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The Balkans Continued Fraction

David Naccache¹ and Ofer Yifrach-Stav¹

DIÉNS, ÉNS, CNRS, PSL University, Paris, France
45 rue d'Ulm, 75230, Paris CEDEX 05, France
ofer.friedman@ens.fr, david.naccache@ens.fr

Abstract. In a previous article we gave a collection of continued fractions involving Catalan's constant. In this note we provide more general formulae governing those continued fractions. Having distinguished different cases associated to regions in the plan, we nickname those continued fractions "The Balkans" as they divide into areas which are related but still different in nature.

Because we do not provide formal proofs of those machine-constructed formulae we do not claim them to be theorems. Still, each and every proposed formula was extensively tested numerically.

1 Introduction

In a previous article [6] we gave a collection of continued fractions involving Catalan's constant. In this note we provide more general formulae governing those continued fractions. Having distinguished different cases associated to regions in the plan, we nickname those continued fractions "The Balkans" as they divide into areas which are related but still different in nature.

Because we do not provide formal proofs of those machine-constructed formulae we do not claim them to be theorems. Still, each and every proposed formula was extensively tested numerically.

All the programs included in this article are *self-contained*, i.e. any code snippet can be run independently of the others to fully illustrate the encoded formula. This renders the code longer but has the great advantage of allowing the reader to run and modify each snippet directly, by just cutting and pasting it into Mathematica without requiring any other module¹. The code was compacted for the sake of concision but loading it into Mathematica's editor re-indents it automatically.

2 Notations

We denote by $n!!$ the semifactorial of, i.e. the product of all the integers from 1 up to n having the same parity as n :

¹ Each snippet ends by a `ClearAll["Global`*"]`; command whose purpose is to make Mathematica "forget" all passed history.

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k) = n(n-2)(n-4) \cdots$$

Because in all the following we will only apply semifactorials to odd numbers, this can be simplified as:

$$n!! = \prod_{k=1}^{\frac{n+1}{2}} (2k-1) = n(n-2)(n-4) \cdots 3 \cdot 1$$

We denote by Catalan's constant by $G = 0.91596559 \dots$ and let C_n be the n -th Catalan number:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^n \frac{n+k}{k} \quad \text{for } n \geq 0$$

The first Catalan numbers are:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

3 The target

We define for odd j and $\kappa, c \in \mathbb{N}$ the following quantity nicknamed "The Balkans" continued fraction:

$$Q_{j,\kappa,c} = j(2-j+2\kappa) + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(c+n)(j+n-1)(1-j+2\kappa+n)}{j(2-j+2\kappa) + (3+4\kappa)n + 3n^2} \right)$$

The question asked is that of finding a general process allowing to compute $Q_{j,\kappa,c}$ while resorting as *little as possible* to numerical simulations or integer relation algorithms. The reason for this is that while integer relation algorithms allow us to "magically" discover relations, they do not provide *general information* about the underlying structure of the constants found.

3.1 How formulae were reverse-engineered

The process that allowed us to reverse-engineer the formulae given in this paper is interesting by its own right. A quick look at many examples of the three quantities a_0, a_1, a_2 forming the fractions:

$$Q_{j,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \quad \text{where } a_0, a_1, a_2 \in \mathbb{Z}$$

showed that the a_i s are products of small prime factors and a few large prime factors. This suggested that a_i s were initially² of some form:

² i.e., before simplification intervenes.

$$a_i = \text{expression}(j, \kappa, c) = \frac{\prod_{i=0}^{u-1} \phi_i(j, \kappa, c)}{\prod_{i=0}^{v-1} \phi_i(j, \kappa, c)}$$

where the ϕ_i are functions such as $(an+b)!$, 2^{an+b} , $(an+b)!!$, C_{an+b} , Pochhammer symbols of linear combinations of the parameters j, κ, c etc. and a few unknown “mixing” functions causing the appearance of the large prime factors, e.g. polynomials or recurrence relations.

Fortunately, integer relation algorithms allow us to collect many instances of such forms for diverse j, κ, c values. Hence the problem at hand consists in inferring which ϕ_i s are compatible with the cancellations due to the division. If a given ϕ is present in the expression then it is reasonably assumed that when tried for many j, κ, c the new expression:

$$\text{expression}_1(j, \kappa, c) = \frac{\text{expression}(j, \kappa, c)}{\phi(j, \kappa, c)}$$

or

$$\text{expression}_1(j, \kappa, c) = \text{expression}(j, \kappa, c) \cdot \phi(j, \kappa, c)$$

will feature less small factors.

The process can hence be repeated with proper back-tracking until all the combinatorial ϕ_i s were peeled-off. Then it remains to detect what the remaining “mixing” functions are which is done by monitoring the average growth rate of those surviving constants to emit hypotheses on the type of recurrence relation (or polynomial) at hand and by resorting to a variety of integer sequence recognition tools allowing to identify the hidden culprits.

- We started the exploration with the simplest case of [Bosnia & Herzegovina](#) where $a_2 = 0$.
- Having reverse-engineered [Bosnia & Herzegovina](#) we moved to [Croatia](#) where $a_2 = 0$ as well.
- The conjectured similarity between those Q regions allowed to guide the software in the formula for [Montenegro](#) which is somewhat simpler than [Kosovo](#) and [Serbia](#) given that [Montenegro](#) is the first case involving G where $j = 1$.
- Having interred [Montenegro](#) we moved on to [Kosovo](#) whose symmetry with [Serbia](#) was quickly noted.

The present work demonstrates the interest of statistical classifiers such as Maximum Likelihood Estimation (MLE) and Support Vector Machines (SVM) in mathematical exploration.

As will be shown here, this “gradient descent” method proved itself very well, although it required a few thousands of computation hours on a very powerful cluster.

We estimate that 80% of the discovery effort was done by the machine, the remaining 20% are human guidance that, we are convinced, is at the reach of today’s most powerful LLMs such as Gemini.

