Exact Generalization Guarantees for (Regularized) Wasserstein Distributionally Robust Models

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Abstract

Wasserstein distributionally robust estimators have emerged as powerful models for prediction and decision-making under uncertainty. These estimators provide attractive generalization guarantees: the robust objective obtained from the training distribution is an exact upper bound on the true risk with high probability. However, existing guarantees either suffer from the curse of dimensionality, are restricted to specific settings, or lead to spurious error terms. In this paper, we show that these generalization guarantees actually hold on general classes of models, do not suffer from the curse of dimensionality, and can even cover distribution shifts at testing. We also prove that these results carry over to the newly-introduced regularized versions of Wasserstein distributionally robust problems.

1 Introduction

1.1 Generalization and (Wasserstein) Distributionally Robust Models

We consider the fundamental question of generalization of machine learning models. Let us denote by f_θ the loss induced by a model parametrized by θ for some uncertain variable ξ (typically a data point). When ξ follows some distribution P, seeking the best parameter θ writes as minimizing the expected loss

$$\min_{\theta\in\Theta} \mathbb{E}_{\xi\sim\mathrm{P}}[f_\theta(\xi)]\,.$$

We usually do not have a direct knowledge of P but rather we have access to samples $(\xi_i)_{i=1}^n$ independently drawn from P. The empirical risk minimization approach then consists in minimizing the expected loss over the associated empirical distribution $\mathrm{P_n} = \frac{1}{n}\sum_{i=1}^n \delta_{\xi_i}$ (as a proxy for the expected loss over P), i.e.,

$$\min_{\theta\in\Theta} \mathbb{E}_{\xi\sim\mathrm{P_n}}[f_\theta(\xi)] \quad \left(= \frac{1}{n}\sum_{i=1}^n f_\theta(\xi_i)\right).$$

Classical statistical learning theory ensures that, with high probability, $\mathbb{E}_{\mathrm{P}}[f_\theta]$ is close to $\mathbb{E}_{\mathrm{P_n}}[f_\theta]$ up to $\mathcal{O}(1/\sqrt{n})$ error terms, see e.g., the monographs Boucheron et al. (2013); Wainwright (2019).

A practical drawback of empirical risk minimization is that it can lead to over-confident decisions (when $\mathbb{E}_{\mathrm{P_n}}[f_\theta] < \mathbb{E}_{\mathrm{P}}[f_\theta]$, the real loss can be higher that the empirical one (Esfahani and Kuhn, 2018)). In addition, this approach is also sensitive to distribution shifts between training and application. To overcome these drawbacks, an approach gaining momentum in machine learning is *distributionally robust* optimization, which consists in minimizing the *worst expectation* of the loss when the distribution lives in a neighborhood of $\mathrm{P_n}$:

$$\min_{\theta\in\Theta} \sup_{\mathrm{Q}\in\mathcal{U}(\mathrm{P_n})} \mathbb{E}_{\xi\sim\mathrm{Q}}[f_\theta(\xi)] \tag{1}$$

Preprint. Under review.

where the inner sup is thus taken over Q in the neighborhood $\mathcal{U}(\mathrm{P_n})$ of $\mathrm{P_n}$ in the space of probability distributions. Popular choices of distribution neighborhoods are based on the Kullback-Leibler (KL) divergence (Laguel et al., 2020; Levy et al., 2020), kernel tools (Zhu et al., 2021a; Staib and Jegelka, 2019; Zhu et al., 2021b), moments (Delage and Ye, 2010; Goh and Sim, 2010), or Wasserstein distance (Shafieezadeh Abadeh et al., 2015; Esfahani and Kuhn, 2018). If $\mathrm{P} \in \mathcal{U}(\mathrm{P_n})$, distributionally robust models can benefit from direct generalization guarantees as,

$$\sup_{\mathrm{Q}\in\mathcal{U}(\mathrm{P_n})} \mathbb{E}_{\xi\sim\mathrm{Q}}[f_\theta(\xi)] \;\geq\; \mathbb{E}_{\xi\sim\mathrm{P}}[f_\theta(\xi)]. \tag{2}$$

Thus, for well-chosen neighborhoods $\mathcal{U}(\mathrm{P_n})$, distributionally robust objectives are able to provide *exact upper-bounds* on the expected loss over distribution P, i.e., the true risk.

Wasserstein distributionally robust optimization (WDRO) problems correspond to (1) with

$$\mathcal{U}(\mathrm{P_n}) = \{\mathrm{Q} \in \mathcal{P}(\Xi) : W(\mathrm{P_n}, \mathrm{Q}) \leq \rho\} \ ,$$

where $W(\mathrm{P_n}, \mathrm{Q})$ denotes the Wasserstein distance between $\mathrm{P_n}$ and Q and $\rho > 0$ controls the required level of robustness around $\mathrm{P_n}$. As a natural metric to compare discrete and absolutely continuous probability distributions, the Wasserstein distance has attracted a lot of interest in both machine learning (Shafieezadeh Abadeh et al., 2015; Sinha et al., 2018; Shafieezadeh-Abadeh et al., 2019; Li et al., 2020; Kwon et al., 2020) and operation research (Zhao and Guan, 2018; Arrigo et al., 2022) communities; see e.g., the review articles Blanchet et al. (2021); Kuhn et al. (2019).

WDRO benefits from out-of-the-box generalization guarantees in the form of (2) since it inherits the concentration properties of the Wasserstein distance. More precisely, under mild assumptions on P (Fournier and Guillin, 2015), $W(\mathrm{P_n}, \mathrm{P}) \leq \rho$ with high probability as soon as $\rho \sim 1/n^{1/d}$ where d denotes the dimension of the samples space. Thus, a major issue is the prescribed radius ρ suffers from the curse of the dimensionality: when d is large, ρ decreases slowly as the number of samples n increases. This scaling is overly conservative for WDRO objectives since Blanchet et al. (2022); Blanchet and Shapiro (2023) prove that a radius behaving as $1/\sqrt{n}$ is asymptotically optimal.

1.2 Contributions and related works

In this paper, we show that WDRO provides exact upper-bounds on the true risk with high probability. More precisely, we prove non-asymptotic generalization bounds of the form of (2), that hold for general classes of functions, and that only require ρ to scale as $1/\sqrt{n}$ and not $1/n^{1/d}$. To do so, we construct an interval for the radius ρ for which it is both sufficiently large so that we can go from the empirical to the true estimator (i.e., at least of the order of $1/\sqrt{n}$) and sufficiently small so that the robust problem does not become degenerate (i.e., smaller than some critical radius, that we introduce as an explicit constant). Our results imply proving concentration results on Wasserstein Distributionally Robust objectives that are of independent interest.

This work is part of a rich and recent line of research about theoretical guarantees on WDRO for machine learning. One of this first results, Lee and Raginsky (2018), provides generalization guarantees, for a general class of models and a fixed ρ, that, however, become degenerate as the radius goes to zero. In the particular case of linear models, WDRO models admit an explicit form that allows Shafieezadeh-Abadeh et al. (2019); Chen and Paschalidis (2018) to provide generalization guarantees (2) with the right scaling for the radius $1/\sqrt{n}$.The case of general classes of models is more intricate. Sinha et al. (2018) showed that a modified version of (2) holds at the price of non-negligible error terms. Gao (2022); An and Gao (2021) made another step towards broad generalization guarantees for WDRO but with error terms that vanish only when ρ goes to zero.

In contrast, our analysis provides exact generalization guarantees without additional error terms, that hold for general classes of functions and allow for a non-vanishing uncertainty radius to cover for distribution shifts at testing. Moreover, our guarantees also carry over to the recently introduced regularized versions of WDRO (Wang et al., 2021; Azizian et al., 2023), whose statistical properties have not been studied yet.

This paper is organized as follows. In Section 2, we introduce notations and our blanket assumptions. In Section 3, we present our main results, an idea of proof, and discussions. The complete proofs are deferred to the appendix.

2 Setup and Assumptions

In this section, we formalize our setting and introduce Wasserstein Distributionally Robust risks.

2.1 Wasserstein Distributionally Robust risk functions

In this paper, we consider as a samples space Ξ a subset of $\mathbb{R}^d$ equipped with the Euclidean norm $\|\cdot\|$. We rely on Wasserstein distances of order 2, in line with the seminal work Blanchet et al. (2022) on generalization of WDRO. This distance is defined for two distributions Q, Q' in the set of probability distributions on Ξ, denoted by $\mathcal{P}(\Xi)$, as

$$W_2(\mathrm{Q}, \mathrm{Q}') := \left(\inf_{\pi \in \mathcal{P}(\Xi \times \Xi), \pi_1 = \mathrm{Q}, \pi_2 = \mathrm{Q}'} \mathbb{E}_{(\xi,\zeta)\sim\pi} \left[\frac{1}{2} \|\xi - \zeta\|^2 \right] \right)^{1/2}$$

where $\mathcal{P}(\Xi \times \Xi)$ is the set of probability distributions in the product space $\Xi \times \Xi$, and π_1 (resp. π_2) denotes the first (resp. second) marginal of π.

We denote by $f : \Xi \to \mathbb{R}$ the loss function of some model over the sample space. The model may depend on some parameter θ, that we drop for now to lighten the notations; instead, we consider a class of functions $\mathcal{F}$ encompassing our various models and losses of interest (we come back to classes of parametric models of the form $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ in Section 3.4).

We define the empirical Wasserstein Distributionally Robust risk $\widehat{\mathcal{R}}_{\rho^2}(f)$ centered on $\mathrm{P_n}$ and similarly the true robust risk $\mathcal{R}_{\rho^2}(f)$ centered on P as

$$\widehat{\mathcal{R}}_{\rho^2}(f) := \sup_{\substack{\mathrm{Q}\in\mathcal{P}(\Xi) \\ W_2^2(\mathrm{P_n},\mathrm{Q})\le\rho^2}} \mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right] \qquad \text{and} \qquad \mathcal{R}_{\rho^2}(f) := \sup_{\substack{\mathrm{Q}\in\mathcal{P}(\Xi) \\ W_2^2(\mathrm{P},\mathrm{Q})\le\rho^2}} \mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right]. \tag{3}$$

Note that $\widehat{\mathcal{R}}_{\rho^2}(f)$, which is based on the empirical distribution $\mathrm{P_n}$, is a computable proxy for the true robust risk $\mathcal{R}_{\rho^2}(f)$. Note also that the true robust risk $\mathcal{R}_{\rho^2}(f)$ immediately upper-bounds the true (non-robust) risk $\mathbb{E}_{\xi\sim\mathrm{P}}\left[f(\xi)\right]$ and also upper-bounds $\mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right]$ for neighboring distributions Q that correspond to distributions shifts of magnitude smaller than ρ in Wasserstein distance.

2.2 Regularized versions

Entropic regularization of WDRO problems was recently used in Wang et al. (2021); Blanchet and Kang (2020); Piat et al. (2022); Azizian et al. (2023). Inspired by the entropic regularization in optimal transport (OT) (Peyré and Cuturi, 2019, Chap. 4), the idea is to regularize the objective by adding a KL divergence, that is defined, for any transport plan $\pi \in \mathcal{P}(\Xi \times \Xi)$ and a fixed reference $\overline{\pi} \in \mathcal{P}(\Xi \times \Xi)$, by

$$\mathrm{KL}(\pi\,|\overline{\pi}) = \begin{cases} \int \log \frac{\mathrm{d}\,\pi}{\mathrm{d}\,\overline{\pi}}\,\mathrm{d}\,\pi & \text{when } \pi \ll \overline{\pi} \\ +\infty & \text{otherwise.} \end{cases}$$

Unlike in OT though, the choice of the reference measure in WDRO is not neutral and introduces a bias in the robust objective (Azizian et al., 2023). For their theoretical convenience, we take reference measures that have Gaussian conditional distributions

$$\pi_\sigma(\mathrm{d}\zeta|\xi) \propto \mathbb{1}_{\zeta\in\Xi}\, e^{-\frac{\|\xi-\zeta\|^2}{2\sigma^2}} \mathrm{d}\zeta, \qquad \text{for all } \xi \in \Xi \tag{4}$$

where $\sigma > 0$ controls the spread of the second marginals, following Wang et al. (2021); Azizian et al. (2023). Then, the regularized version of $\widehat{\mathcal{R}}_{\rho^2}(f)$ (WDRO empirical risk) is given by

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) := \sup_{\substack{\pi\in\mathcal{P}(\Xi\times\Xi),\pi_1=\mathrm{P_n} \\ \mathbb{E}_{(\xi,\zeta)\sim\pi}\left[\frac{1}{2}\|\xi-\zeta\|^2\right]\le\rho^2}} \mathbb{E}_{\xi\sim\pi_2}\left[f(\xi)\right] - \varepsilon\,\mathrm{KL}\left(\pi\,|\pi^n_\sigma\right) \qquad \text{with } \pi^n_\sigma = \mathrm{P_n}(\mathrm{d}\xi)\,\pi_\sigma(\mathrm{d}\zeta|\xi) \tag{5}$$

and similarly, the regularized version of $\mathcal{R}_{\rho^2}(f)$ is given by

$$\mathcal{R}^{\varepsilon}_{\rho^2}(f) := \sup_{\substack{\pi\in\mathcal{P}(\Xi\times\Xi),\pi_1=\mathrm{P} \\ \mathbb{E}_{(\xi,\zeta)\sim\pi}\left[\frac{1}{2}\|\xi-\zeta\|^2\right]\le\rho^2}} \mathbb{E}_{\xi\sim\pi_2}\left[f(\xi)\right] - \varepsilon\,\mathrm{KL}\left(\pi\,|\pi_\sigma\right) \qquad \text{with } \pi_\sigma = \mathrm{P}(\mathrm{d}\xi)\,\pi_\sigma(\mathrm{d}\zeta|\xi). \tag{6}$$

These regularized risks have been studied in terms of computational or approximation properties, but their statistical properties have not been investigated yet. The analysis we develop for WDRO estimators is general enough to carry over to these settings.

2.3 Blanket assumptions

Our analysis is carried under the following set of assumptions that will be in place throughout the paper. First, we assume that the sample space $\Xi \subset \mathbb{R}^d$ is convex and compact, which is in line with previous work, e.g., (Lee and Raginsky, 2018; An and Gao, 2021).

Assumption 1 (On the set Ξ)**.** The sample space Ξ is a compact convex subset of $\mathbb{R}^d$.

Second, we require the class of loss functions $\mathcal{F}$ to be sufficiently regular. In particular, we assume that they have Lipschitz continuous gradients.

Assumption 2 (On the function class)**.** The functions of $\mathcal{F}$ are twice differentiable, uniformly bounded, and their derivatives are uniformly bounded and uniformly Lipschitz.

Finally, we assume that P_n is made of independent and identically distributed (i.i.d.) samples of P and that P is supported on the interior of Ξ (which can be done without loss of generality by slightly enlarging Ξ if needed).

Assumption 3 (On the distributions)**.** $\mathrm{P}_\mathrm{n} = \frac{1}{n}\sum_{i=1}^n \delta_{\xi_i}$ where $\xi_1, \dots, \xi_n$ are i.i.d. samples of P. We further assume that there is some $R > 0$ such that P satisfies $\operatorname{supp} \mathrm{P} + \overline{\mathbb{B}}(0, R) \subset \Xi$.

3 Main results and discussions

The main results of our paper establish that the empirical robust risk provide high probability bounds, of the form of (2), on the true risk. Since the results and assumptions slightly differ between the WDRO models and their regularized counterparts, we present them separately in Section 3.1 and Section 3.2. In Section 3.3, we provide the common outline for the proofs of these results, the proofs themselves being provided in the appendix. Finally, in Section 3.4, we detail some examples.

3.1 Exact generalization guarantees for WDRO models

In this section, we require the two following additional assumptions on the function class. The first assumption is common in the WDRO litterature, see e.g., Blanchet et al. (2022); Blanchet and Shapiro (2023); Gao (2022); An and Gao (2021).

Assumption 4. The quantity $\inf_{f\in\mathcal{F}} \mathbb{E}_\mathrm{P}\left[\|\nabla f\|^2\right]$ is positive.

The second assumption we consider in this section makes use of the notation $d(\xi, A)$, for a set $A \subset \Xi$ and a point $\xi \in \Xi$, to denote the distance between ξ and A, i.e., $d(\xi, A) = \inf_{\zeta\in A}\|\xi - \zeta\|$.

Assumption 5.

1. For any $R > 0$, there exists $\Delta > 0$ such that,
$$\forall f \in \mathcal{F},\, \forall \zeta \in \Xi, \quad d(\zeta, \arg\max f) \geq R \implies f(\zeta) - \max f \leq -\Delta\,.$$
2. The following growth condition holds: there exist $\mu > 0$ and $L > 0$ such that, for all $f \in \mathcal{F}$, $\xi \in \Xi$ and ξ^* a projection of ξ on $\arg\max f$, i.e., $\xi^* \in \arg\min_{\arg\max f}\|\xi - \cdot\|$,
$$f(\xi^*) \geq f(\xi) + \frac{\mu}{2}\|\xi - \xi^*\|^2 - \frac{L}{6}\|\xi - \xi^*\|^3\,.$$

The first item of this assumption has a natural interpretation: we show in Lemma A.7, that it is equivalent to the relative compactness of the function space $\mathcal{F}$ w.r.t. to the distance

$$D(f, g) \coloneqq \|f - g\|_\infty + D_H(\arg\max f, \arg\max g)\,,$$

where D_H denotes the (Hausdorff) distance between sets and $\|f\|_\infty \coloneqq \sup_{\xi\in\Xi}|f(\xi)|$ is the infinity norm. The last one is a structural assumption on the functions $\mathcal{F}$ that is new in our context but is actually very close the so-called parametric Morse-Bott condition, introduced in of bilevel optimization (Arbel and Mairal, 2022), see Section A.5.

We now state our main generalization result for WDRO risks.

Theorem 3.1. *Under Assumptions 4 and 5, there is an explicit constant ρ_c depending only on $\mathcal{F}$ and P such that for any $\delta \in (0,1)$ and $n \geq 1$, if*

$$\mathcal{O}\left(\sqrt{\frac{1+\log 1/\delta}{n}}\right) \leq \rho \leq \frac{\rho_c}{2} - \mathcal{O}\left(\sqrt{\frac{1+\log 1/\delta}{n}}\right) \tag{7}$$

then, there is $\rho_n = \mathcal{O}\left(\sqrt{\frac{1+\log 1/\delta}{n}}\right)$ such that, with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right] \qquad \text{for all } \mathrm{Q} \text{ such that } W_2^2(\mathrm{P},\mathrm{Q}) \leq \rho(\rho-\rho_n)\,. \tag{8}$$

In particular, with probability $1-\delta$, we have

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{P}}\left[f(\xi)\right]. \tag{9}$$

The second part of the result, (9), is an *exact* generalization bound: it is an actual upper-bound on the true risk $\mathbb{E}_{\xi\sim\mathrm{P}}[f(\xi)]$, that we cannot access in general, through a quantity that we can actually compute with P_n. The first part of the result, (8) gives us insight into the robustness guarantees offered by the WDRO risk. Indeed, it tells us that, when ρ is greater than the minimal radius $\rho_n \propto 1/\sqrt{n}$ by some margin, the empirical robust risk $\widehat{\mathcal{R}}_{\rho^2}(f)$ is an upper-bound on the loss even with some perturbations of the true distribution. Hence, as long as ρ is large enough, the WDRO objective enables us to guarantee the performance of our model even in the event of a distribution shift at testing time. In other words, the empirical robust risk is an *exact* upper-bound on the true robust risk $\mathcal{R}_{\rho(\rho-\rho_n)}(f)$ with a reduced radius.

The range of admissible radiuses is described by (7). The lower-bound, roughly proportional to $1/\sqrt{n}$, is optimal, following the results of Blanchet et al. (2022). The upper-bound, almost independent of n, depends on a constant ρ_c, that we call *critical radius* and that has an interesting interpretation, that we formalize in the following remark. Note, finally, that, the big-O notation in this theorem has a slightly stronger meaning[1] than the usual one, being non-asymptotic in n and δ.

Remark 3.2 (Interpretation of critical radius)**.** *The critical radius ρ_c, appearing in (7), is defined by*

$$\rho_c^2 \coloneqq \inf_{f\in\mathcal{F}} \mathbb{E}_{\xi\sim\mathrm{P}}\left[\frac{1}{2}d^2(\xi, \arg\max f)\right].$$

It can be interpreted as the threshold at which the WDRO problem w.r.t. P starts becoming degenerate. Indeed, when $\rho^2 \geq \mathbb{E}_{\xi\sim\mathrm{P}}\left[\frac{1}{2}d^2(\xi, \arg\max f)\right]$ for some $f \in \mathcal{F}$ that we fix, the distribution Q given by the second marginal of the transport plan π defined by,

$$\pi(\mathrm{d}\xi, \mathrm{d}\zeta) \coloneqq \mathrm{P}(\mathrm{d}\xi)\delta_{\zeta^\star(\xi)}(\mathrm{d}\zeta) \quad \textit{where} \quad \zeta^\star(\xi) \in \operatorname*{arg\,min}_{\zeta\in\arg\max f} d^2(\xi,\zeta)\,,$$

satisfies

$$W_2^2(\mathrm{P},\mathrm{Q}) \leq \mathbb{E}_{(\xi,\zeta)\sim\pi}\left[\frac{1}{2}\|\xi-\zeta\|^2\right] = \mathbb{E}_{\xi\sim\mathrm{P}}\left[\frac{1}{2}d^2(\xi,\arg\max f)\right] \leq \rho^2\,.$$

As a consequence, the robust problem is equal to

$$\mathcal{R}_{\rho^2}(f) = \sup_{\substack{\mathrm{Q}\in\mathcal{P}(\Xi)\\ W_2^2(\mathrm{P},\mathrm{Q})\leq\rho^2}} \mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right] = \max_{\xi\in\Xi} f(\xi)\,.$$

Thus, when the radius exceeds ρ_c, there is some f such that the robust problem becomes degenerate as it does not depend on P nor ρ anymore.

Finally, note that we can obtain the same generalization guarantee as Theorem 3.1 without Assumption 5 at the expense of losing the above interpration on the condition on the radius. More precisely, we have the following result.

Theorem 3.3. *Let Assumption 4 hold. For any $\delta \in (0,1)$ and $n \geq 1$, if ρ satisfies (7), and if, in addition, it is smaller than a positive constant which depends only on P, $\mathcal{F}$ and Ξ, then both conclusions of Theorem 3.1 hold.*

[1] Eg., $\rho_n = \mathcal{O}\left(\sqrt{\frac{1+\log 1/\delta}{n}}\right)$ means that $\exists C > 0$ such that $\rho_n \leq C\sqrt{\frac{1+\log 1/\delta}{n}}$ for all $\delta \in (0,1)$ and $n \geq 1$.

This theorem can be compared to existing results, and in particular with Gao (2022); An and Gao (2021). These two papers provide generalization bounds for WDRO under a similar assumption on $\mathcal{F}$ and a weakened version of Assumption 3. However, these generalization bounds involve extra error terms, that require ρ to be vanishing. In comparison, with a similar set of assumptions, Theorem 3.3 improves on these two issues, by allowing ρ not to vanish as $n \to \infty$ and by providing the exact upper-bound (9). Allowing non-vanishing radiuses is an attractive feature of our results that enables us to cover distribution shifts.

3.2 Regularized WDRO models

The analysis that we develop for the standard WDRO estimators is general enough to also cover the regularized versions presented in Section 2.2. We thus obtain the following Theorem 3.4 which is the first generalization guarantee for regularized WDRO. This theorem is very similar to Theorem 3.1 with still a couple of differences. First, the regularization leads to ambiguity sets defined in terms of $W_{2,\tau}(\mathrm{P}, \cdot)$, the *regularized* Wasserstein distance to the true distribution P, defined, for some regularization parameter $\tau > 0$, as

$$W_{2,\tau}^2(\mathrm{P}, \mathrm{Q}) \coloneqq \inf \left\{ \mathbb{E}_\pi \left[\frac{1}{2} \|\xi - \zeta\|^2 \right] + \tau \, \mathrm{KL}(\pi \,|\, \pi_\sigma) : \pi \in \mathcal{P}(\Xi \times \Xi), \pi_1 = \mathrm{P}, \, \pi_2 = \mathrm{Q} \right\}$$

where π_σ appears in the definition of the regularized robust risk (6). Besides, the regularization allows us to avoid Assumptions 4 and 5 to show our generalization result.

Theorem 3.4. *For $\sigma = \sigma_0 \rho$ with $\sigma_0 > 0$, $\varepsilon = \varepsilon_0 \rho$ with $\varepsilon_0 > 0$ such that $\varepsilon_0 / \sigma_0^2$ is small enough depending on $\mathcal{F}$, P, Ξ, there is an explicit constant ρ_c depending only on $\mathcal{F}$, P and Ξ such that for all $\delta \in (0,1)$ and $n \geq 1$, if*

$$\mathcal{O}\left(\sqrt{\frac{1 + \log 1/\delta}{n}} \right) \leq \rho \leq \frac{\rho_c}{2} - \mathcal{O}\left(\frac{1}{\sqrt{n}} \right), \quad \text{and} \quad \rho_c \geq \mathcal{O}\left(\frac{1}{n^{1/6}} + \left(\frac{1 + \log 1/\delta}{n} \right)^{1/4} \right),$$

then, there are $\tau = \mathcal{O}(\varepsilon \rho)$ and $\rho_n = \mathcal{O}\left(\sqrt{\frac{1+\log 1/\delta}{n}} \right)$ such that, with probability at least $1 - \delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}^{\varepsilon}(f) \geq \mathbb{E}_{\xi \sim \mathrm{Q}}\left[f(\xi) \right] \qquad \text{for all } \mathrm{Q} \text{ such that } W_{2,\tau}^2(\mathrm{P}, \mathrm{Q}) \leq \rho(\rho - \rho_n) \tag{10}$$

Furthermore, when σ_0 and σ are small enough depending on P and Ξ, with probability $1 - \delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}^{\varepsilon}(f) \geq \mathbb{E}_{\xi \sim \mathrm{P}} \mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)}\left[f(\zeta) \right].$$

The first part of the theorem, (10), guarantees that the empirical robust risk is an upper-bound on the loss even with some perturbations of the true distribution. As in OT, the regularization added to the Wasserstein metric induces a bias that may prevent $W_{2,\tau}^2(\mathrm{P}, \mathrm{P})$ from being null. As a result, the second part of the theorem involves a smoothed version of the true risk: the empirical robust risk provides an *exact* upper-bound the true expectation of a convolution of the loss with π_σ.

