state our results.

The first result states a condition for the uniform extinction of any solution of the Cauchy problem.

Theorem 1.1. *Assume $\lambda'_1 \geq 0$.*

Then all solutions of the Cauchy problem (KPP)–(IC) satisfy

$$(7) \quad \lim_{t \rightarrow +\infty} \max_{i \in [N]} \sup_{x \in \mathbb{R}^n} u_i(t, x) = 0.$$

As an immediate corollary, when $\lambda'_1 \geq 0$, $\mathbf{0}$ is the only nonnegative bounded entire solution of (KPP). This is no longer true when $\lambda'_1 < 0$, as stated by the following theorem.

Theorem 1.2. *Assume $\lambda'_1 < 0$.*

Then there exists a uniformly positive space-time periodic entire solution \mathbf{u}^ of (KPP).*

Actually, when $\lambda'_1 < 0$ and $z \in \mathbb{R}^n \mapsto \lambda_{1,z}$ is not maximal at $z = 0$, all solutions of the Cauchy problem starting from a sufficiently large initial condition persist locally uniformly, as stated in the next result.

Theorem 1.3. *Assume $\lambda'_1 < 0$ and the existence of $z \in \mathbb{R}^n$ such that:*

- (1) $\lambda_{1,z} < 0$;
- (2) $\zeta \in (0, 2) \mapsto \lambda_{1,\zeta z}$ is increasing in a neighborhood of 1;
- (3) there exists $C > 0$, $B \in \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$ such that $z \cdot x \leq B$, $\min_{i \in [N]} u_{\text{ini},i}(x) \geq C^{-1} e_z(x)$.

Then the solution \mathbf{u} of the Cauchy problem (KPP)–(IC) satisfies

$$(8) \quad \liminf_{t \rightarrow +\infty} \min_{i \in [N]} \inf_{|x| \leq R} u_i(t, x) > 0 \quad \text{for all } R > 0.$$

As stated by the following result, the stronger condition $\lambda_1 < 0$ is sufficient to ensure the locally uniform persistence of any nonzero solution of the Cauchy problem. This type of property is usually referred to as a *hair-trigger effect*. Note that if $\lambda'_1 < 0$ and $z \in \mathbb{R}^n \mapsto \lambda_{1,z}$ is maximal at $z = 0$, then obviously $\lambda_1 < 0$, so that in all cases $\lambda'_1 < 0$ implies the persistence of at least some solutions.

Theorem 1.4. *Assume $\lambda_1 < 0$.*

Then all solutions of the Cauchy problem (KPP)–(IC) satisfy (8).

The next result shows that when $\lambda_1 \geq 0$, solutions of the Cauchy problem starting from a sufficiently small initial condition go extinct locally uniformly. This violation of the hair-trigger effect is especially interesting in the intermediate case $\lambda'_1 < 0 \leq \lambda_1$.

Theorem 1.5. *Assume $\lambda_1 \geq 0$ and the existence of $z \in \mathbb{R}^n$ such that:*

- (1) $\lambda_{1,z} \geq 0$;
- (2) there exists $C > 0$ such that, for all $x \in \mathbb{R}^n$, $\max_{i \in [N]} u_{\text{ini},i}(x) \leq C e_z(x)$.

Then the solution \mathbf{u} of the Cauchy problem (KPP)–(IC) satisfies

$$(9) \quad \lim_{t \rightarrow +\infty} \max_{i \in [N]} \sup_{|x| \leq R} u_i(t, x) = 0 \quad \text{for all } R > 0.$$

This collection of results indicates in particular that solutions evolving from compactly supported initial data persist when $\lambda_1 < 0$ and go extinct at least locally uniformly when $\lambda_1 \geq 0$. It becomes then natural to investigate spreading properties in the case $\lambda_1 < 0$. The last result provides a Freidlin–Gärtner-type formula [7] for the asymptotic spreading speed of such solutions.

We introduce, for any direction $e \in \mathbb{S}^{n-1}$ and any decay rate $\mu > 0$:

$$(10) \quad c_e^\mu = \frac{\lambda_{1,-\mu e}}{-\mu}, \quad c_e^* = \min_{\mu > 0} c_e^\mu, \quad c_e^{\text{FG}} = \min_{\substack{e' \in \mathbb{S}^{n-1} \\ e \cdot e' > 0}} \frac{c_{e'}^*}{e \cdot e'}.$$

The fact that the minima involved in the definition of c_e^* and c_e^{FG} are well-defined is classical in KPP-type problems and will be verified later on.

Theorem 1.6. *Assume $\lambda_1 < 0$ and \mathbf{u}_{ini} is nonzero and compactly supported.*

Then the solution \mathbf{u} of the Cauchy problem (KPP)–(IC) spreads in the direction $e \in \mathbb{S}^{n-1}$ at speed c_e^{FG} , namely

$$(11) \quad \liminf_{t \rightarrow +\infty} \min_{i \in [N]} \inf_{|x| \leq R} u_i(t, x + cte) > 0 \quad \text{for all } R > 0 \text{ and } c \in (0, c_e^{\text{FG}}),$$

$$(12) \quad \lim_{t \rightarrow +\infty} \max_{i \in [N]} \sup_{|x| \leq R} u_i(t, x + cte) = 0 \quad \text{for all } R > 0 \text{ and } c > c_e^{\text{FG}}.$$

1.5. Comments. The first two results show that the sign of λ'_1 is a sharp criterion for the existence of nonnegative nonzero entire solutions. However, when studying the long-time behavior of the Cauchy problem, the knowledge of the sign of λ_1 is also needed, and moreover in the case $\lambda'_1 < 0 \leq \lambda_1$ the outcome depends also on the initial condition.

The case $\lambda_1 = 0 > \lambda'_1$ was stated as an open problem in the scalar case [16], but is actually within reach with the same methods. Since our paper covers the scalar case as the particular case $N = 1$, Theorem 1.5 solves the question raised by [16].

The strict inequality $\lambda'_1 < \lambda_1$ can be induced by, *e.g.*, nonzero advection rates q_i or spatial heterogeneities combined with asymmetries in the matrix \mathbf{L} [13].

The sharpness of the conditions on the size of (the exponential decay of) the initial condition in Theorems 1.3 and 1.5 can be discussed as in [16, p. 1296]. We only mention that uniformly positive initial conditions will always satisfy the condition of Theorem 1.3 while compactly supported initial conditions will always satisfy that of Theorem 1.5.

Heuristically, the results can be summarized as follows. Since we restrict ourselves *a priori* to bounded initial conditions, that is, to bounded perturbations of $\mathbf{0}$, the family $(\lambda_{1,z})$ gives stability criteria depending on the exponential decay z at spatial infinity. When $\mathbf{0}$ is unstable with respect to any exponential decay, that is, $\lambda_1 < 0$, then it is actually unstable with respect to compactly supported initial conditions and the hair-trigger effect holds. But when $\lambda_1 > 0 > \lambda'_1$, then some exponential decays are too strong and make $\mathbf{0}$ stable: the hair-trigger effect does not hold. For instance, for the scalar operator $\mathcal{Q} = \partial_t - \partial_{xx} + \partial_x - 1/8$, the generalized principal eigenvalues satisfy $\lambda_{1,z} = z(1 - z) - 1/8$, and values of z satisfying the monotonicity condition stated in Theorem 1.3 are $z \in (0, 1/2)$. In this interval, the sign change occurs at $z^* = (1 - \sqrt{2}/2)/2$. Applying Theorem 1.3 with $z = z^* - \varepsilon$ and Theorem 1.5 with $z = z^* + \varepsilon$, we find that the zero steady state is, with respect to perturbations of the form $C \min(e_z, e_{z'})$ with $z' \leq 0 \leq z$:

- unstable if $z \in [0, z^*]$;
- stable if $z > z^*$.

Interestingly, this confirms the crucial role of the monotonicity condition of Theorem 1.3: the stability of the zero steady state is fully determined by an arbitrarily small open neighborhood of z^* , and in particular the other sign change of $z \mapsto \lambda_{1,z}$ at $z = (1 + \sqrt{2}/2)/2$ brings no additional stability information. Intuitively, the transport of the solution at speed 1 towards the right, encoded in the advection term $+\partial_x$, washes away all solutions having an initial leftward exponential decay $z > (1 - \sqrt{2}/2)/2$, that is, solutions whose initial conditions are too thin-tailed towards the left.

The Freidlin–Gärtner-type formula of Theorem 1.6 was established in the space-time periodic scalar case ($N = 1$) in [3, Theorem 1.13]. For space-time periodic

cooperative systems satisfying appropriate assumptions, it was established in [5]. In our case, the arguments of proof are very similar to those used to prove the other theorems; roughly speaking, such spreading results are still stability properties and can be understood as persistence/extinction results in moving frames. With similar methods, the spreading speed of solutions evolving from exponentially decaying initial data could be investigated. Here we focus on the compactly supported case, which is more relevant biologically and usually provides in KPP-type problems sub-estimates for more general invasions – it will be clear from the proof that this is again the case here, despite the default of comparison principle. On the contrary, the construction of entire solutions that describe the invasion of open space by positive population densities at constant speed, namely *pulsating traveling waves*, is a different problem that requires other methods. This problem will be investigated in a future sequel. There, we will prove in particular that c_e^* is the minimal wave speed of planar pulsating traveling waves in the direction e , as is standard in KPP-type problems.