4 Kosovo, Serbia, Croatia, Montenegro, Bosnia & Herzegovina

As we will see, we distinguish are five cases that we call **Kosovo**, **Serbia**, **Croatia**, **Montenegro** and **Bosnia & Herzegovina** after the regions of interest in the (j, κ) -space shown in Figure 1. In each region c runs over the integers.

Area	Domain	$Q_{j,\kappa,c}$ is in
Croatia	$j \geq 2\kappa + 5$	\mathbb{Q}
Bosnia & Herzegovina	$j = 2\kappa + 3$	\mathbb{Q}
Serbia	$2\kappa + 1 \geq j \geq \kappa + 3$	\mathbb{R}
Kosovo	$\kappa + 2 \geq j \geq 3$	\mathbb{R}
Montenegro	$j = 1$	\mathbb{R}

Table 1. Areas. We list fractions involving G as being in \mathbb{R} although it is currently unknown if $G \in \mathbb{Q}$ (this is a major open problem).

A first restriction of our study will be to focus on odd j that produce $Q_{j,\kappa,c}$ values involving G . Note that:

$$Q_{j,c,\kappa} = \begin{cases} \frac{a_0}{a_1 + a_2 G} & \text{if } j \text{ is odd} \\ \frac{a_0}{a_1 + a_2 \log 2} & \text{if } j \text{ is even} \end{cases}$$

Note as well that j, κ, c can be negative. In the examples given in the appendix we adopt the notation:

$$\frac{a_0}{a_1 + a_2 \cdot \text{constant}} = T(0) + \mathcal{K}_{n=1}^{\infty} \left(\frac{P(n)}{T(n)} \right)$$

This paper will not treat the even and negative j, κ, c cases (a follow-up paper dealing with those cases being underway).

5 The Montenegro Conjecture

We start with the first j, κ space called **Montenegro**. **Montenegro** corresponds to the case $j = 1$. In other words we consider:

$$Q_{1,\kappa,c} = 2\kappa + 1 + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2(n+2\kappa)(n+c)}{3n^2 + (3+4\kappa)n + 2\kappa + 1} \right)$$

Define the functions:

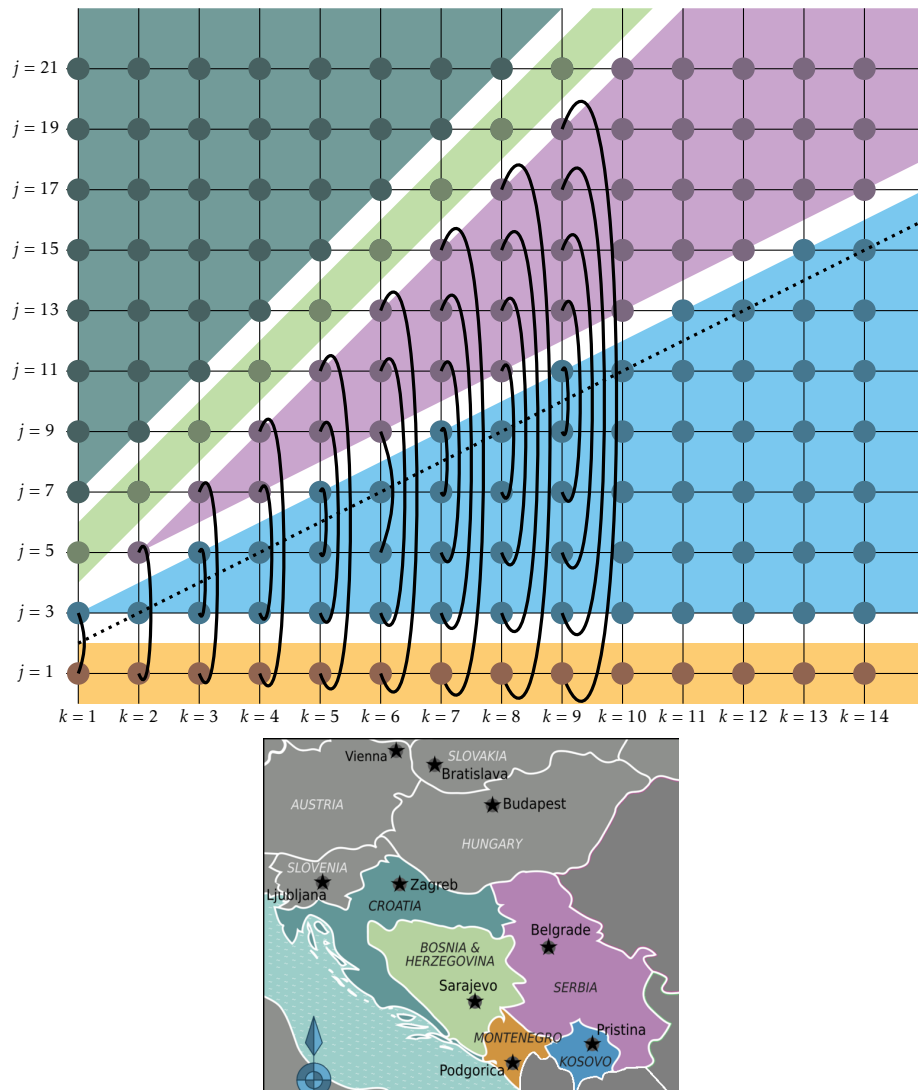


Fig. 1. The Five j,k Areas. The meaning of the arches connecting Kosovo, Serbia, and Montenegro will be clarified later.