A few additional comments are in order:

- Our result prescribes the scaling of the regularization parameters: ε and σ should be taken proportional to ρ.
- The critical radius ρ_c has a slighlty more intricate definition, yet the same interpretation as in the standard WDRO case inRemark 3.2; see Section D.2.
- The regularized OT distances do not suffer from the curse of dimensionality (Genevay et al., 2019). However this property does not directly carry over to regularized WDRO. Indeed, we cannot choose the same reference measure as in OT and we have to fix the measure π_σ, introducing a bias. As a consequence, we have to extend the analysis of the previous section to obtain the claimed guarantees that avoid the curse of dimensionality.

3.3 Idea of the proofs

In this section, we present the main ideas of the proofs of Theorem 3.1, Theorem 3.3, and Theorem 3.4. The full proofs are detailed in appendix; we point to relevant sections along the discussion. First, we recall the duality results for WDRO that play a crucial role in our analysis. Second, we present a rough sketch of proofs that is common to both the standard and the regularized cases. Finally, we provide a refinement of our results that is a by-product of our analysis.

Duality in WDRO. Duality has been a central tool in both the theoretical analyses and computational schemes of WDRO from the onset (Shafieezadeh Abadeh et al., 2015; Esfahani and Kuhn, 2018). The expressions of the dual of WDRO problems for both the standard case (Gao and Kleywegt, 2016; Blanchet and Murthy, 2019) and the regularized case (Wang et al., 2021; Azizian et al., 2023) can be written with the following dual generator function ϕ defined as

$$\phi(f,\xi,\lambda,\varepsilon,\sigma) \coloneqq \begin{cases} \sup_{\zeta\in\Xi}\left\{f(\zeta) - \frac{\lambda}{2}\|\xi-\zeta\|^2\right\} & \text{if } \varepsilon = 0 \\ \varepsilon \log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)} \exp\left(\frac{f(\zeta)-\lambda\|\xi-\zeta\|^2/2}{\varepsilon}\right)\right) & \text{if } \varepsilon > 0\,, \end{cases} \tag{11}$$

where λ is the dual variable associated to the Wasserstein constraint in (3), (5) and (6). The effect of regularization appears here clearly as a smoothing of the supremum. Note also that this function depends on the conditional reference measures $\pi_\sigma(\cdot|\xi)$ but not on other probability distributions. Then, under some general assumptions (specified in Section 2.3 in appendix), the existing strong duality results yield that the (regularized) empirical robust risk writes

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\lambda\geq 0} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P_n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\,, \tag{12}$$

and, similarly, the (regularized) true robust risk writes

$$\mathcal{R}^{\varepsilon}_{\rho^2}(f) = \inf_{\lambda\geq 0} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\,. \tag{13}$$

These expressions for the risks are the bedrock of our analysis.

Sketch of proof. In both the standard case and the regularized case, our proof is built on two main parts: the first part is to obtain a concentration bound on the dual problems that crucially relies on a lower bound of the dual multiplier; the second part then consists in establishing such a lower bound. All the bounds are valid with high probability, and we drop the dependency on the confidence level δ of the theorems for simplicity.

For the first part of the proof (Section B), we assume that there is a deterministic lower-bound $\underline{\lambda} > 0$ on the optimal dual multiplier in (12) that holds with high-probability. As a consequence, we can restrict the range of λ in (12) to obtain:

$$\begin{aligned}
\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) &= \inf_{\lambda\geq\underline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P_n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\right\} \\
&= \inf_{\lambda\geq\underline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda\frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P_n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda}\right\} \\
&\geq \inf_{\lambda\geq\underline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda \sup_{\lambda'\geq\underline{\lambda}} \frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P_n}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)]}{\lambda'}\right\} \\
&\geq \inf_{\lambda\geq\underline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda\overline{\rho}_n^2\right\} \\
&\geq \mathcal{R}^{\varepsilon}_{\rho^2-\overline{\rho}_n^2}(f)\,.
\end{aligned} \tag{14}$$

In the above, we used that the inner supremum, which is random, can be bounded by a deterministic and explicit quantity that we call $\overline{\rho}_n^2$, i.e.,

$$\overline{\rho}_n^2 \geq \sup_{\lambda'\geq\underline{\lambda}} \frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P_n}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)]}{\lambda'} \qquad \text{with high probability.}$$

Hence, we obtain an upper-bound on the robust risk w.r.t. the true distribution with radius $\rho^2 - \overline{\rho}_n^2$. Moreover, we show that $\overline{\rho}_n^2 = \mathcal{O}(1/(\underline{\lambda}\sqrt{n}))$ which highlights the need for a precise lower bound $\underline{\lambda}$ to control the decrease in radius.

The second part of the proof thus consists in showing that the dual variable is indeed bounded away from 0, which means that the Wasserstein constraint is *sufficiently active*. We have to handle two cases differently:

- when ρ is small, i.e., close to ρ_n (Section C),
- when ρ is large, i.e., close to the critical radius ρ_c (Section D). Note that the additional Assumption 5 is required here: where we need to control the behaviors of $f \in \mathcal{F}$ close to their maxima (see (11) for $\varepsilon = 0$ and small λ).

In both cases we obtain that $\underline{\lambda}$ scales as $1/\rho$ for the respective ranges of admissible radiuses. As a consequence $\overline{\rho}_n^2$ is bounded by $\rho\rho_n$ with $\rho_n = \mathcal{O}(1/\sqrt{n})$ and (14) becomes

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \mathcal{R}^{\varepsilon}_{\rho(\rho-\rho_n)}(f)\,, \tag{15}$$

which leads to our main results.

Extension: upper and lower bounds on the empirical robust risk. The proof that we sketched above actually shows that $\mathcal{R}^{\varepsilon}_{\rho(\rho-\rho_n)}(f)$ is a lower bound of $\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f)$. This proof technique also yields an upper bound by exchanging the roles of P and $\mathrm{P_n}$.

Theorem 3.5. *In the setting of either Theorem 3.1, Theorem 3.3 or Theorem 3.4 (with $\varepsilon = 0$ or $\varepsilon > 0$), with probability at least $1-\delta$, it holds that*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}^{\varepsilon}_{\rho(\rho-\rho_n)}(f) \leq \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \leq \mathcal{R}^{\varepsilon}_{\rho(\rho+\rho_n)}(f)\,,$$

with $\rho_n = \mathcal{O}\left(\sqrt{\frac{1+\log 1/\delta}{n}}\right)$.

This result shows how two robust objectives w.r.t. P provide upper and lower bounds on the empirical robust risk, with only slight variations in the radius. Furthermore, when the number of data points n grows, both extremes of the bound converge to the same quantity $\mathcal{R}^{\varepsilon}_{\rho^2}(f)$. Hence our generalization bounds of the form (15) are asymptotically tight.

3.4 Parametric models

Our main theorems Theorems 3.1, 3.3 and 3.4 involve a general class $\mathcal{F}$ of loss functions. We explain in this section how to instantiate our results in the important class of parametric models. We then illustrate this setting with logistic regression and linear regression in Examples 3.6 and 3.7.

Let us consider the class of functions of the form

$$\mathcal{F} = \{\xi \mapsto f(\theta, \xi) : \theta \in \Theta\} \qquad \text{with } f : \Theta \times \Xi \longrightarrow \mathbb{R} \tag{16}$$

where Θ, the parameter space, is a subset of $\mathbb{R}^p$ and Ξ, the sample space, is a subset of $\mathbb{R}^d$.

For instance, this covers the case of linear models of the form $f(\theta, \xi) = \ell(\langle \xi, \theta \rangle)$ with ℓ a convex loss. This class of models is studied by Shafieezadeh-Abadeh et al. (2019); Chen and Paschalidis (2018) in a slightly different setting, where they obtain a closed form for the robust objective and then establish a generalization bound similar to (9).

Let us show how to instantiate our theorems in the case of (16).

- If f is twice continuously differentiable on a neighborhood of $\Theta \times \Xi$ with Θ and Ξ both compact, then Assumption 2 is immediately satisfied. Therefore, Theorem 3.4 can be readily applied and its generalization guarantee hold.
- As for Assumption 4, it is equivalent to: for all $\theta \in \Theta$, $\mathrm{P}(\nabla_\xi f(\theta, \xi) \neq 0) > 0$. This avoids the degenerate case when $\nabla_\xi f(\theta, \xi)$ is null for P-almost every ξ (in particular, when the loss does not depend on ξ). With this assumption, we are in the setting of Theorem 3.3.
- Satisfying Assumption 5, needed for Theorem 3.1, requires some problem-dependent developments. In the next two examples, we detail how to instantiate it for logistic and linear regressions. Note though that the second item of Assumption 5 is directly implied by the parametric Morse-Bott property (Arbel and Mairal, 2022); see Section A.5 for details.

Example 3.6 (Logistic Regression)**.** *For a training sample $(x, y) \in \mathbb{R}^p \times \{-1, +1\}$, the logistic loss for a parameter $\theta \in \mathbb{R}^p$ is given by $\log\big(1 + e^{-y\langle x, \theta\rangle}\big)$. It fits into our framework by defining $f(\theta, \xi) = \log\big(1 + e^{\langle \xi, \theta\rangle}\big)$ with ξ playing the role of $-y \times x$. We assume that Θ is a compact set that does not include the origin, and, for the sake of simplicity, we take Ξ as a closed Euclidean ball, i.e., $\Xi = \overline{\mathbb{B}}(0, r)$. We are going to show that Assumption 5 is satisfied, and, for this, we need the following elements. For any θ, the maximizer of $f(\theta, \cdot)$ over $\Xi = \overline{\mathbb{B}}(0, r)$ is reached at $\xi^* := \frac{r\theta}{\|\theta\|}$. Besides, for any $\xi \in \Xi$, it holds that*

$$r^2 \geq \|\xi\|^2 = \|\xi^*\|^2 + 2\langle \xi^*, \xi - \xi^* \rangle + \|\xi - \xi^*\|^2\,,$$

so that, since $\|\xi^*\| = r$, we have

$$\langle \xi^*, \xi^* - \xi \rangle \geq \frac{1}{2}\|\xi - \xi^*\|^2 . \tag{17}$$

We can now turn to the verification of Assumption 5.

1. *Take some $R > 0$ and some $\xi \in \Xi$ such that $\|\xi - \xi^*\| \geq R$. Then, (17) yields*

$$\langle \theta, \xi \rangle - \langle \theta, \xi^* \rangle = \frac{\|\theta\|}{r}\langle \xi^*, \xi - \xi^* \rangle \leq -\frac{\|\theta\|}{2r}\|\xi - \xi^*\|^2 \leq -\frac{d(0,\Theta)R^2}{2r} . \tag{18}$$

   Since $u \mapsto \log(1 + e^u)$ is increasing, this yields that $f(\theta, \xi) - f(\theta, \xi^*)$ is bounded away from 0 by a negative constant uniformly in θ. The first item of Assumption 5 is thus satisfied.

2. *Fix $\theta \in \Theta$; by Taylor expanding $u \mapsto \log(1 + e^u)$ around $\langle \theta, \xi^* \rangle$ we get*

$$f(\theta, \xi) = f(\theta, \xi^*) + \frac{1}{1 + e^{-\langle \theta, \xi^* \rangle}}\langle \theta, \xi - \xi^* \rangle + \mathcal{O}(\langle \theta, \xi - \xi^* \rangle)^2$$

   where the big-O remainder is uniform over $\theta \in \Theta$. Using the first inequality in (18), we get for ξ close enough to ξ^*

$$f(\theta, \xi) \leq f(\theta, \xi^*) - \frac{1}{2(1 + e^{-\langle \theta, \xi^* \rangle})}\langle \theta, \xi - \xi^* \rangle \leq f(\theta, \xi^*) - \frac{\|\theta\|}{4r}\|\xi - \xi^*\|^2 .$$

   This shows that the second item of Assumption 5 is satisfied locally around ξ^*. It can be made global by using the uniform Lipschitz-continuity of f, which introduces a term of the form $\frac{L}{6}\|\xi - \xi^*\|^3$.

Example 3.7 (Linear Regression)**.** *With samples of the form $\xi = (x, y) \in \mathbb{R}^p \times \mathbb{R}$ and parameters $\theta \in \mathbb{R}^p$, the loss is given by $f(\theta, \xi) = \frac{1}{2}(\langle \theta, x \rangle - y)^2$. Similarly to the previous example, we take Θ as a compact set of $\mathbb{R}^d$ that does not include the origin and Ξ of the form $\overline{\mathbb{B}}(0, r) \times [-r', r']$. The maximizers of $f(\theta, \cdot)$ on Ξ are $\xi_1^* = (r\theta/\|\theta\|, -r')$ and $\xi_2^* = (-r\theta/\|\theta\|, r')$. By symmetry, one can restrict to the case of ξ_1^* and $\langle \theta, x \rangle - y \geq 0$; the same rationale as above can then be applied.*

4 Conclusion and perspectives

In this work, we provide generalization guarantees for WDRO models that improve over existing literature in the following aspects: our results avoid the curse of dimensionality, provide exact upper bounds without spurious error terms, and allow for distribution shifts during testing. We obtained these bounds through the development of an original concentration result on the dual of WDRO.

Our work could be naturally extended in several ways. For instance, it might be possible to relax any of the assumptions (on the sample space, on the sampling process, on the Wasserstein metric, and on the class of functions) at the expense of additional technical work. Another interesting direction would be to refine and complement our results. For instance, Theorem 3.5 gives a first result on the tightness of our generalization guarantees.

Acknowledgments and Disclosure of Funding

This work has been supported by MIAI Grenoble Alpes (ANR-19-P3IA-0003).

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Appendices

We provide here the proofs of our main results Theorems 3.1, 3.3 and 3.4, along with detailed versions that include explicit bounds. We start, in Section A, by referencing preliminary results, reformulate some of our assumptions, and introduce quantities that appear in the final bounds.

As sketched in Section 3.3, our proof is built on two main parts. The first part is to obtain a concentration bound on the dual problems (12) and (13) by leveraging on a lower bound on the dual multiplier. This concentration result is presented in Section B where we assume that such a lower-bound is given. The second part of the proof then consists in establishing the lower-bound. We have to distinguish two cases: when ρ is small (Section C) and when ρ is close to the critical radius ρ_c (Section D). For the latter, we also need to treat separately the cases where the WDRO problem is regularized or not (respectively Section D.2 and Section D.1): this is where the Assumptions 4 and 5, that are not required in the regularized case, come into play. Putting together these two parts, we obtain our precise theorems in Section E and show how they imply our main results. Section F then complements our theorems to obtain Theorem 3.5. Finally, some variations of known results and technical computations are compiled in Section G as standalone lemmas.

A Preliminaries

This section presents preliminary results before we start the proofs in Section B. In the first part of this section Section A.1, we present a weaker and more detailed version of Assumption 2, namely Assumption 6, that will suffice for all the proofs in the appendix. We also introduce several quantities that will appear in the final bounds. Then, in Section A.1 we recall the dual problems introduced in Section 3.3 and justify that strong duality holds. Preliminary approximation results on the dual are then given in Section A.3. We then proceed to show the relative compactness of the class $\mathcal{F}$ w.r.t. several metrics in Section A.4. These properties provide a convenient way of ensuring that quantities involving $\mathcal{F}$ are finite, e.g., complexity measures or supremums over $\mathcal{F}$. Finally, we introduce the so-called parametric Morse-Bott condition of Arbel and Mairal (2022) and show how it implies the second item of Assumption 2 in a Riemannian setting.

A.1 Detailed assumption on the function class and important quantities

Here we present the precise assumptions that we will refer to in the proofs. While Assumptions 1 and 3 are used as presented in the main text, we slightly weaken Assumption 2 to Assumption 6. We also introduce some quantities that we will be of interest for the proofs and the final results.

Assumption 6 (On the function class)**.** Consider $\mathcal{F}$ a set of real-valued non-negative continuous functions on Ξ. We assume that:

- the functions $f \in \mathcal{F}$ are uniformly L_2-smooth;
- the gradients are uniformly bounded, i.e.,
$$G \coloneqq \sup_{f\in\mathcal{F}} \sup_{\xi\in\operatorname{supp} \mathrm{P}} \|\nabla f(\xi)\|_2 < +\infty$$
- when $\varepsilon = 0$, the supremum in (11) is finite, i.e.,
$$\widetilde{F}(\lambda) \coloneqq \sup_{f\in\mathcal{F}} \sup_{\xi\in\operatorname{supp} \mathrm{P}} \sup_{\arg\max\{f-\frac{\lambda}{2}\|\xi-\cdot\|^2\}} f < +\infty\,.$$

Note that the non-negativity assumption is without loss of generality since otherwise, it suffices to consider $\widetilde{\mathcal{F}} \coloneqq \{f - \min f : f \in \mathcal{F}\}$ and our results are invariant by addition of a constant.

The blanket assumptions for the remaining of the appendix will be Assumptions 1, 3 and 6.

The following finite quantities are relevant for the proofs and appear in the quantitative versions of Theorems 3.1, 3.3 and 3.4.

$$C^\star \coloneqq \sup\left\{\frac{1}{2}\|\xi-\xi^\star\|^2 : \xi\in\operatorname{supp}\mathrm{P},\ f\in\mathcal{F},\ \xi^\star\in\arg\max f\right\},$$

$$C(\sigma) \coloneqq \sup_{\xi\in\operatorname{supp}\mathrm{P}} \mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right],$$

$$C_{\mathcal{F}}(\varepsilon,\sigma) \coloneqq \sup_{f\in\operatorname{conv}(\mathcal{F})} \sup_{\xi\in\operatorname{supp}\mathrm{P}} \mathbb{E}_{\zeta\sim\pi_\sigma^{f/\varepsilon}(\cdot|\xi)} \frac{1}{2}\|\xi-\zeta\|^2,$$

$$\text{and}\quad \operatorname{Var}(\varepsilon,\sigma) \coloneqq \sup_{f\in\operatorname{conv}(\mathcal{F})} \sup_{\xi\in\operatorname{supp}\mathrm{P}} \sup_{\lambda\geq 0} \operatorname{Var}_{\zeta\sim\pi_\sigma^{\frac{f-\lambda\|\xi-\cdot\|^2/2}{\varepsilon}}(\cdot|\xi)} \frac{1}{2}\|\xi-\zeta\|^2,$$

where $\varepsilon > 0$, $\sigma > 0$, and π_σ is given by (4).

A.2 Strong duality

As mentioned in Section 3.3, duality plays a central role in our proofs. Let us recall the central notion of dual generator functions, introduced in (11): for any $f \in \mathcal{F}$, $\xi \in \Xi$, $\lambda, \varepsilon \geq 0$ and $\sigma > 0$, the dual generator ϕ is given as

$$\phi(f,\xi,\lambda,\varepsilon,\sigma) \coloneqq \begin{cases} \sup_{\zeta\in\Xi}\left\{f(\zeta) - \frac{\lambda}{2}\|\xi-\zeta\|^2\right\} & \text{if } \varepsilon = 0 \\ \varepsilon\log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)} \exp\left(\frac{f(\zeta)-\lambda\|\xi-\zeta\|^2/2}{\varepsilon}\right)\right) & \text{if } \varepsilon > 0\,. \end{cases}$$

Our proofs are based on the (strong) dual formulations of WDRO, as given by the following lemma that summarizes results of the literature for the regularized and unregularized cases.

Lemma A.1. *Under the blanket assumptions, for $f \in \mathcal{F}$, $\rho > 0$, $\varepsilon \geq 0$ and $\sigma > 0$,*

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\lambda \geq 0} \lambda\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P}_\mathrm{n}}\left[\phi(f, \xi, \lambda, \varepsilon, \sigma)\right]$$
$$\text{and} \quad \mathcal{R}^{\varepsilon}_{\rho^2}(f) = \inf_{\lambda \geq 0} \lambda\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P}}\left[\phi(f, \xi, \lambda, \varepsilon, \sigma)\right] .$$

Proof. See Blanchet and Murthy (2019); Gao and Kleywegt (2016) for the unregularized case and Azizian et al. (2023) for the regularized case. □

A.3 Approximation of the dual generator ϕ

Important preliminary results for our upcoming concentration bounds (in Section B) are quantitative approximations of the dual generator ϕ, namely Proposition A.2 and Lemma A.3. In particular, these results also imply bounds on ϕ in Corollary A.5.

Proposition A.2 (Bounding the distance between ϕ and f)**.** *There are positive constants λ_1, ε_1, σ_1, c_1, c_2 which depend on G, R and d such that taking some $\overline{\lambda} \geq \underline{\lambda} \geq \lambda_1 + L_2$, we have for any $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$, $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, $\varepsilon \in [0, \varepsilon_1]$ and $\sigma \in (0, \sigma_1]$*

$$|\phi(f, \xi, \lambda, \varepsilon, \sigma) - f(\xi)| \leq M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)$$

where

$$M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) \coloneqq \frac{1}{2\underline{\lambda}} G^2 + \frac{\varepsilon d}{2} \log\left(\frac{\overline{\lambda}}{\varepsilon} + \frac{1}{\sigma^2}\right) + \varepsilon \log 2 + \varepsilon d |\log \sigma| + \varepsilon c_1 e^{-c_2 \left(\frac{\underline{\lambda} - L_2}{\varepsilon}\right)^{\frac{1}{3}}} .$$

The proof of this result is based on the following second approximation result which gives a precise approximation of ϕ that will be used several times in the upcoming proof.

More precisely, we want to approximate ϕ by a Taylor development $\overline{\phi}$ defined for any $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$, $\lambda \geq 0$, $\varepsilon \geq 0$ and $\sigma > 0$ as

$$\overline{\phi}(f, \xi, \lambda, \varepsilon, \sigma) \coloneqq f(\xi) + \frac{1}{2\left(\lambda + \frac{\varepsilon}{\sigma^2}\right)} \|\nabla f(\xi)\|_2^2 - \frac{\varepsilon d}{2} \log\left(\frac{\lambda}{\varepsilon} + \frac{1}{\sigma^2}\right) + \varepsilon \log \frac{(2\pi)^{\frac{d}{2}}}{Z(\xi, \sigma)} \tag{19}$$

where $Z(\xi, \sigma) \coloneqq \int_\Xi e^{-\frac{\|\xi - \zeta\|_2^2}{2\sigma^2}} \mathrm{d}\zeta$. The distance between ϕ and $\overline{\phi}$ is then controlled by the following Laplace approximation lemma.

Lemma A.3 (Approximation of ϕ)**.** *There are positive constants λ_1, ε_1, σ_1, c_1, c_2 which depend on G, R and d such that $\varepsilon_1 \leq \lambda_1$ and, when $\varepsilon \in [0, \varepsilon_1]$, $\sigma \in (0, \sigma_1]$ and $\lambda \geq \lambda_1 + L_2$, we have for any $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$*

$$\overline{\phi}(f, \xi, \lambda + L_2, \varepsilon, \sigma) - \varepsilon c_1 e^{-c_2 \left(\frac{\lambda + L_2}{\varepsilon}\right)^{\frac{1}{3}}} \leq \phi(f, \xi, \lambda, \varepsilon, \sigma) \leq \overline{\phi}(f, \xi, \lambda - L_2, \varepsilon, \sigma) + \varepsilon c_1 e^{-c_2 \left(\frac{\lambda - L_2}{\varepsilon}\right)^{\frac{1}{3}}} .$$

Proof. Fix $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$, $\lambda \geq 0$, $\varepsilon \geq 0$ and $\sigma > 0$. To bound the error between ϕ and its approximation $\overline{\phi}$, we introduce an intermediate approximation $\widetilde{\phi}$ defined as

$$\widetilde{\phi}(f, \xi, \lambda, \varepsilon, \sigma) \coloneqq \begin{cases} \varepsilon \log\left(\mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)} \exp\left(\frac{f(\xi) + \langle \nabla f(\xi), \zeta - \xi \rangle - \lambda \|\xi - \zeta\|^2/2}{\varepsilon}\right)\right) & \text{if } \varepsilon > 0 \\ \sup_{\zeta \in \Xi}[f(\xi) + \langle \nabla f(\xi), \zeta - \xi \rangle - \frac{\lambda}{2}\|\xi - \zeta\|^2] & \text{if } \varepsilon = 0 , \end{cases}$$

which corresponds to ϕ applied to the Taylor approximation of f at ξ (instead of f itself). By smoothness of the functions in $\mathcal{F}$ (Assumption 6), we readily have that,

$$\widetilde{\phi}(f, \xi, \lambda + L_2, \varepsilon, \sigma) \leq \phi(f, \xi, \lambda, \varepsilon, \sigma) \leq \widetilde{\phi}(f, \xi, \lambda - L_2, \varepsilon, \sigma) .$$

Now, all that is left to bound, is the error between $\widetilde{\phi}$ and $\overline{\phi}$. Consider first the case where $\varepsilon > 0$ and let us rewrite $\widetilde{\phi}$ by using the definition of π_σ:

$$\begin{aligned} \widetilde{\phi}(f, \xi, \lambda, \varepsilon, \sigma) &= \varepsilon \log\left(\mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)} \exp\left(\frac{f(\xi) + \langle \nabla f(\xi), \zeta - \xi \rangle - \lambda \|\xi - \zeta\|^2/2}{\varepsilon}\right)\right) \\ &= \varepsilon \log\left(\int_\Xi \exp\left(\frac{1}{\varepsilon}\left(f(\xi) + \langle \nabla f(\xi), \zeta - \xi \rangle - \left(\lambda + \frac{\varepsilon}{\sigma^2}\right)\frac{1}{2}\|\xi - \zeta\|^2\right)\right) \mathrm{d}\zeta\right) - \varepsilon \log Z(\xi, \sigma) . \end{aligned}$$

But, looking at the inner expression, we have that

$$f(\xi) + \langle \nabla f(\xi), \zeta - \xi \rangle - \left(\lambda + \frac{\sigma^2}{\varepsilon}\right) \frac{1}{2}\|\xi - \zeta\|^2 = f(\xi) + \frac{1}{2(\lambda + \frac{\sigma^2}{\varepsilon})}\|\nabla f(\xi)\|_2^2 - \frac{1}{2\tau}\|\zeta - \zeta^\star(\tau)\|_2^2 , \tag{20}$$

where we defined $\frac{1}{\tau} \coloneqq \lambda + \frac{\varepsilon}{\sigma^2}$ and $\zeta^\star(\tau) = \xi + \tau \nabla f(\xi)$. Hence,

$$\widetilde{\phi}(f, \xi, \lambda, \varepsilon, \sigma) = \overline{\phi}(f, \xi, \lambda, \varepsilon, \sigma) + \varepsilon \log\left(\int_\Xi \exp\left(-\frac{1}{2\varepsilon\tau}\|\zeta - \zeta^\star(\tau)\|_2^2\right) \mathrm{d}\zeta\right) - \frac{\varepsilon d}{2} \log(2\pi\varepsilon\tau) .$$

Define $\lambda_1' \coloneqq \frac{\sqrt{6}G}{R}$ and $\tau_1 \coloneqq \frac{1}{\lambda_1'} = \frac{R}{2G}$ so that $\lambda \geq \lambda_1'$ implies that $\tau \leq \frac{1}{\lambda_1'} = \tau_1$. Let us now check that the conditions of Lemma G.1 are satisfied.

