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A manufactured solution for a steady capillary free surface flow governed by the Navier-Stokes equations

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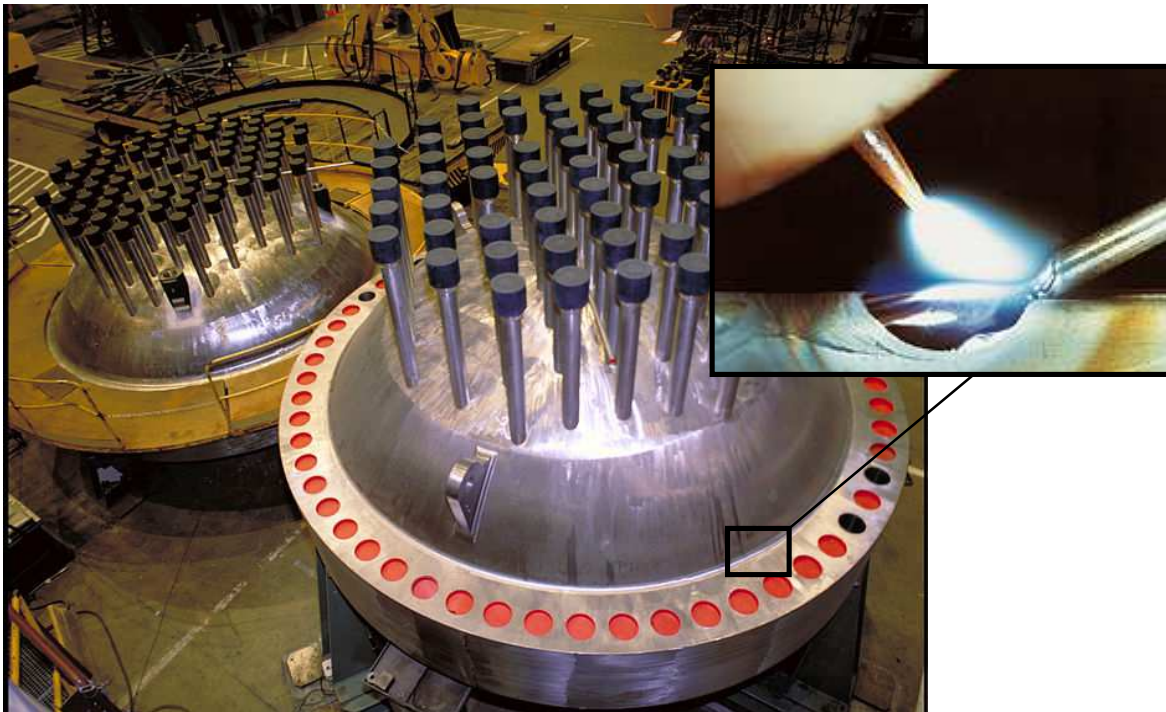
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May 20, 2013



Context : Numerical modeling of welding processes (I)

