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Calculus of Compatible Nilpotents

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Abstract

The concept of *compatible null vectors* reflects in many ways the standard definition of *dual vector spaces* in linear algebra, with one important difference. Whereas the usual concept of duality between vector spaces ties together the structure of these distinct vector spaces in a unique way, the concept of compatible null vectors ties together properties between two geometric algebras defined on the same vector space. The purpose of this work is to further explore the strange properties that reveal themselves when a canonical basis of null vectors is chosen instead of the standard basis of orthonormal vectors in a geometric algebra of a Minkowski space defined by a Lorentz metric.

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Keywords: Clifford geometric algebra, Grassmann algebra, Lorentzian spacetime.

0 Introduction

The concept of *compatible null vectors* reflects in many ways the standard definition of *dual vector spaces* in linear algebra, with one important difference. Whereas the usual concept of duality between vector spaces ties together the structure of these distinct vector spaces in a unique way, the concept of compatible null vectors ties together properties between two geometric algebras defined on the *same vector space*. The purpose of this work is to further explore the strange properties that reveal themselves when a canonical basis of null vectors is chosen instead of the standard basis of orthonormal vectors in a geometric algebra of a Minkowski space defined by a Lorentz metric [1, 2].

Section 1, defines and characterizes the properties of the algebra of positive and negatively correlated nilpotents of the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$. Whereas linear algebra has deep roots in the works of Grassmann, Hamilton, Clifford and Cayley, the importance of early works of Grassmann and Clifford

have not been not been fully recognized until fairly recently [3], [4]. Whereas the invention of multiplication of matrices by Cayley permeates elementary linear algebra today, it is shown here that linear algebra could have developed from Grassmann algebra together with the quite different Multiplication Tables of compatible null vectors.

Section 2, develops the basic properties of the *vector derivative* or *gradient operator*, the basic tool of calculus that has been developed in the early works [5], [6], and other authors [7], and by the Clifford analysis community [8]. A more detailed history can be found in [7]. Whereas usually the vector derivative is employed in the geometric algebra of a Euclidean space, here its properties are developed in the geometric algebra of a Lorentz-Minkowski space. It is shown how a slight perturbation of the metric on Minkowski space leads to a Euclidean structure on the corresponding geometric algebra, perhaps offering a new mathematical formalism for quantum mechanics.

Section 3, sets down basic differential formulas for elementary functions of the position vector x , expressed in the basis of compatible null vectors. All of the formulas developed previously in the geometric algebra of Euclidean space, appropriately modified, can be used in the setting of the geometric algebra of the Lorentz-Minkowski space.

1 The geometric algebras $\mathbb{G}_{n,1}$ and $\mathbb{G}_{1,n}$

- *Nilpotents* are algebraic quantities $x \neq 0$ with the property that $x^2 = 0$. They are added together using the same rules for the *addition and multiplication by scalars* \mathcal{F} , as the real or complex numbers. The trivial nilpotent is denoted by 0.
- A set of nilpotents $\mathcal{A}_n := \{\mathbf{a}_1, \dots, \mathbf{a}_n\}_{\mathcal{F}}$ is said to be *multiplicatively uncorrelated* over \mathcal{F} , if for all $\mathbf{a}_i, \mathbf{a}_j \in \mathcal{A}_n$,

$$\mathbf{a}_i \mathbf{a}_j + \mathbf{a}_j \mathbf{a}_i = 0. \quad (1)$$

A set of uncorrelated nilpotents over a field \mathcal{F} are called *null vectors*, and generate a *Grassmann algebra* $\mathcal{G}_n(\mathcal{F})$, provided they are *linearly independent over* \mathcal{F} , satisfying

$$\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_n \neq 0, \quad (2)$$

[9]. More general fields \mathcal{F} can be considered as long as *characteristic* $\mathcal{F} \neq 2$.

- A set $\mathcal{A}_{n+1}^{\pm} := \{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\}$ of $n+1$ null vectors is said to be *positively* or *negatively correlated* if

$$\mathbf{a}_i \mathbf{a}_j + \mathbf{a}_j \mathbf{a}_i = 2\mathbf{a}_i \cdot \mathbf{a}_j = \pm(1 - \delta_{ij}), \quad (3)$$

respectively. They generate the 2^{n+1} -dimensional *Clifford geometric algebras* $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$, [10].