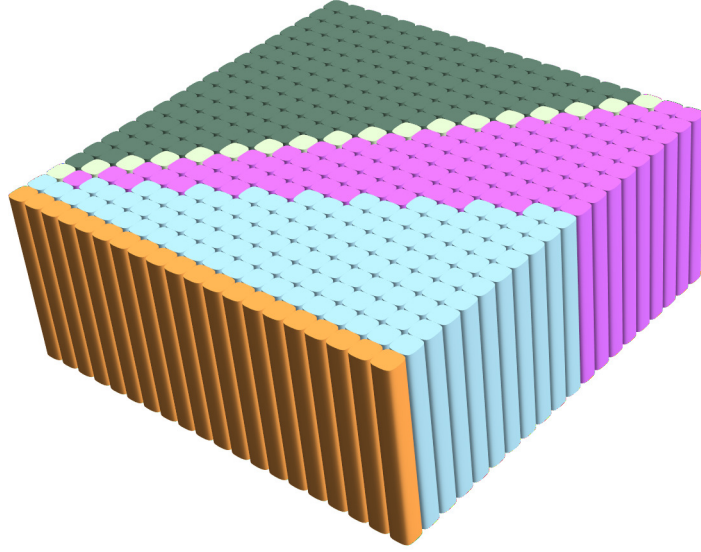


Fig. 2. The areas of Figure 1, with the dimension c added.

$$\Delta_{\kappa,c}(\alpha, \beta) = \begin{cases} \alpha + \beta c & \text{if } c < 2 \\ -2c(2c-1)(2(c-\kappa)-1)^2 \Delta_{\kappa,c-2} & \text{if } c \geq 2 \\ + (8c^2 + (2-8\kappa)c - 2\kappa + 1) \Delta_{\kappa,c-1} & \end{cases}$$

$$\Gamma_{\kappa,c}(\alpha, \beta) = (2c-1)!!^2 G + \Delta_{\kappa,c-1}(\alpha, \beta) \cdot \prod_{i=0}^{\kappa-1} (2(c-i)-1)$$

$$\delta_{\kappa} = \frac{4^{\kappa-1}}{(2\kappa-1)C_{\kappa-1}} \quad \text{and} \quad \rho_{\kappa} = \frac{\delta_{\kappa}(-1)^{\kappa}(1-2\kappa)}{(2\kappa)!(2\kappa-3)!!}$$

$$\alpha_{\kappa} = \rho_{\kappa} \Delta_{1,\kappa-1}(1, -2) \quad \text{and} \quad \beta_{\kappa} = -\rho_{\kappa} (2\kappa-3)^2 \Delta_{2,\kappa-1}(1, 12) - \alpha_{\kappa}$$

$$\text{Then } \forall \kappa, c \in \mathbb{N}^2, \quad Q_{1,\kappa,c} = \frac{\delta_{\kappa}(2c)!}{\Gamma_{\kappa,c}(\alpha_{\kappa}, \beta_{\kappa})}$$

$Q_{1,\kappa,c}$ is hence an explicitly computable fraction of the form:

$$Q_{1,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \quad \text{where } a_0, a_1, a_2 \in \mathbb{Z}$$

The code snippet testing this formula over the square $1 \leq \kappa, c \leq 14$ is entitled "1. Montenegro".

In summary in [Montenegro](#) we do not need to resort to any integer relation algorithms to compute $Q_{1,\kappa,c}$ for all κ, c values.

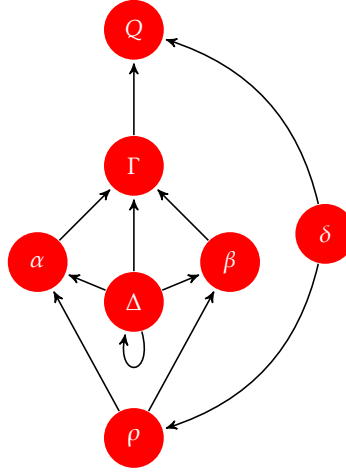


Fig. 3. Functional dependency between the functions computing $Q_{1,\kappa,c}$

Remark 1. Note that, as described here, the complexity of $\Delta_{\kappa,c}$ is exponential in c , however, using classical Fibonacci memoization, this complexity can be reduced to $O(c)$ thereby resulting in a very efficient algorithm for computing $Q_{1,\kappa,c}$.

6 The Bosnia & Herzegovina Conjecture

Bosnia & Herzegovina corresponds to the line $2\kappa = j - 3$, that is:

$$Q_{j, \frac{j-3}{2}, c} = -j + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(n-2)(n+c)(n+j-1)}{3n^2 + (2j-3)n - j} \right)$$

We start by defining:

$$\alpha_j = (-2)^{\frac{3j-11}{2}} \quad \text{and} \quad \beta_j = \alpha_j(15 - 4j)$$

And let:

$$\Delta_{j,c} = \begin{cases} \alpha_j + \beta_j c & \text{if } c < 2 \\ -2c(2c-j)(2c+1)(2c-j+2)\Delta_{j,c-2} & \text{if } c \geq 2 \\ \quad + (8c^2 + (14-4j)c - 3(j-2))\Delta_{j,c-1} & \end{cases}$$

$$g(j,c) = (2c)! 2^{\frac{3j-13}{2}} \prod_{i=1}^{\frac{j-1}{2}} (2c-2i+1)(j-2-2i)$$

$$h(j, c) = (2c - j + 2) \prod_{i=1}^{\frac{j-5}{2}} (2c - 2i - 1)^2$$

Then:

$$Q_{j, \frac{j-3}{2}, c} = \frac{g(j, c)}{\Delta_{j, c-1} \cdot h(j, c)}$$

Hence, in **Bosnia & Herzegovina** as well we do not need to resort to any integer relation algorithms to compute $Q_{j, \frac{j-3}{2}, c}$ which is an explicitly computable fraction in \mathbb{Q} .

The corresponding code snippet is "2. Bosnia". We conjecture that **Bosnia & Herzegovina** is actually a particular case of **Croatia** which values are also in \mathbb{Q} .

7 Roadmap

We are now ready to describe the general scheme that will govern the rest of this paper.

7.1 Areas governed by one running variable

As we have just seen, **Montenegro** and **Bosnia & Herzegovina** follow similar behaviors. This pattern revolves around the "magic" values α, β which are formally known for **Montenegro** and **Bosnia & Herzegovina**. Both regions are lines parameterized by a single running variable (κ for **Montenegro** and j for **Bosnia & Herzegovina**).

7.2 The c -level master formula for all Balkans except **Montenegro**

Except **Montenegro** (whose case was settled) all other areas obey a common c -level master formula that we will provide below. Computing $Q_{j, \kappa, c}$ for any c using this c -level master formula requires knowledge of two rational parameters: $\alpha_{j, \kappa}$ and $\beta_{j, \kappa}$.