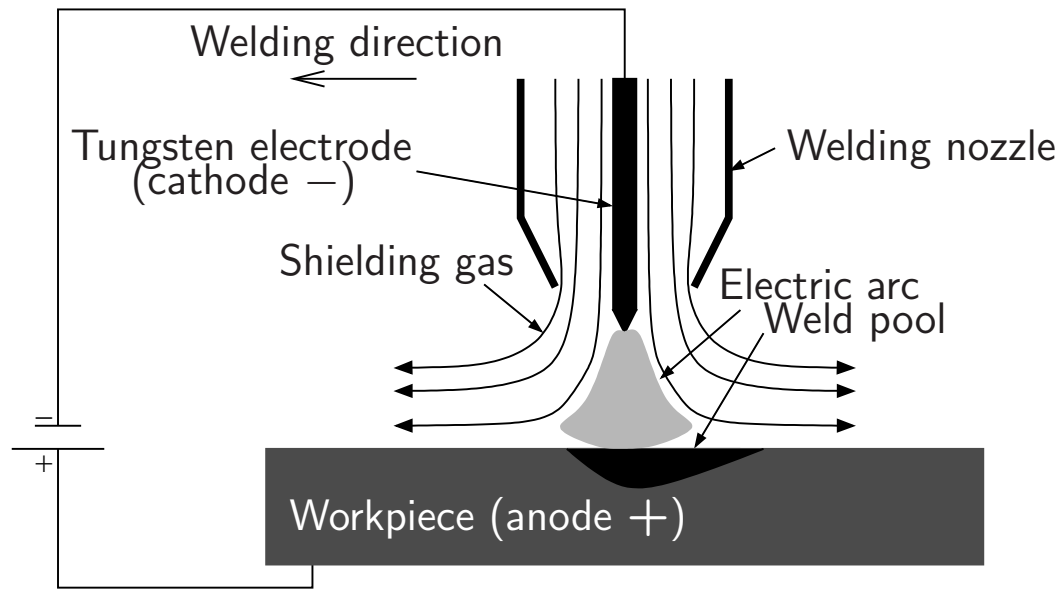


PWR Reactor Vessel Head ©NRC

- Nuclear components (vessel);
- Tungsten Inert Gas (TIG) Arc welding;
- Process \Rightarrow component aging;
- Interest in numerical modeling: MUSICAS national project (CEA, AREVA, PSA,...).

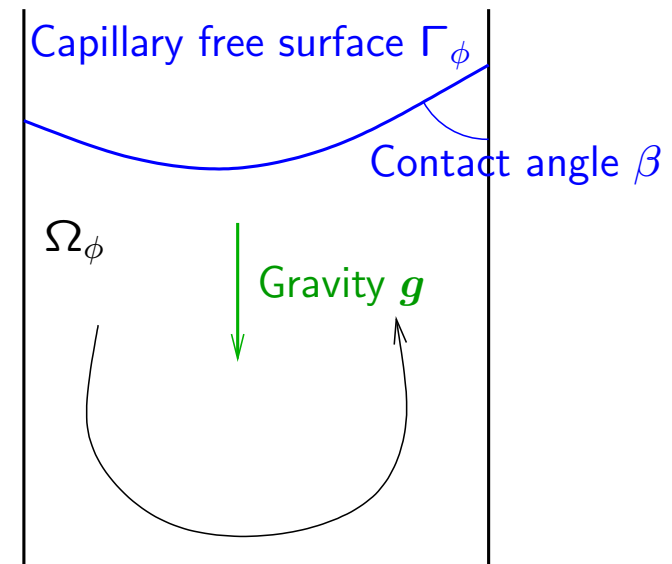
Context : Verification of the code (II)

Real Physics



TIG Welding (from [Brochard, 2009])

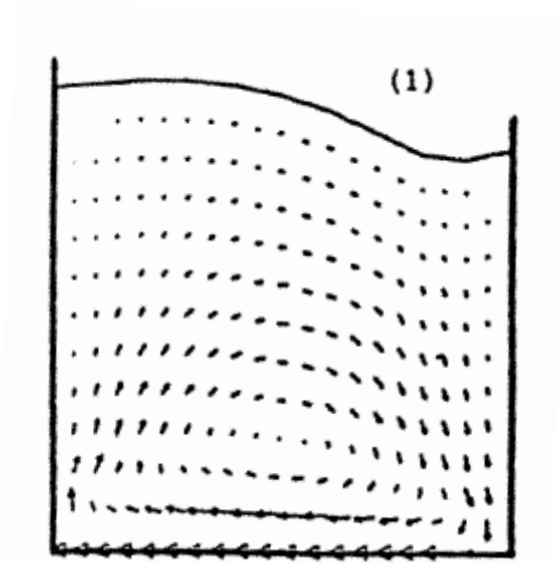
Simplified model



- Interaction focus: driven flow, free capillary surface;
- Important hypothesis: 2D steady-state.

Preliminary work : bibliography (I)

Reference article: [Cuvelier and Schulkes, 1990] Cuvelier, C. and Schulkes, R. (1990). Some numerical methods for the computation of capillary free boundaries governed by the Navier-Stokes equation. *SIAM Review*, 32(3):355–423

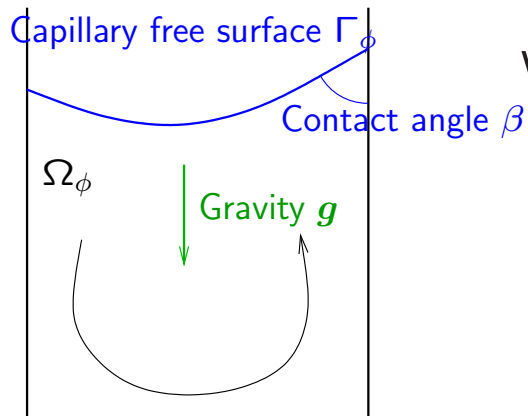


From [Cuvelier and Schulkes, 1990]

- + Very good review article;
- Cannot match his and our numerical solutions ;
- ⇒ Method of manufactured solutions.