Table 1: PC Multiplication table.

	\mathbf{a}_i	\mathbf{a}_j	$\mathbf{a}_i\mathbf{a}_j$	$\mathbf{a}_j\mathbf{a}_i$
\mathbf{a}_i	0	$\mathbf{a}_i\mathbf{a}_j$	0	\mathbf{a}_i
\mathbf{a}_j	$\mathbf{a}_j\mathbf{a}_i$	0	\mathbf{a}_j	0
$\mathbf{a}_i\mathbf{a}_j$	\mathbf{a}_i	0	$\mathbf{a}_i\mathbf{a}_j$	0
$\mathbf{a}_j\mathbf{a}_i$	0	\mathbf{a}_j	0	$\mathbf{a}_j\mathbf{a}_i$

Table 2: NC Multiplication table.

	\mathbf{a}_i	\mathbf{a}_j	$\mathbf{a}_i\mathbf{a}_j$	$\mathbf{a}_j\mathbf{a}_i$
\mathbf{a}_i	0	$\mathbf{a}_i\mathbf{a}_j$	0	$-\mathbf{a}_i$
\mathbf{a}_j	$\mathbf{a}_j\mathbf{a}_i$	0	$-\mathbf{a}_j$	0
$\mathbf{a}_i\mathbf{a}_j$	$-\mathbf{a}_i$	0	$-\mathbf{a}_i\mathbf{a}_j$	0
$\mathbf{a}_j\mathbf{a}_i$	0	$-\mathbf{a}_j$	0	$-\mathbf{a}_j\mathbf{a}_i$

Given below are the multiplication tables for sets of positively (PC), or negatively (NC), correlated null vectors $\mathbf{a}_i, \mathbf{a}_j$, for $1 \leq i < j \leq n+1$, [11]. For a set of positively or negatively correlated null vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\}$, define

$$A_k := \sum_{i=1}^k \mathbf{a}_i.$$

The geometric algebra

$$\mathbb{G}_{1,n} := \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{f}_n),$$

where $\{\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{f}_n\}$ is the *standard basis* of anticommuting orthonormal vectors, with $\mathbf{e}_1^2 = 1$ and $\mathbf{f}_1^2 = \dots = \mathbf{f}_n^2 = -1$, [12, p.71]. Alternatively, the geometric algebra $\mathbb{G}_{1,n}$ can be defined by

$$\mathbb{G}_{1,n} := \mathbb{R}(\mathbf{a}_1, \dots, \mathbf{a}_{n+1}) = \mathcal{A}_{n+1}^+,$$

where $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\}$ is a set of positively correlated null vectors satisfying the multiplication Table 1. In this case, the standard basis vectors of $\mathbb{G}_{1,n}$ can be defined by $\mathbf{e}_1 = \mathbf{a}_1 + \mathbf{a}_2 = A_2$, $\mathbf{f}_1 = \mathbf{a}_1 - \mathbf{a}_2 = A_1 - \mathbf{a}_2$, and for $2 \leq k \leq n$

$$\mathbf{f}_k = \alpha_k \left(A_k - (k-1)\mathbf{a}_{k+1} \right), \quad (4)$$

where $\alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}$.

The geometric algebra

$$\mathbb{G}_{n,1} := \mathbb{R}(\mathbf{f}_1, \mathbf{e}_1, \dots, \mathbf{e}_n),$$

where $\{\mathbf{f}_1, \mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the *standard basis* of anticommuting orthonormal vectors, with $\mathbf{f}_1^2 = -1$ and $\mathbf{e}_1^2 = \dots = \mathbf{e}_n^2 = 1$. Alternatively, the geometric algebra $\mathbb{G}_{n,1}$ can be defined by

$$\mathbb{G}_{n,1} := \mathbb{R}(\mathbf{a}_1, \dots, \mathbf{a}_{n+1}) = \mathcal{A}_{n+1}^-,$$

where $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+1}\}$ is a set of negatively correlated null vectors satisfying the multiplication Table 2. In this case, the standard basis vectors of $\mathbb{G}_{n,1}$ can be defined by $\mathbf{f}_1 = \mathbf{a}_1 + \mathbf{a}_2 = A_2$, $\mathbf{e}_1 = \mathbf{a}_1 - \mathbf{a}_2 = A_1 - \mathbf{a}_2$, and for $2 \leq k \leq n$

$$\mathbf{e}_k = \alpha_k \left(A_k - (k-1)\mathbf{a}_{k+1} \right), \quad (5)$$

where $\alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}$.

