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# A manufactured solution for a steady capillary free surface flow governed by the Navier-Stokes equations

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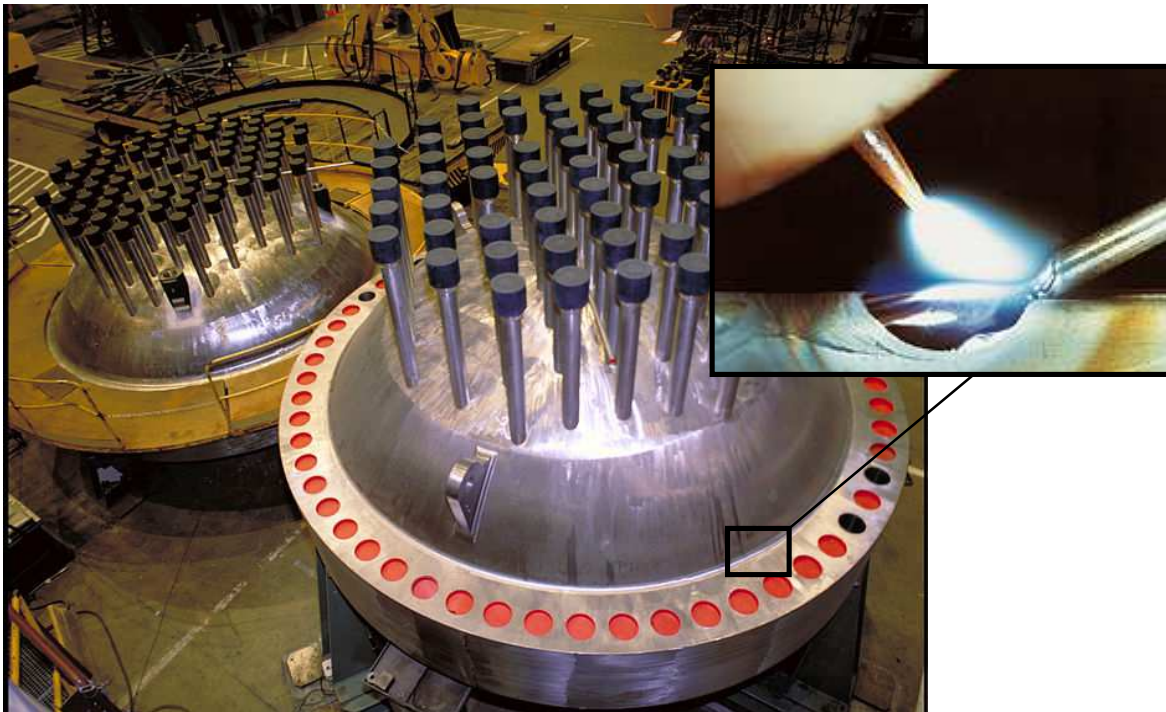
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May 20, 2013