Simplified Model : Equations (I)

{	$(\nabla \mathbf{u}) \cdot \mathbf{u} = -\nabla p_h + \frac{1}{\text{Re}} \nabla \cdot \boldsymbol{\tau}$	Force balance on Ω_ϕ
	$\nabla \cdot \mathbf{u} = 0$	Mass balance on Ω_ϕ
	$\mathbf{u} = \mathbf{u}_0$	Speed Dirichlet BC on $\delta\Omega_\phi \setminus \Gamma_\phi$
	$\mathbf{u} \cdot \mathbf{n} = 0$	Kinematic BC (mass balance on Γ_ϕ)
	$(\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{t} = 0$	Tangential force balance on Γ_ϕ
	$(\text{OhRe})^2 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n} = \frac{1}{\text{Re}_\phi} - \text{Bo} + \lambda_V$	Normal force balance on Γ_ϕ
	$V = V_0$	Volume constraint on Γ_ϕ
$\beta = \beta_0$	Tangential Force balance on $\delta\Gamma$	



with:

$\text{Re} = \rho U L \mu^{-1}$	Reynolds: inertia / viscosity
$\text{Oh} = \mu / \sqrt{\rho \gamma L}$	Ohnesorge: viscosity / $\sqrt{\text{inertia} \times \text{surface tension}}$
$(\text{OhRe})^2 = \text{We}$	Weber: inertia / surface tension
$\text{Bo} = g \rho L^2 \gamma^{-1}$	Bond: gravity / surface tension
$\boldsymbol{\sigma} = -p \mathbf{I} + \frac{1}{\text{Re}} \boldsymbol{\tau}$	
$\boldsymbol{\tau} = \nabla \mathbf{u} + \nabla^t \mathbf{u}$	

Driven lid $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Simplified Model : Numerical method (II)

- Trial Method [Cuvelier and Schulkes, 1990] \Leftrightarrow splitting method :

$$\begin{array}{l}
 \text{First step: sur-} \\
 \text{face } \Gamma_\phi \text{ kept fixed}
 \end{array}
 \left\{ \begin{array}{l}
 (\nabla \mathbf{u}) \cdot \mathbf{u} = -\nabla p_h + \frac{1}{\text{Re}} \nabla \cdot \boldsymbol{\tau} \\
 \nabla \cdot \mathbf{u} = 0 \\
 \mathbf{u} = \mathbf{u}_0 \\
 \mathbf{u} \cdot \mathbf{n} = 0 \\
 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{t} = 0
 \end{array} \right.$$

$$\begin{array}{l}
 \text{Second step: } \mathbf{u} \\
 \text{and } p_h \text{ (in } \boldsymbol{\sigma} \text{) kept} \\
 \text{fixed}
 \end{array}
 \left\{ \begin{array}{l}
 (\text{OhRe})^2 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n} = \frac{1}{R_\phi} - \text{Bo}_y + \lambda_V \\
 V = V_0 \\
 \beta = \beta_0
 \end{array} \right.$$

Two well-posed problems, converges well when surface tension effects are strong.

- Finite Element method $\mathcal{Q}_2\text{-}\mathcal{P}_1^{\text{nc}}$ [Cast3M, 2013].

Simplified Model : Method of Manufactured Solutions (III)

Review article: [Roy, 2005] and books: [Roache, 1998], [Oberkampf and Roy, 2010]

1. True problem

$$P \begin{pmatrix} \phi \\ \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}$$

2. Choose an arbitrary value for the unknown

$$P \begin{pmatrix} \phi_a \\ \mathbf{u}_a \\ p_a \end{pmatrix} = \begin{pmatrix} s_\phi \\ s_u \\ s_p \end{pmatrix}$$

3. Solve the discretized problem and verify convergence to **a**

$$P_d \begin{pmatrix} \phi_d \\ \mathbf{u}_d \\ p_d \end{pmatrix} = \begin{pmatrix} s_\phi \\ s_u \\ s_p \end{pmatrix}$$

$$\left\{ \begin{array}{l} \textcircled{2} \quad \begin{array}{l} (\nabla \mathbf{u}) \cdot \mathbf{u} + \nabla p - \frac{1}{\text{Re}} \nabla \cdot \boldsymbol{\tau} = s_u \\ \nabla \cdot \mathbf{u} = s_p \\ \mathbf{u} = \mathbf{u}_a \\ \mathbf{u} \cdot \mathbf{n} = 0 \end{array} \\ \textcircled{3} \quad \begin{array}{l} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{t} = s_t \\ (\text{OhRe})^2 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n} - \frac{1}{R_\phi} + \text{Bo}_y = 0 \end{array} \\ \textcircled{1} \quad V = V_0 \\ \textcircled{1} \quad \beta = \beta_0 \end{array} \right.$$

①: $\phi(x)$, ②: $\mathbf{u}(x, y)$, ③: $p(x)$.