It is well-known that Clifford's geometric algebras $\mathbb{G}_{p,q}$ are algebraically isomorphic to matrix algebras over the real or complex numbers, [12, Ch.4]. For this reason, matrices over geometric algebra modules is well defined and fully compatible with the usual rules of matrix multiplication and addition.

1.1 Change of basis formulas for $\mathbb{G}_{1,n}$

$$\begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_8 \end{pmatrix} = T_8 \begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_7 \end{pmatrix}$$

for

$$T_8 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{5}}{2\sqrt{2}} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{10}} & \sqrt{\frac{3}{5}} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{10}} & \frac{1}{2\sqrt{15}} & \frac{\sqrt{7}}{2\sqrt{3}} \end{pmatrix},$$

and

$$\begin{pmatrix} e_1 \\ f_1 \\ \cdot \\ \cdot \\ f_7 \end{pmatrix} = T_8^{-1} \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_8 \end{pmatrix},$$

for

$$T_8^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & 2\sqrt{\frac{2}{5}} & 0 & 0 \\ -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \sqrt{\frac{5}{3}} & 0 \\ -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & 2\sqrt{\frac{3}{7}} \end{pmatrix},$$

[11].

We now turn our attention to the study of geometric calculus in the algebra of nilpotents $\mathcal{A}_{n+1}^+ \equiv \mathbb{G}_{1,n}$ over the real numbers \mathbb{R} .

2 Gradient operators in $\mathcal{A}_{n+1}^+ = \{a_1, \dots, a_{n+1}\}$

Note, from here-on null vectors are not denoted in bold face. Let \mathcal{A}_{n+1}^+ be a set of $n + 1$ linearly independent, positively correlated null vectors, satisfying (2) and (3). The positive compatibility property

$$a_i a_j + a_j a_i = 2a_i \cdot a_j = (1 - \delta_{ij}),$$

guarantees that a_i are null vectors, with the inner product of distinct pairs equal to $\frac{1}{2}$, and in addition, they satisfy the multiplication Table 1. For simplicity, we only consider $\mathcal{F} := \mathbb{R}$ the real number system.

Given any two distinct $a_i, a_j \in \mathcal{A}_n^+$, let $e := a_i + a_j$ and $f := a_i - a_j$. It easily follows that

$$e^2 = (a_i + a_j)(a_i + a_j) = a_i^2 + (a_i a_j + a_j a_i) + a_j^2 = 1, \quad (6)$$

and

$$f^2 = (a_i - a_j)(a_i - a_j) = a_i^2 - (a_i a_j + a_j a_i) + a_j^2 = -1. \quad (7)$$

Furthermore, the orthonormal vectors e and f are *anticommutative*,

$$ef = (a_i + a_j)(a_i - a_j) = a_j a_i - a_i a_j = -(a_i a_j - a_j a_i) = -fe. \quad (8)$$

Once the *seed null vectors* a_i and a_j are chosen determining e and f , the other null vectors a_k used to define the successive vectors f_k 's can randomly be chosen using the recursive definition (4).

The fundamental *gradient operator* ∇ in $\mathbb{G}_{1,n}$ is defined by

$$\nabla := e_1 \partial_{s_1} - \sum_{i=1}^n f_i \partial_{s_{i+1}}, \quad (9)$$

for the *position vector*

$$x = s_1 e_1 + \sum_{i=1}^n s_{i+1} f_i \in \mathbb{G}_{1,n}^1, \quad (10)$$

from which it follows that

$$\nabla x = n + 1 \iff \nabla \cdot x = n + 1, \text{ and } \nabla \wedge x = 0.$$

Regardless of whether the gradient, or *vector derivative*, is defined in terms of the Euclidean metric in \mathbb{R}^{n+1} of the geometric algebra \mathbb{G}_{n+1} , or in terms of the Lorentz metric in $\mathbb{R}^{1,n}$ of the geometric algebra $\mathbb{G}_{1,n}$ it is characterized by two fundamental properties:

ONE: The gradient ∇ has the properties of a *vector*

$$x \in \mathbb{G}_{p,q}^1 \text{ and } \nabla x = p + q \quad (11)$$

the dimension of \mathbb{R}^{p+q} , *independent* of the metric in which it is calculated.