Let us mention that, by drawing inspiration from [13] and [17], we could combine elementarily results on the dependence of the generalized principal eigenvalues on the coefficients and the Freidlin–Gärtner formula to obtain dependence results for the spreading speed. We point out in particular that the spreading speed is in general not monotonic with respect to the diffusivity amplitude [17], but is monotonic with respect to the matrix entries $l_{i,j}$ [13]. We also point out that space homogeneity and time homogeneity of the coefficients, supplemented with appropriate specific conditions, lead to simplifications of the formula or to upper or lower estimates [13].

The results we manage to prove in the present paper are analogous to their scalar counterparts [3, 16] but their proofs are carefully improved so that they only exploit the comparison principle of the linearized system $\mathcal{Q} = \mathbf{0}$. This is the main difficulty and novelty of this work. It is actually known that not all results of the scalar case can be generalized in this way; in particular, Liouville-type results on the uniformly positive entire solution are in general false even with constant coefficients [4, 8, 11, 12, 15]. In this regard, our intent is precisely to show what can be generalized to non-cooperative Fisher–KPP systems, and what cannot.

2. PROOFS

Below we prove our results. In order to ease the reading, we first prove the extinction results (Theorems 1.1 and 1.5), that use super-solutions, then the persistence results (Theorems 1.2, 1.4 and 1.3), that use sub-solutions. We conclude with the proof of the Freidlin–Gärtner-type formula (Theorem 1.6).

2.1. Preliminary: global boundedness estimates and absorbing set. In this preparatory section, we establish that a solution of (KPP)–(IC) satisfies a global boundedness estimate that depends only on the initial values, and that becomes uniform with respect to the initial values in long time. This is a direct adaptation of the proof of [9, Theorem 1.2], shortened by the stronger assumption $((A_4))$.

Proposition 2.1. *There exists a constant $K > 0$ such that, for any solution \mathbf{u} of (KPP)–(IC),*

$$(13) \quad \mathbf{u} \leq \left(K + \sup_{x \in \mathbb{R}^n} \max_{i \in [N]} u_{\text{ini}}(x) \right) \mathbf{1} \quad \text{in } [0, +\infty) \times \mathbb{R}^n,$$

$$(14) \quad \limsup_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}^n} \max_{i \in [N]} u_i(t, x) \leq K.$$

Proof. By assumptions $((A_5))$ and $((A_4))$, there exist constants $r, K > 0$ such that, for any $\mathbf{u} \geq \mathbf{0}$,

$$\mathbf{L}\mathbf{u} - \mathbf{C}\mathbf{u} \circ \mathbf{u} \leq r(\mathbf{1}^T \mathbf{u})(K\mathbf{1} - \mathbf{u}).$$

In particular, solutions \mathbf{u} of (KPP)–(IC) satisfy $\text{diag}(\mathcal{P})\mathbf{u} \leq r(\mathbf{1}^T \mathbf{u})(K\mathbf{1} - \mathbf{u})$, or else:

$$\mathcal{P}_i u_i \leq r(K - u_i) \sum_{j=1}^N u_j \quad \text{for each } i \in [N].$$

Whenever $u_i \geq K$, $\mathcal{P}_i u_i \leq r u_i (K - u_i)$. In particular, $\underline{u}_i = \max(u_i, K)$ is a sub-solution of the equation $\mathcal{P}_i u = r u (K - u)$. Moreover it satisfies $\underline{u}_i(0, \cdot) \leq M$, where:

$$M = \sup_{x \in \mathbb{R}^n} \max_{i \in [N]} u_{\text{ini}}(x).$$

Now, still for the equation $\mathcal{P}_i u = r u (K - u)$, consider the space-homogeneous (super-)solution \bar{u}_i satisfying

$$\begin{cases} \mathcal{P}_i \bar{u}_i = r u_i (K - \bar{u}_i) & \text{in } (0, +\infty) \times \mathbb{R}^n, \\ \bar{u}_i(0, \cdot) = K + M & \text{in } \mathbb{R}^n. \end{cases}$$

By virtue of the comparison principle, $\underline{u}_i \leq \bar{u}_i$ globally in $[0, +\infty) \times \mathbb{R}^n$ for each $i \in [N]$. Since $\bar{u}_i \leq K + M$ and $\lim_{t \rightarrow +\infty} \bar{u}_i = K$, this ends the proof. \square

As a consequence, we obtain the following corollary on entire solutions. In particular, this applies to space-time periodic solutions.

Corollary 2.2. *All nonnegative globally bounded entire solutions \mathbf{u} of (KPP) satisfy*

$$(15) \quad \mathbf{u} \leq K\mathbf{1} \quad \text{in } \mathbb{R} \times \mathbb{R}^n.$$

2.2. Global extinction (Theorem 1.1). It is convenient to distinguish two cases: $\lambda'_1 > 0$ on one hand, $\lambda'_1 = 0$ on the other hand. In the first case, the extinction is due to the linear part of the operator (and occurs therefore at an exponential rate). In the second case, however, the extinction is due to the signed quadratic part of the operator (it is conjectured to occur at an algebraic rate, cf. Remark 2.1). The following proofs are straightforward adaptations of [9, 10].

Proof in the case $\lambda'_1 > 0$. The idea is very classical and consists in constructing a super-solution of the form

$$\bar{\mathbf{u}} : (t, x) \mapsto M e^{-\lambda'_1 t} \mathbf{u}_0(t, x),$$

where \mathbf{u}_0 is a positive generalized principal eigenfunction associated with λ'_1 of fixed amplitude and $M > 0$ is a constant so large that $M\mathbf{u}_0 \gg \mathbf{u}_{\text{ini}}$. Then $\bar{\mathbf{u}} - \mathbf{u}$ satisfies

$$\mathcal{Q}(\bar{\mathbf{u}} - \mathbf{u}) = (\mathbf{C}\mathbf{u}) \circ \mathbf{u} \geq \mathbf{0} \quad \text{in } (0, +\infty) \times \mathbb{R}^n,$$

so that by the comparison principle $\bar{\mathbf{u}} \geq \mathbf{u}$ globally in $(0, +\infty) \times \mathbb{R}^n$, and consequently \mathbf{u} vanishes asymptotically in time, uniformly in space, exponentially fast. \square

Proof in the case $\lambda_1' = 0$. This time we use a family of super-solutions of the form

$$\bar{\mathbf{u}}_T : (t, x) \in (T, +\infty) \times \mathbb{R}^n \mapsto M_T \mathbf{u}_0(t, x).$$

Assuming that $M_T > 0$ is defined optimally for each $T \geq 0$, namely

$$M_T = \sup_{x \in \mathbb{R}^n} \max_{i \in [N]} \frac{u_i(T, x)}{u_{0,i}(T, x)},$$

the goal is to prove that M_T decreases to 0 as $T \rightarrow +\infty$.

By the comparison principle, $\bar{\mathbf{u}}_T \geq \mathbf{u}$ globally in $[T, +\infty) \times \mathbb{R}^n$. Therefore, for any $T' > T$, $M_{T'} \leq M_T$, simply by definition of $M_{T'}$. Hence the family $(M_T)_{T \geq 0}$ is nonincreasing.

Of course, if $\mathbf{u}_{\text{ini}} = \mathbf{0}$, then $M_0 = 0$, the family $(M_T)_{T \geq 0}$ is stationary and $\mathbf{u} = \mathbf{0}$, which ends the proof. From now on we discard this case and therefore assume $\mathbf{u}_{\text{ini}} \neq \mathbf{0}$. Under such an assumption, let us prove that $(M_T)_{T \geq 0}$ is actually decreasing.

Assume by contradiction that there exist $0 \leq T < T'$ such that $M_T \leq M_{T'}$. Then, by large monotonicity, $t \mapsto M_t$ is constant in $[T, T']$. Below, we will begin by discarding the possibility that the optimum defining M_T is attained pointwise, and subsequently we will discard the possibility that it is attained asymptotically. We recall the basis for our application of the strong comparison principle: $\mathcal{Q}(M_T \mathbf{u}_0 - \mathbf{u}) \geq \mathbf{0}$.

If there exists $(t^*, x^*, i^*) \in (T, T') \times \mathbb{R}^n \times [N]$ such that $M_T u_{0,i^*}(t^*, x^*) = u_{i^*}(t^*, x^*)$, then by virtue of the strong comparison principle, $\mathbf{u} = M_T \mathbf{u}_0$ globally in $[T, t^*] \times \mathbb{R}^n$, which directly contradicts the fact $\mathcal{Q}\mathbf{u} = \mathbf{C}\mathbf{u} \circ \mathbf{u} \gg \mathbf{0}$.

Consequently, for all $(t, x, i) \in (T, T') \times \mathbb{R}^n \times [N]$, $M_T u_{0,i}(t, x) > u_i(t, x)$ and the equality can only be attained asymptotically at $|x| = \infty$.