The cornerstone of the rest of this paper is thus the quest for $\alpha_{j, \kappa}$ and $\beta_{j, \kappa}$ for all Balkan areas except **Montenegro**.

$\alpha_{j, \kappa}, \beta_{j, \kappa}$ can always be inferred by computing numerically³ Q_{j, κ, c_1} and Q_{j, κ, c_2} for two values $c_1 \neq c_2$ and solving a system of two equations in the unknowns $\alpha_{j, \kappa}, \beta_{j, \kappa}$.

When $\alpha_{j, \kappa}, \beta_{j, \kappa}$ are found $Q_{j, \kappa, c}$ can be computed for any other $c \notin \{c_1, c_2\}$.

Using this process requires resorting twice to integer relation algorithms such as LLL [5], HJLS [4], PSOS [1,2] or PSLQ [3]. This works perfectly in practice but is not entirely satisfactory because the process has a "blind spot" which

³ i.e. using an integer relation algorithm.

is the integer relation oracle. We would like to avoid such blind spots as much as possible and dispose of a fully algebraic process for computing each $Q_{j,\kappa,c}$. The ideal situation being, of course, a direct algebraic computation of $Q_{j,\kappa,c}$ from the data j, κ, c alone.

While we do not achieve this goal completely, we grandly progress towards it by minimizing a lot the call to the integer relation oracle.

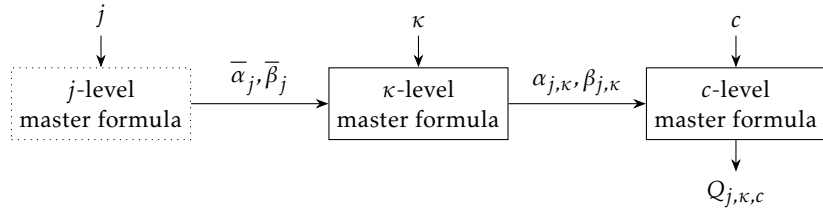


Fig. 4. j , κ and c -level master formulae. j -level master formulae (dotted) are only conjectured to exist. To date we have processes for inferring c -level and κ -level master formulae.

The snippet "3. Northern Balkans" illustrates the process with all areas except **Montenegro**. The snippet validates the c -level master formula on the 3D volume $3 \leq j \leq 13$, $1 \leq \kappa \leq 6$, $1 \leq c \leq 7$ (Figure 10) stretching over parts of **Croatia**, **Bosnia & Herzegovina**, **Serbia** and **Kosovo**. In each case the program derives the corresponding $\alpha_{j,\kappa}, \beta_{j,\kappa}$ and allows the computing of $Q_{j,\kappa,c}$ for any c . The formally computed results are then successfully compared to numerical ones.

As we will later see, for **Serbia** and **Kosovo** we have more powerful master formulae operating at the κ -level.

The c -level master formula⁴ is defined as follows, assuming that we are somehow given the magic constants $\alpha_{j,\kappa}, \beta_{j,\kappa}$.

Let:

$$\Delta_{j,\kappa,c}(\alpha_{j,\kappa}, \beta_{j,\kappa}) = \begin{cases} \alpha_{j,\kappa} + \beta_{j,\kappa}c & \text{if } c < 2 \\ -2c(2c-j)(2c-2\kappa+j-2)(2c-2\kappa-1)\Delta_{j,\kappa,c-2}(\alpha_{j,\kappa}, \beta_{j,\kappa}) & \text{if } c \geq 2 \\ \quad + (8c^2 + (2-8\kappa)c + (j-2)(2\kappa-j))\Delta_{j,\kappa,c-1}(\alpha_{j,\kappa}, \beta_{j,\kappa}) & \end{cases}$$

$$f_{j,\kappa,c} = C_{\frac{j-3}{2}} C_{\kappa-1} (j-2)(2\kappa-1)(2c-1)!!^2 \prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1)(\kappa-i+1)$$

⁴ valid for all areas except **Montenegro** whose case was anyhow previously settled.

$$g_{j,\kappa,c} = (2c)! 2^{\frac{j+4\kappa-7}{2}} \prod_{i=1}^{\frac{j-1}{2}} (2c-2i+1)(2\kappa-2i+1)$$

$$h_{j,\kappa,c} = \prod_{i=0}^{\frac{j-3}{2}} (2c-2i-1) \prod_{i=0}^{\kappa-1} (2c-2i-1)$$

Then:

$$Q_{j,\kappa,c} = \frac{g(j,\kappa,c)}{\Delta_{j,\kappa,c-1}(\alpha_{j,\kappa}, \beta_{j,\kappa}) \cdot h(j,\kappa,c) + f(j,\kappa,c) \cdot G}$$

7.3 Symmetries

Montenegro, Kosovo and Serbia feature internal symmetries illustrated by the arches in Figures 1 where connected values are identical. We will also provide a formula connecting α, β values over the plan. This means that any formula applicable to Kosovo will also settle the case of Serbia (given that Montenegro is settled and given that Montenegro provides the values for the upper border line of Serbia). We will come to that later on.

8 The Kosovo Conjecture

Kosovo is very specific in that it has a κ -level master formula. This means that for each j , knowledge of four upper-level rational constants:

$$\{\bar{\alpha}_j, \bar{\beta}_j\} = \left\{ \left\{ \alpha_j^\alpha, \alpha_j^\beta \right\}, \left\{ \beta_j^\alpha, \beta_j^\beta \right\} \right\}$$

allows to generate $\alpha_{j,\kappa}, \beta_{j,\kappa}$ for all κ values along a j -line within Kosovo.

Finding $\bar{\alpha}_j, \bar{\beta}_j$ for a given j requires solving a system of equations using four known $\alpha_{j,\kappa_1}, \beta_{j,\kappa_1}, \alpha_{j,\kappa_2}, \beta_{j,\kappa_2}$ for some $\kappa_1 \neq \kappa_2$. To determine the two $\alpha_{j,\kappa_i}, \beta_{j,\kappa_i}$ pairs⁵ we can solve two systems of equations (each in two unknowns) using any $Q_{j,\kappa_1,c_1}, Q_{j,\kappa_1,c_2}, Q_{j,\kappa_2,c_3}$ and Q_{j,κ_2,c_4} . Note that there is no opposition to take $c_1 = c_3$ and $c_2 = c_4$. This process is illustrated in Figure 5.