TWO: The *v-directional derivative* is the *scalar differential operator* $v \cdot \nabla$, and applied to the vector

$$x \in \mathbb{G}_{p,q}^1 \text{ gives } v \cdot \nabla x = v \in \mathbb{G}_{p,q}^1, \quad (12)$$

independent of the metric in which it is calculated.¹

Applied to the position vector $x \in \mathbb{R}^{n+1}$ of \mathbb{G}_{n+1} or a position vector $x \in \mathbb{R}^{1,n}$ of $\mathbb{G}_{1,n}$, the gradient of x counts the dimension of the space,

$$\nabla x = \nabla \cdot x + \nabla \wedge x = n + 1 \implies \nabla \wedge x = 0, \quad (13)$$

independent of the metric employed. Similarly, Given a vector $v \in \mathbb{G}_{n+1}^1$ or $v \in \mathbb{G}_{1,n}^1$, respectively,

$$v \cdot \nabla x = v, \quad (14)$$

independent of the metric employed. It follows that all differentiation formulas worked out in any one metric apply equally to any other metric, [6, p.63-66]

The gradient ∇ is closely related to two other *vector derivatives* which make their appearance in \mathcal{A}_{n+1}^+ . The *null gradient*

$$\hat{\nabla} := \sum_{i=1}^{n+1} a_i \partial_i, \quad (15)$$

where $\partial_i := \frac{\partial}{\partial x_i}$, for the position vector

$$x = \sum_{i=1}^{n+1} x_i a_i \in \mathcal{A}_{n+1}^+. \quad (16)$$

¹The idea of the gradient ∇ , or *vector derivative*, being a fundamental tool in the study of linear algebra was first developed in my doctoral thesis at Arizona State University (1971). I have published it on my website at <https://www.garretstar.com/secciones/publicaciones/publicaciones.html>

It follows that $\hat{\nabla}x = 0$, or equivalently, $\hat{\nabla} \cdot x = 0 = \hat{\nabla} \wedge x$. The additive *dual sum gradient*

$$\check{\nabla} := \sum_{i=1}^{n+1} \check{a}_i \partial_i, \quad (17)$$

where $\check{a}_i := A_{n+1} - a_i$ for $1 \leq i \leq n+1$. For example,

$$\check{a}_3 = A_{3+1} - a_3 = a_1 + a_2 + a_4.$$

In working in the calculus of the algebra \mathcal{A}_{n+1}^+ , the *partial differential sum operator*,

$$\partial_{(n+1)} := \sum_{i=1}^{n+1} \partial_i = \partial_1 + \cdots + \partial_{n+1} \quad (18)$$

is of fundamental importance. Noting that for $v = v_1 a_1 + \cdots + v_{n+1} a_{n+1}$,

$$v \cdot A_{n+1} = \sum_{i=1}^{n+1} v_i a_i \cdot A_{n+1} = \sum_{i=1}^{n+1} \frac{n}{2} v_i = \frac{n}{2} \check{v},$$

where we have introduced the *dual-like* notation

$$\check{v} := \sum_{i=1}^{n+1} v_i \quad \text{and} \quad \check{v}_i := v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_{n+1}, \quad (19)$$

for the vector $v = v_1 a_1 + \cdots + v_{n+1} a_{n+1} \in \mathcal{A}_{n+1}^+$. Whereas v is a vector, its dual \check{v} is a scalar, but *both* v_i and its dual \check{v}_i are scalars. Noting that

$$\partial_{(n+1)} x = A_{n+1}, \quad \text{and} \quad A_{n+1} \cdot \hat{\nabla} = \frac{n}{2} \partial_{(n+1)}, \quad (20)$$

it follows that

$$A_{n+1} \cdot \hat{\nabla} x = \frac{n}{2} A_{n+1} \quad \text{and} \quad \partial_{(n+1)} x \cdot A_{n+1} = \frac{(n+1)n}{2}.$$