1. Since $\zeta^\star(0) = \xi$, by Assumption 3, $\overline{\mathbb{B}}(\zeta^\star(0), R)$ is contained in Ξ.
2. For $\tau \leq \tau_1$, we have that $\tau^2\|\nabla f(\xi)\|_2^2 \leq \frac{G^2}{(\lambda_1')^2} = \frac{R^2}{6}$ by definition.

Hence, we can apply Lemma G.1 to get that, for any $\lambda \geq \lambda_1'$

$$\varepsilon \log\left(1 - 6^{d/2} e^{-\frac{R^2}{12\varepsilon\tau}}\right) \leq \widetilde{\phi}(f, \xi, \lambda, \varepsilon, \sigma) - \overline{\phi}(f, \xi, \lambda, \varepsilon, \sigma) \leq \varepsilon \log\left(1 + 6^{d/2} e^{-\frac{R^2}{12\varepsilon\tau}}\right) .$$

Now, using Lemma G.5, we get that there are positive constants ε_1, λ_1, c_1,c_2 depending on R, G and d such that, if $\varepsilon \leq \varepsilon_1$ and $\lambda \geq \lambda_1$, then

$$|\widetilde{\phi}(f, \xi, \lambda, \varepsilon, \sigma) - \overline{\phi}(f, \xi, \lambda, \varepsilon, \sigma)| \leq \varepsilon c_1 e^{-c_2 \left(\frac{\lambda}{\varepsilon}\right)^{\frac{1}{3}}} .$$

Moreover, ε_1 can be reduced so that it is less than λ_1 if it is not the case originally.

To finish the proof, let us now come back to the case $\varepsilon = 0$. First, note that (20) is still valid even with $\varepsilon = 0$ so that we have

$$\widetilde{\phi}(f, \xi, \lambda, 0) = \overline{\phi}(f, \xi, \lambda, 0) - \frac{1}{2\tau} \inf_{\zeta \in \Xi} \|\zeta - \zeta^\star(\tau)\|_2^2 .$$

But as seen above, for $\tau = \lambda^{-1} \leq \tau_1$, $\zeta^\star(\tau)$ is inside $\overline{\mathbb{B}}(\zeta^\star(0), R)$ so that $\widetilde{\phi}(f, \xi, \lambda, 0) = \overline{\phi}(f, \xi, \lambda, 0)$.

We conclude the proof by noticing that the obtained bounds are valid for any $0 < \varepsilon \leq \varepsilon_1$, $\lambda \geq \lambda_1 + L_2$, $f \in \mathcal{F}$ and $\xi \in \operatorname{supp} \mathrm{P}$. □

The following lemma is needed for the proof of Proposition A.2.

Lemma A.4. *There is a positive constant $\sigma_1 > 0$ which depends on R and d such that, for $\sigma \in (0, \sigma_1]$ and $\xi \in \operatorname{supp} \mathrm{P}$,*

$$\left|\log \frac{Z(\xi, \sigma)}{(2\pi)^{d/2}}\right| \leq d|\log \sigma| + \log 2 .$$

Proof. It suffices to show that

$$\frac{(2\pi\sigma^2)^{d/2}}{2} \leq Z(\xi, \sigma) \leq (2\pi\sigma^2)^{d/2} , \tag{21}$$

for any $\sigma \in (0, \sigma_1]$ with some $\sigma_1 > 0$ suitably defined. We prove the right-hand side (RHS) by removing the constraint Ξ in the integral defining $Z(\xi, \sigma)$:

$$Z(\xi, \sigma) = \int_\Xi e^{-\frac{\|\xi - \zeta\|^2}{2\sigma^2}} \mathrm{d}\zeta \leq \int_{\mathbb{R}^d} e^{-\frac{\|\xi - \zeta\|^2}{2\sigma^2}} \mathrm{d}\zeta = (2\pi\sigma^2)^{d/2} .$$

For the left-hand side (LHS), we invoke Lemma G.1 using Assumption 3 to get that $Z(\xi, \sigma) \geq \sigma^d$ when $\sigma \leq \sigma_1$ with $\sigma_1 > 0$ satisfying

$$1 - 6^{d/2} e^{-\frac{R^2}{12\sigma_1^2}} \geq \frac{1}{2} .$$

□

We are now in a position to prove the main result of the section.

Proof of Proposition A.2. Applying Lemmas A.3 and A.4 and using the definition of G readily gives us that

$$|\phi(f,\xi,\lambda,\varepsilon,\sigma) - f(\xi)| \leq \frac{1}{2\lambda}G^2 + \frac{\varepsilon d}{2}|\log\left(\frac{\lambda}{\varepsilon} + \frac{1}{\sigma^2}\right)| + \varepsilon(d|\log\sigma| + \log 2) + \varepsilon\overline{Z}(\sigma) + \varepsilon c_1 e^{-c_2(\frac{\lambda - L_2}{\varepsilon})^{\frac{1}{3}}}.$$

Since λ is always greater or equal than ε, $\log\left(\frac{\lambda}{\varepsilon} + \frac{1}{\sigma^2}\right)$ is always non-negative and, with λ belonging to $[\underline{\lambda}, \overline{\lambda}]$, we get that

$$|\phi(f,\xi,\lambda,\varepsilon,\sigma) - f(\xi)| \leq \frac{1}{2\underline{\lambda}}G^2 + \frac{\varepsilon d}{2}\log\left(\frac{\overline{\lambda}}{\varepsilon} + \frac{1}{\sigma^2}\right) + \varepsilon(d|\log\sigma| + \log 2) + \varepsilon c_1 e^{-c_2(\frac{\underline{\lambda} - L_2}{\varepsilon})^{\frac{1}{3}}},$$

which is the desired result. □

As a consequence of this result, we have the following bound on the dual generator.

Corollary A.5. *For any $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$, $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, $\varepsilon \geq 0$ and $\sigma > 0$, the bound*

$$-a(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) \leq \phi(f,\xi,\lambda,\varepsilon,\sigma) \leq \widetilde{F}(\underline{\lambda}),$$

holds where

$$a(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) := \begin{cases} 0 & \text{when } \varepsilon = 0 \\ M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) & \text{when } 0 < \varepsilon \leq \varepsilon_1,\ 0 < \sigma \leq \sigma_1,\ \underline{\lambda} \geq \lambda_1 + L_2 \\ G\sqrt{2C(\sigma)} + \left(L_2 + \overline{\lambda}\right) C(\sigma) & \text{otherwise.} \end{cases}$$

with $M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)$ the bounding term appearing in Proposition A.2, as well as ε_1, σ_1, λ_1.

Proof. For the upper-bound, it suffices to note that

$$\phi(f,\xi,\lambda,\varepsilon,\sigma) \leq \phi(f,\xi,\lambda,0) \leq \phi(f,\xi,\underline{\lambda},0) \leq \widetilde{F}(\underline{\lambda})$$

by definition of $\widetilde{F}(\underline{\lambda})$. Let us now turn to the lower bound.

When $\varepsilon = 0$, we have that $\phi(f,\xi,\lambda,\varepsilon,\sigma) \geq f(\xi) \geq 0$.

When $0 < \varepsilon \leq \varepsilon_1$ and $\underline{\lambda} \geq \lambda_1 + L_2$, we have from Proposition A.2

$$\phi(f,\xi,\lambda,\varepsilon,\sigma) \geq f(\xi) - M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) \geq -M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma).$$

Otherwise, the bound comes from the smoothness of f and Jensen's inequality as

$$\begin{aligned}
\phi(f,\xi,\lambda,\varepsilon,\sigma) &\geq \varepsilon \log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)} \exp\left(\frac{f(\xi) + \langle\nabla f(\xi), \zeta - \xi\rangle - (L_2 + \lambda)\|\xi - \zeta\|^2/2}{\varepsilon}\right)\right) \\
&\geq \mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[f(\xi) + \langle\nabla f(\xi), \zeta - \xi\rangle - (L_2 + \lambda)\frac{1}{2}\|\xi - \zeta\|^2\right] \\
&\geq -\left(G\sqrt{2C(\sigma)} + (L_2 + \lambda)\, C(\sigma)\right) \geq -\left(G\sqrt{2C(\sigma)} + \left(L_2 + \overline{\lambda}\right) C(\sigma)\right).
\end{aligned}$$

□

A.4 Relative compactness of the class $\mathcal{F}$ of loss functions

In this section we prove the relative compactness of the class $\mathcal{F}$ w.r.t. several metrics. First, we show in Lemma A.6 that, under our blanket assumptions, $\mathcal{F}$ is relatively compact for for the infinity norm over Ξ, defined by i.e., $\|f\|_\infty := \sup_{\xi\in\Xi}|f(\xi)|$. Then, in Lemma A.7, we establish the equivalence between the first item of Assumption 5 and the relative compactness of $\mathcal{F}$ w.r.t. another distance that we introduce, as mentioned below Assumption 5 in Section 3.1. Finally, we leverage these compactness properties to ensure that the Dudley integral of $\mathcal{F}$ w.r.t. those metrics, a standard complexity measure in concentration theory, is finite in Lemma A.10.

Lemma A.6. *$\mathcal{F}$ and $\mathrm{conv}(\mathcal{F})$ are relatively compact for the topology of the uniform convergence.*

Proof. First, the functions of $\mathcal{F}$ are uniformly Lipschitz-continuous: fix $\xi \in \operatorname{supp} \mathrm{P}$, then, for any $\zeta \in \Xi$ $\|\nabla f(\zeta)\| \leq L_2\|\xi-\zeta\|+G \leq L_2 \sup_{(\xi,\zeta)\in\Xi}\|\xi-\zeta\|+G$ which is finite by compactness. Using the compactness of Ξ again, the functions in $\mathcal{F}$ are also uniformly bounded. As a consequence, the functions in $\mathrm{conv}(\mathcal{F})$ are also uniformly Lipschitz-continuous and uniformly bounded. By the Arzelà-Ascoli theorem, see e.g., (Rudin, 1987, Thm. 11.28), $\mathcal{F}$ and $\mathrm{conv}(\mathcal{F})$ are then relatively compact for the topology of uniform convergence. □

Recall that, for a set $A \subset \Xi$ and a point $\xi \in \Xi$, we denote by $d(\xi, A)$ the distance between ξ and A, i.e., $d(\xi, A) = \inf_{\zeta\in A}\|\xi - \zeta\|$.

Lemma A.7. *Consider the distance, defined on continuous functions on Ξ by*

$$D(f, g) \coloneqq \|f - g\|_\infty + D_H(\arg\max f, \arg\max g)$$

where D_H denotes the Hausdorff distance between sets associated to d, i.e., for $A, B \subset \Xi$,

$$D_H(A, B) \coloneqq \max\left(\sup_{\xi\in A} d(\xi, A), \sup_{\xi\in B} d(\xi, B)\right).$$

Under the blanket assumptions, we have that Item 1 of Assumption 5, i.e., that for any $R > 0$, there exists $\Delta > 0$ such that,

$$\forall f \in \mathcal{F},\ \forall \zeta \in \Xi,\ d(\zeta, \arg\max f) \geq R \implies f(\zeta) - \max f \leq -\Delta\,, \tag{22}$$

is equivalent to $\mathcal{F}$ being relatively compact for D.

Proof. ($\implies$) Let us begin by showing that (22) implies the relative compactness of $\mathcal{F}$ for D, i.e., that the adherence of $\mathcal{F}$ is compact for D.

Take $(f_t)_{t=1,2,\ldots}$ a sequence of functions from $\mathcal{F}$, and we will show that there is a subsequence which converges to some function in $\mathcal{F}$ for D. By compactness of $\mathcal{F}$ for the infinity norm, Lemma A.6, there readily is a subsequence of $(f_t)_{t=1,2,\ldots}$ that converges uniformly to some continuous function $f : \Xi \to \mathbb{R}$. Without loss of generality, let us assume that the whole sequence $(f_t)_{t=1,2,\ldots}$ converges uniformly to f, i.e., that $\|f - f_t\|_\infty \to 0$ as $t \to +\infty$. As a consequence, it holds also holds that $\max_\Xi f_t$ converges to $\max_\Xi f$.

We now show that $D_H(\arg\max f_t, \arg\max f)$ converges to 0. $\mathcal{F}$ satisfy (22) by assumption. Hence, for any fixed $\eta > 0$, we can invoke (22) with $R \leftarrow \eta$ and it gives us some $\Delta > 0$. Now, since f is continuous, $\{\zeta \in \Xi : d(\zeta, \arg\max f) \geq \eta\}$ is a closed set inside a compact and therefore is compact as well. Hence, f reaches its maximum over this set and it is strictly less than $\max_\Xi f$ by construction. Substituting Δ with $\min(\Delta, \max_\Xi f - \max\{f(\zeta) : \zeta \in \Xi,\ d(\zeta, \arg\max f) \geq \eta\})$ which is still positive, we get that, for any $\zeta \in \Xi$, both,

$$d(\zeta, \arg\max f) \geq \eta \implies f(\zeta) - \max f \leq -\Delta\,,$$

and, for any $t = 1, 2, \ldots$,

$$d(\zeta, \arg\max f_t) \geq \eta \implies f_t(\zeta) - \max f_t \leq -\Delta\,.$$

By convergence of the sequence, as mentioned above, there is some $T \geq 1$ such that, for any $t \geq T$, $\|f - f_t\|_\infty \leq \Delta/3$ and $|\max_\Xi f_t - \max_\Xi f| \leq \Delta/3$. These two inequalities imply that, for any $\xi \in \arg\max f$,

$$\max_\Xi f_t - f_t(\xi) \leq \max_\Xi f + \frac{\Delta}{3} - f(\xi) + \frac{\Delta}{3} = \frac{2\Delta}{3}\,.$$

Therefore, by definition of Δ, it holds that $d(\xi, \arg\max f_t) < \eta$. Similarly, when $\xi \in \arg\max f_t$, one shows that $\max_\Xi f - f(\xi) \leq 2\Delta/3$ so that we have $d(\xi, \arg\max f) < \eta$ as well. Hence, for any $t \geq T$, $D_H(\arg\max f_t, \arg\max f)$ is at most η.

Therefore, we have shown that $D_H(\arg\max f_t, \arg\max f)$ goes to zero. Since $\|f - f_t\|_\infty$ converges to zero as well by construction, this means that $D(f_t, f)$ converges to zero, which concludes the proof.

($\Longleftarrow$) Let us proceed by contradiction, i.e., assume that there is some $R > 0$, some sequence $(f_t)_{t=1,2,\dots}$ of functions from $\mathcal{F}$ and some sequence $(\xi_t)_{t=1,2,\dots}$ of points from Ξ such that,

$$\forall t = 1, 2, \dots, \; d(\arg\max f_t, \xi_t) \geq R \quad \text{yet} \quad f_t(\xi_t) - \max_{\Xi} f_t \to 0 \text{ as } t \to +\infty \,.$$

Since Ξ is compact and since we assume $\mathcal{F}$ to be relatively compact for D, without loss of generality, we can assume that $(\xi_t)_{t=1,2,\dots}$ converges to some $\xi \in \Xi$ while $(f_t)_{t=1,2,\dots}$ converges to some continuous function f for D. On the one hand, by definition of the Hausdorff distance, we have that, for any $t = 1, 2, \dots$,

$$\begin{aligned} d(\arg\max f, \xi) &\geq d(\arg\max f_t, \xi) - D_H(\arg\max f_t, \arg\max f) \\ &\geq d(\arg\max f_t, \xi_t) - (d(\xi, \xi_t) + D_H(\arg\max f_t, \arg\max f)) \,, \end{aligned}$$

so that, by taking $t \to +\infty$, we get that $d(\arg\max f, \xi) \geq R$. On the other hand, by uniform convergence, one has that

$$f(\xi) - \max f = \lim_{t \to +\infty} f_t(\xi_t) - \max f_t = 0 \,,$$

which yields the contradiction since ξ cannot belong to $\arg\max f$.

□

Note that, for parametric models (Section 3.4), this lemma gives a computation-free approach to verifying the second item of Assumption 5.

Corollary A.8. *Consider Θ a compact subset of $\mathbb{R}^p$ and $f : \Theta \times \Xi \to \mathbb{R}$ a continuous function. If the map $\theta \in \Theta \mapsto f(\theta, \cdot)$ is continuous from Θ to the space of continuous functions on Ξ equipped with the distance D defined in Lemma A.7, then $\mathcal{F} \coloneqq \{f(\theta, \cdot) : \theta \in \Theta\}$ is compact for D.*

In particular, this corollary allows one to easily check that Examples 3.6 and 3.7 satisfy the second item of Assumption 5.

We finally introduce Dudley's integral, which is a standard complexity measure in concentration theory.

Definition A.9. *Dudley's entropy integral $\mathcal{I}(\mathcal{X}, \mathrm{dist})$ is defined for a metric space $(\mathcal{X}, \mathrm{dist})$ as*

$$\mathcal{I}(\mathcal{X}, \mathrm{dist}) \coloneqq \int_0^{+\infty} \sqrt{\log N(t, \mathcal{X}, \mathrm{dist})} \mathrm{d}t$$

where $N(t, \mathcal{X}, \mathrm{dist})$ denotes the t-packing number of $\mathcal{X}$, which is the maximal number of points in $\mathcal{X}$ which are at least at a distance t from each other.

Lemma A.10. *The Dudley integral of $\mathcal{F}$ w.r.t. $\|\cdot\|_\infty$, that we denote by $\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)$, is finite. Under Assumption 5, the Dudley integral of $\mathcal{F}$ w.r.t. D, denoted by $\mathcal{I}(\mathcal{F}, D)$ is finite as well.*

Proof. Lemma A.6 shows that $\mathcal{F}$ is relatively compact for the norm $\|\cdot\|_\infty$ and in particular bounded. Since Dudley's entropy integral is finite for balls (Wainwright, 2019, Ex. 5.18) and $\mathcal{F}$ is now included in some ball for $\|\cdot\|_\infty$, the integral $\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)$ is indeed finite. The second assertion is proven using the same reasoning and Lemma A.7. □

A.5 Parametric Morse-Bott objectives

In this section, we discuss the quadratic growth condition of the second item of Assumption 5 and its relation to the parametric Morse-Bott assumption of Arbel and Mairal (2022). Indeed, in the context of smooth manifolds and parametric models, we prove that the parametric Morse-Bott assumption implies the quadratic growth condition of Assumption 5. In Assumption 7, we introduce the Riemannian and parametric settings that are necessary to formulate the parametric Morse-Bott condition and we then present a version of this condition adapted to our context. We refer to Lee (2018) for definitions relevant to Riemannian geometry. The main result of this section is then Proposition A.11, which relies on Lemma A.12 for its proof.

Assumption 7 (Parametric Morse-Bott)**.** Let $\mathcal{F} = \{\xi \mapsto f(\theta, \xi) : \theta \in \Theta\}$ where :

- Ξ, Θ are smooth compact (connected embedded) submanifolds of $\mathbb{R}^d$ and $\mathbb{R}^p$ respectively, endowed with the induced Euclidean metric.
- $f : \Theta \times \Xi \to \mathbb{R}$ is thrice continuously differentiable on the product manifold.
- f is a parametric Morse-Bott function (Arbel and Mairal, 2022, Def. 2): the set of augmented critical points of $\mathcal{F}$, defined as

$$M \coloneqq \{(\theta, \xi) \in \Theta \times \Xi : \operatorname{grad}_\xi f(\theta, \xi) = 0\}\,.$$

  is a C^2-(embedded) submanifold of $\Theta \times \Xi$ whose dimension at $(\theta, \xi) \in M$ is $\dim_\theta(\Theta) + \dim(\ker \operatorname{Hess}_\xi f(\theta, \xi))$.

Under this assumption Assumption 7, the following result thus guarantees that the quadratic growth condition of Assumption 5 holds.

Proposition A.11. *Under Assumption 7 and the first item of Assumption 5, the second item of Assumption 5 holds, i.e., there exists $\mu, L_3 > 0$ such that, for all $\theta \in \Theta$, $\xi \in \Xi$ and $\xi^* \in \arg\max f$ a projection of ξ on $\arg\max f$, i.e., $\xi^* \in \arg\min_{\arg\max f} \|\xi - \cdot\|$, it holds that*

$$f(\theta, \xi^*) \geq f(\theta, \xi) + \frac{\mu}{2}\|\xi - \xi^*\|^2 - \frac{L_3}{6}\|\xi - \xi^*\|^3\,.$$

To show this result, we rely on the following lemma that relates Assumption 7 to a local quadratic growth condition.

Lemma A.12. *Under Assumption 7, for any $(\theta_0, \xi_0) \in M$ such that ξ_0 is a local maximum of $f(\theta_0, \cdot)$ and any neighborhood $\mathcal{W}$ of (θ_0, ξ_0) in M, there exists a neighborhood $\mathcal{U}$ of (θ_0, ξ_0) in $\Theta \times \xi$ and $\mu > 0$ such that, for any $(\theta, \xi) \in \mathcal{U}$, there exists $\xi^* \in \Xi$ such that $(\theta, \xi^*) \in \mathcal{W}$ and*

$$f(\theta, \xi^*) \geq f(\theta, \xi) + \frac{\mu}{2}\|\xi - \xi^*\|^2\,.$$

Proof. By assumption, the tangent space of M at (θ, ξ) is given by

$$\mathrm{T}_{(\theta,\xi)} M = \mathrm{T}_\theta \Theta \times \ker \operatorname{Hess}_\xi f(\theta, \xi)\,,$$

and so its normal space (in $\Theta \times \Xi$) is equal to

$$\mathrm{N}_{(\theta,\xi)} M = \{0\} \times (\ker \operatorname{Hess}_\xi f(\theta, \xi))^\perp \subset \mathrm{T}_\theta \Theta \times \mathrm{T}_\xi \Xi\,.$$

Since M is a closed subset of the compact set $\Theta \times \Xi$, it is compact. By the tubular neighborhood theorem for compact submanifolds, there exists $\eta > 0$ and a neighborhood $\mathcal{U}_\eta$ of M in $\Theta \times \xi$ such that, with

$$\mathcal{V}_\eta \coloneqq \left\{(\theta, \xi, z) : (\theta, \xi) \in M,\, z \in (\ker \operatorname{Hess}_\xi f(\theta, \xi))^\perp,\, \|z\| < \eta\right\},$$

the normal exponential map $E : (\theta, \xi, z) \in \mathcal{V}_\eta \mapsto (\theta, \exp_\xi(z))$ is a diffeomorphism from $\mathcal{V}_\eta$ to $\mathcal{U}_\eta$. Note that $\mathcal{V}_\eta$ is relatively compact and, as a consequence, the third derivative of $t \in [0, 1] \mapsto f(\theta, \exp_\xi(tz))$ is a continuous function of $t \in [0, 1]$ and $(\theta, \xi, z) \in \mathcal{V}_\eta$ and as a consequence is bounded uniformly by some constant $L_3 > 0$. Fix $(\theta_0, \xi_0) \in M$ such that ξ_0 is a local maximum of $f(\theta, \cdot)$. Consider the map

$$\nu : (\theta, \xi) \mapsto \inf\left\{\langle z, -\operatorname{Hess}_\xi f(\theta, \xi) z\rangle : z \in (\ker \operatorname{Hess} f(\theta, \xi))^\perp,\, \|z\| = 1\right\}.$$

If $\nu(\theta_0, \xi_0)$ is $+\infty$, i.e., if $\ker \operatorname{Hess} f(\theta_0, \xi_0)$ is equal to the whole $\mathrm{T}_{\xi_0} \Xi$, then, since the dimension of a manifold is locally constant, there is a neighborhood of (θ_0, ξ_0) in $\Theta \times \Xi$ on which ν is identically equal to $+\infty$. Otherwise, if $\nu(\theta_0, \xi_0)$ is finite, then it is positive by construction. Hence, the continuity of ν implies there is a positive constant $\mu > 0$ and a neighborhood of (θ_0, ξ_0) in $\Theta \times \Xi$ on which ν is lower-bounded by μ.