Fix temporarily $t_0 \in (T, T')$ and let $i \in [N]$ such that

$$M_T = \sup_{x \in \mathbb{R}^n} \frac{u_i(t_0, x)}{u_{0,i}(t_0, x)}.$$

There exists a sequence $(x_k) \in (\mathbb{R}^n)^\mathbb{N}$ such that $|x_k| \rightarrow +\infty$ and

$$\frac{u_i(t_0, x_k)}{u_{0,i}(t_0, x_k)} \rightarrow M_T \quad \text{as } k \rightarrow +\infty.$$

We intend to use the spatial periodicity and therefore we define, for each $k \in \mathbb{N}$, $y_k \in [0, L]$ and $z_k \in \prod_{\alpha \in [n]} L_\alpha \mathbb{Z}$ such that $x_k = y_k + z_k$. Up to extraction, the sequence (y_k) converges to a limit $y_\infty \in [0, L]$. Then, by classical parabolic estimates [14] and up to a diagonal extraction, the sequence (\mathbf{u}_k) defined by

$$\mathbf{u}^k : (t, x) \mapsto \mathbf{u}(t, x + z_k) \quad \text{for each } k \in \mathbb{N}$$

converges locally uniformly to a solution \mathbf{u}^∞ of (KPP). The solution \mathbf{u}^∞ satisfies moreover $M_T u_{0,i}(t_0, y_\infty) = u_i^\infty(t_0, y_\infty)$ and also $\mathbf{u}^\infty \leq M_T \mathbf{u}_0$ globally in $(T, T') \times \mathbb{R}^n$. Repeating the previous strong comparison argument, we deduce again a contradiction.

Therefore $(M_T)_{T \geq 0}$ is decreasing and converges to a limit $M_\infty \geq 0$. Assume by contradiction $M_\infty > 0$. Then there exist $i \in [N]$ and a sequence (t_k, x_k) such that

$$t_k \rightarrow +\infty, \quad |x_k| \rightarrow +\infty, \quad \frac{u_i(t_k, x_k)}{u_{0,i}(t_k, x_k)} \rightarrow M_\infty \quad \text{as } k \rightarrow +\infty.$$

Arguing exactly as before after passing to the limit $k \rightarrow +\infty$ locally uniformly, a new contradiction arises. In the end, $M_\infty = 0$, and by definition of M_T , \mathbf{u} vanishes asymptotically in time, uniformly in space. \square

Remark 2.1. We are unable to prove the algebraic decay in the case $\lambda'_1 = 0$. It is strongly conjectured in view of the quadratic nonlinearity (by analogy with the ODE $u' = -u^2$), but it remains as an open question. Some technical obstacles are discussed in [9, Section 4.1.1].

2.3. Conditional extinction of small solutions (Theorem 1.5). In this section, we prove that if there exists $z \in \mathbb{R}^n$ such that

- (1) $\lambda_{1,z} \geq 0$;
- (2) there exists $C > 0$ such that, for all $x \in \mathbb{R}^n$, $\max_{i \in [N]} u_{\text{ini},i}(x) \leq C e_z(x)$;

then the solution of the Cauchy problem goes extinct locally uniformly.

(Note that $\lambda_{1,z} \geq 0$ implies $\lambda_1 \geq 0$.)

The proof is very similar to that of Theorem 1.1.

Proof. We use super-solutions of the form

$$\bar{\mathbf{u}} : (t, x) \in (T, +\infty) \times \mathbb{R}^n \mapsto M_T e^{-\lambda_{1,z} t} e^{z \cdot x} \mathbf{u}_z(t, x),$$

where $\mathbf{u}_z \in \mathcal{C}_{\text{per}}^{1,2}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \infty))$ is a positive periodic principal eigenfunction of the operator $e_{-z} \mathcal{Q} e_z$. By definition, it satisfies

$$\mathcal{Q} \bar{\mathbf{u}} = -\lambda_{1,z} \bar{\mathbf{u}} + \lambda_{1,z} \bar{\mathbf{u}} = \mathbf{0} \quad \text{in } (T, +\infty) \times \mathbb{R}^n$$

so that

$$\mathcal{Q}(\bar{\mathbf{u}} - \mathbf{u}) = \mathbf{C} \mathbf{u} \circ \mathbf{u} \geq \mathbf{0} \quad \text{in } (T, +\infty) \times \mathbb{R}^n.$$

Choosing M_T appropriately large so that $\bar{\mathbf{u}}(0, \cdot) \geq \mathbf{u}_{\text{ini}}$, we deduce from the comparison principle that $\bar{\mathbf{u}} \geq \mathbf{u}$ globally.

To conclude, as in the proof of Theorem 1.1, we distinguish two cases:

- in the case $\lambda_{1,z} > 0$, the super-solution with $T = 0$ vanishes locally uniformly so that the solution also vanishes locally uniformly;
- in the case $\lambda_{1,z} = 0$, as in the case $\lambda'_1 = 0$ of the proof of Theorem 1.1, we assume M_T to be optimal,

$$M_T = \sup_{x \in \mathbb{R}^n} \max_{i \in [N]} \frac{u_i(T, x)}{u_{0,i}(T, x)},$$

and show by comparison and limiting arguments that as $T \rightarrow +\infty$ it decreases to zero if \mathbf{u}_{ini} is nonzero. The fact that the super-solution is spatially unbounded is an obstacle easily overcome, thanks to the *a priori* estimates of Proposition 2.1. \square

2.4. Existence of a nonnegative nonzero space-time periodic entire solution (Theorem 1.2). In order to prove Theorem 1.2, we adapt the arguments of the proof of [1, Theorem 2.3], which is a similar result but established under more restrictive assumptions ($N = 2$, the coefficients are space-periodic but time-homogeneous) and with more precise conclusions (the constructed solutions are space-periodic time-independent stationary states).

The proof involves the following bifurcation theorem [1, Theorem 3.1], that we recall for clarity. The notations in the following statement are completely independent from the notations in the rest of the paper.

Theorem 2.3. *Let E a Banach space, $C \subset E$ a closed convex cone with nonempty interior and vertex 0 (i.e., $C \cap -C = \{0\}$). Let $F : \mathbb{R} \times E \rightarrow E$ a continuous and compact operator and let*

$$\mathcal{S} = \overline{\{(\alpha, x) \in \mathbb{R} \times E \setminus \{0\} \mid F(\alpha, x) = x\}}$$

and

$$\mathbb{P}_{\mathbb{R}}\mathcal{S} = \{\alpha \in \mathbb{R} \mid \exists x \in C \setminus \{0\} (\alpha, x) \in \mathcal{S}\}.$$

Assume the following properties.

- (1) For all $\alpha \in \mathbb{R}$, $F(\alpha, 0) = 0$.
- (2) F is Fréchet differentiable near $\mathbb{R} \times \{0\}$ with derivative αT locally uniformly with respect to α .
- (3) T is strongly positive in the sense of the Krein–Rutman theorem: $T(C \setminus \{0\}) \subset \text{int}(C)$. Its Krein–Rutman eigenvalue is denoted $\rho(T) > 0$.
- (4) $\mathcal{S} \cap (\{\alpha\} \times C)$ is bounded locally uniformly with respect to $\alpha \in \mathbb{R}$.
- (5) $\mathcal{S} \cap (\mathbb{R} \times (\partial C \setminus \{0\})) = \emptyset$.

Then, either $(-\infty, \frac{1}{\rho(T)}) \subset \mathbb{P}_{\mathbb{R}}\mathcal{S}$ or $(\frac{1}{\rho(T)}, +\infty) \subset \mathbb{P}_{\mathbb{R}}\mathcal{S}$.

In our case, the Banach space will be $\mathcal{C}_{\text{per}}^{1+\delta/2, 2+\delta}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$, the cone will be $\mathcal{C}_{\text{per}}^{1+\delta/2, 2+\delta}(\mathbb{R} \times \mathbb{R}^n, [\mathbf{0}, \infty))$ ¹, and F will be the mapping $(\alpha, \mathbf{f}) \mapsto \mathbf{u}$ where \mathbf{u} is the space-time periodic solution of $\mathcal{Q}\mathbf{u} + M\mathbf{u} = -\mathbf{C}\mathbf{f} \circ \mathbf{f} + \alpha\mathbf{f}$. In other words, $F(\alpha, \mathbf{f}) = (\mathcal{Q} + M)^{-1}(-\mathbf{C}\mathbf{f} \circ \mathbf{f} + \alpha\mathbf{f})$. The invertibility of $\mathcal{Q} + M$ is obviously false for some values of $M \in \mathbb{R}$, but is true once $M > 0$ is large enough, and this is why this parameter is introduced.

The derivative T at $\mathbb{R} \times \{\mathbf{0}\}$ can be easily identified as $\alpha(\mathcal{Q} + M)^{-1}$ (cf. [1]). Note that the Krein–Rutman eigenvalue $\rho(T)$ of T is related to the generalized principal eigenvalue λ'_1 of \mathcal{Q} via the relation $1/\rho(T) = \lambda'_1 + M$.

Therefore, keeping in mind that, when \mathcal{Q} is replaced by $\mathcal{Q} + M - \alpha$, the extinction case of Theorem 1.1 corresponds to $0 \leq \lambda'_1 + M - \alpha$, i.e. for all $\alpha \leq \lambda'_1 + M$, the conclusion of Theorem 2.3 will read as $(\lambda'_1 + M, +\infty) \subset \mathbb{P}_{\mathbb{R}}\mathcal{S}$. By continuity, any $(\alpha, \mathbf{u}) \in \mathcal{S}$ satisfies $F(\alpha, \mathbf{u}) = \mathbf{u}$. Therefore Theorem 2.3 brings forth a nonnegative nonzero solution \mathbf{u} of $\mathcal{Q}\mathbf{u} + M\mathbf{u} = -\mathbf{C}\mathbf{u} \circ \mathbf{u} + \alpha\mathbf{u}$ for any $\alpha > \lambda'_1 + M$.