## *Context : Numerical modeling of welding processes (I)*

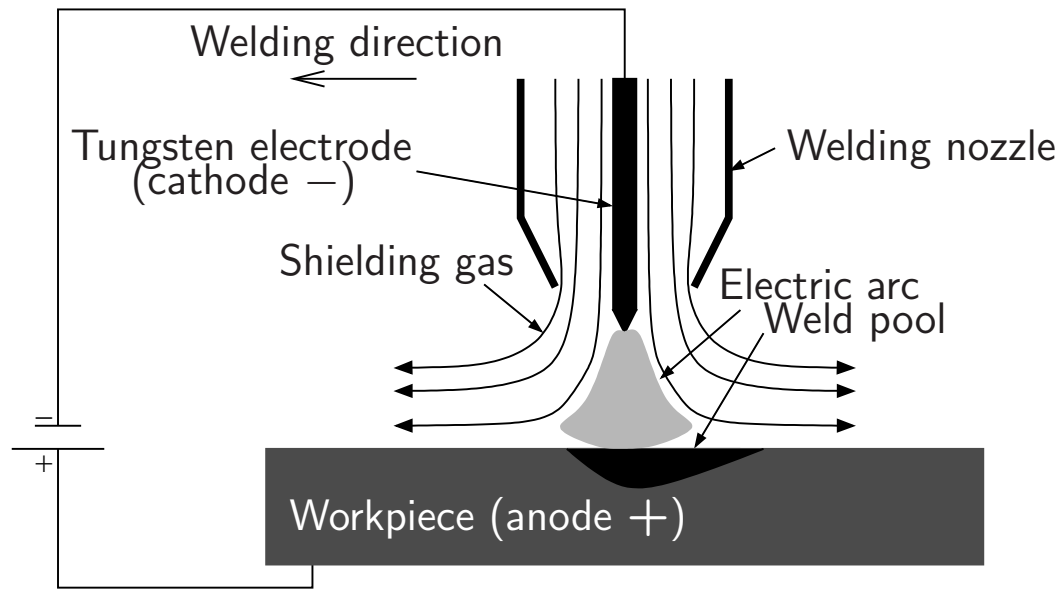


PWR Reactor Vessel Head ©NRC

- Nuclear components (vessel);
- Tungsten Inert Gas (TIG) Arc welding;
- Process  $\Rightarrow$  component aging;
- Interest in numerical modeling: MUSICAS national project (CEA, AREVA, PSA, ...).

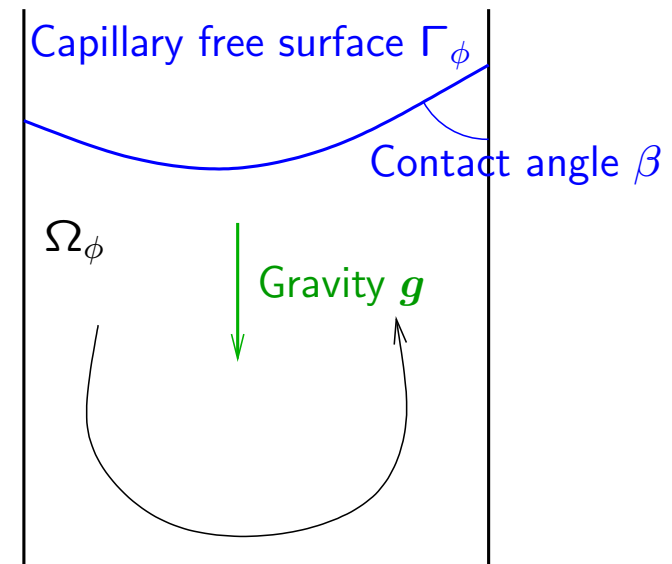
## Context : Verification of the code (II)

### Real Physics



TIG Welding (from [Brochard, 2009])

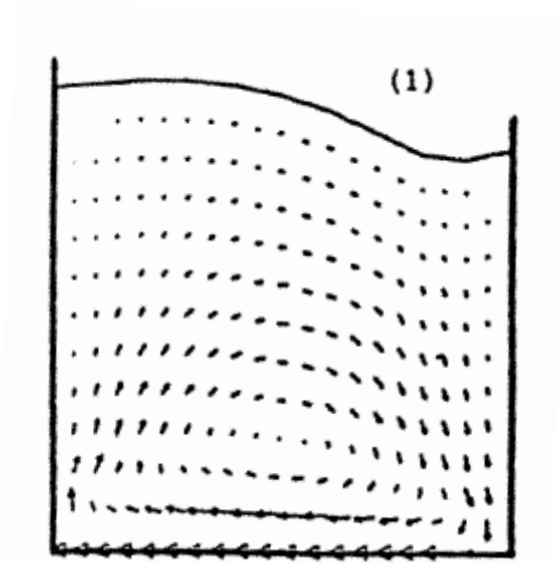
### Simplified model



- Interaction focus: driven flow, free capillary surface;
- Important hypothesis: 2D steady-state.