8.1 The Kosovo κ -level master formula

The κ -level master formula for Kosovo allows to infer $\alpha_{j,\kappa}, \beta_{j,\kappa}$ from $\bar{\alpha}_j, \bar{\beta}_j$ for a fixed j and a variable κ .

In other words, the Kosovo κ -level master formula generates for a fixed j and for a variable κ the data $\alpha_{j,\kappa}, \beta_{j,\kappa}$ necessary to operate the general c -level master formula given in subsection 7.2.

⁵ for $i \in \{1, 2\}$.

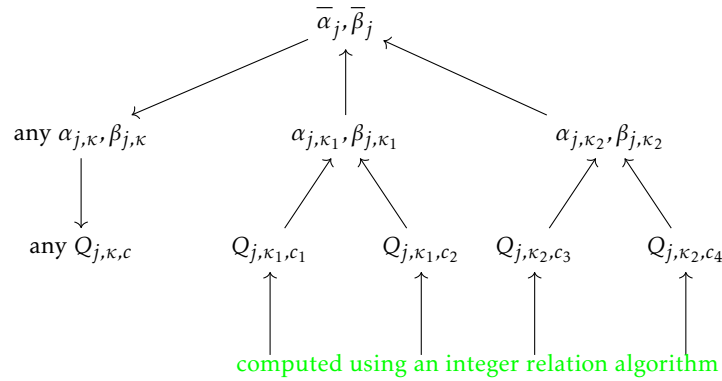


Fig. 5. κ -level resolution process for Kosovo.

Recall that the c -level master formula of subsection 7.2 generates for a fixed j, κ and a variable c a formal expression of the continued fraction $Q_{j,\kappa,c}$, given the auxiliary input $\alpha_{j,\kappa}, \beta_{j,\kappa}$.

Define:

$$\pi(j, \kappa) = \prod_{i=0}^{\frac{j-3}{2}} (\kappa - i)(2\kappa - 2i - 1)^2$$

$$\ell(n, j, \kappa) = \frac{(-1)^{\kappa+1} (2\kappa)!^2}{\kappa! 2^{3\kappa-2} (2\kappa - j)(2\kappa - 1)(n((2\kappa - j - 2)(3 - 2\kappa) - 1) + 1) \cdot \pi(j, \kappa)}$$

$$\eta(n, j, \kappa) = (2\kappa + 2j - 9 - 2n)(2\kappa + j - 8 - 2n)(-2\kappa + 5 - j)(2\kappa + j - 6)$$

$$\phi(n, j, \kappa) = 8\kappa^2 + \kappa(10j - 48 - 8n) + 3j^2 - (28 + 4n)j + 68 + 18n$$

$$\bar{\Delta}_{n,j,\kappa}(\alpha, \beta) = \begin{cases} \alpha + \beta\kappa & \text{if } \kappa < 2 \\ \eta(n, j, \kappa) \cdot \bar{\Delta}_{n,j,\kappa-2} + \phi(n, j, \kappa) \cdot \bar{\Delta}_{n,j,\kappa-1} & \text{if } \kappa \geq 2 \end{cases}$$

We assume that we are given the four constants:

$$\{\bar{\alpha}_j, \bar{\beta}_j\} = \left\{ \left\{ \alpha_j^\alpha, \alpha_j^\beta \right\}, \left\{ \beta_j^\alpha, \beta_j^\beta \right\} \right\}$$

Then:

$$\alpha_{j,\kappa} = \frac{\bar{\Delta}_{0,j,\kappa-j+2}(\alpha_j^\alpha, \alpha_j^\beta)}{\ell(0, j, \kappa)} \quad \text{and} \quad \beta_{j,\kappa} = \frac{\bar{\Delta}_{1,j,\kappa-j+2}(\beta_j^\alpha, \beta_j^\beta)}{\ell(1, j, \kappa)} - \alpha_{j,\kappa}$$

The process is illustrated by the code snippet "4. Kosovo".

The computation of the "magic" lists of constants $\{\bar{\alpha}_j, \bar{\beta}_j\}$ hard-coded in the snippet "4. Kosovo" is done by resorting to integer relation resolution in snippet "5. resolution".

Remark 2. It is interesting to note that thanks to the inner symmetry within [Kosovo](#), it is possible to determine $\bar{\alpha}_{j+2}, \bar{\beta}_{j+2}$ if $\bar{\alpha}_j, \bar{\beta}_j, \bar{\alpha}_{j+4}, \bar{\beta}_{j+4}$ are known. Taking as an example $j = 5$, Figure 1 shows that $Q_{5,3,c} = Q_{3,3,c}$. Hence knowledge of $\bar{\alpha}_3, \bar{\beta}_3$ will be used to compute $Q_{3,3,c}$ which is identical to $Q_{5,3,c}$.

Note that $Q_{5,5,c} = Q_{7,5,c}$. Hence knowledge of $\bar{\alpha}_7, \bar{\beta}_7$ will be used to compute $Q_{7,5,c}$ which is identical to $Q_{5,5,c}$. Finally, having in hand $Q_{5,3,c}, Q_{5,5,c}$ we can solve a system in two unknowns and determine $\bar{\alpha}_5, \bar{\beta}_5$. Unfortunately this process has an information flow that only operates in "sandwich mode" allowing to determine $\bar{\alpha}_{j+2}, \bar{\beta}_{j+2}$ from $\bar{\alpha}_j, \bar{\beta}_j, \bar{\alpha}_{j+4}, \bar{\beta}_{j+4}$. This information flow cannot be reversed into an "escalator" allowing to ascend to level $j + 4$ from levels j and $j + 2$. The situation is illustrated in Figure 6 where $x \rightarrow y$ denotes the relation "y is computable from x".