Hence, in both cases, there is $\mu > 0$ and $\mathcal{V}'$ a neighborhood of (θ_0, ξ_0) in $\Theta \times \Xi$ such that ν is at least greater or equal to μ on $\mathcal{V}'$. Finally, take

$$\mathcal{U} \coloneqq \mathcal{U}_\eta \cap E\left(\left\{(\theta, \xi, z) \in \mathcal{V}_\eta : (\theta, \xi) \in \mathcal{V}' \cap \mathcal{W},\, \|z\| < \frac{3\mu}{4L_3}\right\}\right).$$

We are now in a position to prove the result. Take $(\theta, \xi) \in \mathcal{U}$. Since $\mathcal{U}$ is included in $\mathcal{U}_\eta$, there is some $\xi^* \in \Xi$ and $z \in (\ker \mathrm{Hess}\, f(\theta,\xi))^\perp$ such that $(\theta, \xi^*) \in M \cap \mathcal{V}' \cap \mathcal{W}$, $\|z\| < \frac{3\mu}{4L_3}$ and $\exp_{\xi^*}(z) = \xi$. Let $\gamma(t) \coloneqq \exp_{\xi^*}(tz)$ for $t \in [0,1]$ denotes the geodesic curve going from ξ^* to ξ. Then, by the Taylor inequality applied to $t \mapsto f(\xi, \gamma(t))$ (see Boumal (2020, § 5.9)) and by definition of L_3,

$$\begin{aligned} f(\theta,\xi) \leq & f(\theta,\xi^*) + \langle \mathrm{grad}_\xi\, f(\theta,\xi^*), z\rangle + \frac{1}{2}\langle \mathrm{Hess}_\xi\, f(\theta,\xi^*)z, z\rangle \\ & + \frac{1}{2}\langle \mathrm{grad}_\xi\, f(\theta,\xi^*), \gamma''(0)\rangle + \frac{L_3}{6}\|z\|^3 . \end{aligned}$$

But $\gamma''(t)$ is null since γ is a geodesic and $\mathrm{grad}_\xi\, f(\theta,\xi^*)$ too by definition. Moreover, since $(\theta,\xi^*) \in \mathcal{V}'$ and $z \in (\ker \mathrm{Hess}_\xi\, f(\theta,\xi^*))^\perp$, the term $\langle \mathrm{Hess}_\xi\, f(\theta,\xi^*)z, z\rangle$ is bounded by $-\mu\|z\|^2$. But $\|z\|$ is also equal to $\|\xi - \xi^*\|$ by definition of z so we get,

$$\begin{aligned} f(\theta,\xi) \leq & f(\theta,\xi^*) - \frac{\mu}{2}\|\xi-\xi^*\|^2 + \frac{L_3}{6}\|\xi-\xi^*\|^3 \\ \leq & f(\theta,\xi^*) - \frac{\mu}{4}\|\xi-\xi^*\|^2 , \end{aligned}$$

since $\|z\| = \|\xi - \xi^*\| \leq \frac{3\mu}{4L_3}$, which gives the result. □

We are now ready to prove Proposition A.11.

Proof of Proposition A.11. We build upon the result of Lemma A.12. Fix $(\theta_0, \xi_0) \in M$ such that ξ_0 is a maximum of $f(\theta_0, \cdot)$ and let $r > 0$ such that $\overline{\mathbb{B}}((\theta_0,\xi_0), r) \cap M$ is diffeomorphic to an Euclidean ball. Invoke the first item of Assumption 5, with $R \leftarrow r/2$ and let $\Delta > 0$ be the given positive quantity. Let $\mathcal{U}$, μ be given by Lemma A.12 invoked with $\mathcal{W} \coloneqq \overline{\mathbb{B}}\big((\theta_0,\xi_0), \frac{r}{2}\big) \cap \{(\theta,\xi) \in M : f(\theta,\xi) > \max_\Xi f(\theta,\cdot) - \Delta\}$.

Hence, for any $(\theta,\xi) \in \mathcal{U}$, there is $\xi^* \in \Xi$ such that $(\theta,\xi^*) \in M \cap \mathcal{W}$ and

$$f(\theta,\xi^*) \geq f(\theta,\xi) + \frac{\mu}{2}\|\xi-\xi^*\|^2 . \tag{23}$$

But (θ,ξ^*) also satisfies $f(\theta,\xi^*) > \max_\Xi f(\theta,\cdot) - \Delta$ so that $d(\xi^*, \arg\max_\Xi f(\theta,\cdot)) < \frac{r}{2}$ by definition of Δ, i.e., there exists ξ^{**} that is a maximizer of $f(\theta,\cdot)$ and that is at distance at most $< \frac{r}{2}$ from ξ^*. But then both ξ^* and ξ^{**} belong to $\overline{\mathbb{B}}((\theta_0,\xi_0), r)$ that is diffeomorphic to an Euclidean ball. Hence, since the derivative of $f(\theta,\cdot)$ is null on M, $f(\theta,\xi^*) = f(\theta,\xi^{**}) = \max_\Xi f(\theta,\cdot)$ so that ξ^* is a maximizer of $f(\theta,\cdot)$ too. Therefore, (23) becomes

$$\max_\Xi f(\theta,\cdot) = f(\theta,\xi^*) \geq f(\theta,\xi) + \frac{\mu}{2}\|\xi-\xi^*\|^2 \geq f(\theta,\xi) + \frac{\mu}{2} d^2\left(\xi, \arg\max_\Xi f(\theta,\cdot)\right) .$$

The final statement of the proposition follows by compactness and uniform Lipschitz-continuity of $\mathcal{F}$ (see the proof of Lemma A.6). □

B From empirical to true risk via duality

In this part of the proof, our objective is to show that: if the dual variable λ in (12) can be bounded uniformly in $[\underline{\lambda}, \overline{\lambda}]$ with probability $1-\delta$, then we can concentrate the empirical expectation in (12) towards the one in (13). The concentration error induces a loss in the radius, fortunately, captured by the variable $\overline{\rho}_n^2(\delta, \underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)$ that we take as

$$\overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma) \coloneqq \frac{117}{\sqrt{n}\underline{\lambda}}\left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\Big(\widetilde{F}(\underline{\lambda}), a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\Big)\left(1 + \sqrt{\log\frac{1}{\delta}}\right)\right) , \tag{24}$$

where $\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)$ is the Dudley integral of $\mathcal{F}$ w.r.t. the infinity norm (Definition A.9), $\widetilde{F}$ is defined in Assumption 6, and $a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)$ is the bounding term appearing in Corollary A.5.

The main result of this part is Proposition B.1, stated below, and the remainder of the section will consist in proving it.

Proposition B.1. *for $\rho > 0$, $\varepsilon \geq 0$, $\sigma > 0$ and $\delta \in (0,1)$, assume that there is some $0 < \underline{\lambda} \leq \overline{\lambda} < +\infty$ such that, with probability at least $1 - \frac{\delta}{2}$,*

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda} \leq \lambda \leq \overline{\lambda}} \lambda\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}}\left[\phi(f, \xi, \lambda, \varepsilon, \sigma)\right] . \tag{25}$$

then, when $\rho^2 \geq \overline{\rho}^2_n(\delta, \underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)$, with probability $1 - \delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \mathcal{R}^{\varepsilon}_{\rho^2 - \overline{\rho}^2_n(\delta, \underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)}(f) .$$

The proof of this result mainly consists in verifying that under our standing assumptions, we can apply the concentration result presented in Lemma G.2 in order to concentrate $\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f)$ towards $\mathcal{R}^{\varepsilon}_{\rho^2}(f)$ through their dual formulations.

We begin by showing that the dual generator *divided by* λ is Lipchitz continuous in f and in λ^{-1} (for convenience, we use the notation $\mu = \lambda^{-1}$).

Lemma B.2. *Fix some $\overline{\lambda} \geq \underline{\lambda} > 0$. For any $\xi \in \Xi$, $\varepsilon \geq 0$ and $\sigma > 0$ we have that*

(a) for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, $f \mapsto \lambda^{-1}\phi(f, \xi, \lambda, \varepsilon, \sigma)$ is $\underline{\lambda}^{-1}$-Lipschitz continuous w.r.t. the norm $\|\cdot\|_\infty$;

(b) for any $f \in \mathcal{F}$, $\mu \mapsto \mu\phi(f, \xi, \mu^{-1}, \varepsilon, \sigma)$ is $\max\Big(\widetilde{F}(\underline{\lambda}), a(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)\Big)$-Lipschitz continuous on $\left[\overline{\lambda}^{-1}, \underline{\lambda}^{-1}\right]$.

Proof. Item (a). When $\varepsilon = 0$, $f \mapsto \phi(f, \xi, \lambda, \varepsilon, \sigma)$ is a supremum of 1-Lipschitz functions and is thus 1-Lipschitz. For $\varepsilon > 0$, take $f, g \in \mathcal{F}$ and, for $t \in [0,1]$, define $f_t = f + t(g - f)$. Differentiating $t \mapsto \phi(f_t, \xi, \lambda, \varepsilon, \sigma)$ yields

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\phi(f_t, \xi, \lambda, \varepsilon, \sigma)\right| = \left|\frac{\mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)}\left[(g(\zeta) - f(\zeta))e^{\frac{f_t(\zeta) - \lambda\|\xi - \zeta\|^2/2}{\varepsilon}}\right]}{\mathbb{E}_{\zeta' \sim \pi_\sigma(\cdot|\xi)}\left[e^{\frac{f_t(\zeta) - \lambda\|\xi - \zeta'\|^2/2}{\varepsilon}}\right]}\right| \leq \|g - f\|_\infty ,$$

which gives that $f \mapsto \phi(f, \xi, \lambda, \varepsilon, \sigma)$ is 1-Lipschitz continuous w.r.t. the norm $\|\cdot\|_\infty$.

Since this bound is uniform in λ, we immediately get that $f \mapsto \lambda^{-1}\phi(f, \xi, \lambda, \varepsilon, \sigma)$ is $\underline{\lambda}^{-1}$-Lipschitz continuous for all $\lambda \geq \underline{\lambda}$.

Item (b). Fix $f \in \mathcal{F}$, $\xi \in \Xi$, $\varepsilon \geq 0$, $\sigma > 0$ and define $g(\lambda) \coloneqq \lambda \mapsto \lambda^{-1}\phi(f, \xi, \lambda, \varepsilon, \sigma)$.

Let us first begin with the case $\varepsilon = 0$. Take $\lambda, \lambda' \in [\underline{\lambda}, \overline{\lambda}]$. Without loss of generality, we can suppose that $g(\lambda) \geq g(\lambda')$. Since f is continuous and Ξ is a compact set, choose $\zeta \in \arg\max_{\zeta \in \Xi}\left\{f(\zeta) - \frac{\lambda}{2}\|\xi - \zeta\|^2\right\}$. Then, the claim comes from the fact that

$$0 \leq g(\lambda) - g(\lambda') \leq \lambda^{-1}f(\zeta) - \frac{1}{2}\|\xi - \zeta\|^2 - \left(\lambda'^{-1}f(\zeta) - \frac{1}{2}\|\xi - \zeta\|^2\right) \leq |\lambda^{-1} - \lambda'^{-1}|\, \widetilde{F}(\underline{\lambda}) ,$$

where we use that since f is non-negative by assumption, $\left|\widetilde{F}(\lambda)\right| = \widetilde{F}(\lambda) \leq \widetilde{F}(\underline{\lambda})$.

Let us now turn to the case where $\varepsilon > 0$, for which g is differentiable on $[\underline{\lambda}, \overline{\lambda}]$ with derivative

$$g'(\lambda) = -\frac{1}{\lambda^2}\phi(f, \xi, \lambda, \varepsilon, \sigma) - \frac{1}{\lambda}\frac{\mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)}\left[\frac{1}{2}\|\xi - \zeta'\|^2 e^{\frac{f(\zeta) - \lambda\|\xi - \zeta\|^2/2}{\varepsilon}}\right]}{\mathbb{E}_{\zeta' \sim \pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta') - \lambda\|\xi - \zeta'\|^2/2}{\varepsilon}}\right]} .$$

Since the claimed result is the Lipchitz continuity of $h : \mu \mapsto g(\mu^{-1})$, it suffices to bound its derivative, i.e., to bound $-\lambda^{-2}g'(\lambda)$ for all $\overline{\lambda} \geq \lambda \geq \underline{\lambda}$. On the one hand, thanks to Lemma G.7, it is bounded above as

$$\begin{aligned}
-\frac{1}{\lambda^2}g'(\lambda) &\leq \frac{\mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)}\left[(f(\zeta) - \frac{\lambda}{2}\|\xi - \zeta\|^2)e^{\frac{f(\zeta) - \lambda\|\xi - \zeta\|^2/2}{\varepsilon}}\right]}{\mathbb{E}_{\zeta' \sim \pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta') - \lambda\|\xi - \zeta'\|^2/2}{\varepsilon}}\right]} + \lambda\frac{\mathbb{E}_{\zeta \sim \pi_\sigma(\cdot|\xi)}\left[\frac{1}{2}\|\xi - \zeta\|^2 e^{\frac{f(\zeta) - \lambda\|\xi - \zeta\|^2/2}{\varepsilon}}\right]}{\mathbb{E}_{\zeta' \sim \pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta') - \lambda\|\xi - \zeta'\|^2/2}{\varepsilon}}\right]} \\
&\leq \phi(f, \xi, \lambda, 0) \leq \widetilde{F}(\lambda) \leq \widetilde{F}(\underline{\lambda}) .
\end{aligned}$$

On the other hand, invoking Corollary A.5 also yields that

$$-\frac{1}{\lambda^2}g'(\lambda) \geq \varepsilon \log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)} e^{\frac{f(\zeta)-\lambda\|\xi-\zeta\|^2/2}{\varepsilon}}\right) \geq -a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\,,$$

which concludes the proof. ☐

We can now apply standard concentration for bounded Lipschitz quantities to bound the difference between the expectation of the dual generator over the empirical distribution $\mathrm{P_n}$ and true one P.

Lemma B.3. *For $\rho > 0$, $\varepsilon \geq 0$, $\sigma > 0$, $\delta \in (0,1)$ and some $0 < \underline{\lambda} \leq \overline{\lambda} < +\infty$, we have with probability at least $1 - \frac{\delta}{2}$ that*

$$\sup_{(f,\lambda)\in\mathcal{F}\times[\underline{\lambda},\overline{\lambda}]} \left\{ \frac{\mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim\mathrm{P_n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \leq \overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\,.$$

Proof. Our objective is to bound the quantity

$$\begin{aligned}
&\sup_{(f,\lambda)\in\mathcal{F}\times[\underline{\lambda},\overline{\lambda}]} \left\{ \frac{\mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim\mathrm{P_n}}[\phi(f,\mathrm{P_n},\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \\
&= \sup_{(f,\mu)\in\mathcal{F}\times\left[\overline{\lambda}^{-1},\underline{\lambda}^{-1}\right]} \left\{ \mathbb{E}_{\xi\sim\mathrm{P}}\left[\mu\,\phi(f,\xi,\mu^{-1},\varepsilon,\sigma)\right] - \mathbb{E}_{\xi\sim\mathrm{P_n}}\left[\mu\,\phi(f,\xi,\mu^{-1},\varepsilon,\sigma)\right] \right\} \\
&= \sup_{(f,\mu)\in\mathcal{X}} \left\{ \mathbb{E}_{\xi\sim\mathrm{P}}[X((f,\mu),\xi)] - \mathbb{E}_{\xi\sim\mathrm{P_n}}[X((f,\mu),\xi)] \right\},
\end{aligned}$$

where we used again the notation $\mu = \lambda^{-1}$ and defined

$$\mathcal{X} \coloneqq \mathcal{F} \times \left[\overline{\lambda}^{-1},\underline{\lambda}^{-1}\right] \quad \text{and} \quad X((f,\mu),\xi) \coloneqq \mu\,\phi(f,\xi,\mu^{-1},\varepsilon,\sigma)\,.$$

Let us endow $\mathcal{X}$ with the distance,

$$\mathrm{dist}((f,\mu),(f',\mu')) \coloneqq \underline{\lambda}^{-1}\|f-f'\|_\infty + \max\left(\widetilde{F}(\underline{\lambda}), a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\right)|\mu-\mu'|\,.$$

We now wish to apply Lemma G.2 and check its three requirements:

1. For any $(f,\mu) \in \mathcal{F} \times \left[\overline{\lambda}^{-1},\underline{\lambda}^{-1}\right]$, $X((f,\mu),\cdot)$ is measurable since the functions of $\mathcal{F}$ are continuous and thus *a fortiori* measurable;
2. By Lemma B.2, for any $\varepsilon \geq 0$ and any $\xi \in \operatorname{supp}\mathrm{P}$, $X(\cdot,\xi)$ is 1-Lipschitz w.r.t. dist;
3. Thanks to Corollary A.5, for any $(f,\mu) \in \mathcal{X}$, $\xi \in \operatorname{supp}\mathrm{P}$, $\varepsilon \geq 0$ and $\sigma > 0$, we have

$$-\frac{a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)}{\underline{\lambda}} \leq X((f,\mu),\xi) \leq \frac{\widetilde{F}(\underline{\lambda})}{\underline{\lambda}}\,.$$

As a consequence, applying statement (b) of Lemma G.2 yields that, with probability at least $1-\frac{\delta}{2}$,

$$\begin{aligned}
&\sup_{(f,\lambda)\in\mathcal{F}\times[\underline{\lambda},\overline{\lambda}]} \left\{ \frac{\mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim\mathrm{P_n}}[\phi(f,\mathrm{P_n},\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \\
&\leq \frac{48\mathcal{I}(\mathcal{X},\mathrm{dist})}{\sqrt{n}} + \frac{2}{\underline{\lambda}}\left(\widetilde{F}(\underline{\lambda}) + a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\right)\sqrt{\frac{\log\frac{2}{\delta}}{2n}}\,.
\end{aligned}$$

We now proceed to bound $\mathcal{I}(\mathcal{X},\mathrm{dist})$. Exploiting the product space structure of $\mathcal{X}$ and dist with Lemma G.3, one has that,

$$\begin{aligned}
\mathcal{I}(\mathcal{X},\mathrm{dist}) &\leq \underline{\lambda}^{-1}\mathcal{I}(\mathcal{F},\|\cdot\|_\infty) + \max\left(\widetilde{F}(\underline{\lambda}), a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\right)\mathcal{I}([0,\underline{\lambda}^{-1}],|\cdot|) \\
&\leq \underline{\lambda}^{-1}\left(\mathcal{I}(\mathcal{F},\|\cdot\|_\infty) + \max\left(\widetilde{F}(\underline{\lambda}), a(\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\right)\frac{1+2\log 2}{2}\right),
\end{aligned}$$

where we used Lemma G.4. Hence, we have shown that with probability at least $1 - \frac{\delta}{2}$,

$$\sup_{(f,\lambda)\in\mathcal{F}\times[\underline{\lambda},\overline{\lambda}]} \left\{ \frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \leq \overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)$$

where some numerical constants have been simplified. □

Proof of Proposition B.1. Building on Lemma B.3, we can now conclude the main result of this section. Using our boundedness assumption on λ, we have that, with probability $1-\delta$, the two following statements hold simultaneously

- $\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]$;
- $\sup_{(f,\lambda)\in\mathcal{F}\times[\underline{\lambda},\overline{\lambda}]} \left\{ \frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \leq \overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\,.$

As a consequence, on this event, for any $f \in \mathcal{F}$,

$$\begin{aligned}
\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) &= \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\right\} \\
&= \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda\frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda}\right\} \\
&\geq \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda \sup_{\underline{\lambda}\leq\lambda'\leq\overline{\lambda}} \frac{\mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)]}{\lambda'}\right\} \\
&\geq \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda\overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)\right\} \\
&\geq \mathcal{R}^{\varepsilon}_{\rho^2-\overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)}(f)
\end{aligned}$$

where $\rho^2 - \overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma) \geq 0$ by assumption. □

Remark B.4. *Note that the proof of Proposition B.1 actually gives us the slightly stronger result at the penultimate equation: with probability at least $1-\delta$, for any $f \in \mathcal{F}$,*

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{\lambda(\rho^2 - \overline{\rho}_n^2(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)) + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\right\}\,,$$

that we will require later.

C Dual bound when ρ is small

In this section, we show how the condition (25) of Proposition B.1 can be obtained when the robustness radius ρ is small enough. The results of this section cover both the standard WDRO setting of Theorems 3.1 and 3.3 and the regularized case of Theorem 3.4.

In the following Assumption 8, we precise how small ρ has to be; we also take ε and σ proportional to ρ in order to get close to the true risk with ρ, ε and σ "small" at the same time. The main result of this section is Proposition C.1, whose proof relies on Lemma C.2.

Assumption 8 (ρ is small)**.** Take $\varepsilon = \varepsilon_0\rho$, $\sigma = \sigma_0\rho$ with $\varepsilon_0 \geq 0$, $\sigma_0 > 0$ and define

$$\lambda_0^* \coloneqq \varepsilon_0 d + \sqrt{(\varepsilon_0 d)^2 + 8\inf_{f\in\mathcal{F}} \mathbb{E}_{\mathrm{P}}\|\nabla f\|_2^2} \qquad \mu^* \coloneqq \frac{8\inf_{f\in\mathcal{F}}\mathbb{E}_{\mathrm{P}}\|\nabla f\|_2^2}{(\lambda_0^*)^3} + \frac{2\varepsilon_0 d}{(\lambda_0^{*2})}\,.$$

Moreover, assume that ε_0 and σ_0 satisfy

$$\frac{\varepsilon_0}{{\sigma_0}^2} \leq \frac{\lambda_0^*}{8}\,.$$

Assume that $\rho > 0$ is small enough so that,

$$\rho \leq \min\left(\frac{\varepsilon_1}{\varepsilon_0}, \frac{\lambda_0^*}{32(\lambda_1+L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096L_2}, \sqrt{\frac{c_2^3\lambda_0^*}{8\varepsilon_0}}\left(\log\left(\frac{4096\varepsilon_0 c_1}{\mu^*(\lambda_0^*)^2}\right)\right)_+^{-\frac{3}{2}}\right)$$

where $\lambda_1, \varepsilon_1, c_1, c_2$ are positive constant given by Lemma A.3 and σ comes from (4).

Note that λ_0^* and μ^* are both always positive, be it thanks to Assumption 5 or the regularization with $\varepsilon_0 > 0$. For such values of ρ, the main result of this section Proposition C.1 shows that the dual variable of (12) can be bounded with high probability.

Proposition C.1. *Let Assumption 8 hold and fix a threshold $\delta \in (0,1)$. Assume in addition that*

$$\rho \geq \frac{8192}{\sqrt{n}\,\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + (\widetilde{F}(0) + \overline{M}(\rho))\sqrt{1+\log\frac{1}{\delta}}\right)$$

where $\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)$, $\widetilde{F}$ are defined in Section A.1 and $M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)$, the bounding term appearing in Proposition A.2, is used to define

$$\overline{M}(\rho) \coloneqq \sup_{\rho' \in (0,\rho]} M\left(\max\left(\frac{\lambda_0^*}{32\rho'}, \lambda_1 + L_2\right), \frac{\lambda_0^*}{2\rho'}, \varepsilon_0\rho', \sigma_0\rho'\right).$$

Then, with probability at least $1-\delta$, we have

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\frac{\lambda_0^*}{32\rho} \leq \lambda \leq \frac{\lambda_0^*}{2\rho}} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right].$$

To show Proposition C.1, we need the following helper lemma.