## Preliminary work : bibliography (I)

Reference article: [Cuvelier and Schulkes, 1990] Cuvelier, C. and Schulkes, R. (1990). Some numerical methods for the computation of capillary free boundaries governed by the Navier-Stokes equation. *SIAM Review*, 32(3):355–423

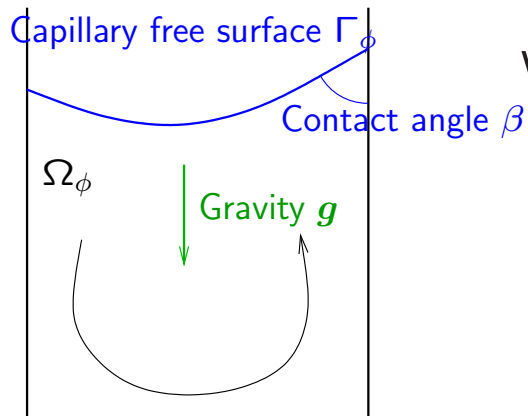


From [Cuvelier and Schulkes, 1990]

- + Very good review article;
- Cannot match his and our numerical solutions ;
- ⇒ Method of manufactured solutions.

# Simplified Model : Equations (I)

$$\left\{ \begin{array}{ll}
 (\nabla \mathbf{u}) \cdot \mathbf{u} = -\nabla p_h + \frac{1}{\text{Re}} \nabla \cdot \boldsymbol{\tau} & \text{Force balance on } \Omega_\phi \\
 \nabla \cdot \mathbf{u} = 0 & \text{Mass balance on } \Omega_\phi \\
 \mathbf{u} = \mathbf{u}_0 & \text{Speed Dirichlet BC on } \delta\Omega_\phi \setminus \Gamma_\phi \\
 \mathbf{u} \cdot \mathbf{n} = 0 & \text{Kinematic BC (mass balance on } \Gamma_\phi) \\
 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{t} = 0 & \text{Tangential force balance on } \Gamma_\phi \\
 (\text{OhRe})^2 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n} = \frac{1}{R_\phi} - \text{Bo} + \lambda_V & \text{Normal force balance on } \Gamma_\phi \\
 V = V_0 & \text{Volume constraint on } \Gamma_\phi \\
 \beta = \beta_0 & \text{Tangential Force balance on } \delta\Gamma
 \end{array} \right.$$



with:

$$\begin{array}{ll}
 \text{Re} = \rho U L \mu^{-1} & \text{Reynolds: inertia / viscosity} \\
 \text{Oh} = \mu / \sqrt{\rho \gamma L} & \text{Ohnesorge: viscosity / } \sqrt{\text{inertia} \times \text{surface tension}} \\
 (\text{OhRe})^2 = \text{We} & \text{Weber: inertia / surface tension} \\
 \text{Bo} = g \rho L^2 \gamma^{-1} & \text{Bond: gravity / surface tension} \\
 \boldsymbol{\sigma} = -p \mathbf{I} + \frac{1}{\text{Re}} \boldsymbol{\tau} & \\
 \boldsymbol{\tau} = \nabla \mathbf{u} + \nabla^t \mathbf{u} &
 \end{array}$$

Driven lid  $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

## Simplified Model : Numerical method (II)

- Trial Method [Cuvelier and Schulkes, 1990]  $\Leftrightarrow$  splitting method :

$$\begin{array}{l}
 \text{First step: sur-} \\
 \text{face } \Gamma_\phi \text{ kept fixed}
 \end{array}
 \left\{ \begin{array}{l}
 (\nabla \mathbf{u}) \cdot \mathbf{u} = -\nabla p_h + \frac{1}{\text{Re}} \nabla \cdot \boldsymbol{\tau} \\
 \nabla \cdot \mathbf{u} = 0 \\
 \mathbf{u} = \mathbf{u}_0 \\
 \mathbf{u} \cdot \mathbf{n} = 0 \\
 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{t} = 0
 \end{array} \right.$$
  

$$\begin{array}{l}
 \text{Second step: } \mathbf{u} \\
 \text{and } p_h \text{ (in } \boldsymbol{\sigma} \text{) kept} \\
 \text{fixed}
 \end{array}
 \left\{ \begin{array}{l}
 (\text{OhRe})^2 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n} = \frac{1}{R_\phi} - \text{Bo}_y + \lambda_V \\
 V = V_0 \\
 \beta = \beta_0
 \end{array} \right.$$