Remark 3. It is also possible to directly infer the $\alpha_{j,\kappa}, \beta_{j,\kappa}$ through symmetry. Note that the relation below also works for negative j, κ values.

Define:

$$\zeta(j, u) = \frac{1}{2^u} \prod_{i=0}^{u-1} (2 + 2i - j)$$

$$\tau_{j,u} = \begin{cases} \frac{\zeta(j, u)}{|\zeta(j, u)|} \cdot (2j - 2u - 3)!!(2u - j)!!(-2)^u & \text{if } 2u > j - 1 \\ \zeta(j, u) \cdot (-4)^u & \text{otherwise} \end{cases}$$

Then:

$$\frac{\alpha_{j,j-u-1}}{\alpha_{j-2u,j-u-1}} = \frac{\beta_{j,j-u-1}}{\beta_{j-2u,j-u-1}} = \tau_{j,u}$$

In particular $\tau_{j,j-1} = \frac{1}{2^{j-4}}$. The code is snippet "6. symmetry".

9 The Croatia Conjecture

We suspect that [Bosnia & Herzegovina](#) is actually a particular border case of [Croatia](#).

Our automated software detected the following **stunning** behavior providing a κ -level formula for [Croatia](#).

Let:

$$\mu_{i,j} = -(-2)^{\frac{3j-11-4i}{2}} \prod_{q=1}^i (j - 2q - 2)$$

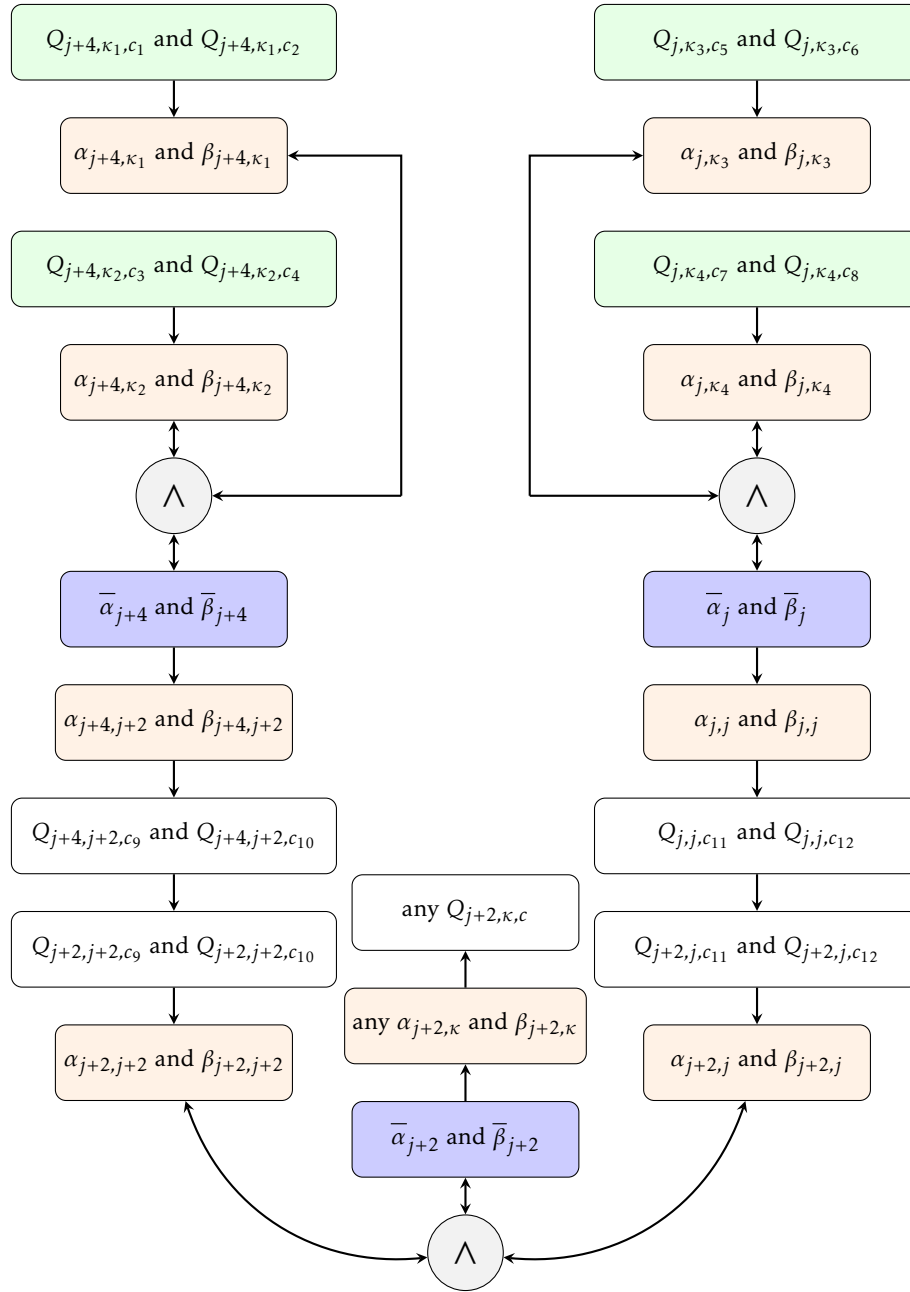


Fig. 6. Inferring $\bar{\alpha}_{j+2}, \bar{\beta}_{j+2}$ from $\bar{\alpha}_j, \bar{\beta}_j$ and $\bar{\alpha}_{j+4}, \bar{\beta}_{j+4}$. The $Q_{j, \kappa, c}$ in green boxes are determined using integer relation algorithms.