Another calculation gives the basic result

$$\check{\nabla} x = \binom{n+1}{2} := \frac{(n+1)n}{2}.$$

2.1 Basic decomposition formulas

The dual sum and null gradients satisfy the basic property

$$\check{\nabla} + \hat{\nabla} = A_{n+1} \partial_{(n+1)}, \quad (21)$$

which easily follow from the definitions. In view of (21), we introduce the *vector partial dual sum gradient*

$$\nabla_{(n+1)} := A_{n+1} \partial_{(n+1)}, \quad (22)$$

giving the fundamental relationship

$$\check{\nabla} + \hat{\nabla} = \nabla_{(n+1)}$$

between the three vector derivative operators.

Squaring this last equation gives

$$(\check{\nabla} + \hat{\nabla})^2 = \check{\nabla}^2 + 2\check{\nabla} \cdot \hat{\nabla} + \hat{\nabla}^2 = \nabla_{(n+1)}^2 = \frac{(n+1)n}{2} \partial_{(n+1)}^2, \quad (23)$$

where

$$\hat{\nabla}^2 = \sum_{1 \leq i < j \leq n+1} \partial_i \partial_j, \quad (24)$$

$$\check{\nabla}^2 = \left(\sum_{i=1}^{n+1} \check{a}_i \partial_i \right)^2 = \frac{n(n-1)}{2} \sum_{i=1}^{n+1} \partial_i^2 + (n^2 - n + 1) \sum_{i < j}^{n+1} \partial_i \partial_j, \quad (25)$$

and

$$\check{\nabla} \cdot \hat{\nabla} = \frac{n}{2} \sum_{i=1}^{n+1} \partial_i^2 + (n-1) \sum_{i < j}^{n+1} \partial_i \partial_j. \quad (26)$$

In doing the above calculations for (23)-(26), we have used the fundamental relationships

$$\partial_{(n+1)}^2 = \left(\partial_1 + \cdots + \partial_{n+1} \right)^2 = \sum_{i=1}^{n+1} \partial_i^2 + 2 \sum_{i < j}^{n+1} \partial_i \partial_j \quad (27)$$

and

$$\check{a}_i^2 = \frac{n(n-1)}{2}, \quad \sum_{i < j}^{n+1} \check{a}_i \cdot \check{a}_j = \frac{n^2 - n + 1}{2}. \quad (28)$$

The *directional dual sum* and *directional null* gradients are given by

$$v \cdot \check{\nabla} = \sum_{i,j=1}^{n+1} v_i a_i \cdot \check{a}_j \partial_j = \sum_{i,j=1}^{n+1} \left(v_i a_i \cdot A_{n+1} - v_i a_i \cdot a_j \right) \partial_j = \frac{n}{2} \check{v} \partial_{(n+1)} - v \cdot \hat{\nabla}, \quad (29)$$

and

$$v \cdot \hat{\nabla} = \sum_{i=1}^{n+1} \frac{1}{2} \left(\check{v}_i \partial_1 + \cdots + \check{v}_{n+1} \partial_{n+1} \right) = \frac{1}{2} \left(\check{v} \partial_{(n+1)} - (v_1 \partial_1 + \cdots + v_{n+1} \partial_{n+1}) \right), \quad (30)$$

respectively. Applying (29) and (30) to the position vector x , gives

$$v \cdot \hat{\nabla} x = \frac{1}{2} \left(\check{v} A_{n+1} - v \right) = \hat{\nabla} x \cdot v, \quad (31)$$

and

$$v \cdot \check{\nabla} x = \frac{n-1}{2} \check{v} A_{n+1} + \frac{1}{2} v = \check{\nabla} x \cdot v, \quad (32)$$

since $\hat{\nabla} \wedge x = \check{\nabla} \wedge x = 0$. Taking the sum of these last two expressions, gives

$$v \cdot \nabla_{(n+1)} x = \frac{n}{2} A_{n+1} - \frac{1}{2} v = \nabla_{(n+1)} x \cdot v, \quad (33)$$

for the directional derivative of the dual sum gradient, since $\nabla_{(n+1)} \wedge x = 0$.

