Verification of a CFD benchmark solution of transient low Mach number flows with Richardson extrapolation procedure

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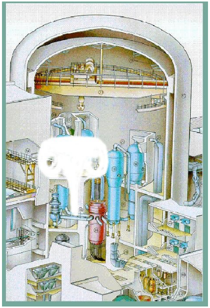
Motivation & Physical problem

Nuclear reactor safety: containment pressurization due to loss-of-coolant accident in nuclear reactor

- Fluid injection at low Mach number into an axisymmetric cavity
- Mixed convection flow (hot fluid injected in cold atmosphere)



• **H. Paillère** *et al.*, "Modelling of natural convection flows with large temperature differences : A Benchmark problem for low Mach number solvers. *ESAIM: Math. Mod. & Num. Anal.*, 2005; 39:617-621

• **A. Beccantini** *et al.*, Numerical simulations of transient injection flow at low Mach number regime. *Int. J. Numer. Meth. Engng* 2008; 76:662-696.

• **S. Benteboula**, Modeling of ideal gas injection into a closed domain at low Mach number: Fractional step and pressure based solvers. Tech. Report: CEA - SFME/LTMF/RT08-034/A 2008.

Objectives

1. Modeling flow injection with large density gradients caused by thermal effects
2. Computing numerical benchmark solution of the flow with two different CFD codes
3. Estimating numerical errors and the method convergence order
4. Applying Richardson extrapolation (R.E) to determine a reference solution

Low Mach number model

Compressibility effects are characterized by the Mach number $M = \dfrac{v}{c}$

1. **Compressible flow** $(M > 0.3)$: evolution equation for ρ, $\rho(p, T)$
2. **Incompressible flow** $(M \to 0)$: $\nabla \cdot v = 0$ constraint, ρ constant

3. **Low Mach flow** $M < 0.3$

   Low Mach formulation (A. Majda and J.Sethian (1985))

   Asymptotic analysis $\gamma M^2 \ll 1 \to$ Navier-Stokes equations of order zero

Asymptotic results

- Pressure decomposition: thermodynamic $P(t)$, dynamic $p''(x, t)$
- Density variations decoupled from pressure gradients, $\rho(T)$, $\nabla \cdot v \neq 0$

Advantages

- Allows large variations of ρ due to temperature gradients
- Removes acoustic waves

CFD codes modeling

Model 1: Conservative unsteady axisymmetric N-S equations

S. Benteboula, G. Lauriat,
Numerical simulations of anisothermal laminar vortex rings with large density variations.
Int. J. Heat & Fluid Flow 2009; 30:186-197.

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \\
\dfrac{\partial \rho \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla p'' + \dfrac{1}{Re} \nabla \cdot \underline{\tau} \\
\qquad\qquad\qquad\qquad + \dfrac{1}{\mathrm{Fr}} (\rho - \rho_0) \vec{z} \\
\dfrac{\partial T}{\partial t} + \gamma T \nabla \cdot \vec{u} - \dfrac{T}{\rho} \nabla \cdot (\rho \vec{u}) = \\
\qquad\qquad\qquad\qquad \dfrac{\gamma}{\mathrm{RePr}} \nabla \cdot (\lambda \nabla T) \\
P = \rho T
\end{cases}
$$

Model 2: Non-conservative unsteady axisymmetric N-S equations

A.Beccantini, et.al,
Numerical simulations of transient injection flow at low Mach number regime.
Int. J. Numer. Meth. Engng 2008; 76:662-696.

$$
\begin{cases}
\dfrac{1}{\gamma P} \dfrac{\mathrm{d}P}{\mathrm{d}t} + \nabla \cdot \vec{u} = \dfrac{\gamma - 1}{\gamma P} \nabla \cdot (\lambda \nabla T) \\
\dfrac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = \dfrac{RT}{P} \left(-\nabla p'' + \nabla \cdot \underline{\tau} \right) \\
\qquad\qquad\qquad\qquad + \vec{g} \left(1 - \dfrac{T}{T_0} \dfrac{P_0}{P} \right) \\
\dfrac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \dfrac{\gamma - 1}{\gamma} \dfrac{T}{P} \dfrac{\mathrm{d}P}{\mathrm{d}t} \\
\qquad\qquad\qquad\qquad + \dfrac{\gamma - 1}{\gamma} \dfrac{T}{P} \nabla \cdot (\lambda \nabla T) \\
P = r \rho T
\end{cases}
$$

Numerical method 1: FD1

- **Finite Difference** 2^{nd} centered
- Staggered grid (MAC)
- Continuous projection method
- Time integration: Explicit 2^{nd} order A-B/A-M predictor-corrector scheme

Numerical method 2: FE2

- **Finite Element** Q2/P1
- Regular mesh
- Algebraic projection (Quarteroni)
- Time integration: Implicit BDF2 scheme

Test case - Hot fluid injection

Physical parameters

- **Computational domain**

  $R_j = 0.5$ m, $L_z = 2$ m, $L_z = 7$ m.