For every $i = 0, 1, 2, \dots$ there exist two polynomials in j , denoted $\phi_1(i, j)$ and $\phi_2(i, j)$ such that for $j \geq 2i + 5$ we have:

$$\alpha_{j, \frac{j-2i-3}{2}} = \frac{\psi_1(i, j)}{\mu(i, j)} \quad \text{and} \quad \beta_{j, \frac{j-2i-3}{2}} = \frac{\psi_2(i, j)}{\mu(i, j)}$$

Tables 2 and 3 provide the first values of the polynomials ψ_1, ψ_2 .
See code snippet "7. Croatia".

i	$\psi_1(i, j)$
0	-1
1	$14 - j$
2	$-464 + 58j - 3j^2$
3	$27936 - 4692j + 432j^2 - 15j^3$
4	$-2659968 + 542256j - 67836j^2 + 4260j^3 - 105j^4$
5	$367568640 - 86278560j + 13203480j^2 - 1139700j^3 + 51450j^4 - 945j^5$

Table 2. $\psi_1(i, j)$ for $0 \leq i \leq 5$.

i	$\psi_2(i, j)$
0	$-15 + 4j$
1	$306 - 95j + 4j^2$
2	$-13360 + 4646j - 357j^2 + 12j^3$
3	$999648 - 379692j + 40368j^2 - 2457j^3 + 60j^4$
4	$-113885568 + 46449360j - 6124164j^2 + 513228j^3 - 22935j^4 + 420j^5$
5	$18333538560 - 7933530720j + 1224286440j^2 - 126833100j^3 + 7864950j^4 - 266175j^5 + 3780j^6$

Table 3. $\psi_2(i, j)$ for $0 \leq i \leq 5$.

Remark 4.

The leading coefficients of $\psi_1(i, j)$ (i.e. 1, 1, 3, 15, 105, 945, ...) are $(2i-1)!!$ whereas the leading coefficients of $\psi_2(i, j)$ (i.e. 4, 4, 12, 60, 420, 3780, ...) are $4(2i-1)!!$.

Remark 5. The ψ polynomials can always be written under a nested form, e.g.:

$$\begin{aligned} \psi_1(j, 6) = & -14487726825 - (104826150 + (452605725 \\ & + (121200300 + (13697775 + (640710 \\ & + 10395 \cdot (j - 27))(j - 25))(j - 23))(j - 21))(j - 19))(j - 17) \end{aligned}$$

$$\begin{aligned} \psi_2(j, 6) = & 3198013886925 + (145296572850 + (5207427225 \\ & + (4353102000 + (877052475 + (78210090 + (3023055 \\ & + 41580 \cdot (j - 29))(j - 27))(j - 25))(j - 23))(j - 21))(j - 19))(j - 17) \end{aligned}$$

Remark 6. The GCD between the u -th coefficient of $\psi_1(i, j)$ and the u -th coefficient of $\psi_2(i, j)$ is always smooth as illustrated in the code snippet "8. coefficients".

10 Balkans Knowledge Summary

In summary:

- For **Kosovo** and **Croatia** we have κ -level formulae⁶.
- Any $Q_{j,\kappa,c}$ value in **Montenegro** is algebraically computable.
- Any $Q_{j,\kappa,c}$ value in **Bosnia & Herzegovina** is algebraically computable.
- **Serbia** is fully determined by our knowledge of **Montenegro** and **Kosovo**.

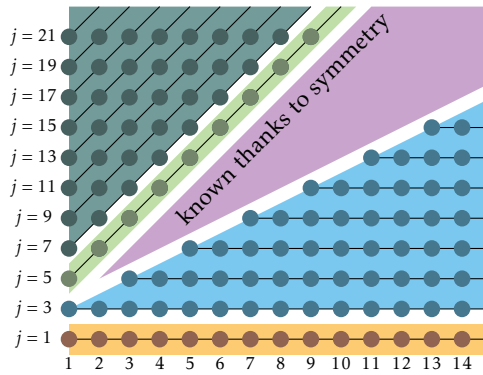


Fig. 7. The axes along which the κ -level master formulae operate in each area. We hence see that a finite amount of “magic” information per one dimension allows to algebraically compute $Q_{j,\kappa,c}$ over the two remaining dimensions.

Area	j -level formula	κ -level formula	c -level formula
Kosovo+Serbia	unknown	✓	✓
Croatia	unknown	✓	✓
Montenegro	not needed	✓	✓
Bosnia & Herzegovina	not needed	✓	✓

Table 4. Knowledge Summary

⁶ This means that oracle knowledge of $\bar{\alpha}_j, \bar{\beta}_j$ is required to generate $Q_{j,\kappa,c}$ for all κ, c . $\bar{\alpha}_j, \bar{\beta}_j$ can be inferred from any four $Q_{j,\kappa_1,c_1}, Q_{j,\kappa_1,c_2}, Q_{j,\kappa_2,c_3}, Q_{j,\kappa_2,c_3}$ values.

We conjecture that there exists (possibly very complex) j -level master formulae for all regions allowing to algebraically access any $Q_{j,\kappa,c}$ value. Another challenge, on which the authors are currently working, is characterizing the case of even j values (that yield formulae involving $\log 2$) and negative j, κ, c values.

As a motivational example let us unveil the simple $\log 2$ example $Q_{2,\kappa,0}$. In this case:

$$Q_{2,\kappa,0} = \frac{(-1)^{\kappa+1} a_\kappa}{-b_\kappa + a_\kappa \log 2}$$

Where a_κ is the LCM of the list of κ integers starting with κ (in Mathematica: `Table[Apply[LCM, Table[i, {i, k, 2k-1}]], {k, 1, 100}]`) while the b_κ are the numerators of the coefficients in the power series for $-\log(1+x)\log(1-x)$.

Evidently, providing formal proofs of the formulae provided in this paper is a challenge by its own right. To date, attempts to code automated substitution-simplification-induction proofs were not successful.

The following observations are hints that may serve in future quests:

Remark 7. Denoting:

$$Q_{j,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \text{ where } a_0, a_1, a_2 \in \mathbb{Z}$$

The following formula (code snippet "9. ratio") is valid all over areas:

$$\frac{a_0}{a_2} = \frac{(2c)! \cdot 2^{\lfloor 1/j \rfloor + 2\kappa} \cdot \prod_{i=1}^{\frac{j-1}{2}} \frac{(4c-4i+2) \cdot (2\kappa-2i+1)}{(\kappa-i+1) \cdot (2c+2i-2\kappa-1)}}{8 \cdot C_{\frac{j-3}{2}} \cdot (j-2) \cdot (2c-1)!!^2 \cdot (2\kappa-1) \cdot C_{\kappa-1}}$$

Where for $j=1$ the value of the Π product is taken to be 1 by definition.