Two well-posed problems, converges well when surface tension effects are strong.

- Finite Element method  $\mathcal{Q}_2\text{-}\mathcal{P}_1^{\text{nc}}$  [Cast3M, 2013].

# Simplified Model : Method of Manufactured Solutions (III)

Review article: [Roy, 2005] and books: [Roache, 1998], [Oberkampf and Roy, 2010]

1. True problem

$$P \begin{pmatrix} \phi \\ \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix}$$

2. Choose an arbitrary value for the unknown

$$P \begin{pmatrix} \phi_a \\ \mathbf{u}_a \\ p_a \end{pmatrix} = \begin{pmatrix} s_\phi \\ s_u \\ s_p \end{pmatrix}$$

3. Solve the discretized problem and verify convergence to  $\mathbf{a}$

$$P_d \begin{pmatrix} \phi_d \\ \mathbf{u}_d \\ p_d \end{pmatrix} = \begin{pmatrix} s_\phi \\ s_u \\ s_p \end{pmatrix}$$

$$\left\{ \begin{array}{l} \textcircled{2} \quad (\nabla \mathbf{u}) \cdot \mathbf{u} + \nabla p - \frac{1}{\text{Re}} \nabla \cdot \boldsymbol{\tau} = s_u \\ \quad \quad \quad \nabla \cdot \mathbf{u} = s_p \\ \quad \quad \quad \mathbf{u} = \mathbf{u}_a \\ \quad \quad \quad \mathbf{u} \cdot \mathbf{n} = 0 \\ \textcircled{3} \quad (\sigma \cdot \mathbf{n}) \cdot \mathbf{t} = s_t \\ \textcircled{1} \quad (\text{OhRe})^2 (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{n} - \frac{1}{R_\phi} + \text{Bo}_y = 0 \\ \textcircled{1} \quad V = V_0 \\ \quad \quad \beta = \beta_0 \end{array} \right.$$

①:  $\phi(x)$ , ②:  $\mathbf{u}(x, y)$ , ③:  $p(x)$ .



## Simplified Model : Method of Manufactured Solutions (IV)

$$\phi(x) = \alpha \cos\left(\frac{\pi x}{2}\right) - \frac{2\alpha}{\pi} + 1$$

$$u(x, y) = \left(y - \alpha \cos\left(\frac{\pi x}{2}\right) + \frac{2\alpha}{\pi} - 1\right)^2$$