Lemma C.2. *Let Assumption 8 hold. Then,*

$$\begin{aligned}
&\left(\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2} + L_2\right)\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\phi\left(f,\xi,\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2} + L_2, \varepsilon\right)\right] + \frac{\rho\mu^*}{1024}(\lambda_0^*)^2 \\
&\quad\leq \min\left(\left(\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0\rho}{2\sigma^2} - L_2\right)\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\phi\left(f,\xi,\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0\rho}{2\sigma^2} - L_2, \varepsilon\right)\right],\right. \\
&\qquad\left.\left(\frac{\lambda_0^*}{2\rho} - \frac{2\varepsilon_0\rho}{\sigma^2} + L_2\right)\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\phi\left(f,\xi,\frac{\lambda_0^*}{2\rho} - \frac{2\varepsilon_0\rho}{\sigma^2} + L_2, \varepsilon\right)\right]\right) \\
&\textit{and}\quad \max\left(\frac{\lambda_0^*}{32\rho}, \lambda_1 + L_2\right) \leq \frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0\rho}{2\sigma^2} - L_2 \leq \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2} + L_2 \leq \frac{\lambda_0^*}{2\rho}
\end{aligned}$$

Proof. Fix $f \in \mathcal{F}$. Consider the function $\overline{\psi_\rho} : \lambda \mapsto \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\overline{\phi}(f,\xi,\lambda,\varepsilon,\sigma)\right]$ where $\overline{\phi}$ is defined in (19). By Lemma G.8 invoked with $a \leftarrow \rho^2$, $b \leftarrow \frac{1}{2}\mathbb{E}_\mathrm{P}\left[\|\nabla f\|_2^2\right]$, $c \leftarrow \frac{\varepsilon d}{2}$ and $r \leftarrow \frac{\varepsilon}{\sigma^2}$, its unique minimizer is

$$\lambda^\star \coloneqq \left[\frac{\varepsilon_0 d + \sqrt{(\varepsilon_0 d)^2 + 8\mathbb{E}_\mathrm{P}\|\nabla f\|_2^2}}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2}\right]_+ = \left[\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2}\right]_+.$$

where we used that $\varepsilon = \varepsilon_0\rho$. And, since $\frac{\varepsilon_0\rho^2}{\sigma^2} = \frac{\varepsilon_0}{{\sigma_0}^2} \leq \frac{\lambda_0^*}{8}$ by Assumption 8, $\lambda^\star$ actually satisfies

$$\frac{\lambda_0^*}{8\rho} \leq \lambda^\star \leq \frac{\lambda_0^*}{4\rho}. \tag{26}$$

Moreover, Lemma G.8 also shows that, on $[0, 2\lambda^\star]$, $\overline{\psi_\rho}$ is strongly convex with modulus

$$\frac{\mathbb{E}_\mathrm{P}\left[\|\nabla f\|_2^2\right]}{(2\lambda^\star + \frac{\varepsilon_0\rho}{\sigma^2})^3} + \frac{\varepsilon_0 d\rho}{2(2\lambda^\star + \frac{\varepsilon_0\rho}{\sigma^2})^2} = \frac{\mathbb{E}_\mathrm{P}\left[\|\nabla f\|_2^2\right]}{(\frac{\lambda_0^*}{2\rho} - \frac{\varepsilon_0\rho}{\sigma^2})^3} + \frac{\varepsilon_0 d\rho}{2(\frac{\lambda_0^*}{2\rho} - \frac{\varepsilon_0\rho}{\sigma^2})^2} \geq \rho^3\mu^*.$$

Now, we notice that $\varepsilon = \varepsilon_0\rho \leq \varepsilon_1$ by Assumption 8. Then, if $\lambda \in [\lambda_1 + L_2, 2\lambda^\star - L_2]$, then Lemma A.3 (applied twice) and the strong convexity of $\overline{\psi_\rho}$ yield

$$\begin{aligned}
&\lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] \\
&\geq \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\overline{\phi}(f,\xi,\lambda+L_2,\varepsilon,\sigma)\right] - \varepsilon c_1 e^{-c_2(\frac{\lambda+L_2}{\varepsilon})^{\frac{1}{3}}} \\
&\geq \lambda^\star\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\overline{\phi}(f,\xi,\lambda^\star,\varepsilon,\sigma)\right] - \rho^2 L_2 + \frac{\rho^3\mu^*}{2}(\lambda^\star - (\lambda+L_2))^2 - \varepsilon c_1 e^{-c_2(\frac{\lambda+L_2}{\varepsilon})^{\frac{1}{3}}} \\
&\geq \lambda^\star\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda^\star+L_2,\varepsilon,\sigma)] - \rho^2 L_2 + \frac{\rho^3\mu^*}{2}(\lambda^\star - (\lambda+L_2))^2 - \varepsilon c_1 e^{-c_2(\frac{\lambda+L_2}{\varepsilon})^{\frac{1}{3}}} - \varepsilon c_1 e^{-c_2(\frac{\lambda^\star}{\varepsilon})^{\frac{1}{3}}}.
\end{aligned} \tag{27}$$

We first wish to choose $\lambda = \frac{\lambda^\star}{2} - L_2$. By (26), since $\rho \leq \frac{\lambda_0^*}{32(\lambda_1+L_2)}$ by Assumption 8, this choice of λ is indeed greater than or equal to $\lambda_1 + L_2$ and (27) leads to

$$\begin{aligned}
&\left(\frac{\lambda^\star}{2} - L_2\right)\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\frac{\lambda^\star}{2} - L_2,\varepsilon,\sigma)\right] \\
&\geq \lambda^\star\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda^\star + L_2,\varepsilon,\sigma)] - \rho^2 L_2 + \frac{\rho^3\mu^*}{8}(\lambda^\star)^2 - \varepsilon c_1 e^{-c_2\left(\frac{\lambda^\star}{2\varepsilon}\right)^{\frac{1}{3}}} - \varepsilon c_1 e^{-c_2\left(\frac{\lambda^\star}{\varepsilon}\right)^{\frac{1}{3}}} \\
&\geq (\lambda^\star + L_2)\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda^\star + L_2,\varepsilon,\sigma)] - 2\rho^2 L_2 + \frac{\rho\mu^*}{512}(\lambda_0^*)^2 - 2\varepsilon c_1 e^{-c_2\left(\frac{\lambda_0^*}{8\varepsilon\rho}\right)^{\frac{1}{3}}}. \quad (28)
\end{aligned}$$

where we used (26) again for the last inequality.

To obtain the other inequality we pick $\lambda = 2\lambda^\star - L_2$, which is greater or equal to $\lambda_1 + L_2$ by Assumption 8 and (26) as above. Then, (27) yields

$$\begin{aligned}
&(2\lambda^\star - L_2)\,\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,2\lambda^\star - L_2,\varepsilon,\sigma)] \\
&\geq \lambda^\star\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda^\star + L_2,\varepsilon,\sigma)] - \rho^2 L_2 + \frac{\rho^3\mu^*}{2}(\lambda^\star)^2 - \varepsilon c_1 e^{-c_2\left(\frac{2\lambda^\star}{\varepsilon}\right)^{\frac{1}{3}}} - \varepsilon c_1 e^{-c_2\left(\frac{\lambda^\star}{\varepsilon}\right)^{\frac{1}{3}}} \\
&\geq (\lambda^\star + L_2)\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda^\star + L_2,\varepsilon,\sigma)] - 2\rho^2 L_2 + \frac{\rho\mu^*}{512}(\lambda_0^*)^2 - 2\varepsilon c_1 e^{-c_2\left(\frac{\lambda_0^*}{8\varepsilon\rho}\right)^{\frac{1}{3}}}
\end{aligned}$$

where we used (26) again, and degraded the constants to match those of (28).

Thus, we have that $\lambda^\star = \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2}$ and

$$\begin{aligned}
&(\lambda^\star + L_2)\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,\lambda^\star + L_2,\varepsilon,\sigma)] - 2\rho^2 L_2 + \frac{\rho\mu^*}{512}(\lambda_0^*)^2 - 2\varepsilon c_1 e^{-c_2\left(\frac{\lambda_0^*}{8\varepsilon\rho}\right)^{\frac{1}{3}}} \\
&\leq \min\left(\left(\frac{\lambda^\star}{2} - L_2\right)\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\frac{\lambda^\star}{2} - L_2,\varepsilon,\sigma)\right], (2\lambda^\star - L_2)\,\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}[\phi(f,\xi,2\lambda^\star - L_2,\varepsilon,\sigma)]\right)
\end{aligned}$$

All that is left to show for the main result of the lemma is that

$$\begin{aligned}
&-2\rho^2 L_2 + \frac{\rho\mu^*}{512}(\lambda_0^*)^2 - 2\varepsilon c_1 e^{-c_2\left(\frac{\lambda_0^*}{8\varepsilon\rho}\right)^{\frac{1}{3}}} \geq \frac{\rho\mu^*}{1024}(\lambda_0^*)^2 \\
\Leftrightarrow \quad &2\rho L_2 + 2\varepsilon_0 c_1 e^{-c_2\left(\frac{\lambda_0^*}{8\varepsilon_0\rho^2}\right)^{\frac{1}{3}}} \leq \frac{\mu^*}{1024}(\lambda_0^*)^2. \quad (29)
\end{aligned}$$

This is a consequence of Assumption 8 which states that

$$\rho \leq \frac{\mu^*}{4096 L_2}(\lambda_0^*)^2 \quad \text{and} \quad \rho \leq \sqrt{\frac{c_2^3\lambda_0^*}{8\varepsilon_0}}\left(\log\left(\frac{4096\varepsilon_0 c_1}{\mu^*(\lambda_0^*)^2}\right)\right)_+^{-\frac{3}{2}},$$

which imply that $2\rho L_2 \leq \frac{\mu^*}{2048}(\lambda_0^*)^2$, and $2\varepsilon_0 c_1 e^{-c_2\left(\frac{\lambda_0^*}{8\varepsilon_0\rho^2}\right)^{\frac{1}{3}}} \leq \frac{\mu^*}{2048}(\lambda_0^*)^2$,

so that (29) indeed holds, concluding the proof of the first part of the result.

The supplementary bounds follow directly from (26) and our assumptions on ρ. □

We are now in a position to show our main result when ρ is small, namely Proposition C.1.

Proof of Proposition C.1. Let us first take any $\lambda \in [\max\left(\frac{\lambda_0^*}{32\rho}, \lambda_1 + L_2\right), \frac{\lambda_0^*}{2\rho}]$. We want to instante Lemma G.2 with $X(f,\xi) \leftarrow \phi(f,\xi,\lambda,\varepsilon,\sigma)$, $(\mathcal{X},\mathrm{dist}) \leftarrow (\mathcal{F},\|\cdot\|_\infty)$, whose requirements are checked since:

1. For any $f \in \mathcal{F}$, $\phi(f,\xi,\lambda,\varepsilon,\sigma)$ is measurable since the functions of $\mathcal{F}$ are continuous and thus *a fortiori* measurable;

2. By the proof of Lemma B.2(a), we have that for any $\varepsilon \geq 0$, $\sigma > 0$ and any $\xi \in \operatorname{supp} \mathrm{P}$, $f \mapsto \phi(f, \xi, \lambda, \varepsilon, \sigma)$ is 1-Lipschitz continuous w.r.t. the norm $\|\cdot\|_\infty$;

3. With Proposition A.2 with $\underline{\lambda} \leftarrow \max\left(\frac{\lambda_0^*}{32\rho}, \lambda_1 + L_2\right)$, $\overline{\lambda} \leftarrow \frac{\lambda_0^*}{2\rho}$, for any $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$ and $\varepsilon \in [0, \varepsilon_1]$ (by Assumption 8), we have

$$-\overline{M}(\rho) \leq f(\xi) - M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) \leq \phi(f, \xi, \lambda, \varepsilon, \sigma) \leq f(\xi) + M(\underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma) \leq \widetilde{F}(0) + \overline{M}(\rho)$$

where $\overline{M}(\rho)$ is defined in Proposition C.1.

Since $\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2 \geq \lambda_1 + L_2$ by Lemma C.2, we can apply statement (b) of Lemma G.2 with $\lambda \leftarrow \frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2$ and $\delta \leftarrow \frac{\delta}{4}$ to have that, with probability at least $1 - \frac{\delta}{4}$, for all $f \in \mathcal{F}$

$$\begin{aligned}
&\mathbb{E}_{\xi \sim \mathrm{P}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2, \varepsilon\right)\right] - \mathbb{E}_{\xi \sim \mathrm{P_n}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2, \varepsilon\right)\right] \\
&\quad \leq \frac{48\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\sqrt{n}} + 4(\widetilde{F}(0) + \overline{M}(\rho))\sqrt{\frac{\log\frac{4}{\delta}}{2n}}\,.
\end{aligned}$$

Similarly, we can apply statement (a) of Lemma G.2 with $\lambda \leftarrow \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{\sigma^2} + L_2$ and $\delta \leftarrow \frac{\delta}{4}$ to get that, with probability at least $1 - \frac{\delta}{4}$, for all $f \in \mathcal{F}$,

$$\begin{aligned}
&\mathbb{E}_{\xi \sim \mathrm{P_n}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{\sigma^2} + L_2, \varepsilon\right)\right] - \mathbb{E}_{\xi \sim \mathrm{P}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} + L_2, \varepsilon\right)\right] \\
&\quad \leq \frac{48\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\sqrt{n}} + 4(\widetilde{F}(0) + \overline{M}(\rho))\sqrt{\frac{\log\frac{4}{\delta}}{2n}}\,.
\end{aligned}$$

Combining the two statements above and using Lemma C.2, we get that, with probability at least $1 - \frac{\delta}{2}$, for any $f \in \mathcal{F}$,

$$\begin{aligned}
&\left(\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2\right)\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2, \varepsilon\right)\right] \\
&\geq \left(\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2\right)\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0 \rho}{2\sigma^2} - L_2, \varepsilon\right)\right] \\
&\qquad - \frac{48\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\sqrt{n}} - 4(\widetilde{F}(0) + \overline{M}(\rho))\sqrt{\frac{\log\frac{4}{\delta}}{2n}} \\
&\geq \left(\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{\sigma^2} + L_2\right)\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{\sigma^2} + L_2, \varepsilon\right)\right] \\
&\qquad + \frac{\rho\mu^*}{1024}(\lambda_0^*)^2 - \frac{48\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\sqrt{n}} - 4(\widetilde{F}(0) + \overline{M}(\rho))\sqrt{\frac{\log\frac{4}{\delta}}{2n}} \\
&\geq \left(\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{\sigma^2} + L_2\right)\rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}}\left[\phi\left(f, \xi, \frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0 \rho}{\sigma^2} + L_2, \varepsilon\right)\right] \\
&\qquad + \frac{\rho\mu^*}{1024}(\lambda_0^*)^2 - \frac{96\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\sqrt{n}} - 8(\widetilde{F}(0) + \overline{M}(\rho))\sqrt{\frac{\log\frac{4}{\delta}}{2n}}\,.
\end{aligned}$$

Noting that the assumption on ρ in Proposition C.1 implies that

$$\frac{\rho\mu^*}{1024}(\lambda_0^*)^2 \geq \frac{96\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\sqrt{n}} + 8\left(\widetilde{F}(0) + \overline{M}(\rho)\right)\sqrt{\frac{\log\frac{4}{\delta}}{2n}}$$

we have proven that, with probability at least $1 - \frac{\delta}{2}$, for any $f \in \mathcal{F}$,

$$\psi_\rho\left(\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0\rho}{2\sigma^2} - L_2\right) \geq \psi_\rho\left(\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2} + L_2\right)$$

where $\psi_\rho : \lambda \mapsto \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P_n}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)]$. Now, since ψ_ρ is convex, this means that its minimizers on $\mathbb{R}_+$ are greater than

$$\frac{\lambda_0^*}{8\rho} - \frac{\varepsilon_0\rho}{2\sigma^2} - L_2 \geq \frac{\lambda_0^*}{32\rho}$$

where the inequality comes from Lemma C.2.

Using the same reasoning, one can get that with probability at least $1 - \frac{\delta}{2}$ the minimizers are no greater than

$$\frac{\lambda_0^*}{4\rho} - \frac{\varepsilon_0\rho}{\sigma^2} + L_2 \leq \frac{\lambda_0^*}{2\rho}.$$

Thus, we have shown that with probability at least $1 - \frac{\delta}{2}$, for any $f \in \mathcal{F}$,

$$\begin{aligned}\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) &= \inf_{\lambda\geq 0} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P_n}}\left[\phi(f, \xi, \lambda, \varepsilon, \sigma)\right] \\ &= \inf_{\frac{\lambda_0^*}{32\rho}\leq\lambda\leq\frac{\lambda_0^*}{2\rho}} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P_n}}\left[\phi(f, \xi, \lambda, \varepsilon, \sigma)\right] .\end{aligned}$$

□

D Dual bound when ρ is close to this maximal radius

Complementary to the previous section Section C, we consider the case where ρ is close than the critical radius. Though the bounds of this section are much worse that the one of Section C when ρ goes to zero, they hold for the whole ranges of ρ considered in the theorems.

As mentioned in Remark 3.2, as ρ grows, the Wasserstein ball constraint can stop being active, leading to a null dual variable. Thus, it is essential that ρ be lower then the critical radius to stay in the distributionally robust regime and to avoid the worst-case regime. In that case, we are able to lower-bound the dual multiplier λ.

We defined the critical radius in standard WDRO case in Remark 3.2 and we extend it here to cover the regularized case:

$$\rho_c^2(\varepsilon, \sigma) := \begin{cases} \inf_{f\in\mathcal{F}} \mathbb{E}_{\xi\sim \mathrm{P}}\left[\mathbb{E}_{\zeta\sim\pi_\sigma^{f/\varepsilon}(\cdot|\xi)}\left[\frac{1}{2}\|\xi - \zeta\|^2\right]\right] & \text{if } \varepsilon > 0 \\ \inf_{f\in\mathcal{F}} \mathbb{E}_{\xi\sim \mathrm{P}}\left[\min\{\frac{1}{2}\|\xi - \zeta\|^2 : \zeta \in \arg\max f\}\right] & \text{otherwise} \end{cases} \tag{30}$$

where $\pi_\sigma^g(\mathrm{d}\zeta|\xi) \propto e^{g(\zeta)}\,\pi_\sigma(\mathrm{d}\zeta|\xi)$ is a conditional probability distribution parametrized by an $\Xi \to \mathbb{R}$ function g, i.e.,

$$\mathbb{E}_{\zeta\sim\pi_\sigma^g(\cdot|\xi)}\left[h(\xi, \zeta)\right] = \frac{\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[e^{g(\zeta)}h(\xi, \zeta)\right]}{\mathbb{E}_{\zeta'\sim\pi_\sigma(\cdot|\xi)}\left[e^{g(\zeta')}\right]}.$$

For this part of the proof, the case when $\varepsilon = 0$ differs from the regularized one $\varepsilon > 0$. We thus present them in separate sections Sections D.1 and D.2.

D.1 Standard WDRO case

The main result of this section in the standard WDRO case is Proposition D.1 below.

Proposition D.1. *Let Assumption 5 hold and fix a threshold $\delta \in (0, 1)$. Assume that*

$$\rho^2 \leq \rho_c^2(0, 0) - \frac{2B(\delta)}{\sqrt{n}}$$

$$\textit{with} \quad L \coloneqq \frac{16 \sup_{f \in \mathcal{F}} \mathbb{E}_{\xi \sim \mathrm{P}} \left[\frac{1}{2} d^2(\xi, \arg\max f) \right]}{\mu}$$
$$\textit{and} \quad B(\delta) \coloneqq 48 \mathcal{I}(\mathcal{F}, D) + 2\sqrt{C^\star \log 1/\delta}\,.$$

where $\mathcal{I}(\mathcal{F}, D)$ and D are defined in Section A.4, and $\lambda_2 > 0$ is a constant depending on Ξ, $\mathcal{F}$, L_3, μ and $C^\star$.

Then, with probability at least $1 - \delta$, we have

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda} \le \lambda} \lambda \rho^2 + \mathbb{E}_{\xi \sim \mathrm{P}_{\mathrm{n}}} \left[\phi(f, \xi, \lambda, 0) \right]$$

where the dual bound $\underline{\lambda}$ is defined as

$$\underline{\lambda} \coloneqq \min\left(\lambda_2, \frac{\rho_c^2(0,0) - \rho^2}{2L} \right).$$

Before proceeding with the proof, we need to prove the following lemma which leverages Assumption 5.

Lemma D.2. *Fix $f \in \mathcal{F}$ and $\xi \in \operatorname{supp} \mathrm{P}$. There exists a constant $\lambda_2 > 0$ depending on Ξ, $\mathcal{F}$, L_3, μ and $C^\star$ such that, for $\lambda \in [0, \lambda_2]$,*

$$\min\left\{ \frac{1}{2}\|\xi - \zeta\|^2 : \zeta \in \arg\max_{\Xi} f - \frac{\lambda}{2}\|\xi - \cdot\|^2 \right\} \ge \left(1 - \frac{16\lambda}{\mu}\right) \min\left\{ \frac{1}{2}\|\xi - \zeta\|^2 : \zeta \in \arg\max_{\Xi} f \right\}.$$

Proof. Fix $f \in \mathcal{F}$ and $\xi \in \operatorname{supp} \mathrm{P}$. Define, for convenience, $\Xi^\star \coloneqq \arg\max f$ and

$$Y(\lambda) \coloneqq \min\left\{ \frac{1}{2}\|\xi - \zeta\|^2 : \zeta \in \arg\max_{\Xi} f - \frac{\lambda}{2}\|\xi - \cdot\|^2 \right\}.$$

Step 1: Localization in a $\mathcal{O}(1)$-neighborhood of $\Xi^\star$. For a fixed $R^* > 0$ that will be chosen later, we show that, for λ small enough, $Y(\lambda)$ is equal to

$$\min\left\{ \frac{1}{2}\|\xi - \zeta\|^2 : \zeta \in \arg\max_{(\Xi^\star)^{R^*}} f - \frac{\lambda}{2}\|\xi - \cdot\|^2 \right\} \quad \text{where} \quad (\Xi^\star)^{R^*} \coloneqq \{\xi \in \Xi : d(\xi, \Xi^\star) \le R^*\}\,.$$

Indeed, by Assumption 5, there is some $\Delta(R^*) > 0$ such that for all $f \in \mathcal{F}$ and $\zeta \in \Xi \setminus (\Xi^\star)^{R^*}$,

$$f(\zeta) - \max f - \frac{\lambda}{2}\|\xi - \zeta\|^2 \le f(\zeta) - \max f \le -\Delta(R^*)\,,$$

while, for any $\xi^\star \in \Xi^\star$,

$$f(\xi^\star) - \max f - \frac{\lambda}{2}\|\xi - \xi^\star\|^2 = -\frac{\lambda}{2}\|\xi - \xi^\star\|^2 \ge -\lambda C^\star\,.$$

Hence, for $\lambda \le \frac{\Delta(R^*)}{C^\star}$, $f(\zeta) - \frac{\lambda}{2}\|\xi - \zeta\|^2 \le -\Delta(R^*) + \max f \le -\lambda C^\star + \max f \le f(\xi^\star) - \frac{\lambda}{2}\|\xi - \xi^\star\|^2$. This means that points in $\Xi \setminus (\Xi^\star)^{R^*}$ cannot maximize $f - \frac{\lambda}{2}\|\xi - \cdot\|^2$ and so it suffices to consider the $\arg\max$ over $(\Xi^\star)^{R^*}$ in the definition of $Y(\lambda)$.

Step 2: Localization in a $\mathcal{O}(\lambda)$-neighborhood of $\Xi^\star$. Take $\zeta^\star \in \arg\max_{(\Xi^\star)^{R^*}} f - \frac{\lambda}{2}\|\xi - \cdot\|^2$. Since $d(\zeta^\star, \Xi^\star) \le R^*$, the Euclidean projection of $\zeta^\star$ on $\Xi^\star$, that we denote by $\xi^\star$, is at most at distance R^* of $\zeta^\star$ and $\zeta^\star - \xi^\star \in \widehat{\mathrm{NC}}_{\xi^\star}(\Xi^\star)$, see e.g., Rockafellar and Wets (1998, Thm. 6.12). By the growth condition of Assumption 5, we get that

$$f(\xi^\star) \ge f(\zeta^\star) + \frac{\mu}{2}\|\zeta^\star - \xi^\star\|^2 - \frac{L_3}{6}\|\zeta^\star - \xi^\star\|^3\,. \tag{31}$$

But, by definition of $\zeta^\star$, we also have that

$$f(\zeta^\star) - \frac{\lambda}{2}\|\xi - \zeta^\star\|^2 \ge f(\xi^\star) - \frac{\lambda}{2}\|\xi - \xi^\star\|^2\,.$$

Plugging (31) we get that

$$-\frac{\lambda}{2}\|\xi-\zeta^\star\|^2 \geq -\frac{\lambda}{2}\|\xi-\xi^\star\|^2 + \frac{\mu}{2}\|\zeta^\star-\xi^\star\|^2 - \frac{L_3}{6}\|\zeta^\star-\xi^\star\|^3 .$$

Rearranging and developing $\frac{1}{2}\|\xi-\xi^\star\|^2$ yields

$$\frac{L_3}{6}\|\zeta^\star-\xi^\star\|^3 + \lambda\langle \xi-\zeta^\star, \zeta^\star-\xi^\star\rangle \geq \frac{\lambda+\mu}{2}\|\xi^\star-\zeta^\star\|^2 ,$$

which gives, by Cauchy-Schwarz inequality,

$$\frac{L_3}{6}\|\zeta^\star-\xi^\star\|^3 + \lambda\|\xi-\zeta^\star\|\|\zeta^\star-\xi^\star\| \geq \frac{\lambda+\mu}{2}\|\xi^\star-\zeta^\star\|^2 , \tag{32}$$

We now wish to obtain a bound on $u^\star \coloneqq \|\zeta^\star-\xi^\star\|$. If it is zero, there is nothing to do. Otherwise, assuming that it is positive, (32) gives the inequation

$$\frac{\mu+\lambda}{2}u^\star \leq \frac{L_3}{6}(u^\star)^2 + \lambda\|\xi-\xi^\star\| .$$

When $\frac{(\mu+\lambda)^2}{4} - \frac{2L_3\lambda}{3}\|\xi-\xi^\star\|$ is non-negative, this inequation is satisfied for

$$u^\star \notin \left[\frac{(\mu+\lambda) \pm \sqrt{(\mu+\lambda)^2 - 8L_3\lambda\|\xi-\xi^\star\|/3}}{L_3/3}\right] .$$

Hence, in particular, if $u^\star \leq \frac{3\mu}{L_3}$, then $u^\star$ must be less or equal than

$$\frac{(\mu+\lambda) - \sqrt{(\mu+\lambda)^2 - 8L_3\lambda\|\xi-\xi^\star\|/3}}{L_3/3} = \frac{3(\mu+\lambda)}{L_3}\left(1-\sqrt{1-\frac{8L_3\lambda\|\xi-\xi^\star\|}{3(\mu+\lambda)^2}}\right) \leq \frac{8\lambda\|\xi-\xi^\star\|}{\mu+\lambda}$$

when $\frac{8L_3\lambda\|\xi-\xi^\star\|}{3(\mu+\lambda)^2} \leq 1$, using that $1-\sqrt{1-x} \leq x$ for $x \in [0,1]$.

Thus, assuming that λ is small enough so that $8L_3\lambda C^\star \leq 3(\mu)^2$ and choosing $R^* \coloneqq \frac{3\mu}{L_3}$ so that $u^\star \leq \frac{3\mu}{L_3}$ by construction, we have that for any $\zeta^\star \in \arg\max_{(\Xi^\star)^{R^*}} f - \frac{\lambda}{2}\|\xi-\cdot\|^2$, there is a point $\xi^\star \in \Xi^\star$ such that

$$\|\zeta^\star-\xi^\star\| \leq \frac{8\lambda\|\xi-\xi^\star\|}{\mu} .$$

Step 3: Conclusion. Defining the constant

$$\lambda_2 \coloneqq \min\left(\frac{\Delta(R^*)}{C^\star}, \frac{3\mu^2}{8L_3C^\star}, \frac{\mu}{16}\right),$$

and using the previous steps, we have for any $\lambda \in [0,\lambda_2]$ and any $\zeta^\star \in \arg\max_\Xi f - \frac{\lambda}{2}\|\xi-\cdot\|^2$

$$\begin{aligned}\frac{1}{2}\|\xi-\zeta^\star\|^2 = \frac{1}{2}\|\xi-\zeta^\star\|^2 &= \frac{1}{2}\|\xi-\xi^\star\|^2 + \frac{1}{2}\|\xi^\star-\zeta^\star\|^2 - \langle \xi-\xi^\star, \xi^\star-\zeta^\star\rangle \\ &\geq \frac{1}{2}\|\xi-\xi^\star\|^2 - \|\xi-\xi^\star\|\|\xi^\star-\zeta^\star\| \\ &\geq \left(1-\frac{16\lambda}{\mu}\right)\frac{1}{2}\|\xi-\xi^\star\|^2\end{aligned}$$

which concludes the proof. □

We can now turn to the proof of our proposition.