In order to conclude the proof, it will suffice then to observe that:

- (1) $M > \lambda'_1 + M$;
- (2) all nonnegative nonzero solutions \mathbf{u} of $\mathcal{Q}\mathbf{u} = -\mathbf{C}\mathbf{u} \circ \mathbf{u}$ are actually uniformly positive (as a particular case of assumption 5 in Theorem 2.3).

In view of this sketch of the proof, it only remains to verify the assumptions of Theorem 2.3. The first one is immediate; the second one is just standard calculus [1]; the third one is a standard consequence of the maximum principle for cooperative fully coupled systems once M is chosen sufficiently large (e.g., [2, 13]). Assumption 4 (the fixed points of $F(\alpha, \cdot)$ are locally bounded) is satisfied by virtue of Corollary 2.2 (the parameters α and M change nothing to the argument and the obtained bound is indeed locally uniform with respect to α). It only remains to verify assumption 5 (there is no fixed point of $F(\alpha, \cdot)$ on the boundary of the cone). This is the object

¹In [1], the authors chose the Banach space $L_{\text{per}}^{\infty}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^N)$. However the cone of nonnegative functions in this space has empty interior. More generally, it is known that the Krein–Rutman theorem cannot be directly applied in L^p spaces. This is why usually a space of Hölder-continuous functions is used instead. Indeed, replacing L^{∞} by $\mathcal{C}^{2+\delta}$ all along the proof in [1] corrects it without further issues.

of the following lemma. For ease of reading, we only state the case $\alpha = M$, without loss of generality.

Lemma 2.4. *All nonnegative nonzero space-time periodic solutions \mathbf{u} of (KPP) satisfy:*

$$(16) \quad \min_{(t,x) \in \overline{\Omega_{\text{per}}}} \min_{i \in [N]} u_i(t, x) > 0.$$

Proof. We rewrite the semilinear operator $\mathbf{u} \mapsto \mathcal{Q}\mathbf{u} + \mathbf{C}\mathbf{u} \circ \mathbf{u}$ as a space-time heterogeneous linear operator $\mathcal{Q} + \text{diag}(\mathbf{C}\mathbf{u})$. Provided \mathbf{u} is a space-time periodic classical solution, this linear operator has space-time periodic globally bounded coefficients and is cooperative and fully coupled, and therefore we are in a position to apply the strong maximum principle. The conclusion follows directly. \square

2.5. Hair-trigger effect (Theorem 1.4). The sketch of this proof follows that of [9, Theorem 1.3]. It uses a refined Harnack inequality established in [13, Proposition 2.4]. It also uses the limit $\lambda_1 = \lim_{R \rightarrow +\infty} \lambda_{1, \text{Dir}}(B(0, R))$ of time-periodic Dirichlet principal eigenvalues in balls of increasing radius.

We fix $R > 0$ once and for all and intend to prove that

$$\liminf_{t \rightarrow +\infty} \min_{i \in [N]} \min_{|x| \leq R} u_i(t, x) > 0.$$

If $R > 0$ is large enough, then there exists $\theta \geq \max(T, L_1, \dots, L_n)$ such that $[-3\theta/2, 3\theta/2]^n \subset B(0, R)$ and also $\lambda_{1, \text{Dir}}(B(0, R)) < 0$. Without loss of generality, we assume that R is accordingly large.

We begin with a lemma showing that the solution cannot stay too small in the ball for too long.

Lemma 2.5. *Let $\lambda = \lambda_{1, \text{Dir}}(B(0, R))$ and $\alpha > 0$ such that, for any $\mathbf{v} \in [0, \alpha \mathbf{1}]$ and any $(t, x) \in \Omega_{\text{per}}$, $\mathbf{0} \leq \mathbf{C}(t, x)\mathbf{v} \leq \frac{-\lambda}{2}\mathbf{1}$.*

Let $\mu \in (0, \alpha)$.

Then there exists a duration $\tau_\mu > 0$ such that, for all $t_0 \geq 0$, $t_1 > t_0$, and for any nonnegative solution \mathbf{u} of (KPP) such that $\mu \mathbf{1} \leq \mathbf{u}(t_0, \cdot) \leq \alpha \mathbf{1}$ in $B(0, R)$,

$$t_1 > t_0 + \tau_\mu \implies \max_{(t,x) \in [t_0, t_1] \times \overline{B(0, R)}} \max_{i \in [N]} u_i(t, x) > \alpha.$$

Proof. Equivalently, we prove that if $\mathbf{u} \leq \alpha \mathbf{1}$ in $[t_0, t_1] \times B(0, R)$, then $t_1 - t_0 \leq \tau_\mu$ for some duration τ_μ , uniform with respect to t_0, t_1, \mathbf{u} , to be exhibited.

We define $\underline{\mathbf{u}} : (t, x) \mapsto \mu e^{\frac{-\lambda}{2}(t-t_0)} \mathbf{u}_{\text{Dir}}(t, x)$ where \mathbf{u}_{Dir} solves in $\mathbb{R} \times \overline{B(0, R)}$:

$$\begin{cases} \text{diag}(\mathcal{P})\mathbf{u}_{\text{Dir}} - \mathbf{L}\mathbf{u} = \lambda \mathbf{u}_{\text{Dir}} & \text{in } \mathbb{R} \times B(0, R), \\ \mathbf{u}_{\text{Dir}} = \mathbf{0} & \text{on } \mathbb{R} \times \partial B(0, R), \\ \mathbf{u}_{\text{Dir}} \text{ time-periodic,} \\ \max_{(t,x) \in [0, T] \times \overline{B(0, R)}} \max_{i \in [N]} u_{\text{Dir}, i}(t, x) = 1. \end{cases}$$

and is extended to $\mathbb{R} \times \mathbb{R}^n$ by setting $\mathbf{u}_{\text{Dir}}(\cdot, x) = \mathbf{0}$ if $|x| > R$.

Clearly, $\underline{\mathbf{u}}(t_0, \cdot) \leq \mathbf{u}(t_0, \cdot)$ in $B(0, R)$, and

$$\mathcal{Q}\underline{\mathbf{u}} = \frac{-\lambda}{2}\underline{\mathbf{u}} + \lambda \underline{\mathbf{u}} = \frac{\lambda}{2}\underline{\mathbf{u}},$$

whence

$$\mathcal{Q}(\mathbf{u} - \underline{\mathbf{u}}) = -(\mathbf{C}\mathbf{u}) \circ \mathbf{u} + \frac{-\lambda}{2}\underline{\mathbf{u}} \geq -\frac{-\lambda}{2}(\mathbf{u} - \underline{\mathbf{u}}).$$

Therefore, by applying the comparison principle to the cooperative fully coupled operator $\mathcal{Q} + \frac{-\lambda}{2}$ in $[t_0, t_1] \times B(0, R)$, we deduce that $\mathbf{u} \geq \underline{\mathbf{u}}$ in $[t_0, t_1] \times B(0, R)$.

By virtue of the Harnack inequality [13, Proposition 2.4] and of the global boundedness estimates $\mathbf{0} \leq \mathbf{u} \leq \alpha \mathbf{1}$, there exists a constant $\kappa > 0$, independent of t_0, t_1, \mathbf{u} , such that:

$$\min_{(t,x) \in \mathbb{R} \times [-\theta/2, \theta/2]^n} \min_{i \in [N]} u_{\text{Dir},i}(t, x) \geq \kappa \max_{(t,x) \in \mathbb{R} \times [-\theta/2, \theta/2]^n} \max_{i \in [N]} u_{\text{Dir},i}(t, x).$$

We denote

$$\beta = \min_{(t,x) \in \mathbb{R} \times [-\theta/2, \theta/2]^n} \min_{i \in [N]} u_{\text{Dir},i}(t, x) \in (0, 1].$$

It does not depend on t_0, t_1 or \mathbf{u} .

Consequently,

$$\mu \kappa \beta e^{-\frac{\lambda}{2}(t-t_0)} \mathbf{1} \leq \mathbf{u} \leq \alpha \mathbf{1} \quad \text{in } [t_0, t_1] \times B(0, R).$$

It follows directly that

$$t_1 - t_0 \leq \frac{2}{-\lambda} \ln \left(\frac{\alpha}{\mu \beta \kappa} \right)$$

and τ_μ is therefore exhibited. Remark that $\tau_\mu \rightarrow +\infty$ as $\mu \rightarrow 0$. \square

With this lemma we are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Let

$$\mu_{\text{ini}} = \min_{x \in \overline{B(0,R)}} \min_{i \in [N]} u_{\text{ini},i}(x).$$

Although μ_{ini} might or might not be smaller than α , in both cases, by a first application of Lemma 2.5 if necessary, there exists $t_0 \in [0, \tau_{\mu_{\text{ini}}} + 1]$ such that

$$\sup_{x \in B(0,R)} \max_{i \in [N]} u_i(t_0, x) > \alpha.$$

We also define:

$$s_0 = \inf \left\{ t > t_0 \mid \max_{x \in \overline{B(0,R)}} \max_{i \in [N]} u_i(t, x) \leq \alpha \right\}.$$

Either $s_0 = +\infty$ or $s_0 < +\infty$.

Then we construct by induction three sequences (μ_k) , (t_k) and (s_k) .

If $s_k = +\infty$, then we stop the construction and the three sequences are finite.