Simplified Model : Method of Manufactured Solutions (IV)

$$\phi(x) = \alpha \cos\left(\frac{\pi x}{2}\right) - \frac{2\alpha}{\pi} + 1$$

$$u(x, y) = \left(y - \alpha \cos\left(\frac{\pi x}{2}\right) + \frac{2\alpha}{\pi} - 1\right)^2$$

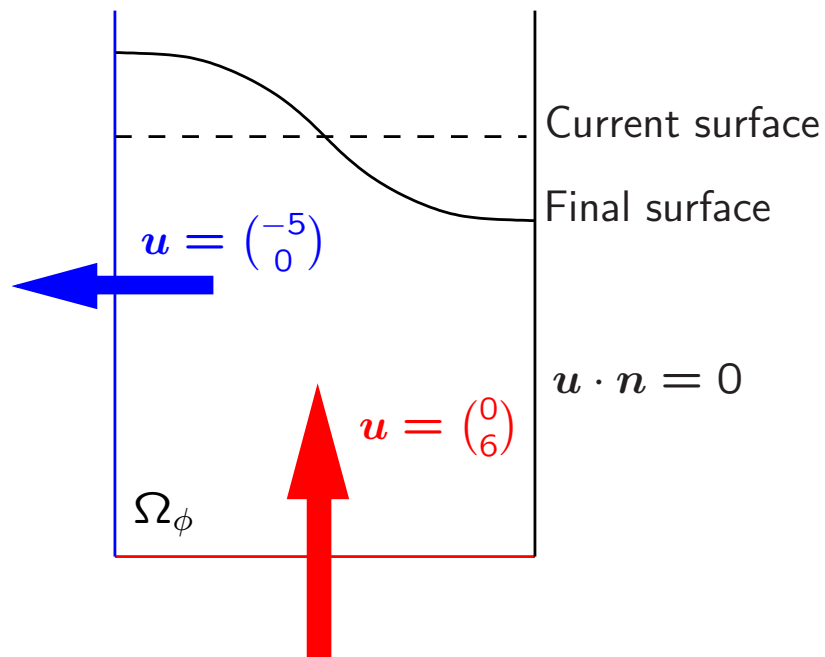
$$v(x, y) = \frac{\pi \alpha \sin\left(\frac{\pi x}{2}\right) \left(y - \alpha \cos\left(\frac{\pi x}{2}\right) + \frac{2\alpha}{\pi} - 1\right)^2}{2}$$

$$p(x) = \frac{\left(\left(\pi^3 \alpha^3 Bo \cos\left(\frac{\pi x}{2}\right) + \left(\pi^3 \alpha^2 - 2\pi^2 \alpha^3\right) Bo\right) \sin^2\left(\frac{\pi x}{2}\right) + 4\pi \alpha Bo \cos\left(\frac{\pi x}{2}\right) + (4\pi - 8\alpha) Bo\right) + 2\pi^3 \alpha \cos\left(\frac{\pi x}{2}\right)}{\left(\pi^3 \alpha^2 Oh^2 Re^2 \sin^2\left(\frac{\pi x}{2}\right) + 4\pi Oh^2 Re^2\right)}$$

$$s_u = \dots$$

Source terms computed with the Computer Algebra System (CAS) Maxima. Output to fortran and T_EX.

Difficulties : the divergence constraint (I)