Proof of Proposition D.1. Let $0 \leq \lambda \leq \underline{\lambda}$. For $f \in \mathcal{F}$ and $\lambda \geq 0$, we define $\hat{\psi}_\rho : \lambda \mapsto \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P_n}}[\phi(f,\xi,\lambda,0)]$ and its (right-sided) derivative $\partial_\lambda\hat{\psi}_\rho$. This derivative is given by,

$$\begin{aligned}\partial_\lambda\hat{\psi}_\rho(\lambda) &= \rho^2 - \mathbb{E}_{\xi\sim \mathrm{P_n}}\left[\min\left\{\frac{1}{2}\|\xi-\zeta\|^2 : \zeta \in \arg\max_\Xi f - \frac{\lambda}{2}\|\xi-\cdot\|^2\right\}\right] \\ &\leq \rho^2 - \left(1-\frac{16\lambda}{\mu}\right)\mathbb{E}_{\xi\sim \mathrm{P_n}}\left[\min\left\{\frac{1}{2}\|\xi-\zeta\|^2 : \zeta \in \arg\max_\Xi f\right\}\right] \end{aligned} \tag{33}$$

where we used Lemma D.2 with $\lambda \leq \underline{\lambda} \leq \lambda_2$.

We then instantiate Lemma G.2 with $X(f, \xi) \leftarrow \frac{1}{2} d^2(\xi, \arg\max f)$, $(\mathcal{X}, \mathrm{dist}) \leftarrow (\mathcal{F}, D)$, whose requirements are checked since:

1. For any $f \in \mathcal{F}$, $X(f, \cdot)$ is measurable since the functions of $\mathcal{F}$ are continuous and thus $\arg\max f$ is *a fortiori* measurable;
2. By definition of D, for any $\xi \in \operatorname{supp} \mathrm{P}$, $f \mapsto d(\xi, \arg\max f)$ is 1-Lipschitz w.r.t. this distance so that $X(\xi, \cdot)$ is $\sqrt{2C^\star}$-Lipschitz.
3. By construction, the range of values X is included in $[0, C^\star]$.

We can thus apply statement (b) of Lemma G.2 to have that, with probability at least $1 - \delta$, for all $f \in \mathcal{F}$,

$$\mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\frac{1}{2} d^2(\xi, \arg\max f) \right] \geq \mathbb{E}_{\xi \sim \mathrm{P}} \left[\frac{1}{2} d^2(\xi, \arg\max f) \right] - \frac{B(\delta)}{\sqrt{n}}$$

Hence, putting this bound together with (33) yields

$$\begin{aligned} \partial_\lambda \hat{\psi}_\rho(\lambda) &\leq \rho^2 - \left(1 - \frac{16\lambda}{\mu}\right) \mathbb{E}_{\xi \sim \mathrm{P}} \left[\frac{1}{2} d^2(\xi, \arg\max f) \right] + \frac{B(\delta)}{\sqrt{n}} \\ &\leq \rho^2 - \rho_c^2 + L\lambda + \frac{B(\delta)}{\sqrt{n}}, \end{aligned}$$

which is non-negative for $\lambda \leq \underline{\lambda}$. □

D.2 Regularized case

The main bound on λ of this section are given by Proposition D.3.

Proposition D.3. *Fix a threshold $\delta \in (0, 1)$. Assume that $\varepsilon_0 > 0$ and that*

$$\rho^2 \leq \rho_c^2(\varepsilon, \sigma) - \left(\frac{48\sqrt{\mathrm{Var}(\varepsilon, \sigma)} \mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon \sqrt{n}} + 2C_{\mathcal{F}}(\varepsilon, \sigma) \sqrt{\frac{\log \frac{1}{\delta}}{2n}} \right)$$

where $\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)$ is defined in Section A.1.

Then, with probability at least $1 - \delta$, we have

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda}_n \leq \lambda \leq \overline{\lambda}} \lambda \rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\phi(f, \xi, \lambda, \varepsilon, \sigma) \right]$$

where the dual bounds are defined by

$$\underline{\lambda}_n := \frac{\varepsilon}{\mathrm{Var}(\varepsilon, \sigma)} \left(\rho_c^2(\varepsilon, \sigma) - \rho^2 - \left(\frac{48\sqrt{\mathrm{Var}(\varepsilon, \sigma)} \mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon \sqrt{n}} + 2C_{\mathcal{F}}(\varepsilon, \sigma) \sqrt{\frac{\log \frac{1}{\delta}}{2n}} \right) \right)$$

and $\overline{\lambda} := \max\left(\frac{12\varepsilon}{R^2} \log(2 \times 6^{d/2}), e^{\frac{\sup_{f \in \mathcal{F}} \|f\|_\infty}{\varepsilon}} \frac{\varepsilon_0}{\rho} \right).$

Proof. Lower-bound: By Assumption 6, for any $f \in \mathcal{F}$, $\xi \in \operatorname{supp} \mathrm{P}$, $\lambda \mapsto \phi(f, \xi, \lambda, \varepsilon, \sigma)$ is twice differentiable and its derivatives are for any $\lambda \geq 0$

$$\begin{aligned} \partial_\lambda \phi(f, \xi, \lambda, \varepsilon, \sigma) &= -\mathbb{E}_{\zeta \sim \pi_\sigma^{\frac{f - \lambda \|\xi - \cdot\|^2/2}{\varepsilon}}(\cdot|\xi)} \left[\frac{1}{2} \|\xi - \zeta\|^2 \right] \\ \partial_\lambda^2 \phi(f, \xi, \lambda, \varepsilon, \sigma) &= \frac{1}{\varepsilon} \mathrm{Var}_{\zeta \sim \pi_\sigma^{\frac{f - \lambda \|\xi - \cdot\|^2/2}{\varepsilon}}(\cdot|\xi)} [\frac{1}{2} \|\xi - \zeta\|^2], \end{aligned}$$

and using $\mathrm{Var}(\varepsilon, \sigma)$ which is defined in Section A.1, we get that, for any $\lambda \geq 0$,

$$0 \leq \partial_\lambda^2 \phi(f, \xi, \lambda, \varepsilon, \sigma) \leq \frac{1}{\varepsilon} \mathrm{Var}(\varepsilon, \sigma) \,.$$

As a consequence,

$$\begin{aligned}
\partial_\lambda \left\{ \lambda \rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\phi(f, \xi, \lambda, \varepsilon, \sigma) \right] \right\} &= \rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\partial_\lambda \phi(f, \xi, \lambda, \varepsilon, \sigma) \right] \\
&\leq \rho^2 + \mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\partial_\lambda \phi(f, \xi, 0, \varepsilon, \sigma) \right] + \frac{\lambda}{\varepsilon} \mathrm{Var}(\varepsilon, \sigma) \\
&= \rho^2 - \mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\mathbb{E}_{\zeta \sim \pi_\sigma^{f/\varepsilon}(\cdot|\xi)} \left[\frac{1}{2} \|\xi - \zeta\|^2 \right] \right] + \frac{\lambda}{\varepsilon} \mathrm{Var}(\varepsilon, \sigma) \,.
\end{aligned} \tag{34}$$

Now, we want to instante Lemma G.2 with $X(f, \xi) \coloneqq \mathbb{E}_{\zeta \sim \pi_\sigma^{f/\varepsilon}(\cdot|\xi)} \frac{1}{2} \|\xi - \zeta\|^2$, $(\mathcal{X}, \mathrm{dist}) \leftarrow (\mathcal{F}, \|\cdot\|_\infty)$, whose requirements are checked since:

1. For any $f \in \mathcal{F}$, $\mathbb{E}_{\zeta \sim \pi_\sigma^{f/\varepsilon}(\cdot|\xi)} \frac{1}{2} \|\xi - \zeta\|^2$ is measurable since the functions of $\mathcal{F}$ are continuous and thus *a fortiori* measurable;

2. To show that $f \mapsto X(f, \xi)$ is $\frac{1}{\varepsilon}\sqrt{\mathrm{Var}(\varepsilon, \sigma)}$-Lipschitz, we take $f, g \in \mathcal{F}$ and define, for $t \in [0, 1]$, $f_t = f + t(g - f)$. Since, $\|f - g\|_\infty < +\infty$ and $\sup_{\xi \in \mathrm{supp\,P}} \mathbb{E}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} \frac{1}{2} \|\xi - \zeta\|^2 < +\infty$ by compactness of Ξ, Assumption 1, $t \mapsto X(f_t, \xi)$ is differentiable with derivative,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} X(f_t, \xi) &= \frac{1}{\varepsilon} \mathbb{E}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} \left[\frac{1}{2} \|\xi - \zeta\|^2 (g(\zeta) - f(\zeta)) \right] \\
&\quad - \frac{1}{\varepsilon} \mathbb{E}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} \left[\frac{1}{2} \|\xi - \zeta\|^2 \right] \mathbb{E}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} [g(\zeta) - f(\zeta)] \\
&= \frac{1}{\varepsilon} \mathbb{E}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} \left[\left(\frac{1}{2} \|\xi - \zeta\|^2 - \mathbb{E}_{\zeta' \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} \left[\frac{1}{2} \|\xi - \zeta'\|^2 \right] \right) \right. \\
&\quad \left. \times \left((g(\zeta) - f(\zeta)) - \mathbb{E}_{\zeta' \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} [g(\zeta') - f(\zeta')] \right) \right] \,.
\end{aligned}$$

   By using Cauchy-Schwarz inequality, we get that,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} X(f_t, \xi) &\leq \frac{1}{\varepsilon} \sqrt{\mathrm{Var}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} [\frac{1}{2} \|\xi - \zeta\|^2]} \sqrt{\mathrm{Var}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} [g(\zeta) - f(\zeta)]} \\
&\leq \frac{1}{\varepsilon} \sqrt{\mathrm{Var}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} [\frac{1}{2} \|\xi - \zeta\|^2]} \sqrt{\mathbb{E}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} \left[(g(\zeta) - f(\zeta))^2 \right]} \\
&\leq \frac{1}{\varepsilon} \sqrt{\mathrm{Var}_{\zeta \sim \pi_\sigma^{f_t/\varepsilon}(\cdot|\xi)} [\frac{1}{2} \|\xi - \zeta\|^2]} \|g - f\|_\infty \,,
\end{aligned}$$

   which gives the desired Lipschitz condition;

3. The random variables $X(f, \xi)$ lie between 0 and $C_{\mathcal{F}}(\varepsilon, \sigma)$, which is defined in Section A.1.

We can thus apply statement (b) of Lemma G.2 to have that, with probability at least $1 - \delta$, for all $f \in \mathcal{F}$

$$\begin{aligned}
&\mathbb{E}_{\xi \sim \mathrm{P}} \left[\mathbb{E}_{\zeta \sim \pi_\sigma^{f/\varepsilon}(\cdot|\xi)} \frac{1}{2} \|\xi - \zeta\|^2 \right] - \mathbb{E}_{\xi \sim \mathrm{P_n}} \left[\mathbb{E}_{\zeta \sim \pi_\sigma^{f/\varepsilon}(\cdot|\xi)} \frac{1}{2} \|\xi - \zeta\|^2 \right] \\
&\quad \leq \frac{48 \sqrt{\mathrm{Var}(\varepsilon, \sigma)} \mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon \sqrt{n}} + 2 C_{\mathcal{F}}(\varepsilon, \sigma) \sqrt{\frac{\log \frac{1}{\delta}}{2n}} \,.
\end{aligned} \tag{35}$$

Combining (34) and (35), we obtain that with probability at least $1-\delta$

$$\begin{aligned}
&\partial_\lambda \left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\right\}\\
&\leq \rho^2 - \mathbb{E}_{\xi\sim\mathrm{P}}\left[\mathbb{E}_{\zeta\sim\pi_\sigma^{f/\varepsilon}(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right]\right] + \frac{\lambda}{\varepsilon}\operatorname{Var}(\varepsilon,\sigma) + \frac{48\sqrt{\operatorname{Var}(\varepsilon,\sigma)}\mathcal{I}(\mathcal{F},\|\cdot\|_\infty)}{\varepsilon\sqrt{n}} + 2C_{\mathcal{F}}(\varepsilon,\sigma)\sqrt{\frac{\log\frac{1}{\delta}}{2n}}\\
&\leq \rho^2 - \rho_c^2(\varepsilon,\sigma) + \frac{\lambda}{\varepsilon}\operatorname{Var}(\varepsilon,\sigma) + \frac{48\sqrt{\operatorname{Var}(\varepsilon,\sigma)}\mathcal{I}(\mathcal{F},\|\cdot\|_\infty)}{\varepsilon\sqrt{n}} + 2C_{\mathcal{F}}(\varepsilon,\sigma)\sqrt{\frac{\log\frac{1}{\delta}}{2n}}\\
&= \frac{1}{\varepsilon}\operatorname{Var}(\varepsilon,\sigma)\left(\lambda - \underline{\lambda}_n\right)
\end{aligned}$$

where $\underline{\lambda}_n \geq 0$ is as defined in the statement of the result.

Hence, for all $0 \leq \lambda \leq \underline{\lambda}_n$, the derivative of $\lambda \mapsto \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]$ is negative; and since this function is convex, this means that its minimizers are greater than $\underline{\lambda}_n$ with probability at least $1-\delta$ which is our result.

Upper-bound: Almost surely, for any $f \in \mathcal{F}$, let us begin by bounding the $\partial_\lambda\phi(f,\xi,\lambda,\varepsilon,\sigma)$ for $\lambda \geq 0$, $f \in \mathcal{F}$ and $\xi \in \operatorname{supp}\mathrm{P}$. Its expression is given by

$$-\partial_\lambda\phi(f,\xi,\lambda,\varepsilon,\sigma) = \mathbb{E}_{\zeta\sim\pi_\sigma^{\frac{f-\lambda\|\xi-\cdot\|^2/2}{\varepsilon}}(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right] \leq e^{\frac{\|f\|_\infty}{\varepsilon}}\frac{\int_\Xi \frac{1}{2}\|\xi-\zeta\|^2 e^{-\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)\frac{1}{2}\|\xi-\zeta\|^2}\mathrm{d}\zeta}{\int_\Xi e^{-\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)\frac{1}{2}\|\xi-\zeta\|^2}\mathrm{d}\zeta}\,.$$

On the one hand, we lower-bound the denominator using Lemma G.1 and Assumption 3 as

$$\frac{1}{(2\pi)^{d/2}}\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)^{d/2}\int_\Xi e^{-\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)\frac{1}{2}\|\xi-\zeta\|^2}\mathrm{d}\zeta \geq 1 - 6^{d/2}e^{-\frac{R^2}{12}\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)} \geq \frac{1}{2}\,,$$

where we used that $\lambda \geq \frac{12\varepsilon}{R^2}\log(2\times 6^{d/2})$.

On the other hand, the denominator is upper-bounded as

$$\frac{1}{(2\pi)^{d/2}}\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)^{d/2}\int_\Xi \frac{1}{2}\|\xi-\zeta\|^2 e^{-\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)\frac{1}{2}\|\xi-\zeta\|^2}\mathrm{d}\zeta \leq \frac{1}{2}\left(\frac{\lambda}{\varepsilon}+\frac{1}{\sigma^2}\right)^{-1} \leq \frac{\varepsilon}{2\lambda}\,.$$

Hence, we have shown that $-\partial_\lambda\phi(f,\xi,\lambda,\varepsilon,\sigma) \leq e^{\frac{\|f\|_\infty}{\varepsilon}}\frac{\varepsilon}{\lambda}$ and, as a consequence,

$$\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}_\mathrm{n}}[\partial_\lambda\phi(f,\xi,\lambda,\varepsilon,\sigma)] \geq \rho^2 - e^{\frac{\|f\|_\infty}{\varepsilon}}\frac{\varepsilon}{\lambda}\,,$$

which is non-negative for $\lambda \geq e^{\frac{\|f\|_\infty}{\varepsilon}}\frac{\varepsilon_0}{\rho}$.

Hence, for

$$\lambda \geq \overline{\lambda} := \max\left(\frac{12\varepsilon}{R^2}\log(2\times 6^{d/2}), e^{\frac{\sup_{f\in\mathcal{F}}\|f\|_\infty}{\varepsilon}}\frac{\varepsilon_0}{\rho}\right),$$

the derivative of $\lambda \mapsto \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}_\mathrm{n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]$ is non-negative, which means that its minimizers are smaller than $\overline{\lambda}$. □

E Proof of the main results

In this section, we present our main results with explicit constants. In Section E.1 we treat the case of standard WDRO, i.e., the setting of Theorems 3.1 and 3.3, while in Section E.2 we handle the regularized setting of Theorem 3.4.

E.1 Standard WDRO case

The main results of this section are Theorems E.1 and E.3 which are more precise versions of Theorems 3.1 and 3.3 respectively.

Theorem E.1 (Extended version of Theorem 3.1)**.** *Under Assumptions 1, 3 and 6 and the additional Assumptions 4 and 5, with $\rho_c = \rho_c(0,0)$ defined in (30) for any $\delta \in (0,1)$ and $n \geq 1$, if*

$$\max\left(\rho_n, \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho_c)\right)\sqrt{1+\log\frac{4}{\delta}}\right)\right) \leq \rho$$

$$\text{and} \qquad \rho \leq \frac{\rho_c}{2} - \frac{96\mathcal{I}(\mathcal{F}, D) + 4\sqrt{C^\star \log 1/\delta}}{\sqrt{n}}\,.$$

where

$$\rho_{\text{thres}} \coloneqq \min\left(\frac{\lambda_0^*}{8(\lambda_1+L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096 L_2}\right)$$

$$\overline{a} \coloneqq \sup_{0<\rho'\leq\rho_{\text{thres}}} a\left(\frac{1}{\rho'}\min\left(\frac{\lambda_0^*}{32}, \rho_{\text{thres}}\lambda_2, \frac{3\rho_c^2\rho_{\text{thres}}}{8L}\right), \frac{\lambda_0^*}{2\rho'}, 0, 0\right)$$

$$\rho_n \coloneqq \frac{117\left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\left(\widetilde{F}\left(\frac{\lambda_0^*}{32\rho_c}\right), \overline{a}\right)\left(1+\sqrt{\log\frac{1}{\delta}}\right)\right)}{\sqrt{n}\min\left(\frac{\lambda_0^*}{32}, \rho_{\text{thres}}\lambda_2, \frac{3\rho_c^2\rho_{\text{thres}}}{8L}\right)}\,,$$

then, with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right] \qquad \text{for all } \mathrm{Q} \text{ such that } W_2^2(\mathrm{P},\mathrm{Q}) \leq \rho(\rho-\rho_n)\,.$$

In particular, with probability $1-\delta$, we have

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{P}}\left[f(\xi)\right].$$

The proof of Theorem E.1 relies on Lemma E.2 that combines the results of the previous sections, namely propositions B.1,C.1 and, D.1.

Lemma E.2. *Under the blanket assumptions Assumptions 1, 3 and 6 and with the additional Assumption 5, for any threshold $\delta \in (0,1)$, define*

$$\underline{\lambda}(\rho) = \begin{cases} \frac{\lambda_0^*}{32\rho} \text{ if } \rho \leq \rho_{\text{thres}} = \min\left(\frac{\lambda_0^*}{32(\lambda_1+L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096 L_2}\right) \\ \min\left(\lambda_2, \frac{\rho_c^2(0,0)-\rho^2}{2L}\right) \text{ otherwise} \end{cases} \tag{36}$$

$$\overline{\lambda}(\rho) = \frac{\lambda_0^*}{2\rho}\,.$$

Assume that

$$\rho \geq \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho)\right)\sqrt{1+\log\frac{4}{\delta}}\right), \tag{37}$$

and that

$$\rho^2 \leq \rho_c^2(0,0) - \frac{2B(\delta)}{\sqrt{n}}\,.$$

Then, with probability at least $1-\frac{\delta}{2}$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) = \inf_{\underline{\lambda}(\rho)\leq\lambda\leq\overline{\lambda}(\rho)} \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,0)\right]$$

and when $\rho^2 \geq \overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), 0)$, with probability $1-\delta$, it holds,

$$\widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathcal{R}_{\rho^2-\overline{\rho}_n^2(\delta,\underline{\lambda}(\rho),\overline{\lambda}(\rho),0)}(f),.$$

Furthermore, with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \sup\left\{\mathbb{E}_\mathrm{Q}[f] : \mathrm{Q} \in \mathcal{P}(\Xi)\,, W_2^2\left(\mathrm{P},\mathrm{Q}\right) \leq \rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), 0)\right\}.$$

Proof. This result is a consequence of Propositions C.1 and D.1 both applied with $\delta \leftarrow \delta/4$ and of Proposition B.1. Note that the upper-bound on the dual variable given by Proposition C.1 holds for any ρ since the optimal dual variable is non-increasing as a function of ρ. □

Proof of Theorem E.1. The proof consists in simplifying both the assumptions and the result of Lemma E.2.

We begin by showing that $\underline{\lambda}(\rho)$ can always be lower-bounded by a quantity proportional to $1/\rho$. Indeed, by definition of $\underline{\lambda}(\rho)$, (36) in Lemma E.2, and using that ρ is in particular less than $\frac{\rho_c}{2}$, it holds that,

$$\underline{\lambda}(\rho) \geq \frac{1}{\rho}\min\left(\frac{\lambda_0^*}{32}, \rho_{\text{thres}}\lambda_2, \frac{3\rho_c^2\rho_{\text{thres}}}{8L}\right) \tag{38}$$

Let us now turn our attention to the condition $\rho^2 \geq \overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), 0, 0)$, whose RHS was defined by (24) in Section B. We have that, by definition (Assumption 6), $\sup_{0<\rho\leq\rho_c} \widetilde{F}(\lambda_0^*/(32\rho)) = \widetilde{F}(\lambda_0^*/(32\rho_c)) < +\infty$ and,

$$\begin{aligned}\sup_{0<\rho\leq\rho_c} a\big(\underline{\lambda}_n(\rho), \overline{\lambda}(\rho), 0, 0\big) &\leq \sup_{0<\rho'\leq\rho_{\text{thres}}} a\left(\frac{1}{\rho}\min\left(\frac{\lambda_0^*}{32}, \rho_{\text{thres}}\lambda_2, \frac{3\rho_c^2\rho_{\text{thres}}}{8L}\right), \frac{\lambda_0^*}{2\rho'}, 0, 0\right) \\ &= \overline{a} < +\infty\,,\end{aligned}$$

by definition and non-decreasingness of a in its first argument (see Corollary A.5) and (38). Hence, the following bound holds

$$\begin{aligned}&\overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma) \\ &\leq \frac{117}{\sqrt{n}\underline{\lambda}(\rho)}\left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\left(\widetilde{F}\left(\frac{\lambda_0^*}{32\rho_c}\right), \overline{a}\right)\left(1 + \sqrt{\log\frac{1}{\delta}}\right)\right) \\ &\leq \rho_n\rho\,,\end{aligned}$$

where we plugged (38).

Finally, since ρ is in particular bounded by ρ_c, the condition (37) is implied by

$$\rho \geq \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho_c)\right)\sqrt{1 + \log\frac{4}{\delta}}\right),$$

with $\overline{M}(\rho_c) < +\infty$ by definition (Proposition C.1). □

Theorem E.3 (Extended version of Theorem 3.3)**.** *Under Assumptions 1, 3 and 6, for any $\delta \in (0,1)$ and $n \geq 1$, if*

$$\max\left(\rho_n, \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho_c)\right)\sqrt{1 + \log\frac{2}{\delta}}\right)\right) \leq \rho$$

and $$\rho \leq \min\left(\rho_{\text{thres}}, \frac{\rho_c}{2} - \frac{96\mathcal{I}(\mathcal{F}, D) + 4\sqrt{C^\star \log 1/\delta}}{\sqrt{n}}\right).$$

where

$$\begin{aligned}\rho_{\text{thres}} &:= \min\left(\frac{\lambda_0^*}{8(\lambda_1 + L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096 L_2}\right) \\ \overline{a} &:= \sup_{0<\rho'\leq\rho_{\text{thres}}} a\left(\frac{\lambda_0^*}{32\rho'}, \frac{\lambda_0^*}{2\rho'}, 0, 0\right) \\ \rho_n &:= \frac{3744\left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\left(\widetilde{F}\left(\frac{\lambda_0^*}{32\rho_{\text{thres}}}\right), \overline{a}\right)\left(1 + \sqrt{\log\frac{1}{\delta}}\right)\right)}{\sqrt{n}\lambda_0^*},\end{aligned}$$

then, with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{Q}}\left[f(\xi)\right] \qquad \text{for all } \mathrm{Q} \text{ such that } W_2^2(\mathrm{P}, \mathrm{Q}) \leq \rho(\rho - \rho_n)\,.$$

In particular, with probability $1-\delta$, we have

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{P}}\left[f(\xi)\right].$$

The proof of Theorem E.3 leverages results from the previous sections, combined in Lemma E.4.