On the contrary, if $s_k < +\infty$, then we define:

$$\mu_k = \min_{x \in \overline{B(0,R)}} \min_{i \in [N]} u_i(s_k, x),$$

$$t_{k+1} = \inf \left\{ t > s_k \mid \max_{x \in \overline{B(0,R)}} \max_{i \in [N]} u_i(t, x) > \alpha \right\},$$

which satisfies $t_{k+1} < s_k + \tau_{\mu_k} + 1 < +\infty$ by virtue of Lemma 2.5, and

$$s_{k+1} = \inf \left\{ t > t_{k+1} \mid \max_{x \in \overline{B(0,R)}} \max_{i \in [N]} u_i(t, x) \leq \alpha \right\},$$

which can be finite or infinite. We then iterate.

The construction being achieved, we are in one of the following two cases:

- (1) either the sequences are finite;
- (2) or they are infinite.

If they are finite, let $k^* \in \mathbb{N}$ such that $s_{k^*-1} < +\infty = s_{k^*}$. By construction,

$$\max_{x \in B(0, R)} \max_{i \in [N]} u_i(t, x) > \alpha \quad \text{for any } t \in [t_{k^*}, +\infty).$$

Let $\tilde{\theta} > \theta$ such that $B(0, R) \subset [-\tilde{\theta}/2, \tilde{\theta}/2]^n$. By virtue of the Harnack inequality [13, Proposition 2.4] and of the global boundedness estimates (cf. Proposition 2.1), there exists another constant $\tilde{\kappa} > 0$, that does not depend on \mathbf{u} but only on some arbitrary constant $C \geq \sup_{x \in \mathbb{R}^n} \max_{i \in [N]} u_{i, \text{ini}, i}(x)$, such that, for any $t' \geq \max(t_{k^*}, 2\tilde{\theta})$,

$$\min_{(t, x) \in [5\tilde{\theta}, 6\tilde{\theta}] \times [-\tilde{\theta}/2, \tilde{\theta}/2]^n} \min_{i \in [N]} u_i(t + t', x) \geq \tilde{\kappa} \max_{(t, x) \in [0, 2\tilde{\theta}] \times [-\tilde{\theta}/2, \tilde{\theta}/2]^n} \max_{i \in [N]} u_i(t + t', x),$$

whence

$$\min_{(t, x) \in [5\tilde{\theta}, 6\tilde{\theta}] \times B(0, R)} \min_{i \in [N]} u_i(t + t', x) \geq \tilde{\kappa}\alpha,$$

whence

$$\min_{x \in B(0, R)} \min_{i \in [N]} u_i(t, x) \geq \tilde{\kappa}\alpha \quad \text{for all } t \geq \max(t_{k^*}, 2\tilde{\theta}) + 5\tilde{\theta}.$$

If the sequences are infinite, then similarly we can use the Harnack inequality to bound the solution from below.

First, we show that the sequence (μ_k) is bounded from below away from zero. By contradiction, assume $\mu_k \rightarrow 0$. By classical parabolic estimates [14], up to a diagonal extraction, the sequence of functions $(\mathbf{u}^k)_{k \in \mathbb{N}}$ defined by $\mathbf{u}^k : (t, x) \mapsto \mathbf{u}(s_k + t, x)$ converges locally uniformly to an entire solution \mathbf{u}^∞ of KPP that satisfies

$$\min_{i \in [N]} \min_{x \in B(0, R)} u_i^\infty(0, x) = 0, \quad \max_{i \in [N]} \max_{x \in B(0, R)} u_i^\infty(0, x) = \alpha.$$

This immediately contradicts the strong comparison principle.

Since $\inf_{k \in \mathbb{N}} \mu_k > 0$, it follows that $\sup_{k \in \mathbb{N}} \tau_{\mu_k} < +\infty$. Defining $\bar{\tau} = \sup_{k \in \mathbb{N}} \tau_{\mu_k} + 1$, we find $t_{k+1} - s_k < \bar{\tau}$ for all $k \in \mathbb{N}$. In other words, the inequality $\mathbf{u} \leq \alpha \mathbf{1}$ in $B(0, R)$ can only be true in disjoint time intervals of duration shorter than $\bar{\tau}$. Consequently, with a similar application of the Harnack inequality [13, Proposition 2.4], and up to increasing $\tilde{\theta}$ so that $2\tilde{\theta} \geq \bar{\tau}$, we find that

$$\min_{x \in B(0, R)} \min_{i \in [N]} u_i(t, x) \geq \tilde{\kappa}\alpha \quad \text{for all } t \geq 7\tilde{\theta}.$$

This ends the proof. \square

Remark that in the above proof, the uniform bound from below in large times $\tilde{\kappa}\alpha$ depends on C and on R through the Harnack constant $\tilde{\kappa}$. To get rid of these dependencies, more work is needed. On one hand, thanks to the absorbing set estimates of Proposition 2.1, up to changing $t = 0$ into a later time, we can choose without loss of generality a constant C independent of the actual initial values (for instance, $C = 2K$). On the other hand, the dependency on $R > 0$ can be handled by fixing a sufficiently large radius and then translating the spatial point of reference $x = 0$ in order to cover the larger ball that is actually considered. Since these arguments are already detailed in [9] and since they are strictly unchanged by the space-time periodicity, we just state the enhanced result that could be established.

Proposition 2.6. *Assume $\lambda_1 < 0$. Then there exists a constant $\nu > 0$ such that all solutions \mathbf{u} of the Cauchy problem (KPP)–(IC) satisfy*

$$(17) \quad \liminf_{t \rightarrow +\infty} \min_{i \in [N]} \min_{|x| \leq R} u_i(t, x) \geq \nu \quad \text{for all } R > 0.$$

The following corollary could also be established.

Corollary 2.7. *Assume $\lambda_1 < 0$. Then all nonnegative nonzero globally bounded time-periodic solutions \mathbf{u} of (KPP) satisfy*

$$(18) \quad \mathbf{u} \geq \nu \mathbf{1} \quad \text{in } \mathbb{R} \times \mathbb{R}^n.$$

2.6. Conditional persistence of large solutions (Theorem 1.3). Let us recall the assumptions of Theorem 1.3. There exists $z \in \mathbb{R}^n$ such that:

- (1) $\lambda_{1,z} < 0$;
- (2) $\zeta \in (0, 2) \mapsto \lambda_{1,\zeta z}$ is increasing in a neighborhood of 1;
- (3) there exists $C > 0$, $B \in \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$ such that $z \cdot x \leq B$, $\min_{i \in [N]} u_{\text{ini},i}(x) \geq C^{-1} e_z(x)$.

The proof is similar in spirit to that of Theorem 1.4, yet we use a slightly different argument, that exploits the Harnack inequality and the global boundedness estimates in another way.

By [13, Proposition 2.4] and Proposition 2.1, for all $M > 0$, there exists a constant $\kappa_M > 1$ such that, for any solution \mathbf{u} of (KPP)–(IC) satisfying

$$\sup_{x \in \mathbb{R}^n} \max_{i \in [N]} u_{\text{ini},i}(x) \leq M,$$

the following inequalities hold true:

$$(19) \quad \forall i, j \in [N] \quad \frac{1}{\kappa_M} u_i \leq u_j \leq \kappa_M u_i \quad \text{in } [1, +\infty) \times \mathbb{R}^n.$$

Hence, once the solution \mathbf{u} is given, we can compare from below and starting from $t = 1$ the reaction term $\mathbf{L}\mathbf{u} - (\mathbf{C}\mathbf{u}) \circ \mathbf{u}$ to a cooperative reaction term $\mathbf{L}\mathbf{u} - D\mathbf{u} \circ \mathbf{u}$, $D > 0$. By the maximum principle applied to this new cooperative reaction term, any solution of $\mathbf{Q}\mathbf{u} = -D\mathbf{u} \circ \mathbf{u}$ with $\mathbf{u}(1, \cdot) \leq \mathbf{u}(1, \cdot)$ will satisfy $\mathbf{u} \leq \mathbf{u}$ globally in space at any time $t \geq 1$. Therefore it only remains to prove that:

- (1) $\min_{i \in [N]} u_i(1, x) \geq \tilde{C}^{-1} e_z(x)$ in $\{z \cdot x \leq \tilde{B}\}$ for some $\tilde{C} > 0$, $\tilde{B} \in \mathbb{R}$;
- (2) the persistence result is true for the cooperative system $\mathbf{Q}\mathbf{v} = -D\mathbf{v} \circ \mathbf{v}$.