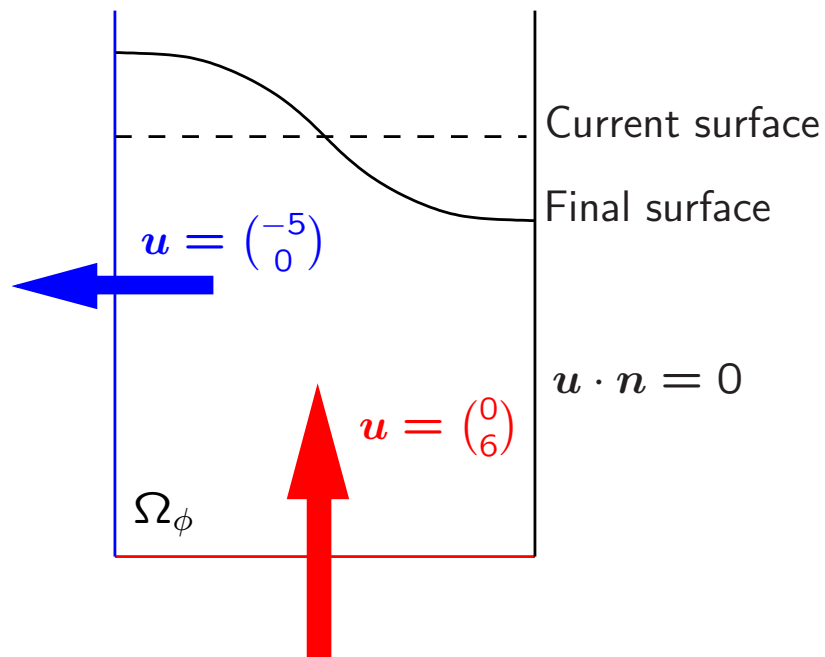
$$v(x, y) = \frac{\pi \alpha \sin\left(\frac{\pi x}{2}\right) \left(y - \alpha \cos\left(\frac{\pi x}{2}\right) + \frac{2\alpha}{\pi} - 1\right)^2}{2}$$

$$p(x) = \frac{\left(\left(\pi^3 \alpha^3 Bo \cos\left(\frac{\pi x}{2}\right) + \left(\pi^3 \alpha^2 - 2\pi^2 \alpha^3\right) Bo\right) \sin^2\left(\frac{\pi x}{2}\right) + 4\pi \alpha Bo \cos\left(\frac{\pi x}{2}\right) + (4\pi - 8\alpha) Bo\right) + 2\pi^3 \alpha \cos\left(\frac{\pi x}{2}\right)}{\left(\pi^3 \alpha^2 Oh^2 Re^2 \sin^2\left(\frac{\pi x}{2}\right) + 4\pi Oh^2 Re^2\right)}$$

$$s_u = \dots$$

Source terms computed with the Computer Algebra System (CAS) Maxima. Output to fortran and T<sub>E</sub>X.

## Difficulties : the divergence constraint (I)



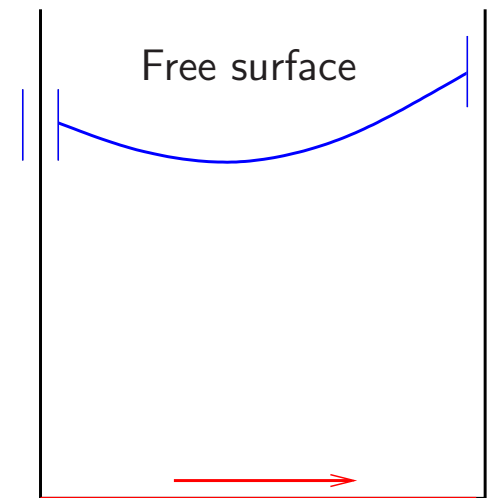
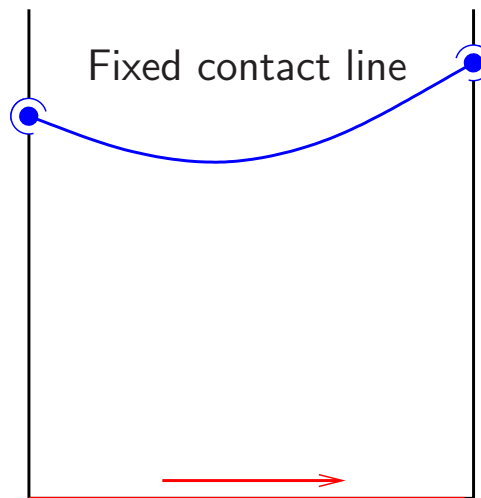
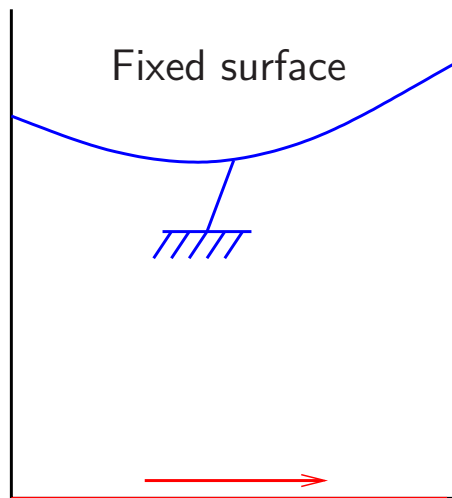
- The divergence constraint couples speed  $\mathbf{u}$  and surface position  $\phi$ :