We are now in a position to state the fundamental relationship between the dual sum and null gradients to the gradient (9) in $\mathbb{G}_{1,n}$,

$$\nabla = 2\check{\nabla} - \frac{2(n-1)}{n} \nabla_{(n+1)} = -2\hat{\nabla} + \frac{2}{n} \nabla_{(n+1)} = \check{\nabla} - \hat{\nabla} - \frac{n-2}{n} \nabla_{(n+1)}. \quad (34)$$

Other relationships between the three gradients that easily follow are

$$\hat{\nabla} = -\frac{1}{2} \nabla + \frac{1}{n} \nabla_{(n+1)}, \quad \text{and} \quad \check{\nabla} = \frac{1}{2} \nabla + \frac{n-1}{n} \nabla_{(n+1)}.$$

The *Laplacian* ∇^2 can now be expressed in terms the scalar differential operator $\partial_{(n+1)}$ and the *dual-null Laplacian* $\hat{\nabla}^2$, [12, p.121]. Using (34),

$$\begin{aligned} \nabla^2 &= \frac{4}{n^2} \nabla_{(n+1)}^2 - \frac{8}{n} A_{n+1} \cdot \hat{\nabla} \partial_{(n+1)} + 4\hat{\nabla}^2 \\ &= \frac{2}{n} (n+1) \partial_{(n+1)}^2 - 4 \partial_{(n+1)}^2 + 4\hat{\nabla}^2 = 4\hat{\nabla}^2 - \frac{2(n-1)}{n} \partial_{(n+1)}^2. \end{aligned} \quad (35)$$

It follows that the Laplacian reduces to 4 times the null Laplacian for any function $f(x)$ with the property that $\partial_{(n+1)}^2 f(x) = 0$. In addition, the Laplacian factors into the product of two first order differential operators:

$$\nabla^2 = \left(2\hat{\nabla} + \sqrt{\frac{2(n-1)}{n}} \partial_{(n+1)} \right) \left(2\hat{\nabla} - \sqrt{\frac{2(n-1)}{n}} \partial_{(n+1)} \right), \quad (36)$$

opening the door to a new class of solutions to the Laplace equation in $\mathbb{G}_{1,n}$, [13, 14, 15].

For $n = 2$,

$$\nabla^2 = \left(2\hat{\nabla} + \partial_{(3)} \right) \left(2\hat{\nabla} - \partial_{(3)} \right), \quad (37)$$

and for $n = 3$, the case of Dirac algebra of spacetime, (36) reduces to

$$\nabla^2 = 4 \left(\hat{\nabla} + \frac{1}{\sqrt{3}} \partial_{(4)} \right) \left(\hat{\nabla} - \frac{1}{\sqrt{3}} \partial_{(4)} \right), \quad (38)$$

giving a new factorization of the Laplacian in \mathcal{A}_4^+ .

2.2 The Euclidean geometric algebra \mathbb{G}_{n+1}^*

Using the dual-like notation introduced in (19), the geometric product of two vectors $v, w \in \mathcal{A}_{n+1}^+$ takes the interesting form

$$vw = \frac{1}{2} \sum_{i,j=1}^{n+1} v_i w_j (1 - \delta_{ij}) + \sum_{i < j}^{n+1} (v_i w_j - v_j w_i) a_i \wedge a_j$$

$$= \frac{1}{2} \left(\overset{\vee}{v}\overset{\vee}{w} - \sum_{i=1}^{n+1} v_i w_i \right) + \sum_{i < j}^{n+1} (v_i w_j - v_j w_i) a_i \wedge a_j. \quad (39)$$

It follows that

$$v \cdot w = \frac{1}{2} (vw + wv) = \frac{1}{2} \left(\overset{\vee}{v}\overset{\vee}{w} - \sum_{i=1}^{n+1} v_i w_i \right) = \frac{1}{2} \sum_{i < j}^{n+1} (v_i w_j + v_j w_i), \quad (40)$$

and

$$v \wedge w = \frac{1}{2} (vw - wv) = \sum_{i < j}^{n+1} (v_i w_j - v_j w_i) a_i \wedge a_j. \quad (41)$$