Proof of the exponential estimates at $t = 1$. The exponential estimates at $t = 1$ are easily derived. Indeed, there exists a large $E > 0$ such that each u_i admits a (rough) sub-solution $v_i \leq u_i$ satisfying

$$\begin{cases} \mathcal{P}_i v_i = -E v_i & \text{in } (0, +\infty) \times \mathbb{R}^n \\ v_i(0, x) = \frac{1}{C} e_z(x) & \text{in } \{z \cdot x \leq B\} \\ v_i(0, x) = 0 & \text{in } \{z \cdot x > B\}. \end{cases}$$

Such sub-solutions can be related to solutions of

$$\begin{cases} (e_{-z} \mathcal{P}_i)(e_z w_i) = -E w_i & \text{in } (0, +\infty) \times \mathbb{R}^n \\ w_i(0, x) = \frac{1}{C} & \text{in } \{z \cdot x \leq B\} \\ w_i(0, x) = 0 & \text{in } \{z \cdot x > B\} \end{cases}$$

through the formula $v_i = e_z w_i$. Hence we only have to show that, for each $i \in [N]$, $\liminf_{z \cdot x \rightarrow -\infty} w_i(1, x) > 0$. Recall that the operator $e_{-z} \mathcal{P}_i e_z$ has the following form:

$$e_{-z} \mathcal{P}_i e_z = \mathcal{P}_i - ((A_i + A_i^T)z \cdot \nabla + z \cdot A_i z + \nabla \cdot (A_i z) - q_i \cdot z).$$

Let $(x_k)_{k \in \mathbb{N}}$ such that $z \cdot x_k \rightarrow -\infty$ and such that $w_i(1, x_k) \rightarrow \liminf_{z \cdot x \rightarrow -\infty} w_i(1, x)$. By classical parabolic estimates [14] and a diagonal extraction, up to a subsequence,

the sequence of functions $(t, x) \mapsto w_i(t, x - x_n)$ converges locally uniformly to the solution w_i^∞ of

$$\begin{cases} (e_{-z}\mathcal{P}_i)(e_z w_i^\infty) = -E w_i^\infty & \text{in } (0, +\infty) \times \mathbb{R}^n \\ w_i^\infty(0, \cdot) = \frac{1}{C} & \text{in } \mathbb{R}^n \end{cases}$$

By constructing yet another sub-solution that solves a mere ODE of the form $w' = -Fw$, we deduce easily that $w_i^\infty(1, 0) > 0$. This ends the proof of this step. \square

The last part of the proof is its core and is a straightforward adaptation of the proof in the scalar case [16]. Easing the notations, we consider the cooperative system $\mathcal{Q}\mathbf{v} = -D\mathbf{v} \circ \mathbf{v}$ with nonnegative globally bounded initial conditions \mathbf{v}_{ini} satisfying $\min_{i \in [N]} v_{\text{ini}, i}(x) \geq C^{-1}e_z(x)$ in $\{z \cdot x \leq B\}$. For this system, let us prove that, for any fixed $R > 0$,

$$\liminf_{t \rightarrow +\infty} \min_{i \in [N]} \min_{|x| \leq R} v_i(t, x) > 0.$$

Proof of the cooperative persistence result. Let $\zeta \in (0, 1)$ such that

$$\lambda_{1, z} < \lambda_{1, (1+\zeta)z} < 0.$$

We define:

$$\underline{\mathbf{u}} = \left[\frac{1}{C}e_z \mathbf{u}_z - A e_{(1+\zeta)z} \mathbf{u}_{(1+\zeta)z} \right]^+.$$

Here, the notation $[\cdot]^+$ refers to the component-by-component positive part of a vector. Namely, for each $i \in [N]$, $([\cdot]^+)_i = \max(\cdot, 0)$. Beware that the locii of positivity of each \underline{u}_i , hereafter denoted $\underline{u}_i^{-1}((0, +\infty))$, do not, in general, coincide.

On one hand, whatever $A > 0$ is, $\underline{\mathbf{u}}(0, \cdot) \leq \mathbf{v}_{\text{ini}}$ in $\{z \cdot x \leq B\}$. On the other hand, $A > 0$ can be chosen so large that $\underline{\mathbf{u}}(0, \cdot) \leq \mathbf{v}_{\text{ini}}$ in $\{z \cdot x > B\}$. Hence, with such an appropriate choice of A ,

$$\underline{\mathbf{u}}(0, \cdot) \leq \mathbf{v}_{\text{ini}} \quad \text{globally in } \mathbb{R}^n.$$

Let $\varepsilon \in (0, |\lambda_{1, z}|)$. By renormalizing appropriately the family $(\mathbf{u}_{z'})_{z' \in \mathbb{R}^n}$ if necessary, $\underline{\mathbf{u}}$ satisfies $D\underline{\mathbf{u}} \leq \varepsilon \mathbf{1}$.

In the interior of $\bigcap_{i \in [N]} \underline{u}_i^{-1}(\{0\})$, obviously

$$\mathcal{Q}\underline{\mathbf{u}} = \mathcal{Q}\mathbf{0} = \mathbf{0} \leq \mathbf{0} = -D\underline{\mathbf{u}} \circ \underline{\mathbf{u}}.$$

In the interior of $\bigcap_{i \in [N]} \underline{u}_i^{-1}((0, +\infty))$,

$$\begin{aligned} \mathcal{Q}\underline{\mathbf{u}} &= \frac{1}{C} \lambda_{1, z} e_z \mathbf{u}_z - A \lambda_{1, (1+\zeta)z} e_{(1+\zeta)z} \mathbf{u}_{(1+\zeta)z} \\ &\leq -\varepsilon \underline{\mathbf{u}} + \varepsilon \underline{\mathbf{u}} + \lambda_{1, z} \left(\frac{1}{C} e_z \mathbf{u}_z - A e_{(1+\zeta)z} \mathbf{u}_{(1+\zeta)z} \right) \\ &\leq -D\underline{\mathbf{u}} \circ \underline{\mathbf{u}} + (\varepsilon + \lambda_{1, z}) \underline{\mathbf{u}} \\ &\leq -D\underline{\mathbf{u}} \circ \underline{\mathbf{u}}. \end{aligned}$$

When there exists a subset $I \subset [N]$, nonempty and different from $[N]$ itself, such that $\bigcap_{i \in I} \underline{u}_i^{-1}(\{0\}) \cup \bigcap_{i \in [N] \setminus I} \underline{u}_i^{-1}((0, +\infty))$ is nonempty, then in the interior of this locus, for each $i \in I$,

$$(\mathcal{Q}\underline{\mathbf{u}})_i = - \sum_{j \in [N] \setminus I} l_{i, j} \underline{u}_j \leq 0 = \underline{u}_i,$$

and for each $i \notin I$, by simply adding nonnegative terms and then repeating the previous calculations,

$$\begin{aligned} (\mathcal{Q}\mathbf{u})_i &= \mathcal{P}_i \underline{u}_i - \sum_{i \in [N] \setminus I} l_{i,j} \underline{u}_j \\ &\leq \mathcal{P}_i \underline{u}_i - \left(\mathbf{L} \left[\frac{1}{C} e_z \mathbf{u}_z - A e_{(1+\zeta)z} \mathbf{u}_{(1+\zeta)z} \right] \right)_i \\ &\leq \left(\mathcal{Q} \left[\frac{1}{C} e_z \mathbf{u}_z - A e_{(1+\zeta)z} \mathbf{u}_{(1+\zeta)z} \right] \right)_i \\ &\leq -D \underline{u}_i^2 \end{aligned}$$

It is a standard fact for scalar reaction–diffusion equations that the minimum of two sub-solutions is a sub-solution. Here, similarly, due to the special “positive part” form of the function, the preceding differential inequalities are sufficient to establish that $\underline{\mathbf{u}}$ is indeed a global sub-solution.

Hence, by virtue of the strong comparison principle applied to the semilinear cooperative operator $\mathcal{Q}\mathbf{v} + D\mathbf{v} \circ \mathbf{v}$, the inequality $\mathbf{v} \geq \underline{\mathbf{u}}$ is satisfied globally in $[0, +\infty) \times \mathbb{R}^n$.

By the special form of $\underline{\mathbf{u}}$, there exists $x_0 \in \mathbb{R}^n$ such that

$$\min_{i \in [N]} \inf_{t \geq 0} \underline{u}_i(t, x_0) > 0.$$

Up to increasing R without loss of generality, we can assume that $x_0 \in B(0, R)$. Therefore, by virtue of the Harnack inequality [13, Proposition 2.4], there exists a constant $\kappa > 0$ such that, for all $t \geq 1$,

$$\min_{i \in [N]} \min_{x \in B(0, R)} v_i(t, x) \geq \kappa \max_{i \in [N]} \max_{x \in B(0, R)} v_i(t, x) \geq \kappa \min_{i \in [N]} \inf_{t' \geq 0} \underline{u}_i(t', x_0) > 0.$$

This ends the proof. \square

2.7. The Freidlin–Gärtner-type formula (Theorem 1.6). In this section, we assume that $\lambda_1 < 0$ and \mathbf{u}_{ini} is compactly supported, and we prove (11) and (12).

To this end, we fix once and for all $e \in \mathbb{S}^{n-1}$.

First, we confirm that the minima involved in the definitions of c_e^* and c_e^{FG} are indeed well-defined.

Lemma 2.8. *There exists a unique $\mu^* > 0$ such that*

$$c_e^{\mu^*} = \inf_{\mu > 0} c_e^\mu = \min_{\mu > 0} c_e^\mu = \min_{\mu > 0} \frac{-\lambda_{1, -\mu e}}{\mu}.$$

Moreover, for any $c > c_e^$, there exists $\mu_1, \mu_2 > 0$ such that $\mu_1 < \mu_2$ and $c_e^{\mu_1} = c_e^{\mu_2} = c$, whereas for any $c \in (0, c_e^*)$, there exists no $\mu > 0$ such that $c_e^\mu = c$.*

Proof. We consider the function $\psi : \mu \in [0, +\infty) \mapsto -\lambda_{1, -\mu e} - c\mu$. By continuity and strict concavity of $\mu \mapsto \lambda_{1, -\mu e}$ [13, Corollary 3.6] and by virtue of the quadratic growth of $|\lambda_{1, -\mu e}|$ as $\mu \rightarrow +\infty$ [13, Corollary 3.12]), ψ is continuous, strictly convex, and satisfies

$$\psi(0) = -\lambda_1' \geq -\lambda_1 > 0, \quad \lim_{\mu \rightarrow +\infty} \psi(\mu) = +\infty.$$

It admits a global minimum in $[0, +\infty)$. Since

$$\psi(0) = -\lambda_1', \quad \psi(1) = -\lambda_{1, -e} - c,$$

the minimum is negative if c is large enough. Since

$$\psi(\mu) \geq -\lambda_1 > 0 \quad \text{for any } \mu \geq 0 \quad \text{if } c = 0,$$

the minimum is positive if c is close enough to 0. Moreover, the minimum is either located at $\mu = 0$, in which case its value is $-\lambda_1' > 0$, or it is located at some $\mu^* > 0$, in which case its value is decreasing with respect to c . By continuity, the minimum is, as a function of c , positive and constant in some interval $[0, c^\dagger)$, with $c^\dagger \geq 0$, and decreasing in $[c^\dagger, +\infty)$, with a positive value at $c = c^\dagger$ and with limit $-\infty$ as $c \rightarrow +\infty$.

