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# CONVERGENCE OF MESSAGE PASSING GRAPH NEURAL NETWORKS WITH GENERIC AGGREGATION ON RANDOM GRAPHS

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## 1. MOTIVATIONS

A classical method to study the expressivity of Graph Neural Networks (GNNs) is to compare them to the Weisfeiler-Lehman test for the graph isomorphism problem. Nevertheless, this approach becomes questionable for very large graphs. Real large graphs are rarely fully known, and their structure can change quickly over time. In such cases, we rather focus on global tendencies for which useful tools are random graphs. Let  $(\mathcal{X} \subset \mathbb{R}^d, P)$  be a compact probability space, from a measurable kernel  $W : \mathcal{X}^2 \rightarrow [0, 1]$  one can sample random graphs. Specifically,  $G_n$  drawn from this model has vertices  $\{X_1, \dots, X_n\} \subset \mathcal{X}$  and weights  $w_{ij}$  where :

$$X_1, \dots, X_n \stackrel{iid}{\sim} P, \quad w_{ij} = w_{ji} = W(X_i, X_j).$$

Our purpose is to examine whether a message passing GNN (MPGNN) with an unspecified aggregation on a large  $G_n$  is close to a continuous counterpart on the underlying random graph model. We give sufficient conditions under which the discrete model is a good approximation of the continuous analogue. Moreover, assuming some regularity on the aggregation, we provide explicit non asymptotic bounds for the deviation between a discrete MPGNN and its continuous counterpart. Our main contribution involves considering a generic aggregation function while previous related work has focused on particular cases, such as Graph Convolutional Network [?, ?], or degree normalized mean [?].

## 2. MPGNN

A  $L$ -layer MPGNN iteratively propagates a signal over a graph. At each step, the current representations of every node’s neighbors are gathered, transformed, and combined to update the node’s representation. Let  $G$  be a graph on the vertex set  $\{1, \dots, n\}$ , and  $Z = Z^{(0)} \in \mathbb{R}^{n \times d_0}$  be a signal on it. There are  $L$  operators  $(F^{(l)})_{1 \leq l \leq L}$  such that, at each layer, the update  $Z^{(l+1)}$  of the signal is computed node-wise by:

$$z_i^{(l+1)} = F^{(l+1)} \left( z_i^{(l)}, \left\{ \left( z_j^{(l)}, w_{i,j} \right) \right\}_{j \in \mathcal{N}(i)} \right) \in \mathbb{R}^{d_{l+1}}. \quad (1)$$

The  $F^{(l)}$  are referred to as *aggregations*. They are the core of

a MPGNN and their fundamental property is to be invariant to the permutation of the neighbors, so that the full network is consistent with graph isomorphism. This is ensured in (1) since a multiset (double braces) is taken as argument.

Typically, the aggregation first transforms the  $z_j^{(l)}$  through a learnable transformation  $\psi^{(l+1)}$ . Then combine them using a weighted mean  $\bigoplus$  (in a broad sense, such as arithmetic mean, point-wise maximum, *etc.*) with weights

$$c_{i,j}^{(l+1)} = c^{(l+1)} \left( z_i^{(l)}, z_j^{(l)}, w_{ij} \right) \quad (2)$$

such that (1) can be rewritten as

$$z_i^{(l+1)} = \bigoplus \left( \left\{ \left( \psi^{(l+1)}(z_j^{(l)}), c_{i,j}^{(l+1)} \right) \right\}_{j \in \mathcal{N}(i)} \right).$$

## Examples of different aggregations

1. Convolution :

$$z_i^{(l+1)} = \frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} w_{i,j} \psi^{(l+1)}(z_j^{(l)}).$$

2. Max Convolution :

$$z_i^{(l+1)} = \max_{j \in \mathcal{N}(i)} w_{i,j} \psi^{(l+1)}(z_j^{(l)}).$$

3. Degree Normalized Convolution :

$$z_i^{(l+1)} = \sum_{j \in \mathcal{N}(i)} \frac{w_{i,j}}{\sum_{k \in \mathcal{N}(i)} w_{i,k}} \psi^{(l+1)}(z_j^{(l)}).$$

4. Graph Attention (GAT): the  $c^{(l)}$  from (2) are learnable

$$: z_i^{(l+1)} = \sum_{j \in \mathcal{N}(i)} \frac{c_{i,j}^{(l+1)}}{\sum_{k \in \mathcal{N}(i)} c_{i,k}^{(l+1)}} \psi^{(l+1)}(z_j^{(l)}).$$

## 3. CONTINUOUS MPGNN

Let  $(W, P)$  be a random graph model, a  $L$ -layer continuous MPGNN (cMPPGNN) propagates a function over the latent space  $\mathcal{X}$ . For an input  $f = f^{(0)} \in \mathcal{C}(\mathcal{X}, \mathbb{R}^{d_0})$ ,  $f^{(l+1)}$  is computed by:

$$f^{(l+1)}(x) = \mathcal{F}_P^{(l+1)} \left( f^{(l)}(x), \left( f^{(l)}, W(x, \cdot) \right) \right) \in \mathbb{R}^{d_{l+1}}. \quad (3)$$