Lemma E.4. *Under the blanket assumptions Assumptions 1, 3 and 6, for any threshold $\delta \in (0,1)$, define*

$$\underline{\lambda}(\rho) = \frac{\lambda_0^*}{32\rho}, \qquad \overline{\lambda}(\rho) = \frac{\lambda_0^*}{2\rho}.$$

Assume that

$$\rho \geq \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho)\right)\sqrt{1+\log\frac{2}{\delta}}\right), \tag{39}$$

and that

$$\rho^2 \leq \min\left(\rho_c^2(0,0) - \frac{2B(\delta)}{\sqrt{n}}, \min\left(\frac{\lambda_0^*}{32(\lambda_1+L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096 L_2}\right)^2\right).$$

Then, with probability at least $1-\frac{\delta}{2}$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) = \inf_{\underline{\lambda}(\rho)\leq\lambda\leq\overline{\lambda}(\rho)} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}\left[\phi(f,\xi,\lambda,0)\right]$$

and when $\rho^2 \geq \overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), 0)$, with probability $1-\delta$, it holds,

$$\widehat{\mathcal{R}}_{\rho^2}(f) \geq \mathcal{R}_{\rho^2 - \overline{\rho}_n^2(\delta,\underline{\lambda}(\rho),\overline{\lambda}(\rho),0)}(f),.$$

Furthermore, with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \geq \sup\left\{\mathbb{E}_\mathrm{Q}[f] : \mathrm{Q} \in \mathcal{P}(\Xi), W_2^2(\mathrm{P},\mathrm{Q}) \leq \rho^2 - \overline{\rho}_n^2(\delta,\underline{\lambda}(\rho),\overline{\lambda}(\rho),0)\right\}.$$

Proof. This result follows directly from Proposition C.1 that we invoke with $\delta \leftarrow \delta/2$ and of Proposition B.1. □

Proof of Theorem E.3. The proof consists in simplifying both the assumptions and the result of Lemma E.4 and follows the same structure as the proof of Theorem E.1.

We begin by examining the condition $\rho^2 \geq \overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), 0, 0)$, whose RHS was defined by (24) in Section B. We have that, by definition (Assumption 6), $\sup_{0<\rho\leq\rho_c} \widetilde{F}(\lambda_0^*/(32\rho)) = \widetilde{F}(\lambda_0^*/(32\rho_c)) < +\infty$ and,

$$\sup_{0<\rho\leq\rho_\mathrm{thres}} a\big(\underline{\lambda}_n(\rho), \overline{\lambda}(\rho), 0, 0\big) = \sup_{0<\rho'\leq\rho_\mathrm{thres}} a\left(\frac{\lambda_0^*}{32\rho'}, \frac{\lambda_0^*}{2\rho'}, 0, 0\right) = \overline{a} < +\infty$$

by definition (see Corollary A.5). Hence, we have that

$$\begin{aligned}
&\overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), 0, 0) \\
&\leq \frac{117}{\sqrt{n}\underline{\lambda}(\rho)}\left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\left(\widetilde{F}\left(\frac{\lambda_0^*}{32\rho_c}\right), \overline{a}\right)\left(1+\sqrt{\log\frac{1}{\delta}}\right)\right) \\
&\leq \rho_n \rho,
\end{aligned}$$

by definition of ρ_n and with $117 \times 32 = 3744$.

Finally, m (39) is implied by

$$\rho \geq \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho_\mathrm{thres})\right)\sqrt{1+\log\frac{2}{\delta}}\right),$$

since $\rho \leq \rho_\mathrm{thres}$ and $\overline{M}(\rho_\mathrm{thres}) < +\infty$ by definition (Proposition C.1). □

E.2 Regularized WDRO case

Theorem E.5 (Extended version of Theorem 3.4)**.** *For $\sigma = \sigma_0\rho$ with $\sigma_0 > 0$, $\varepsilon = \varepsilon_0\rho$ with $\varepsilon_0 > 0$ such that $\varepsilon_0/\sigma_0^2 \leq \lambda_0^*/8$, and for any $\delta \in (0,1)$ and $n \geq 1$, define,*

$$\rho_c := \inf\{\rho_c(\varepsilon_0\rho', \sigma_0\rho') : \rho_{\text{thres}} \leq \rho' \leq \rho_c(\varepsilon_0\rho_{\text{thres}}, \sigma_0\rho_{\text{thres}})\}$$

and

$$\rho_{\text{thres}} := \min\left(\frac{\varepsilon_1}{\varepsilon_0}, \frac{\lambda_0^*}{32(\lambda_1+L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096L_2}, \sqrt{\frac{c_2^3\lambda_0^*}{8\varepsilon_0}}\left(\log\left(\frac{4096\varepsilon_0 c_1}{\mu^*(\lambda_0^*)^2}\right)\right)_+^{-\frac{3}{2}}\right)$$

$$\overline{\text{Var}} := \sup_{\rho_{\text{thres}} \leq \rho' \leq \rho_c} \text{Var}(\varepsilon_0\rho', \sigma_0\rho')$$

$$\overline{C}_{\mathcal{F}} := \sup_{\rho_{\text{thres}} \leq \rho' \leq \rho_c} C_{\mathcal{F}}(\varepsilon_0\rho', \sigma_0\rho')$$

$$\overline{a} := \sup_{0 < \rho' \leq \rho_c} a\left(\frac{1}{\rho'}\min\left(\frac{\lambda_0^*}{32}, \frac{\varepsilon_0{\rho_{\text{thres}}}^2\rho_c^2}{4\overline{\text{Var}}}\right), \max\left(\frac{\lambda_0^*}{2\rho'}, \frac{12\varepsilon_0\rho_c\log(2\times 6^{d/2})}{R^2}, \frac{e^{\frac{\sup_{f\in\mathcal{F}}\|f\|_\infty}{\varepsilon_0\rho_{\text{thres}}}}\varepsilon_0}{\rho_{\text{thres}}}\right), \varepsilon_0\rho', \sigma_0\rho'\right)$$

$$\rho_n := \frac{117\left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\left(\widetilde{F}\left(\frac{\lambda_0^*}{32\rho_c}\right), \overline{a}\right)\left(1+\sqrt{\log\frac{1}{\delta}}\right)\right)}{\sqrt{n}\min\left(\frac{\lambda_0^*}{32}, \frac{\varepsilon_0{\rho_{\text{thres}}}^2\rho_c^2}{4\overline{\text{Var}}}\right)};$$

when

$$\max\left(\rho_n, \frac{8192}{\mu^*(\lambda_0^*)^2\sqrt{n}}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho_c)\right)\sqrt{\log\frac{4}{\delta}}\right), \frac{384\sqrt{\overline{\text{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0\rho_c^2\sqrt{n}}\right) \leq \rho$$

$$\rho \leq \frac{\rho_c}{2} - \frac{384\sqrt{\overline{\text{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0\rho_c^2\sqrt{n}}$$

and $$\rho_c \geq \max\left(\left(\frac{192\sqrt{\overline{\text{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0\sqrt{n}}\right)^{1/3}, 2\sqrt{\overline{C}_{\mathcal{F}}}\left(\frac{\log\frac{4}{\delta}}{2n}\right)^{1/4}\right),$$

then, with probability at least $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{Q}}[f(\xi)] \qquad \text{for all } \mathrm{Q} \text{ such that } W^2_{2,\tau(\rho)}(\mathrm{P}, \mathrm{Q}) \leq \rho(\rho - \rho_n),$$

where $\tau(\rho) \leq \frac{\varepsilon\rho}{\min\left(\frac{\lambda_0^*}{32}, \frac{\varepsilon_0{\rho_{\text{thres}}}^2\rho_c^2}{4\overline{\text{Var}}}\right)}$. Furthermore, when $\sigma_0 \leq 1$ and $\sigma \leq \sigma_1$ (defined in Proposition A.2), with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \mathbb{E}_{\xi\sim\mathrm{P}}\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}[f(\zeta)].$$

The proof of Theorem E.5 relies on Lemma E.6 that makes the regularized Wasserstein distance appear. It also uses Lemma E.7, to guarantee that a smoothed version of the true distribution is inside the right neighborhood.

Lemma E.6. *Fix a confidence threshold $\delta \in (0,1)$, take $\varepsilon = \varepsilon_0\rho$, $\sigma = \sigma_0\rho$ with ε_0 and σ_0 positive constants satisfying ${\varepsilon_0}/{\sigma_0}^2 \leq \lambda_0^*/8$ and, define $\underline{\lambda}_n(\rho)$ and $\overline{\lambda}(\rho)$ as functions of ρ by*

- *If*

$$\rho \leq \min\left(\frac{\varepsilon_1}{\varepsilon_0}, \frac{\lambda_0^*}{32(\lambda_1+L_2)}, \frac{\mu^*(\lambda_0^*)^2}{4096L_2}, \sqrt{\frac{c_2^3\lambda_0^*}{8\varepsilon_0}}\left(\log\left(\frac{4096\varepsilon_0 c_1}{\mu^*(\lambda_0^*)^2}\right)\right)_+^{-\frac{3}{2}}\right),$$

  then $\underline{\lambda}_n(\rho) = \frac{\lambda_0^*}{32\rho}$ and $\overline{\lambda}(\rho) = \frac{\lambda_0^*}{2\rho}$,

- *Otherwise,*

$$\underline{\lambda}_n(\rho) = \frac{\varepsilon_0\rho}{\text{Var}(\varepsilon_0\rho, \sigma_0\rho)}\left(\rho_c(\varepsilon_0\rho, \sigma_0\rho)^2 - \rho^2 - \left(\frac{48\sqrt{\text{Var}(\varepsilon_0\rho, \sigma_0\rho)}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0\rho\sqrt{n}} + 2C_{\mathcal{F}}(\varepsilon_0\rho, \sigma_0\rho)\sqrt{\frac{\log\frac{4}{\delta}}{2n}}\right)\right)$$

$$\overline{\lambda}(\rho) = \max\left(\frac{12\varepsilon_0\rho}{R^2}\log(2\times 6^{d/2}), e^{\frac{\sup_{f\in\mathcal{F}}\|f\|_\infty}{\varepsilon_0\rho}}\frac{\varepsilon_0}{\rho}\right).$$

Assume that

$$\rho \geq \frac{8192}{\sqrt{n}\mu^*(\lambda_0^*)^2}\left(12\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho)\right)\sqrt{1 + \log\frac{4}{\delta}}\right), \tag{40}$$

Then, with probability at least $1 - \frac{\delta}{2}$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda}_n(\rho) \leq \lambda \leq \overline{\lambda}(\rho)} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}}\left[\phi(f, \xi, \lambda, \varepsilon, \sigma)\right]$$

and when $\rho^2 \geq \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma)$*, with probability* $1 - \delta$*, it holds,*

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \mathcal{R}^{\varepsilon}_{\rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma)}(f), .$$

Furthermore, with probability $1 - \delta$*,*

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \sup\left\{\mathbb{E}_{\mathrm{Q}}[f] : \mathrm{Q} \in \mathcal{P}(\Xi), W^2_{2,\tau(\rho)}(\mathrm{P}, \mathrm{Q}) \leq \rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma)\right\},$$

with $\tau(\rho) \coloneqq \frac{\varepsilon_0\rho}{\underline{\lambda}_n(\rho)}$.

Proof. The first part of this result is a consequence of the combination of Propositions C.1 and D.3, both applied with $\delta \leftarrow \delta/4$, and of Proposition B.1. For the second part, note that Remark B.4 implies that the above argument actually gives the slightly stronger result: with probability $1 - \delta$, for any $f \in \mathcal{F}$,

$$\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) \geq \inf_{\underline{\lambda}_n(\rho) \leq \lambda \leq \overline{\lambda}(\rho)} \lambda(\rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma)) + \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)]$$

Next, take $\mathrm{Q} \in \mathcal{P}(\Xi)$ such that $W^2_{2,\tau(\rho)}(\mathrm{P}, \mathrm{Q}) \leq \rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \varepsilon, \sigma)$. With a similar argument as in the proof of Proposition B.1, we get that

$$\begin{aligned}
\widehat{\mathcal{R}}^{\varepsilon}_{\rho^2}(f) &\geq \inf_{\underline{\lambda}_n(\rho) \leq \lambda \leq \overline{\lambda}(\rho)} \lambda(\rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma) + \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)] \\
&= \mathbb{E}_{\mathrm{Q}}[f] + \inf_{\underline{\lambda}_n(\rho) \leq \lambda \leq \overline{\lambda}(\rho)} \lambda(\rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma) - \{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)]\} \\
&= \mathbb{E}_{\mathrm{Q}}[f] + \inf_{\underline{\lambda}_n(\rho) \leq \lambda \leq \overline{\lambda}(\rho)} \lambda(\rho^2 - \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma)) - \sup_{f'\in\mathcal{F}}\{\mathbb{E}_{\mathrm{Q}}[f'] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f', \xi, \lambda, \varepsilon, \sigma)]\}.
\end{aligned}$$

We now proceed to show, and this will conclude the proof, that

$$\sup_{f\in\mathcal{F}}\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)]\} \leq \lambda W^2_{2,\tau(\rho)}(\mathrm{P}, \mathrm{Q}),$$

for $\lambda \geq \underline{\lambda}_n(\rho)$.

Indeed,

$$\begin{aligned}
\sup_{f\in\mathcal{F}}\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)]\} &\leq \sup_{f\in\mathcal{C}(\Xi)}\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f, \xi, \lambda, \varepsilon, \sigma)]\} \\
&= \sup_{f\in\mathcal{C}(\Xi)}\left\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}\left[\log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta) - \lambda\|\xi-\zeta\|^2/2}{\varepsilon}}\right]\right)\right]\right\} \\
&= \lambda\sup_{f\in\mathcal{C}(\Xi)}\left\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}\left[\log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta) - \|\xi-\zeta\|^2/2}{\varepsilon/\lambda}}\right]\right)\right]\right\}.
\end{aligned} \tag{41}$$

where we performed the change of variable $f \leftarrow f/\lambda$. We now show the following equality that will allow us to rewrite the RHS of (41).

$$-\mathbb{E}_{\xi\sim\mathrm{P}}\left[\log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta) - \frac{1}{2}\|\xi-\zeta\|^2}{\varepsilon/\lambda}}\right]\right)\right] = \sup_{g\in\mathcal{C}(\Xi)}\mathbb{E}_{\mathrm{P}}[g] - \frac{\varepsilon}{\lambda}\left(\mathbb{E}_{(\xi,\zeta)\sim\pi_\sigma}\left[e^{\frac{g(\xi)+f(\zeta)-\frac{1}{2}\|\xi-\zeta\|^2}{\varepsilon/\lambda}}\right] - 1\right). \tag{42}$$

Solving the optimality condition of the concave problem of the RHS of (42) gives that its maximum is reached for

$$g(\xi) = -\log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta)-\frac{1}{2}\|\xi-\zeta\|^2}{\varepsilon/\lambda}}\right]\right)$$

so that (42) holds. Hence, we get that

$$\begin{aligned}
&\sup_{f\in\mathcal{C}(\Xi)}\left\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}\left[\log\left(\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[e^{\frac{f(\zeta)-\frac{1}{2}\|\xi-\zeta\|^2}{\varepsilon/\lambda}}\right]\right)\right]\right\}\\
&= \sup_{f,g\in\mathcal{C}(\Xi)}\left\{\mathbb{E}_{\mathrm{Q}}[f] + \mathbb{E}_{\mathrm{P}}[g] - \frac{\varepsilon}{\lambda}\left(\mathbb{E}_{(\xi,\zeta)\sim\pi_\sigma}\left[e^{\frac{g(\xi)+f(\zeta)-\frac{1}{2}\|\xi-\zeta\|^2}{\varepsilon/\lambda}}\right]-1\right)\right\}\\
&= W^2_{2,\varepsilon/\lambda}\left(\mathrm{P},\mathrm{Q}\right)\,,
\end{aligned}$$

by the duality formula for regularized OT (Peyré and Cuturi, 2019)[2].

Combining this equality with the bound of (41) gives

$$\sup_{f\in\mathcal{F}}\{\mathbb{E}_{\mathrm{Q}}[f] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]\} \le \lambda W^2_{2,\varepsilon/\lambda}\left(\mathrm{P},\mathrm{Q}\right),$$

which yields the result since $W^2_{2,\tau}(\mathrm{P},\mathrm{Q})$ is non-decreasing in τ. □

Lemma E.7. *In the setting of Theorem E.5, when Q_σ denotes the second marginal of*

$$\mathrm{P}(\mathrm{d}\xi)\,\pi_\sigma(\mathrm{d}\zeta|\xi)\,,$$

and when $\sigma \le \sigma_1$, it holds

$$W^2_{2,\tau(\rho)}\left(\mathrm{P},\mathrm{Q}_\sigma\right) \le \sigma^2\,.$$

Proof. Consider the transport plan $\pi = \mathrm{P}(\mathrm{d}\xi)\,\pi_\sigma(\mathrm{d}\zeta|\xi)$. To show this lemma, it suffices to prove that

$$\mathbb{E}_\pi\left[d^2\right] + \tau(\rho)\,\mathrm{KL}(\pi\,|\,\pi) = \mathcal{O}\big(\sigma^2\big)\,,$$

i.e., that $\mathbb{E}_\pi\left[d^2\right] = \mathcal{O}\big(\sigma^2\big)$. Let us first fix $\xi \in \operatorname{supp}\mathrm{P}$ and consider $\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right]$, which is equal to

$$\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right] = \frac{\int_\Xi \frac{1}{2}\|\xi-\zeta\|^2 e^{-\frac{\|\xi-\zeta\|^2}{2\sigma^2}}\mathrm{d}\zeta}{\int_\Xi e^{-\frac{\|\xi-\zeta\|^2}{2\sigma^2}}\mathrm{d}\zeta}\,.$$

The numerator can be upper-bounded as follows:

$$\int_\Xi \frac{1}{2}\|\xi-\zeta\|^2 e^{-\frac{\|\xi-\zeta\|^2}{2\sigma^2}}\mathrm{d}\zeta \le \int_{\mathbb{R}^d} \frac{1}{2}\|\xi-\zeta\|^2 e^{-\frac{\|\xi-\zeta\|^2}{2\sigma^2}}\mathrm{d}\zeta = (2\pi\sigma^2)^{d/2}\frac{\sigma^2}{2}\,.$$

For the denominator, we have seen in the proof of Lemma A.4, and more precisely (21), that

$$\left(\int_\Xi e^{-\frac{\|\xi-\zeta\|^2}{2\sigma^2}}\mathrm{d}\zeta\right)^{-1} \le \frac{2}{(2\pi\sigma^2)^{d/2}}\,,$$

when $\sigma \le \sigma_1$. Hence, we have the bound

$$\mathbb{E}_{\zeta\sim\pi_\sigma(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right] \le \sigma^2\,,$$

and integrating w.r.t. $\xi \sim \mathrm{P}$ yields the result. □

Proof of Theorem E.5. Since we will only consider radii in particular bounded by ρ_c, the condition (40) is implied by

$$\rho \ge \frac{8192}{\mu^*(\lambda_0^*)^2\sqrt{n}}\left(12\mathcal{I}(\mathcal{F},\|\cdot\|_\infty) + \left(\widetilde{F}(0) + \overline{M}(\rho_c)\right)\sqrt{\log\frac{4}{\delta}}\right),$$

[2]To get this exact result for a regularization w.r.t. an arbitrary measure, one can readily combine Paty and Cuturi (2020, Cor. 1) and Feydy et al. (2019, Prop. 7). Also, note that we essentially reproved the semi-duality formula of Genevay et al. (2016, Prop. 2.1) except that the regularization is taken w.r.t. a general measure.

with $\overline{M}(\rho_c) < +\infty$.

We now show that $\underline{\lambda}_n(\rho)$ can always be lower-bounded by a quantity proportional to $1/\rho$, i.e., that

$$\underline{\lambda}_n(\rho) \geq \frac{1}{\rho} \min\left(\frac{\lambda_0^*}{32}, \frac{\varepsilon_0 {\rho_{\mathrm{thres}}}^2 \rho_c^2}{4\overline{\mathrm{Var}}} \right). \tag{43}$$

Let us discuss separately the cases where $\rho \leq \rho_{\mathrm{thres}}$ holds or not.

- When $\rho \leq \rho_{\mathrm{thres}}$, (43) holds by definition of $\underline{\lambda}_n(\rho)$.
- When $\rho > \rho_{\mathrm{thres}}$, by definition, $\underline{\lambda}_n(\rho)$ is lower bounded as

$$\underline{\lambda}_n(\rho) \geq \frac{\varepsilon_0 \rho}{\overline{\mathrm{Var}}} \left(\rho_c^2 - \rho^2 - \left(\frac{48\sqrt{\overline{\mathrm{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0 \rho \sqrt{n}} + 2\overline{C}_{\mathcal{F}} \sqrt{\frac{\log \frac{4}{\delta}}{2n}} \right) \right).$$

  Applying Lemma G.9 with $\overline{\rho} \leftarrow \frac{\rho_c}{2}$ and $c \leftarrow \frac{48\sqrt{\overline{\mathrm{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0 \sqrt{n}}$, we obtain that, when

$$\rho_c \geq \left(\frac{192\sqrt{\overline{\mathrm{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0 \sqrt{n}} \right)^{1/3}$$

  and

$$\frac{384\sqrt{\overline{\mathrm{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0 \rho_c^2 \sqrt{n}} \leq \rho \leq \frac{\rho_c}{2} - \frac{384\sqrt{\overline{\mathrm{Var}}}\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty)}{\varepsilon_0 \rho_c^2 \sqrt{n}},$$

  the following lower-bound holds,

$$\underline{\lambda}_n(\rho) \geq \frac{\varepsilon_0 \rho}{\overline{\mathrm{Var}}} \left(\frac{3\rho_c^2}{4} - 2\overline{C}_{\mathcal{F}} \sqrt{\frac{\log \frac{4}{\delta}}{2n}} \right) \geq \frac{\varepsilon_0 \rho \rho_c^2}{4\overline{\mathrm{Var}}} \geq \frac{\varepsilon_0 {\rho_{\mathrm{thres}}}^2 \rho_c^2}{4\overline{\mathrm{Var}}\rho},$$

  where we used successively that $\frac{\rho_c^2}{2} \geq 2\overline{C}_{\mathcal{F}} \sqrt{\frac{\log \frac{4}{\delta}}{2n}}$ and $\rho \geq \rho_{\mathrm{thres}}$. This concludes the proof of (43). Note that it implies the bound on $\tau(\rho)$ in the statement.

Let us finally turn our attention to the condition $\rho^2 \geq \overline{\rho}_n^2(\delta, \underline{\lambda}_n(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma)$. Since $\sup_{0<\rho\leq\rho_c} \widetilde{F}(\lambda_0^*/(32\rho)) = \widetilde{F}(\lambda_0^*/(32\rho_c)) < +\infty$ by definition (Assumption 6) and

$$\begin{aligned}
&\sup_{0<\rho'\leq\rho_c} a\big(\underline{\lambda}_n(\rho'), \overline{\lambda}(\rho'), \varepsilon_0\rho', \varepsilon_0\sigma\big) \\
&\leq \sup_{0<\rho'\leq\rho_c} a\left(\frac{1}{\rho'} \min\left(\frac{\lambda_0^*}{32}, \frac{\varepsilon_0 {\rho_{\mathrm{thres}}}^2 \rho_c^2}{4\overline{\mathrm{Var}}} \right), \max\left(\frac{\lambda_0^*}{2\rho'}, \frac{12\varepsilon_0\rho_c \log(2\times 6^{d/2})}{R^2}, \frac{e^{\frac{\sup_{f\in\mathcal{F}}\|f\|_\infty}{\varepsilon_0\rho_{\mathrm{thres}}}}\varepsilon_0}{\rho_{\mathrm{thres}}} \right), \varepsilon_0\rho', \sigma_0\rho' \right) \\
&= \overline{a} < +\infty
\end{aligned}$$

where we used the monotonicity properties of a (Corollary A.5) and (43).

In conclusion, along with (43), we obtain that

$$\begin{aligned}
&\overline{\rho}_n^2(\delta, \underline{\lambda}(\rho), \overline{\lambda}(\rho), \varepsilon, \sigma) \\
&\leq \frac{117}{\sqrt{n}\underline{\lambda}(\rho)} \left(\mathcal{I}(\mathcal{F}, \|\cdot\|_\infty) + \max\left(\widetilde{F}\left(\frac{\lambda_0^*}{32\rho_c} \right), \overline{a} \right) \left(1 + \sqrt{\log \frac{1}{\delta}} \right) \right) \\
&\leq \rho_n \rho,
\end{aligned}$$

by definition of ρ_n. The last part of the statement then follows from Lemma E.7. ∎

F Upper-bound on the empirical robust risk

In this section we prove Theorem 3.5 that complements the main results by providing both a lwoer and an upper bound on the empirical robus risk. In view of the previous section, the missing part is ther upper-bound, that we establish in this section.

The proof of the upper-bound is similar to the proof of our main results, yet simpler. Indeed, the bounds on the dual variable are required for the true distribution P, which is fixed, instead of the empirical distribution $\mathrm{P_n}$. We slightly modify our main concentration result (Proposition B.1) in Proposition F.1. We simplify our bounds on the dual multiplier when the radius is close to the critical radius (Propositions D.1 and D.3) in Propositions F.2 and F.5.