By continuity, strict convexity, strict monotonicity, there exists a threshold $\hat{c} > 0$ such that the equation $\psi(\mu) = 0$ admits therefore:

- no solution if $c \in (0, \hat{c})$;
- exactly one solution μ^* if $c = \hat{c}$;
- exactly two isolated solutions $\mu_1^* < \mu_2^*$ if $c > \hat{c}$.

In view of the sign of $\psi(0)$, these solutions, if any, are positive. Hence the image of $\mu \in (0, +\infty) \mapsto -\lambda_{1, -\mu e}/\mu$ contains $[\hat{c}, +\infty)$ and does not contain $[0, \hat{c})$; in other words, it is exactly $[\hat{c}, +\infty)$, whence $\hat{c} = c_{e^\wedge}^{\mu^*} = \min_{\mu > 0} c_e^\mu$. \square

Lemma 2.9. *There exists $e^\wedge \in \mathbb{S}^{n-1}$ such that*

$$\frac{c_{e^\wedge}^*}{e \cdot e^\wedge} = \inf_{\substack{e' \in \mathbb{S}^{n-1} \\ e \cdot e' > 0}} \frac{c_{e'}^*}{e \cdot e'} = \min_{\substack{e' \in \mathbb{S}^{n-1} \\ e \cdot e' > 0}} \frac{c_{e'}^*}{e \cdot e'}.$$

Proof. It is sufficient to show that $(c_{e'}^*)_{e' \in \mathbb{S}^{n-1}}$ is uniformly positive and globally bounded, so that the quantity $c_{e'}^*/e \cdot e'$ is positive if $e \cdot e' > 0$ and tends to $+\infty$ if in addition $e \cdot e' \rightarrow 0$.

To this end, it suffices to observe that $e \in \mathbb{S}^{n-1} \mapsto c_e^*$ is continuous, defined on a compact set, and pointwise positive (due to $c_e^* \geq -\lambda_1/\mu^* > 0$). \square

The main idea of the forthcoming proof is to compare, again, (KPP) with cooperative systems for which the Freidlin–Gärtner formula is easier to establish.

Just as in the proof of Theorem 1.3, by virtue of [13, Proposition 2.4] and Proposition 2.1, for all $M > 0$, there exists a constant $\kappa_M > 1$ such that, for any solution \mathbf{u} of (KPP)–(IC) satisfying

$$\sup_{x \in \mathbb{R}^n} \max_{i \in [N]} u_{\text{ini}, i}(x) \leq M,$$

the following inequalities hold true:

$$(20) \quad \forall i, j \in [N] \quad \frac{1}{\kappa_M} u_i \leq u_j \leq \kappa_M u_i \quad \text{in } [1, +\infty) \times \mathbb{R}^n.$$

Hence, once the solution \mathbf{u} is given, we can compare from below and starting from $t = 1$ the reaction term $\mathbf{L}\mathbf{u} - (\mathbf{C}\mathbf{u}) \circ \mathbf{u}$ to a cooperative reaction term $\mathbf{L}\mathbf{u} - D\mathbf{u} \circ \mathbf{u}$, $D > 0$. By the maximum principle applied to this new cooperative reaction term, any solution of $\mathbf{Q}\mathbf{u} = -D\mathbf{u} \circ \mathbf{u}$ with $\mathbf{u}(1, \cdot) \leq \mathbf{u}(1, \cdot)$ will satisfy $\mathbf{u} \leq \mathbf{u}$ globally in space at any time $t \geq 1$.

Similarly, by nonnegativity of \mathbf{C} (cf. ((A₄))), we can compare from above and starting from $t = 0$ the reaction term $\mathbf{L}\mathbf{u} - (\mathbf{C}\mathbf{u}) \circ \mathbf{u}$ to another cooperative reaction term $\mathbf{L}\mathbf{u} - D'\mathbf{u} \circ \mathbf{u}$, $D' > 0$. By the maximum principle, any solution of $\mathbf{Q}\bar{\mathbf{u}} = -D'\bar{\mathbf{u}} \circ \bar{\mathbf{u}}$ with $\bar{\mathbf{u}}(0, \cdot) \geq \mathbf{u}(0, \cdot)$ will satisfy $\bar{\mathbf{u}} \geq \mathbf{u}$ globally in space at any time $t \geq 0$.

Consequently, (KPP) can be compared from above and from below to cooperative systems of the form

$$\mathcal{Q}\mathbf{v} = -g\mathbf{v} \circ \mathbf{v} \quad \text{for some } g > 0.$$

Since these semilinear systems share the same linear part $\mathcal{Q}\mathbf{v}$, and since their nonlinear part $-g\mathbf{v} \circ \mathbf{v}$ has a constant negative sign, it is expected that they all satisfy the Freidlin–Gärtner formula, and therefore that they all have the same spreading speed c_e^{FG} (independent of g). Proving this claim will end our proof. To do so, we will use recent results from Du, Li and Shen [5] as well as a delicate construction from Berestycki, Hamel and Nadin [3].

Lemma 2.10. *Let $\mathbf{g} \in C_{\text{per}}^{\delta/2, \delta}(\mathbb{R} \times \mathbb{R}^n, (\mathbf{0}, \infty))$.*

Then any solution \mathbf{v} of $\mathcal{Q}\mathbf{v} = -\text{diag}(\mathbf{g})(\mathbf{v} \circ \mathbf{v})$ in $(0, +\infty) \times \mathbb{R}^n$ with nonnegative nonzero compactly supported initial data \mathbf{v}_{ini} spreads in the direction e at speed c_e^{FG} , namely

$$(21) \quad \liminf_{t \rightarrow +\infty} \min_{i \in [N]} \inf_{|x| \leq R} v_i(t, x + cte) > 0 \quad \text{for all } R > 0 \text{ and } c \in (0, c_e^{\text{FG}}),$$

$$(22) \quad \lim_{t \rightarrow +\infty} \max_{i \in [N]} \sup_{|x| \leq R} v_i(t, x + cte) = 0 \quad \text{for all } R > 0 \text{ and } c > c_e^{\text{FG}}.$$

Proof. Let

$$\kappa \geq \max \left(\frac{-\lambda_1'}{\min_{(i,t,x) \in [N] \times \mathbb{R} \times \mathbb{R}^n} g_i(t, x) u_{0,i}(t, x)}, \max_{(i,x) \in [N] \times \mathbb{R}^n} v_{\text{ini},i}(x) \right) > 0.$$

Then the function $\kappa \mathbf{u}_0$ satisfies straightforwardly

$$\mathcal{Q}(\kappa \mathbf{u}_0) + \text{diag}(\mathbf{g})(\kappa \mathbf{u}_0 \circ \kappa \mathbf{u}_0) \geq \mathbf{0},$$

and, by virtue of the comparison principle for cooperative systems, it is a supersolution and $\kappa \mathbf{u}_0 \geq \mathbf{v}$ in $[0, +\infty) \times \mathbb{R}^n$. Hence \mathbf{v} is globally bounded, in a way that only depends on $\max_{(i,x) \in [N] \times \mathbb{R}^n} v_{\text{ini},i}(x)$. Furthermore, a similar application of the comparison principle and a classical minimization of the parameter κ show that space-time periodic solutions of $\mathcal{Q}\mathbf{v} = -\text{diag}(\mathbf{g})(\mathbf{v} \circ \mathbf{v})$ are *a priori* uniformly globally bounded, in a way reminiscent of Corollary 2.2. Then, by arguments very similar to those proving Theorem 1.2, there exists a nonnegative nonzero space-time periodic entire solution \mathbf{v}^* . Similarly, the proof of Theorem 1.4 can be readily adapted to show that all solutions \mathbf{v} of the Cauchy problem persist locally uniformly, namely they satisfy (8).