- **Injected fluid**

  $T_j = 600$ K,

  $\dot{m}_j = 1.$ kg/m^2/s,

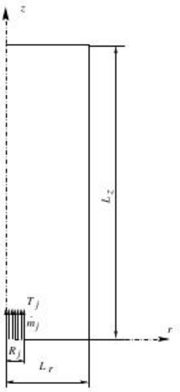
  $V_j = \dfrac{\dot{m}_j}{\rho_j} = 1.722$ m/s.

  $(\mathrm{Re} = 200, \mathrm{Pe} = 142$ and $\mathrm{Fr} = 0.302)$

  $t_{injection} = 6.$ s

- **Cavity atmosphere**

  $T_a = 300$ K, $P = 10^5$Pa.



Boundary conditions

- **Inlet**: $T = T_j$, $\rho u_z = \dfrac{6\dot{m}_j}{R_j^2}\left(R_j^2 - r^2\right)$.

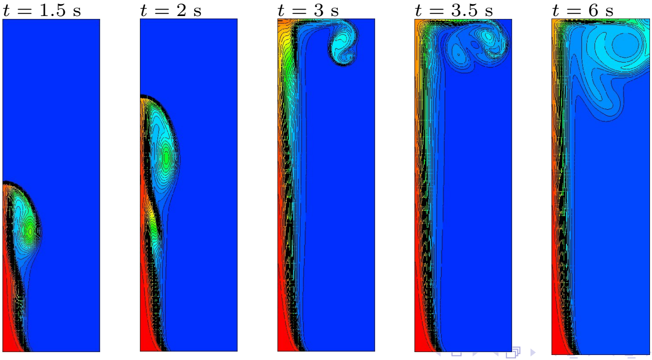
- **Walls**: $\dfrac{\partial T}{\partial n} = 0$, $\rho\vec{u} = 0$.

Numerical parameters & Results

Numerical parameters

Method	$N_r \times N_z$	Δt	Consistency order	User time
FD1	100×350	8.610E-04	2 for $\rho u_r, \rho u_z, T, p$	0.51 hour
	200×700	4.305E-04		5.45 hours
	400×1400	2.152E-04		54.53hours
FE2	108×378	1.075E-03	3 for u_r, u_z, T	
	144×504	8.065E-04	2 for p	
	192×672	6.049E-04		528 hours

Instantaneous temperature fields



Verification method

Richardson Extrapolation (RE): Assumptions

1. f_{exact} regular enough to apply Taylor expansion

$$f_h = f_{exact} + C_\alpha h^\alpha + O(h^{\alpha+1})$$

2. h_i small enough to get asymptotic convergence

$$C_\alpha h_i^\alpha \gg O(h^{\alpha+1})$$

3. Uniform space and time refinement, so for 3 grids $\frac{h_1}{h_2} = \frac{h_2}{h_3}$:

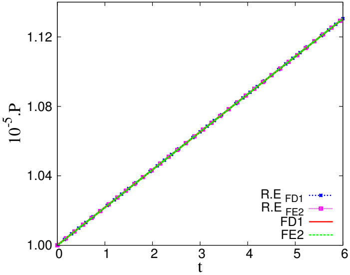
$$\widetilde{\alpha} = \ln\left(\frac{f_{h_1} - f_{h_2}}{f_{h_2} - f_{h_3}}\right) / \ln\left(\frac{h_1}{h_2}\right), \quad \widetilde{C}_\alpha = \frac{f_{h_2} - f_{h_3}}{h_2^{\widetilde{\alpha}} - h_3^{\widetilde{\alpha}}}, \quad \widetilde{f}^{ex} = f_{h_3} - \widetilde{C}_\alpha h_3^{\widetilde{\alpha}}$$

R.E advantages

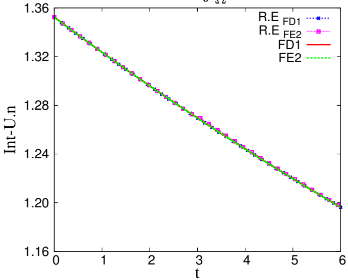
- **Increase of accuracy** by eliminating the truncation errors
- **Error estimate** corresponding to $\widetilde{C}_\alpha h_{fine}^{\widetilde{\alpha}}$
- **Method verification** by estimating the convergence orders