Using (40) of $\mathbb{G}_{1,n}$, a new Euclidean *inner product* $\langle v, w \rangle$ can be defined,

$$\langle v, w \rangle := \sum_{i=1}^{n+1} v_i w_i = \overset{\vee}{v}\overset{\vee}{w} - 2v \cdot w = \overset{\vee}{v}\overset{\vee}{w} - \sum_{i < j}^{n+1} (v_i w_j + v_j w_i). \quad (42)$$

Using this \star -inner product the \star -geometric algebra $\mathbb{G}_{n+1,0}^*$ makes its appearance by defining the \star -geometric product of $v, w \in \mathbb{G}_{n+1}^*$ by

$$v \star w := \langle v, w \rangle + v \wedge w = \overset{\vee}{v}\overset{\vee}{w} - \sum_{i < j}^{n+1} (v_i w_j + v_j w_i) + \sum_{i < j}^{n+1} (v_i w_j - v_j w_i) a_i \wedge a_j,$$

or more simply,

$$v \star w := (\overset{\vee}{v}\overset{\vee}{w} - 2v \cdot w) + v \wedge w = (\overset{\vee}{v}\overset{\vee}{w} - 3v \cdot w) + vw, \quad (43)$$

for which case the basis null vectors $\{a_1, \dots, a_{n+1}\} \in \mathbb{G}_{1,n}$ take on the roll of the orthonormal basis vectors of \mathbb{G}_{n+1}^* .

It is interesting to note that the *grading* of the elements of these respective geometric algebras remains the same, i.e., a k -vector in $\mathbb{G}_{1,n}$ remains the same k -vector in \mathbb{G}_{n+1}^* , but the algebras have completely different structures. Other interesting issues arise as well. Recalling the definition (1) of a null vector, as well as the random selection of the seed null vectors a_i and a_j in (6) and (7), new possibilities arise. In Hestenes' STA algebra $\mathbb{G}_{1,3}$, algebraically isomorphic to the Dirac algebra, each time-like unit vector $e = \gamma_0$ determines a unique *rest frame* or *inertial system* in Minkowski spacetime [16].

At the level of the even Pauli sub-algebra, choosing a γ_0 determines a unique splitting of the algebra into the *relative* components of a *vector* and a *bivector* of an *observer* [12, p.123], [17, 18], [19, 20]. On the other hand, the scalar inner product structure in (42) of such a splitting determines the star Euclidean geometric algebra \mathbb{G}_4^* , [21, 22]. The intriguing question arises whether this might be the missing link to the interpretation of the wave-particle dual nature of matter in quantum mechanics, and in addition provide the key to unification of quantum mechanics with Einstein's general relativity? See Figure 1. Further speculation suggests interpreting the geometric algebra $\mathbb{G}_{3,1} \equiv \mathcal{A}_{3,1}^-$ as being related to antimatter, [23, 24].

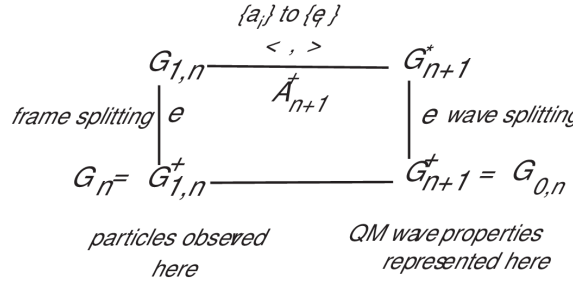


Figure 1: The bottom left side is the even sub-geometric algebra of the rest frame of an observer (particle side). The right side is the quantum mechanical Euclidean geometric algebra of that rest frame (wave side).

3 Differentiation of basic functions

In the previous section, basic relationships between the four fundamental differential operators, the gradient or vector derivative ∇ , the null or hat gradient $\hat{\nabla}$, the dual sum gradient $\check{\nabla}$, and the *scalar sum partial derivative* $\partial_{(n+1)}$ have been established. In order to efficiently proceed, differential formulas for elementary functions need to be developed.