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \int_{\delta\Omega_\phi} \mathbf{u} \cdot \mathbf{n} \, d\delta\Omega_\phi = 0$$

- The trial method uncouples them!
- One possible solution: relax the divergence constraint.

$$\nabla \cdot \mathbf{u} = B \quad \text{with} \quad B = \int_{\delta\Omega_\phi} \mathbf{u} \cdot \mathbf{n} \, d\delta\Omega_\phi$$

## *Difficulties : the tangential force balance at the contact line (II)*



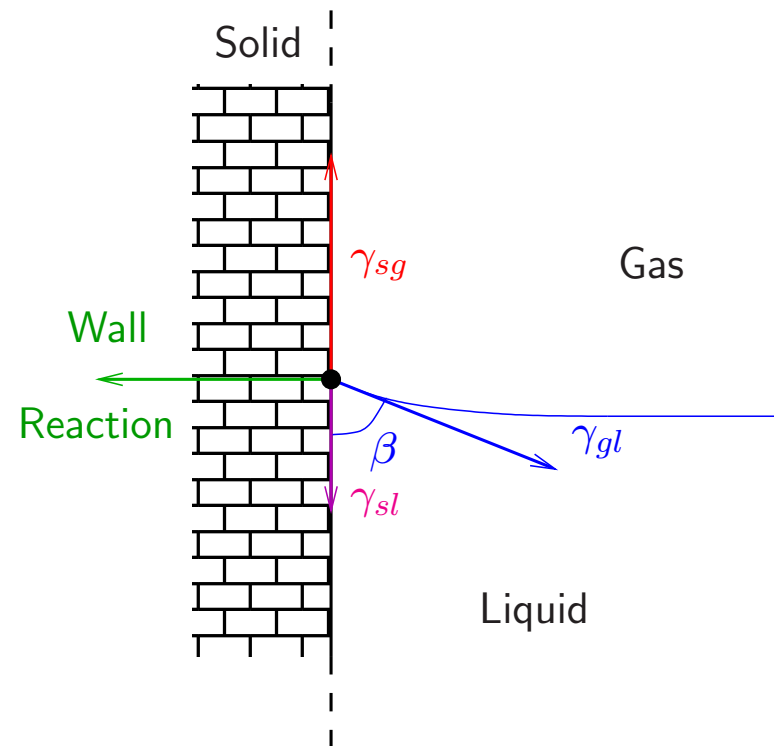
Unknown	Cvg order (theo.)
$u$	4 (3)
$p$	2 (2)