Results & Verifications: Time dependent quantities

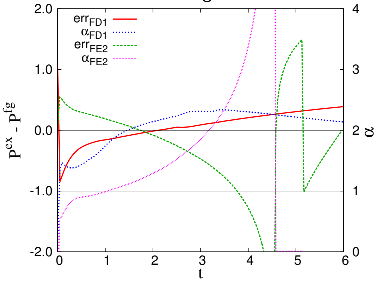
Thermodynamic pressure: $P = r\rho T$



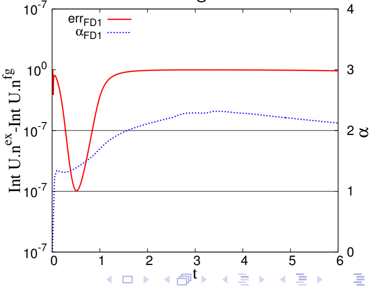
Divergence integral: $\int_\Omega (\nabla \cdot \mathbf{u})\ dV$



Errors & convergence order

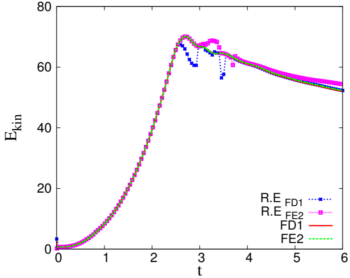


Errors & convergence order

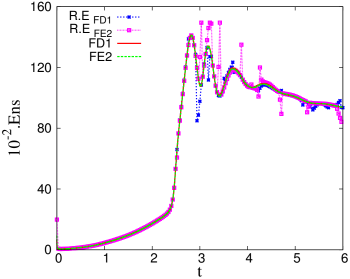


Results & Verifications: Time dependent quantities

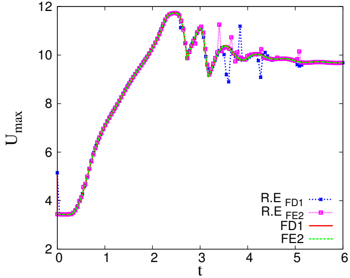
Kinetic energy: $\frac{1}{2}\int_{\Omega} \rho \mathbf{u}^2 \; dV$



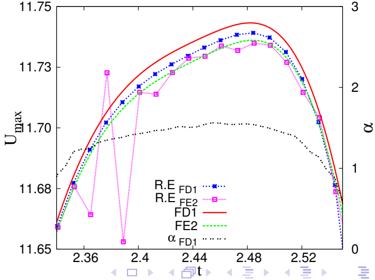
Enstrophy: $\int_{\Omega} (\nabla \mathbf{u})^2 \; dV$



Maximum velocity



Zoom on the maximum velocity



Results & Verifications

At $t = 2s$

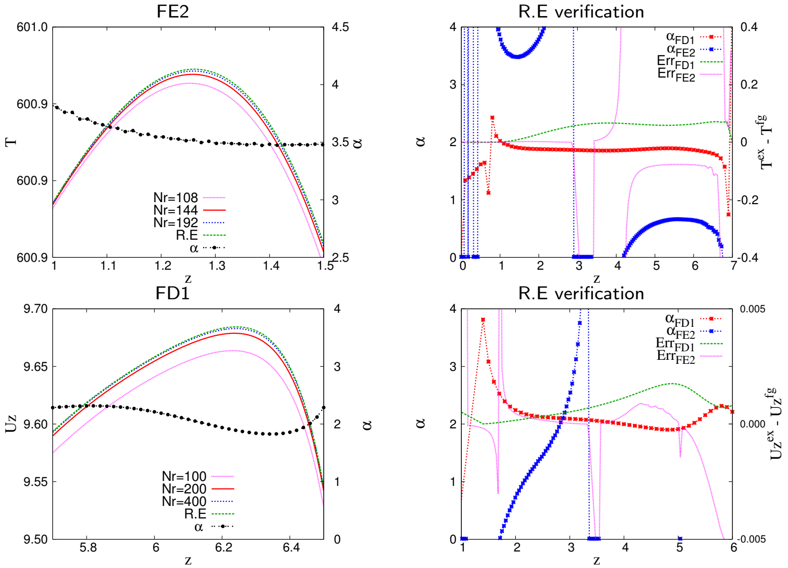
Method		P	$\int \nabla \cdot \mathbf{u}$	E_{kin}	Ens	U_{max}
FD1	fine grid	**104356**.1	**1.29**6001778	**39.3**3128	**172**8.252	**10.29**879
	R.E	104356.8	1.296001779	39.33629	1727.218	10.29893
	$\lvert\Delta f\rvert/f_{fg}$	6.71 E-6	7.72 E-10	1.27E-4	5.98E-4	1.36E-5
	α	**2.49**	**2.49**	1.91	1.45	(4.11)
FE2	fine grid	**104356**.6	**1.29**5949	**39.3**7172	**172**6.572	**10.29**514
	R.E	104357.6	1.295972	39.35677	1726.390	10.29805
	$\lvert\Delta f\rvert/f_{fg}$	3.83E-6	2.31E-5	3.80E-4	1.05E-4	1.82E-4
	α	1.29	1.10	1.65	2.16	1.83