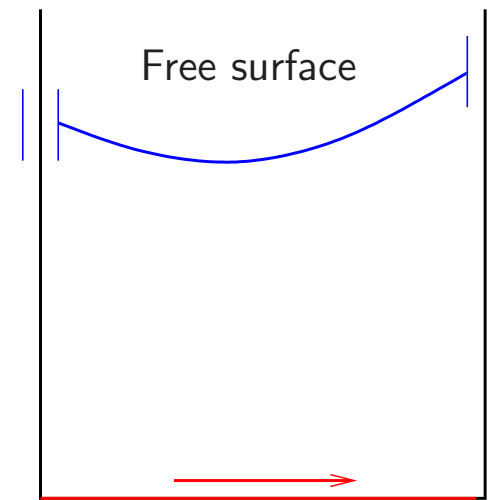
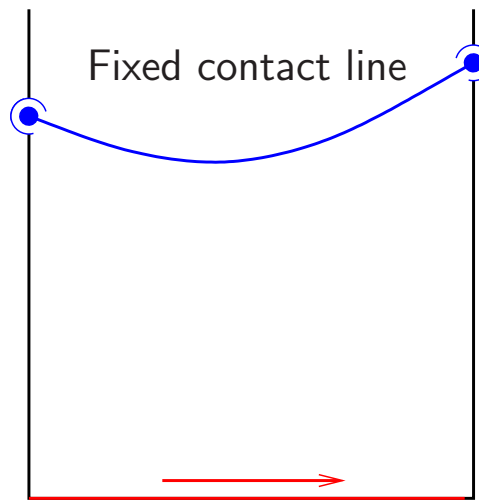
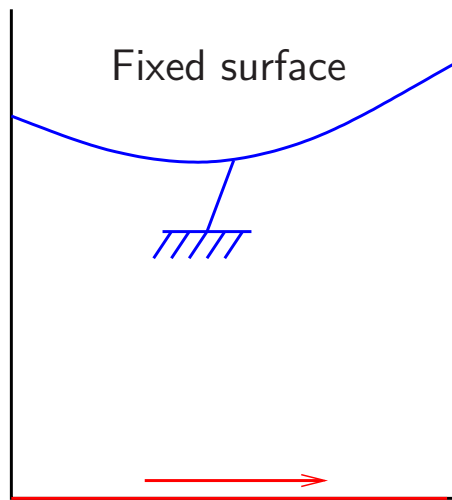
- The divergence constraint couples speed \mathbf{u} and surface position ϕ :

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \int_{\delta\Omega_\phi} \mathbf{u} \cdot \mathbf{n} \, d\delta\Omega_\phi = 0$$

- The trial method uncouples them!
- One possible solution: relax the divergence constraint.

$$\nabla \cdot \mathbf{u} = B \quad \text{with} \quad B = \int_{\delta\Omega_\phi} \mathbf{u} \cdot \mathbf{n} \, d\delta\Omega_\phi$$

Difficulties : the tangential force balance at the contact line (II)



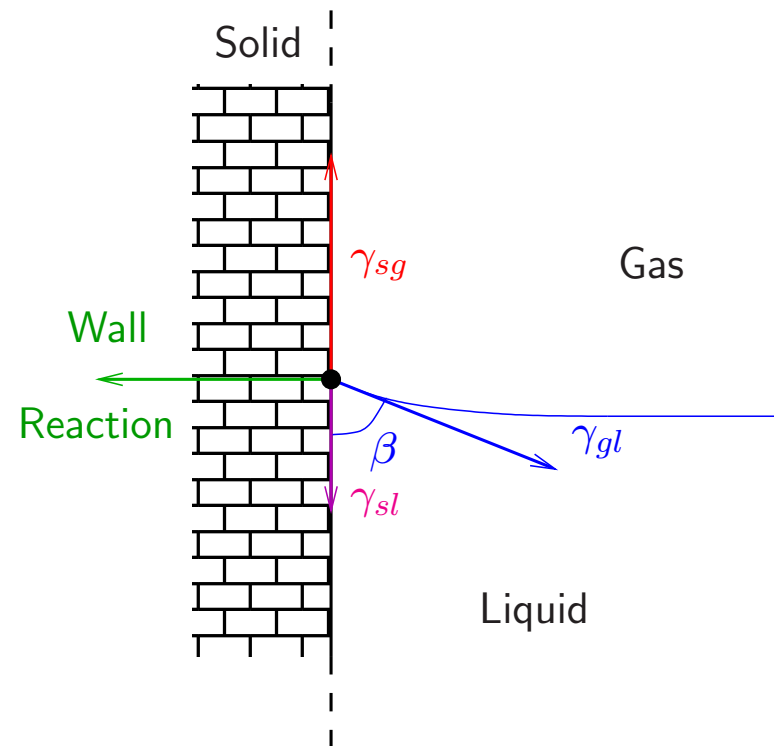
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u	4 (3)
p	2 (2)

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u	4
p	2

Unknown	Cvg order
u	1
p	1

Difficulties : the tangential force balance at the contact line (III)