We begin with a sample of formulas which have been developed elsewhere for the vector derivative ∇ in Euclidean space, [6, p.66]. In adapting the formulas in this reference, the dimension of the space \mathbb{R}^{1+n} is taken to be $n + 1$, for the position vector $x = x_1 a_1 + \cdots + x_{n+1} a_{n+1} \in \mathcal{A}_{n+1}^+$, and the unit vector

$$\hat{x} := \nabla|x| = \frac{x}{\sqrt{|x^2|}} \iff x = |x|\hat{x}, \quad (44)$$

where $|x| := \sqrt{|x^2|}$. The absolute value $|x^2|$ is necessary because only in the case of the positive definite Euclidean space is $x^2 \geq 0$ for all $x \in \mathbb{R}^{n+1}$. For simplicity, the formulas given here are only valid for more general spaces \mathbb{R}^{p+q} when $x^2 \geq 0$, when x is a point in the *positive light cone* of the metric space.

$$\nabla|x|^k = k|x|^{k-1}\hat{x}, \quad \nabla|x|^k x = (n+k+1)|x|^k. \quad (45)$$

$$v \cdot \nabla|x|^k = k|x|^{k-1}v \cdot \hat{x}, \quad v \cdot \nabla|x|^k x = k|x|^{k-1}v \cdot \hat{x} x + |x|^k v. \quad (46)$$

$$\nabla \ln|x| = \frac{x}{|x|^2}, \quad \nabla e^x = e^x + n \frac{\sinh|x|}{|x|}. \quad (47)$$

$$\nabla v x = -(n-1)v, \quad \nabla(v \wedge w)x = (n-3)v \wedge w. \quad (48)$$

The same sample of differential formulas is now given for the null, dual sum gradients, as well as scalar differential sum operator. For the null gradient $\hat{\nabla}$, with the help of (32) and (34), we first calculate $\hat{\nabla}x^2 = \check{x}A_{n+1} - x$. It follows that

$$\hat{\nabla}|x^2| = \hat{\nabla}x^2 = \check{x}A_{n+1} - x = 2|x|\hat{\nabla}|x| \iff \hat{\nabla}|x| = \frac{\check{x}A_{n+1} - x}{2|x|} \quad (49)$$

whenever x is on the *positive light cone* for which case $x^2 > 0$. If we evaluate the right side of (49) at $x = A_{n+1} + y_1 a_1$, for a constant $y_1 \in \mathbb{R}$, we find that

$$\hat{\nabla}|x| = \frac{(n+1+y_1)A_{n+1} - A_{n+1} - y_1 a_1}{2|A_{n+1} + y_1 a_1|} = \frac{(n+y_1)A_{n+1} - y_1 a_1}{2|A_{n+1} + y_1 a_1|}, \quad (50)$$

which is quite different than the definition of the unit vector \hat{x} found in (44).

Using that

$$(A_{n+1} + y_1 a_1)^2 = A_{n+1}^2 + 2y_1 a_1 \cdot A_{n+1} = \frac{1}{2}(n(n+2y_1+1)),$$

it follows that (50) simplifies to

$$\hat{\nabla}|x| = \frac{(n+y_1)A_{n+1} - y_1 a_1}{2|A_{n+1} + y_1 a_1|} = \frac{(n+y_1)A_{n+1} - y_1 a_1}{\sqrt{2}\sqrt{n(n+2y_1+1)}}. \quad (51)$$

It is also interesting to calculate

$$(\hat{\nabla}|x|)^2 = \frac{(n+y_1)(n^2 + (1+y_1)n - y_1)}{4(n+2y_1+1)}, \quad (52)$$

which reduces to $\frac{n^2}{4}$ when $y_1 = 0$ and $x = A_{n+1}$. In this case

$$\left(\hat{\nabla}|x|\right)_{x=A_{n+1}} = \frac{\sqrt{n}A_{n+1}}{\sqrt{2}\sqrt{n+1}}.$$

With (49) and (51) in hand, similar formulas like (45)-(48) easily follow for the other vector derivative operators.

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