## Examples (continuous equivalents of 1, 2, 3 and 4) :

- a. Convolution :

$$f^{(l+1)}(x) = \int_y W(x, y) \psi^{(l+1)}(f^{(l)}(y)) dP.$$

b. Max Convolution :

$$f^{(l+1)}(x) = \text{esssup}_P W(x, \cdot) \psi^{(l+1)}(f^{(l)}(\cdot)).$$

c. Degree Normalized Convolution :

$$f^{(l+1)}(x) = \int_y \frac{W(x,y)}{\int_t W(x,t) dP(t)} \psi^{(l+1)}(f^{(l)}(y)) dP.$$

d. GAT :

$$f^{(l+1)}(x) = \int_y \frac{c^{(l+1)}(x,y)}{\int_t c^{(l+1)}(x,t) dP(t)} \psi^{(l+1)}(f^{(l)}(y)) dP.$$

#### 4. CONVERGENCE ON DENSE RANDOM GRAPHS

We are interested in a MPGNN on a random graph  $G_n$  drawn from  $(W, P)$ . For such a discrete MPGNN, we shall define its continuous counterpart on  $(W, P)$ . Omitting technical details

**Definition 4.1.** We say that  $\mathcal{F}$  is the continuous counterpart of  $F$  if for any  $f$ , for any  $P$ ,

$$x \mapsto \mathcal{F}_P(f(x), (f, W(x, \cdot)))$$

is the uniform limit, as  $n$  tends to infinity, of

$$x \mapsto \mathbb{E}[F(f(x), \{(f(X_k), W(x, X_k))\}_{1 \leq k \leq n})]$$

where  $X_k \stackrel{iid}{\sim} P$ . The rate of this convergence is called  $a_n$ .

For each example calculate 1, 2, 3 or 4, we prove that the continuous equivalent is indeed a, b, c or d, and we calculate the corresponding  $a_n$ .

Denote  $X = (X_1, \dots, X_n)$  the vertices of  $G_n$  and  $S_X$  the sampling operator at  $X$ . For an input function  $f$ ,  $S_X(f)$  is a signal on  $G_n$ . To compare the output of this signal through the discrete network to the output of  $f$  through the continuous counterpart, we shall use the Maximum Absolute Error (MAE) defined by :

$$MAE(f) = \max_i \left\| (S_X(f))_i^{(L)} - f^{(L)}(X_i) \right\|_{\infty}.$$

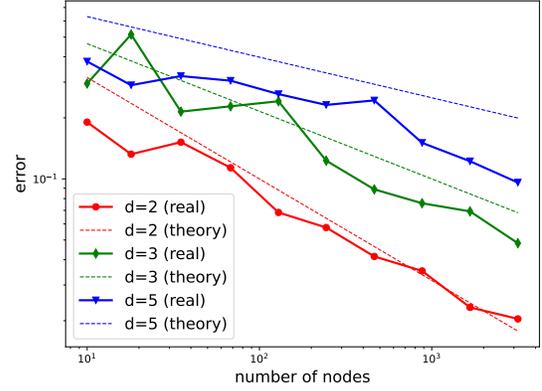
Our main result is a sufficient condition for this MAE to be arbitrary small with high probability. We provide a non asymptotic bound with high probability that relies on the so called McDiarmid inequality.

**Theorem 4.1.** Call  $D_n^{(l)}$  the bounded differences of the  $F^{(l)}$  and  $D_n$  their maximum over  $l$ . Call  $d_{max}$  the maximum of the  $d_l$  over  $l$ . Then, up to some regularity assumptions on the  $F^{(l)}$ , for any  $\rho > 0$ , with probability at least  $1 - \rho$  :

$$MAE(f) \lesssim LD_n \sqrt{n \ln \left( \frac{n 2^L d_{max}}{\rho} \right)} + a_n. \quad (4)$$

Where  $\lesssim$  hides multiplicative constants independent of  $n$ .

This bound suggests that if the bounded differences are sharp enough, typically  $D_n = o(1/\sqrt{n \ln n})$ , the MAE is small.



**Fig. 1:** illustration of (5) for different values of  $d$ , the dashed line represented  $(1/n)^{1/d}$  and the plain line the MAE. The plot is log scaled.

**Corollary 4.1.1.** If  $D_n = o(1/\sqrt{n \ln n})$  then

$$\max_i \left\| (S_X(f))_i^{(L)} - f^{(L)}(X_i) \right\|_{\infty} \rightarrow 0 \text{ in probability.}$$

We check this condition on the examples. It appears that it is verified by 1, 3 and 4 with a  $D_n = O(1/n)$  but not by 2. Nevertheless, the convergence remains true for that case with a different bound.

**Theorem 4.2.** For a Max Conv. MPGNN, assume that  $P \geq C\lambda$  for some  $C > 0$  where  $\lambda$  is the Lebesgue measure. Then for any  $\rho > 0$ , we have with probability at least  $1 - \rho$ ,

$$MAE(f) \lesssim L \left( \frac{1}{n} \ln \left( \frac{2^{L-1} n d_{max}}{\rho} \right) \right)^{1/d}. \quad (5)$$

This bound indicates a different behaviour when the aggregation is a maximum. It depends on the input space's dimension and decreases significantly slower than (4) for a large  $d$ .

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