F.1 From empirical to true risk

Proposition F.1. *For $\rho > 0$, $\varepsilon \geq 0$, $\sigma > 0$ and $\delta \in (0,1)$, assume that there is some $0 < \underline{\lambda} \leq \overline{\lambda} < +\infty$ such that,*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda} \leq \lambda \leq \overline{\lambda}} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] .$$

Then, when $\rho^2 \geq \overline{\rho}^2_n(\delta, \underline{\lambda}, \overline{\lambda}, \varepsilon, \sigma)$, with probability $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2 - \overline{\rho}^2_n(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)}(f) \leq \mathcal{R}^{\varepsilon}_{\rho^2}(f) .$$

Proof. This proof closely mimics the one of Proposition B.1 but switches the roles of P and $\mathrm{P_n}$. First, note that by following the proof of Lemma B.3 with and replacing P by $\mathrm{P_n}$ and *vice versa* (and using statement (a) of Lemma G.2 instead of (b)) yields the following

$$\sup_{(f,\lambda)\in\mathcal{F}\times[\underline{\lambda},\overline{\lambda}]} \left\{ \frac{\mathbb{E}_{\xi\sim\mathrm{P_n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \leq \overline{\rho}^2_n(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma) .$$

We can now follow the last part of the proof of Proposition B.1. On the event above, for any $f \in \mathcal{F}$,

$$\begin{aligned}
\mathcal{R}^{\varepsilon}_{\rho^2}(f) &= \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{ \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] \right\} \\
&= \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{ \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P_n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda \frac{\mathbb{E}_{\xi\sim\mathrm{P_n}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda,\varepsilon,\sigma)]}{\lambda} \right\} \\
&\geq \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{ \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P_n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda \sup_{\underline{\lambda}\leq\lambda'\leq\overline{\lambda}} \frac{\mathbb{E}_{\xi\sim\mathrm{P_n}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)] - \mathbb{E}_{\xi\sim\mathrm{P}}[\phi(f,\xi,\lambda',\varepsilon,\sigma)]}{\lambda'} \right\} \\
&\geq \inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} \left\{ \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P_n}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right] - \lambda\overline{\rho}^2_n(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma) \right\} \\
&\geq \widehat{\mathcal{R}}^{\varepsilon}_{\rho^2-\overline{\rho}^2_n(\delta,\underline{\lambda},\overline{\lambda},\varepsilon,\sigma)}(f) .
\end{aligned}$$

□

F.2 Standard WDRO case

Proposition F.2. *Let Assumption 5 hold and fix a threshold $\delta \in (0,1)$. Assume that $\rho^2 \leq \rho^2_c(0,0)$. Then, we have,*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}_{\rho^2}(f) = \inf_{\underline{\lambda}\leq\lambda} \lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\phi(f,\xi,\lambda,0)\right]$$

where the dual bound $\underline{\lambda}$ is defined as

$$\underline{\lambda} := \min\left(\lambda_2, \frac{\rho^2_c(0,0) - \rho^2}{L}\right),$$

and L is defined in Proposition D.1.

Proof. Let $0 \leq \lambda \leq \underline{\lambda}$. By Lemma D.2 and the dominated convergence theorem, one has that,

$$\begin{aligned}\partial_\lambda \psi_\rho(\lambda) &= \rho^2 - \mathbb{E}_{\xi\sim \mathrm{P}}\left[\min\left\{\frac{1}{2}\|\xi-\zeta\|^2 : \zeta \in \arg\max_{\Xi} f - \frac{\lambda}{2}\|\xi - \cdot\|^2\right\}\right] \\ &\leq \rho^2 - \left(1 - \frac{16\lambda}{\mu}\right)\mathbb{E}_{\xi\sim \mathrm{P}}\left[\min\left\{\frac{1}{2}\|\xi-\zeta\|^2 : \zeta \in \arg\max_{\Xi} f\right\}\right] \\ &\leq \rho^2 - \rho_c^2(0,0) + L\lambda\,,\end{aligned}$$

which is non-negative by definition of $\underline{\lambda}$ and thus concludes the proof. ☐

We can now state analogues of Theorems E.1 and E.3. Note that the bounds $\underline{\lambda}(\rho)$ that we obtained in this section are better than the ones we got in the main proof. For the sake of simplicity, we give up this additional precision and use the same bounds as in Theorems E.1 and E.3.

Corollary F.3. *In the same setting as Theorem E.1, with probability* $1-\delta$*, it holds,*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}_{\rho^2}(f) \geq \widehat{\mathcal{R}}_{\rho(\rho-\rho_n)}(f)\,.$$

Proof. This result is obtained as a combination of Propositions F.1–F.2, which gives the desired result with probability at least $1-\frac{\delta}{2}$ and *a fortiori* $1-\delta$. ☐

Corollary F.4. *In the same setting as Theorem E.3, with probability* $1-\delta$*, it holds,*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}_{\rho^2}(f) \geq \widehat{\mathcal{R}}_{\rho(\rho-\rho_n)}(f)\,.$$

Proof. This result follows by combining Propositions F.1 and C.1, which gives the desired result with probability at least $1-\frac{\delta}{2}$ and *a fortiori* $1-\delta$. ☐

To conclude, in the context of Theorem E.1 (resp. Theorem E.3), Corollary F.3 (resp. Corollary F.4) with $\rho \leftarrow \rho + \rho_n$ yields, with probability at least $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho(\rho+\rho_n)}(f) \leq \mathcal{R}_{(\rho+\rho_n)^2}(f)\,,$$

so that, since $\rho \geq \rho_n$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}_{\rho^2}(f) \leq \mathcal{R}_{\rho(\rho+3\rho_n)}(f)\,,$$

Combining this bound with Theorem E.1 (resp. Theorem E.3) completes the bound of Theorem 3.5.

F.3 Regularized case

In the regularized case, the bound simplifies as well compared to Proposition D.3.

Proposition F.5. *Fix a threshold* $\delta \in (0,1)$*. When* $\rho^2 \leq \rho_c^2(\varepsilon,\sigma)$*, we have,*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}^{\varepsilon}_{\rho^2}(f) = \inf_{\underline{\lambda}_n \leq \lambda \leq \overline{\lambda}} \lambda\rho^2 + \mathbb{E}_{\xi\sim \mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]$$

where the dual bounds are defined by

$$\underline{\lambda} \coloneqq \frac{\varepsilon}{\mathrm{Var}(\varepsilon,\sigma)}\left(\rho_c^2(\varepsilon,\sigma) - \rho^2\right)$$

$$\text{and} \quad \overline{\lambda} \coloneqq \max\left(\frac{12\varepsilon}{R^2}\log(2\times 6^{d/2}), e^{\frac{\sup_{f\in\mathcal{F}}\|f\|_\infty}{\varepsilon}}\frac{\varepsilon_0}{\rho}\right),$$

and λ_0^**,* μ^* *were defined in Assumption 8.*

Proof. The proof of the upper-bound is exactly the same as in Proposition D.3 so we focus on the lower-bound. Following the same reasoning as the one to get (34) in Proposition D.3 but with P instead of $\mathrm{P_n}$ we get that

$$\begin{aligned}\partial_\lambda\left\{\lambda\rho^2 + \mathbb{E}_{\xi\sim\mathrm{P}}\left[\phi(f,\xi,\lambda,\varepsilon,\sigma)\right]\right\} &\leq \rho^2 - \mathbb{E}_{\xi\sim\mathrm{P}}\left[\mathbb{E}_{\zeta\sim\pi_\sigma^{f/\varepsilon}(\cdot|\xi)}\left[\frac{1}{2}\|\xi-\zeta\|^2\right]\right] + \frac{\lambda}{\varepsilon}\mathrm{Var}(\varepsilon,\sigma) \\ &= \rho^2 - \rho_c^2(\varepsilon,\sigma) + \frac{\lambda}{\varepsilon}\mathrm{Var}(\varepsilon,\sigma)\,,\end{aligned}$$

which is non-positive when $\lambda \leq \underline{\lambda}$. ☐

Corollary F.6. *In the same setting as Theorem E.5, with probability $1-\delta$, it holds,*

$$\forall f \in \mathcal{F}, \quad \mathcal{R}^{\varepsilon}_{\rho^2}(f) \geq \widehat{\mathcal{R}}^{\varepsilon}_{\rho(\rho-\rho_n)}(f)\,.$$

Proof. This result follows by combining Propositions F.1 and C.1 and Proposition F.5, which gives the desired result with probability at least $1-\frac{\delta}{2}$ and *a fortiori* $1-\delta$. □

To conclude, in the context of Theorem E.5, Corollary F.6 with $\rho \leftarrow \rho+\rho_n$ and $\varepsilon_0 \leftarrow \frac{\varepsilon_0\rho}{\rho+\rho_n}$, i.e., $\varepsilon \leftarrow \frac{\varepsilon_0\rho}{\rho+\rho_n} \times (\rho+\rho_n)$, yields,[3] with probability at least $1-\delta$,

$$\forall f \in \mathcal{F}, \quad \widehat{\mathcal{R}}^{\varepsilon_0\rho}_{\rho(\rho+\rho_n)}(f) \leq \mathcal{R}^{\varepsilon_0\rho}_{(\rho+\rho_n)^2}(f)\,, \quad \text{and, in particular,} \quad \widehat{\mathcal{R}}^{\varepsilon_0\rho}_{\rho^2}(f) \leq \mathcal{R}^{\varepsilon_0\rho}_{\rho(\rho+3\rho_n)}(f)\,.$$

Combining this bound with Theorem E.5 completes the bound of Theorem 3.5.

G Technical lemmas

In this section, we recall and adapt known results, as well as establish technical facts, all useful in our developments. They are presented in self-contained lemmas and are arranged in four thematic subsections.

G.1 Laplace approximation

Lemma G.1 (Restriction to Ξ)**.** *Consider $\Xi \subset \mathbb{R}^d$, $\varepsilon_1, \tau_1 > 0$ and a map $\zeta^\star : [0,\tau_1] \to \Xi$ defined by $\zeta^\star(\tau) = \xi + \tau g$ with $\xi \in \Xi$, $g \in \mathbb{R}^d$ and assume that there is a positive radius R such that,*

1. *The closed ball $\overline{\mathbb{B}}(\zeta^\star(0), R)$ is included in Ξ.*
2. *R, τ_1 and $\|g\|$ satisfy $\frac{R^2}{6} \geq \tau_1^2\|g\|^2$.*

Then, for $(\varepsilon,\tau) \in [0,\varepsilon_1] \times [0,\tau_1]$,

$$\left|(2\pi\varepsilon\tau)^{-\frac{d}{2}} \int_{\Xi} \exp\left(-\frac{\|\zeta-\zeta^\star(\tau)\|_2^2}{2\varepsilon\tau}\right)\mathrm{d}\zeta - 1\right| \leq 6^{d/2} e^{-\frac{R^2}{12\varepsilon\tau}}\,,$$

Proof. The quantity to bound rewrites

$$\left|(2\pi\varepsilon\tau)^{-\frac{d}{2}} \int_{\Xi} \exp\left(-\frac{\|\zeta-\zeta^\star(\tau)\|_2^2}{2\varepsilon\tau}\right)\mathrm{d}\zeta - 1\right| = (2\pi\varepsilon\tau)^{-\frac{d}{2}} \int_{\mathbb{R}^d\setminus\Xi} \exp\left(-\frac{\|\zeta-\zeta^\star(\tau)\|_2^2}{2\varepsilon\tau}\right)\mathrm{d}\zeta\,,$$

so let us bound this integral. Since $\overline{\mathbb{B}}(\zeta^\star(0), R)$ is inside Ξ, this means that, for any $\zeta \notin \Xi$, $\|\zeta-\xi\|$ is at least equal to R. Hence, for any $\zeta \notin \Xi$, one has that

$$\begin{aligned}
\|\zeta-\zeta^\star(\tau)\|^2 &\geq \frac{1}{2}\|\zeta-\xi\|^2 - \tau^2\|g\|^2 \\
&\geq \frac{1}{6}\|\zeta-\xi\|^2 + \frac{1}{6}R^2 + \frac{1}{6}R^2 - \tau^2\|g\|^2 \\
&\geq \frac{1}{6}\|\zeta-\xi\|^2 + \frac{1}{6}R^2\,,
\end{aligned}$$

so that we get the bound

$$\begin{aligned}
(2\pi\varepsilon\tau)^{-\frac{d}{2}} \int_{\mathbb{R}^d\setminus\Xi} \exp\left(-\frac{\|\zeta-\zeta^\star(\tau)\|_2^2}{2\varepsilon\tau}\right)\mathrm{d}\zeta &\leq e^{-\frac{R^2}{12\varepsilon\tau}} \times (2\pi\varepsilon\tau)^{-\frac{d}{2}} \int_{\mathbb{R}^d\setminus\Xi} \exp\left(-\frac{\|\zeta-\zeta\|_2^2}{12\varepsilon\tau}\right)\mathrm{d}\zeta \\
&= 6^{d/2} e^{-\frac{R^2}{12\varepsilon\tau}}\,.
\end{aligned}$$

□

[3] Though ε_0 now formally depends on ρ', the same bounds still hold and do not become degenerate since ε_0 lies $[\varepsilon_0/2, \varepsilon_0]$ that avoids zero.

G.2 Concentration

We rely on standard concentration tools that we encapsulate in the following lemma for convenience.

Lemma G.2. *Let $(\mathcal{X}, \mathrm{dist})$ be a (totally bounded) separable metric space, P a probability distribution on a probability space Ξ and $\mathrm{P_n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\xi_i}$ with $\xi_1, \dots, \xi_n \sim \mathrm{P}$ i.i.d.. Consider a mapping $X : \mathcal{X} \times \Xi \to \mathbb{R}$ and assume that,*

1. *For each $x \in \mathcal{X}$, $\xi \mapsto X(x, \xi)$ is measurable;*
2. *There is a constant $L > 0$ such that, for each $\xi \in \Xi$, $x \mapsto X(x, \xi)$ is L-Lipschitz;*
3. *X almost surely belongs to $[a, b]$.*

Then, for any $\delta \in (0, 1)$,

(a) With probability at least $1 - \delta$,

$$\forall x \in \mathcal{X}, \quad \mathbb{E}_{\xi \sim \mathrm{P_n}}[X(x, \xi)] - \mathbb{E}_{\xi \sim \mathrm{P}}[X(x, \xi)] \leq \frac{48 L \mathcal{I}(\mathcal{X}, \mathrm{dist})}{\sqrt{n}} + 2(b-a)\sqrt{\frac{\log \frac{1}{\delta}}{2n}}\,.$$

(b) With probability at least $1 - \delta$,

$$\forall x \in \mathcal{X}, \quad \mathbb{E}_{\xi \sim \mathrm{P}}[X(x, \xi)] - \mathbb{E}_{\xi \sim \mathrm{P_n}}[X(x, \xi)] \leq \frac{48 L \mathcal{I}(\mathcal{X}, \mathrm{dist})}{\sqrt{n}} + 2(b-a)\sqrt{\frac{\log \frac{1}{\delta}}{2n}}\,.$$

Proof. First, let us note that we can assume that $\mathbb{E}_{\xi \sim \mathrm{P}}[X(x, \xi)] = 0$ provided that we prove the bound above with the left-hand side divided by a factor two. Indeed, considering the random variables $Y(x, \xi) \coloneqq X(x, \xi) - \mathbb{E}_{\zeta \sim \mathrm{P}}[X(x, \zeta)]$, we see that Y satisfy the assumptions of the lemma, albeit with the constants $L \leftarrow 2L$, $a \leftarrow a - b$ and $b \leftarrow b - a$. Moreover, we only prove the assertion (a) since the (b) follows from (a) with $X \leftarrow -X$.

Step 1: Bound on the expectation. First, we focus on bounding the expectation of the quantity

$$\sup_{x \in \mathcal{X}} \{\mathbb{E}_{\xi \sim \mathrm{P_n}} X(x, \xi)\}\,.$$

By the symmetrization principle (e.g., (Boucheron et al., 2013, Lem. 11.4)), with $\mathrm{s}_1, \dots, \mathrm{s}_n$ i.i.d. Rademacher random variables,

$$\mathbb{E}\left[\sup_{x \in \mathcal{X}} \{\mathbb{E}_{\xi \sim \mathrm{P_n}} X(x, \xi)\}\right] \leq 2\mathbb{E}\left[\sup_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} \mathrm{s}_i\, X(x, \xi_i)\right].$$

Take $x, x' \in \mathcal{X}$. By the Lipschitz property of X, with $\xi \sim \mathrm{P}$, for any $i = 1, \dots, n$, the random variable

$$\frac{\mathrm{s}_i (X(x, \xi) - X(x', \xi))}{\sqrt{n} L} \tag{44}$$

is bounded, in absolute value, by $\frac{\mathrm{dist}(x, x')}{\sqrt{n}}$ and as such it is sub-Gaussian with parameter $\frac{\mathrm{dist}(x, x')^2}{n}$ by Hoeffding's lemma (e.g., (Boucheron et al., 2013, Lem. 2.2)). As a consequence, by independence, the random variable

$$\sum_{i=1}^{n} \frac{\mathrm{s}_i (X(x, \xi) - X(x', \xi))}{\sqrt{n} L}$$

is sub-Gaussian with parameter $\mathrm{dist}(x, x')^2$. Since, in addition, it is zero-mean, we can invoke Dudley's bound (e.g., (Boucheron et al., 2013, Cor. 13.2)) to get that,

$$\mathbb{E}\left[\sup_{x \in \mathcal{X}} \frac{1}{\sqrt{n} L} \sum_{i=1}^{n} \mathrm{s}_i\, X(x, \xi_i)\right] \leq 12 \mathcal{I}(\mathcal{X}, \mathrm{dist})\,,$$

or, in other words, by (44), that

$$\mathbb{E}\left[\sup_{x\in\mathcal{X}}\{\mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}} X(x,\xi)\}\right] \leq \frac{24L\mathcal{I}(\mathcal{X},\mathrm{dist})}{\sqrt{n}}\,. \tag{45}$$

Step 2: Concentration inequality. Since the functions X are uniformly bounded, $\sup_{x\in\mathcal{X}}\{\mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}} X(x,\xi)\}$, seen as a function of $(\xi_1,\dots,\xi_n)$, satisfies the bounded difference property with constant $b-a$. Therefore, the bounded difference inequality (e.g., (Boucheron et al., 2013, Thm. 6.2)) readily yields that, with probability at least $1-\delta$,

$$\begin{aligned}\sup_{x\in\mathcal{X}}\{\mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}} X(x,\xi)\} &\leq \mathbb{E}\sup_{x\in\mathcal{X}}\{\mathbb{E}_{\xi\sim \mathrm{P}_\mathrm{n}} X(x,\xi)\} + (b-a)\sqrt{\frac{\log\frac{1}{\delta}}{2n}} \\ &\leq \frac{24L\mathcal{I}(\mathcal{X},\mathrm{dist})}{\sqrt{n}} + (b-a)\sqrt{\frac{\log\frac{1}{\delta}}{2n}}\end{aligned}$$

where we plugged in (45), the bound on the expectation from the first step. □

G.3 Dudley's integral bounds

Lemma G.3. *Let $(\mathcal{X}_1,\mathrm{dist}_1)$ and $(\mathcal{X}_2,\mathrm{dist}_2)$ be two metric spaces, and consider $\mathcal{X} \coloneqq \mathcal{X}_1\times\mathcal{X}_2$ equipped with the distance $\mathrm{dist} \coloneqq c_1\,\mathrm{dist}_1 + c_2\,\mathrm{dist}_2$ with $c_1,c_2>0$. Then*

$$\mathcal{I}(\mathcal{X},\mathrm{dist}) \leq c_1\mathcal{I}(\mathcal{X}_1,\mathrm{dist}_1) + c_2\mathcal{I}(\mathcal{X}_2,\mathrm{dist}_2)\,.$$

Proof. Note that, for any $t>0$, the inequality $N(t,\mathcal{X},\mathrm{dist}) \leq N(t,\mathcal{X}_1,c_1\,\mathrm{dist}_1)\times N(t,\mathcal{X}_2,c_2\,\mathrm{dist}_2)$ holds, so that, by subdadditivity of the square root,

$$\begin{aligned}\mathcal{I}(\mathcal{X},\mathrm{dist}) &= \int_0^{+\infty}\sqrt{\log N(t,\mathcal{X},\mathrm{dist})}\mathrm{d}t \\ &\leq \int_0^{+\infty}\sqrt{\log N(t,\mathcal{X}_1,c_1\,\mathrm{dist}_1)}\mathrm{d}t + \int_0^{+\infty}\sqrt{\log N(t,\mathcal{X}_2,c_2\,\mathrm{dist}_2)}\mathrm{d}t \\ &= \int_0^{+\infty}\sqrt{\log N(t/c_1,\mathcal{X}_1,\mathrm{dist}_1)}\mathrm{d}t + \int_0^{+\infty}\sqrt{\log N(t/c_2,\mathcal{X}_2,\mathrm{dist}_2)}\mathrm{d}t \\ &= c_1\mathcal{I}(\mathcal{X}_1,\mathrm{dist}_1) + c_2\mathcal{I}(\mathcal{X}_2,\mathrm{dist}_2)\,,\end{aligned}$$

where we performed changes of variable in to obtain the last equality. □

Lemma G.4. *For $c>0$,*

$$\mathcal{I}([0,c],|\cdot|) \leq \frac{c}{2}(1+2\log 2)\,.$$

Proof. Noticing that $N(t,[0,c],|\cdot|)=1$ whenever $t\geq c$, we get that

$$\begin{aligned}\mathcal{I}([0,c],|\cdot|) &= \int_0^{c}\sqrt{\log N(t,[0,c],|\cdot|)}\mathrm{d}t \\ &\leq \int_0^{c}(1+\log N(t,[0,c],|\cdot|))\mathrm{d}t\,.\end{aligned}$$

Now, a rough bound on $N(t,[0,c],|\cdot|)$ is $1+\frac{c}{t}$ which fits our purpose and yields

$$\mathcal{I}([0,c],|\cdot|) \leq c + \int_0^{c}\log\Big(1+\frac{c}{t}\Big)\mathrm{d}t = c(1+2\log 2)\,.$$

□

G.4 Auxiliary results

We conclude these sections with auxiliary technical results.

The following lemma recalls basic inequalities with the logarithm function.

Lemma G.5. *For $0 \leq x \leq \frac{1}{2}$, the following inequalities hold,*

$$\log(1-x) \geq -2x \qquad \text{and} \qquad \log(1+x) \leq x\,.$$

Lemma G.6. *For $\alpha > 0$, the function $x \mapsto \frac{\log(\alpha+x)}{x}$ is non-increasing on $([e^{W(1)} - \alpha]_+, +\infty)$.*

Proof. Denote by $f : x \mapsto \frac{\log(\alpha+x)}{x}$ this function, defined on $(0, +\infty)$. Its derivative is $f' : x \mapsto \frac{1}{x}\Big(\frac{1}{x+\alpha} - \log(x+\alpha)\Big)$. But the function $x \mapsto \frac{1}{x+\alpha} - \log(x+\alpha)$ is non-increasing, goes to $-\infty$ at infinity and its only potential zero is $e^{W(1)} - \alpha$ if it is positive,[4] which yields the result. □

Lemma G.7. *For $\Xi \subset \mathbb{R}^d$ a compact set, $g \in \mathcal{C}(\Xi)$ and, $\mathrm{Q} \in \mathcal{P}(\Xi)$,*

$$\log \mathbb{E}_{\xi\sim\mathrm{Q}}\Big[e^{g(\xi)}\Big] \leq \frac{\mathbb{E}_{\xi\sim\mathrm{Q}}\big[g(\xi)e^{g(\xi)}\big]}{\mathbb{E}_{\xi\sim\mathrm{Q}}\big[e^{g(\xi)}\big]}\,.$$

Proof. Define $\phi : t \mapsto \log \mathbb{E}_{\xi\sim\mathrm{Q}}\big[e^{tg(\xi)}\big]$ which is convex and differentiable, since g is continuous on the compact set Ξ. Hence,

$$0 = \phi(0) \geq \phi(1) + \phi'(1)(0-1)\,,$$

so that $\phi'(1) \geq \phi(1)$ which is the desired inequality. □

Lemma G.8. *For $a, b, c, r > 0$ fixed, consider the function defined on $\mathbb{R}_+$ by*

$$\phi(\lambda) = a\lambda + \frac{b}{\lambda + r} - c\log(\lambda + r)\,.$$

Then, for any $\overline{\lambda} > 0$, ϕ is strongly convex on $\big[0, \overline{\lambda}\big]$ with strong convexity constant

$$\mu^* := \frac{2b}{(\overline{\lambda}+r)^3} + \frac{c}{(\overline{\lambda}+r)^2}\,.$$

and the unique solution to the minimization problem

$$\min_{\lambda \geq 0} \phi(\lambda)$$

is given by,

$$\lambda^\star = \left[\frac{c + \sqrt{c^2 + 4ab}}{2a} - r\right]_+ .$$

Proof. ϕ is twice differentiable and its derivatives are, for $\lambda \geq 0$,

$$\phi'(\lambda) = a - \frac{b}{(\lambda+r)^2} - \frac{c}{\lambda+r}$$
$$\phi''(\lambda) = \frac{2b}{(\lambda+r)^3} + \frac{c}{(\lambda+r)^2}\,,$$

which shows that ϕ is strictly convex on $\mathbb{R}_+$ and yields its strong convexity on compact intervals. Then, the first order optimality condition $\phi'(\lambda) = 0$ gives us that

$$a(\lambda+r)^2 - c(\lambda+r) - b = 0\,, \tag{46}$$

which has an unique solution satisfying $\lambda + r \geq 0$ which is given by,

$$\lambda^\star = \frac{c + \sqrt{c^2 + 4ab}}{2a} - r\,.$$

If $\lambda^\star \geq 0$, then this is the solution we are looking for. If is not, this means that both roots of (46) are non-positive and therefore $\phi'(0) \geq 0$ which means that 0 is the solution to the minimization problem. □

[4] W denotes the Lambert function, i.e., the inverse of the map $x \mapsto xe^x$.

Lemma G.9. *For $c > 0$, $\overline{\rho} > 0$ such that $\overline{\rho} \geq (4c)^{1/3}$, the inequality*

$$\overline{\rho}^2 - \rho^2 - \frac{c}{\rho} \geq 0$$

holds in particular when

$$\frac{2c}{\overline{\rho}^2} \leq \rho \leq \overline{\rho} - \frac{2c}{\overline{\rho}^2}$$

Proof. When $0 < \rho \leq \overline{\rho}$, the inequation $\overline{\rho}^2 - \rho^2 - \frac{c}{\rho} \geq 0$ is implied by

$$\overline{\rho}^2\rho - \overline{\rho}\rho^2 - c \geq 0\,.$$

Solving the latter yields the interval

$$\left[\frac{\overline{\rho}^2 - \sqrt{\overline{\rho}(\overline{\rho}^3 - 4c)}}{2\overline{\rho}}, \frac{\overline{\rho}^2 + \sqrt{\overline{\rho}(\overline{\rho}^3 - 4c)}}{2\overline{\rho}}\right]$$

and the inequality $1 - u \leq \sqrt{1-u}$ for $u \in [0,1]$ yields the result. □