By rewriting the unknown \mathbf{v} as $\mathbf{v} = \tilde{\mathbf{v}} \circ \mathbf{v}^*$, we find that each \tilde{v}_i , $i \in [N]$, satisfies:

$$\mathcal{P}_i \tilde{v}_i - 2 \frac{A_i \nabla v_i^*}{v_i^*} \cdot \nabla \tilde{v}_i = -\tilde{v}_i \left(\sum_{j \in [N]} l_{i,j} \frac{v_j^*}{v_i^*} - g_i v_i^* \right) + \sum_{j \in [N]} l_{i,j} \frac{v_j^*}{v_i^*} \tilde{v}_j - g_i v_i^* (\tilde{v}_i)^2$$

By setting

$$\begin{aligned} \tilde{q}_i &= q_i - 2 \frac{A_i \nabla v_i^*}{v_i^*}, \\ \tilde{g}_i &= g_i v_i^*, \\ \tilde{\mathbf{L}} &= \left(l_{i,j} \frac{v_j^*}{v_i^*} \right)_{(i,j) \in [N]^2} + \text{diag} \left(g_i v_i^* - \sum_{j \in [N]} l_{i,j} \frac{v_j^*}{v_i^*} \right)_{i \in [N]}, \\ \tilde{\mathcal{Q}} &= \text{diag}(\partial_t - \nabla \cdot (A_i \nabla) + \tilde{q}_i) - \tilde{\mathbf{L}}, \end{aligned}$$

we obtain a new system

$$\tilde{\mathcal{Q}}\tilde{\mathbf{v}} = -\text{diag}(\tilde{\mathbf{g}})(\tilde{\mathbf{v}} \circ \tilde{\mathbf{v}})$$

of the same form but where we have without loss of generality replaced \mathbf{v}^* by $\mathbf{1}$. In particular let us emphasize that the matrix $\tilde{\mathbf{L}}$ satisfies the structural assumptions (A_2) , (A_3) .

Let us verify that $\lambda_1(\tilde{\mathcal{Q}}) < 0$. Assume by contradiction $\lambda_1(\tilde{\mathcal{Q}}) \geq 0$. Then the proof of Theorem 1.5 can be readily adapted to construct solutions $\tilde{\mathbf{v}}$ that vanish locally uniformly. But this, in turn, implies the existence of solutions \mathbf{v} of the original system $\mathcal{Q}\mathbf{v} = -\text{diag}(\mathbf{g})(\mathbf{v} \circ \mathbf{v})$ that vanish locally uniformly, contradicting the locally uniform persistence of all solutions.

Let us verify now that $\mathbf{1}$ is globally attractive for solutions $\tilde{\mathbf{v}}$ whose initial data are uniformly positive, space-periodic and valued in $[\mathbf{0}, \mathbf{1}]$. For any $T \geq 0$, let

$$M_T = \min_{i \in [N]} \min_{x \in [0, L]} \tilde{v}_i(T, x),$$

so that

$$M_T \mathbf{1} \leq \tilde{\mathbf{v}}(T, \cdot) \leq \mathbf{1} \quad \text{in } \mathbb{R}^n.$$

It can be easily verified that $M_T \mathbf{1}$ is a sub-solution starting from $t = T$ whereas $\mathbf{1}$ is a global super-solution; hence it suffices to prove $M_T \rightarrow 1$ as $T \rightarrow +\infty$. By applying the strong comparison principle in a way similar to the proof of Theorem 1.1 (case $\lambda'_1 = 0$), we deduce that $T \mapsto M_T$ is increasing. Hence it converges to a limit $M_\infty \in (0, 1]$. If $M_\infty < 1$, then by a limiting argument again similar to that of the proof of Theorem 1.1, we find a new space-periodic entire solution $\tilde{\mathbf{v}}_\infty$ valued in $[M_\infty \mathbf{1}, \mathbf{1}]$, and then by comparison with the sub-solution $M_\infty \mathbf{1}$, a contradiction arises, just as in the proof of Theorem 1.1. Hence $M_\infty = 1$ and $\mathbf{1}$ is indeed globally attractive for uniformly positive, space-periodic solutions in $[\mathbf{0}, \mathbf{1}]$.

Therefore the transformed system $\tilde{\mathcal{Q}}\tilde{\mathbf{v}} = -\text{diag}(\tilde{\mathbf{g}})(\tilde{\mathbf{v}} \circ \tilde{\mathbf{v}})$ satisfies the assumptions **(A1)**–**(A6)** of [5, Theorem 2.1]. Consequently, its solutions with planar Heaviside-like initial data in some direction $e' \in \mathbb{S}^{n-1}$, namely initial data in

$$\mathbf{H}_{e'} = \left\{ \tilde{\mathbf{v}}_{\text{ini}} \in L^\infty(\mathbb{R}^n, [\mathbf{0}, \mathbf{1}]) \mid \liminf_{x \cdot e' \rightarrow -\infty} \tilde{\mathbf{v}}_{\text{ini}} \gg \mathbf{0}, \exists B \in \mathbb{R} \quad (\tilde{\mathbf{v}}_{\text{ini}})|_{\{x \cdot e' \geq B\}} = \mathbf{0} \right\},$$

spread at least at speed $c_{\text{inf}}(e')$ and at most at speed $c_{\text{sup}}(e')$, where

$$(23) \quad c_{\text{inf}}(e') = \sup \left\{ c \geq 0 \mid \tilde{\mathbf{v}}_{\text{ini}} \in \mathbf{H}_{e'} \implies \lim_{t \rightarrow +\infty} \inf_{x \cdot e' \leq ct} \tilde{\mathbf{v}}(t, x) = \mathbf{1} \right\},$$

$$(24) \quad c_{\text{sup}}(e') = \inf \left\{ c \geq 0 \mid \tilde{\mathbf{v}}_{\text{ini}} \in \mathbf{H}_{e'} \implies \lim_{t \rightarrow +\infty} \sup_{x \cdot e' \geq ct} \tilde{\mathbf{v}}(t, x) = \mathbf{0} \right\}.$$

Let us prove now that

$$c_{\text{inf}}(e') = c_{\text{sup}}(e') = c_{e'}^* \quad \text{for any } e' \in \mathbb{S}^{n-1}.$$

More precisely, since $c_{\text{inf}}(e') \leq c_{\text{sup}}(e')$ is clear, we are going to prove $c_{\text{sup}}(e') \leq c_{e'}^*$ on one hand and $c_{\text{inf}}(e') \geq c_{e'}^*$ on the other hand.

The inequality $c_{\text{sup}}(e') \leq c_{e'}^*$ follows from the super-solution

$$\bar{\mathbf{v}} : (t, x) \mapsto M e^{-\lambda_1 z^* t} e^{z^* \cdot x} \mathbf{u}_{z^*}(t, x),$$

where $z^* = -\mu_{e'}^* e'$ and with $M > 0$ so large that $\bar{\mathbf{v}}(0, \cdot) \geq \mathbf{v}_{\text{ini}}$. Here $\mu_{e'}^*$ denotes obviously the unique μ^* associated with direction e' given by Lemma 2.8. In view

of the invertible change of unknown $\mathbf{v} = \tilde{\mathbf{v}} \circ \mathbf{v}^*$ and of the above calculations, the inequality $\tilde{\tilde{\mathbf{v}}} \geq \tilde{\mathbf{v}}$ is clear globally in $[0, +\infty) \times \mathbb{R}^n$. Moreover, by definition of $c_{e'}^*$,

$$\exp(-\lambda_{1,z^*} t) \exp(-\mu_{e'}^* x \cdot e') = \exp(-\mu_{e'}^* (x \cdot e' - c_{e'}^* t)),$$

whence the super-solution spreads exactly at speed $c_{e'}^*$ and, in view of its exponential decay at $x \cdot e' = +\infty$, this implies $c_{\text{sup}}(e') \leq c_{e'}^*$.

The inequality $c_{\text{inf}}(e') \geq c_{e'}^*$ follows similarly from the construction of a Heaviside-like sub-solution that spreads at some speed $c < c_{e'}^*$. Since this construction is quite long – density of directions e' meeting the spatial periodicity network, approximation in straight cylinders in direction e' , existence of principal eigenfunctions in such cylinders by complex analysis arguments, and finally continuation of the spreading speed estimate for directions e' that do not meet the spatial periodicity network – but analogous to the scalar construction in [3, Section 4.2], we do not detail this construction. Let us just mention two specificities of the vector setting worthy of attention:

- The spectral approximation in cylinders of increasing radius, stated in [3, Proposition 4.4], is proved thanks to ratios of scalar quantities whose vector generalization seems *a priori* unclear. It can however be noticed that this proof is analogous to that of the spectral approximation in balls of increasing radius, that we already generalized to the vector setting in [13, Proposition 3.9]. Adapting the proof of [13, Proposition 3.9] to cylinders is straightforward.
- Different components of the oscillating eigenfunction might vanish at different locations, so that by taking the positive part we do not obtain in general the solution of a linear Dirichlet problem. However, we already showed in the proof of the cooperative persistence result in Section 2.6 above that the coexistence, at some location, of components that are locally identically zero and of components that are locally positive is actually not an obstacle when constructing sub-solutions.

Consequently,

$$c_{\text{inf}}(e') = c_{\text{sup}}(e') = c_{e'}^* \quad \text{for all } e' \in \mathbb{S}^{n-1}.$$

We are now in a position to apply [5, Theorem 2.3] and obtain the Freidlin–Gärtner formula for the spreading speed c_e^{FG} in the direction e of solutions of the cooperative system $\tilde{\mathcal{Q}}\tilde{\mathbf{v}} = -\text{diag}(\tilde{\mathbf{g}})(\tilde{\mathbf{v}} \circ \tilde{\mathbf{v}})$ with compactly supported initial data. Going back to the original unknown \mathbf{v} , we deduce as claimed the Freidlin–Gärtner formula for the spreading speed in the direction e of solutions of the cooperative system $\mathcal{Q}\mathbf{v} = -\text{diag}(\mathbf{g})(\mathbf{v} \circ \mathbf{v})$ with compactly supported initial data. \square

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