Remarks

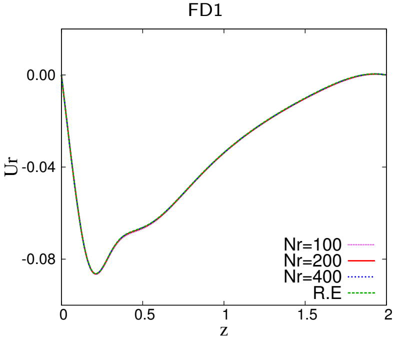
- Relative errors of the same order of magnitude for both **FD1** and **FE2** solver solutions
- **FD1** solver: convergence order given by R.E close to the consistency order of the scheme (2) obtained for P, $\int \nabla \cdot \mathrm{u}$ and E_{kin}
- **FE2** solver: convergence order given by R.E different from the consistency order of the scheme (3)

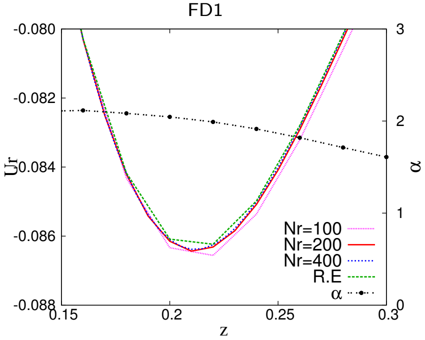
Results & Verifications: Local variables $(t = 6\ s)$

Temperature and axial velocity on the axis



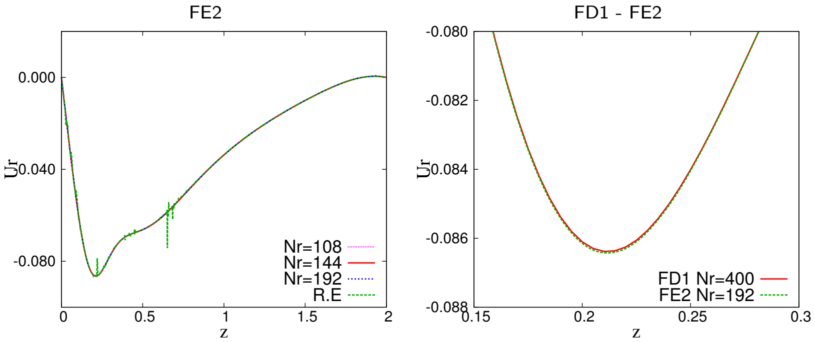
Radial velocity in the crosswise direction at $z = 3$





- R.E works better for variable profiles far enough from the inlet and the impact area (cavity ceiling)

Radial velocity in the crosswise direction at $z = 3$



- Fine grid solutions obtained with FD1 and FE2 solvers are in good agreement even if the R.E doesn't work evrywhere in the computational domain

Conclusions

1. Numerical solution of low Mach number fluid injection is computed with two CFD codes using:
   - Two different formulations of the low Mach number Navier-Stokes equations
   - Two different numerical methods
2. Very good agreement is obtained for local and global quantities characterizing the flow computed with **FD1** and **FE2** solvers
3. Difficulties in applying Richardson extrapolation on the whole computing domain since the physical problem doesn't satisfy the R.E assumptions:
   - Stiffness in time: impulsive injection s
   - Stiffness in space: large gradients in particular at the jet impact regions
   - Discontinuity of the boundary conditions on temperature at the inlet (Dirichlet for $0 < r < R_j$, Neumann for $R_j < r < L_r$)
   - Intersection of the variable profiles resulting from the successively refined meshes
4. **FD1** solver: in it's validity domain, R.E gives convergence order about 2 (formal order) for primitive variables, P and $\int \nabla \cdot \mathbf{u}$
5. **FE2** solver: even if R.E results are not relevant, the computed solution is quite accurate