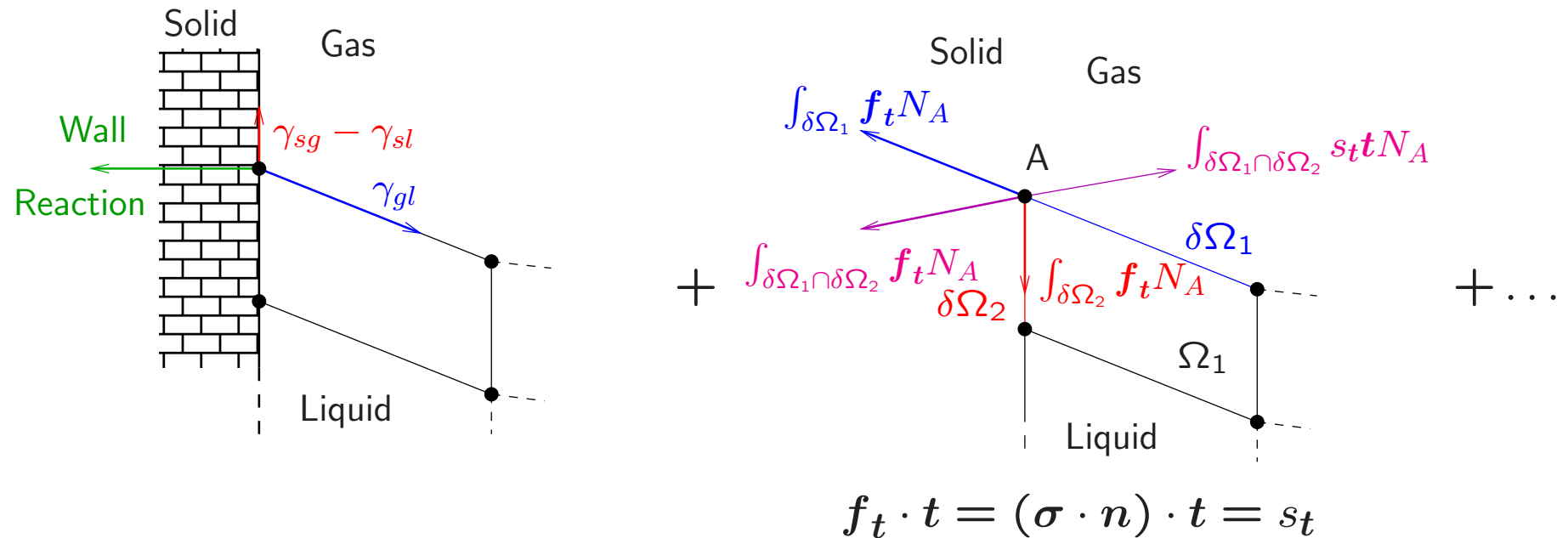
Continuous setting (2D)



$$\text{Dupre-Young Law } (\gamma \text{ in } N.m^{-1}): \cos \beta = \frac{\gamma_{sl} - \gamma_{sg}}{\gamma_{gl}}$$

Difficulties : the tangential force balance at the contact line (IV)

Discrete setting (2D finite element)



Discrete Dupre–Young Law (Weak Neumann BC at A):

$$\underbrace{\gamma_{gl} \cos \beta - \gamma_{sl} + \gamma_{sg}}_{\text{Lineic force terms}} + \underbrace{\int_{\delta\Omega_1 \cap \delta\Omega_2} \dots}_{\text{Surfacic force terms in } O(h)} + \underbrace{\int_{\Omega_1} \dots}_{\text{Volumic force terms in } O(h^2)} = 0$$

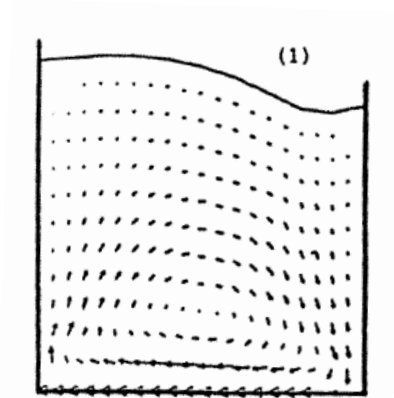
Conclusion

Results obtained:

- a manufactured solution for a capillary free-surface flow;
- bring out two difficulties related to the MMS and the trial algorithm;
- optimal order found numerically.

Perspectives:

- we are now in a better position to trust the code...
- ...and propose verified solutions to Cuvelier's test cases.



The end

Thank you for your attention.

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