Unknown	Cvg order
$u$	4
$p$	2

Unknown	Cvg order
$u$	1
$p$	1

# Difficulties : the tangential force balance at the contact line (III)

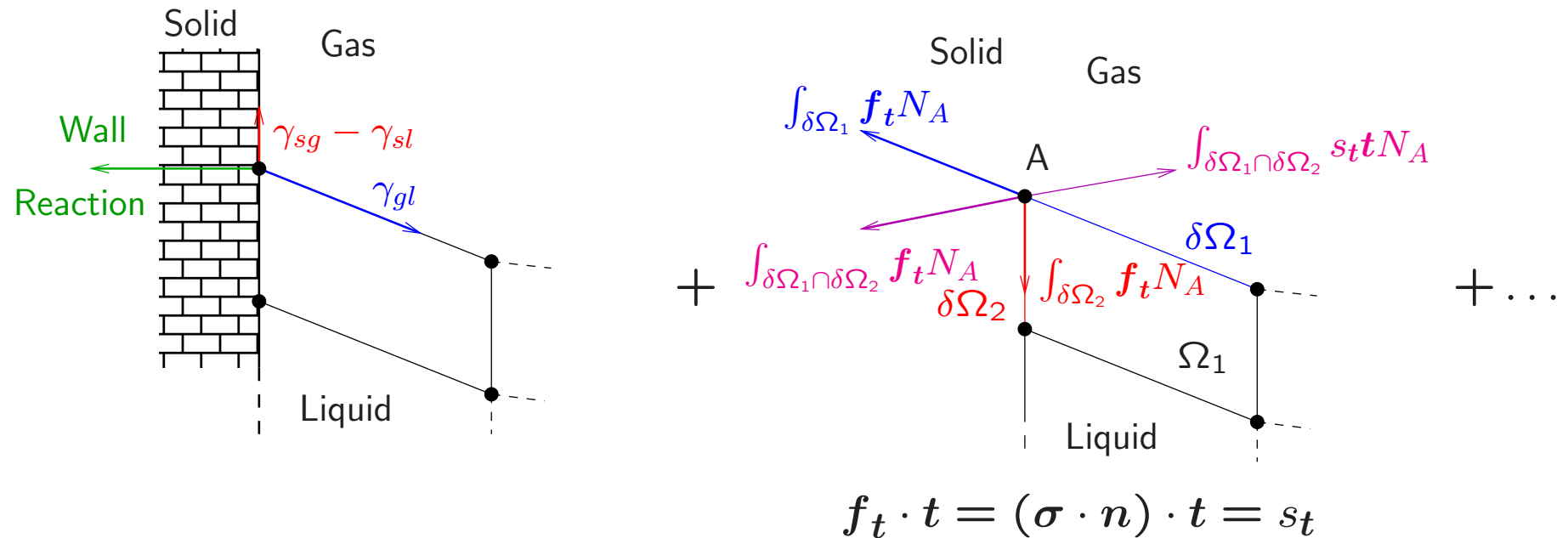
## Continuous setting (2D)



$$\text{Dupre-Young Law } (\gamma \text{ in } N.m^{-1}): \cos \beta = \frac{\gamma_{sl} - \gamma_{sg}}{\gamma_{gl}}$$

*Difficulties : the tangential force balance at the contact line (IV)*

**Discrete setting (2D finite element)**



Discrete Dupre–Young Law (Weak Neumann BC at  $A$ ):

$$\underbrace{\gamma_{gl} \cos \beta - \gamma_{sl} + \gamma_{sg}}_{\text{Lineic force terms}} + \underbrace{\int_{\delta\Omega_1 \cap \delta\Omega_2} \dots}_{\text{Surfacic force terms in } O(h)} + \underbrace{\int_{\Omega_1} \dots}_{\text{Volumic force terms in } O(h^2)} = 0$$

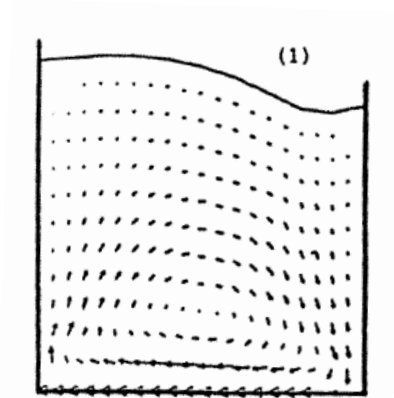
## Conclusion

### Results obtained:

- a manufactured solution for a capillary free-surface flow;
- bring out two difficulties related to the MMS and the trial algorithm;
- optimal order found numerically.

### Perspectives:

- we are now in a better position to trust the code...
- ...and propose verified solutions to Cuvelier's test cases.



*The end*

Thank you for your attention.

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