Remark 8. Although possibly unrelated, we note that low-degree continued fractions involving $\log 2$ can be also obtained with lower degree polynomials, e.g. (See code snippet "10. log2-a"):

$$\frac{2}{L\left(\frac{1}{2}, 1, c-1\right)} = \frac{2}{\sum_{n=0}^{\infty} \frac{e^{\pi i n}}{(n+c-1)}} = \frac{1}{2^{c-2} \log(2) - \sum_{j=1}^{c-2} \frac{2^{c-j-2}}{j}} = c + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2}{3n+c} \right)$$

We get a similar behavior for:

$$R_c = c + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2 - 2n}{3n+c} \right)$$

where $2^{c-4} \cdot (c-3) \cdot a_0 = a_2$ and for which we provide numerical examples in Table 9.

See code snippet "11. log2-b".

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
-7351344	-32375839	46558512	$n(3+n)(17+n)$	$72 + 43n$
-2450448	-1768477	2450448	$n(1+n)(17+n)$	$36 + 39n$
-1081080	-16147379	23279256	$n(5+n)(15+n)$	$96 + 43n$
-793800	-232217	322560	$(4+n)(9+n)^2$	$100 + 39n$
-504504	-16140515	23279256	$n(7+n)(13+n)$	$112 + 43n$
-436590	7989199	-11531520	$(3+n)(9+n)(11+n)$	$120 + 43n$
-180180	-1001393	1441440	$(2+n)(5+n)(13+n)$	$84 + 39n$
-174636	-8069449	11639628	$n(9+n)(11+n)$	$120 + 43n$
-72072	-850133	1225224	$n(5+n)(13+n)$	$84 + 39n$
-72072	-251099	360360	$n(3+n)(13+n)$	$56 + 35n$
-72072	-52279	72072	$n(1+n)(15+n)$	$32 + 35n$
-45045	124048	-180180	$(1+n)(3+n)(13+n)$	$56 + 35n$
-28028	-999391	1441440	$(2+n)(7+n)(13+n)$	$112 + 43n$
-27720	-20417	27720	$n(1+n)(11+n)$	$24 + 27n$
-22050	27649	-40320	$(3+n)(7+n)(9+n)$	$80 + 35n$
-19305	424423	-612612	$(1+n)(5+n)(15+n)$	$96 + 43n$
-17640	-250007	360360	$n(7+n)(9+n)$	$80 + 35n$
-14700	-153907	221760	$(2+n)(7+n)(9+n)$	$80 + 35n$
-10395	124741	-180180	$(1+n)(5+n)(11+n)$	$72 + 35n$
-9450	76691	-110880	$(1+n)(5+n)(9+n)$	$60 + 31n$
-7350	-62563	90090	$n(7+n)^2$	$64 + 31n$
-7007	424566	-612612	$(1+n)(7+n)(13+n)$	$112 + 43n$
-6300	-14087	20160	$(2+n)(5+n)(9+n)$	$60 + 31n$
-6006	99839	-144144	$(1+n)(5+n)(13+n)$	$84 + 39n$
-4900	-21043	30240	$(2+n)(7+n)^2$	$64 + 31n$
-2520	-1879	2520	$n(1+n)(9+n)$	$20 + 23n$
-2450	28781	-41580	$(1+n)(7+n)^2$	$64 + 31n$
-1225	367	-560	$(3+n)(7+n)^2$	$64 + 31n$
-525	-2413	3465	$n(5+n)(7+n)$	$48 + 27n$
-450	1151	-1680	$(1+n)(5+n)^2$	$36 + 23n$
-350	1739	-2520	$(1+n)(5+n)(7+n)$	$48 + 27n$
-180	-299	420	$n(3+n)(5+n)$	$24 + 19n$
-105	142	-210	$(1+n)(3+n)(7+n)$	$32 + 23n$
-18	7	-12	$(1+n)(3+n)^2$	$16 + 15n$
-6	-5	6	$n(1+n)(3+n)$	$8 + 11n$

Table 5. Examples of convergence to $\frac{a_0}{a_1 + a_2 \log 2}$

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
1	1	-1	$n(1+n)^2$	$4 + 7n$
9	11	-15	$n(3+n)^2$	$16 + 15n$
50	147	-210	$n(5+n)^2$	$36 + 23n$
60	47	-60	$n(1+n)(5+n)$	$12 + 15n$
90	-79	120	$(1+n)(3+n)(5+n)$	$24 + 19n$
420	319	-420	$n(1+n)(7+n)$	$16 + 19n$
420	887	-1260	$n(3+n)(7+n)$	$32 + 23n$
900	361	-480	$(2+n)(5+n)^2$	$36 + 23n$
1890	-3443	5040	$(1+n)(3+n)(9+n)$	$40 + 27n$
2100	2377	-3360	$(2+n)(5+n)(7+n)$	$48 + 27n$
5544	50035	-72072	$n(5+n)(11+n)$	$72 + 35n$
6468	-249713	360360	$(1+n)(7+n)(11+n)$	$96 + 39n$
7560	19409	-27720	$n(3+n)(9+n)$	$40 + 27n$
8316	-19031	27720	$(1+n)(3+n)(11+n)$	$48 + 31n$
15444	-49705	72072	$(1+n)(3+n)(15+n)$	$64 + 39n$
22050	-499279	720720	$(1+n)(7+n)(9+n)$	$80 + 35n$
24255	-76586	110880	$(3+n)(7+n)(11+n)$	$96 + 39n$
25740	200107	-288288	$(2+n)(5+n)(15+n)$	$96 + 43n$
37800	250427	-360360	$n(5+n)(9+n)$	$60 + 31n$
38808	849671	-1225224	$n(7+n)(11+n)$	$96 + 39n$
39690	-1997851	2882880	$(1+n)(9+n)^2$	$100 + 39n$
41580	154327	-221760	$(2+n)(5+n)(11+n)$	$72 + 35n$
58212	3997025	-5765760	$(2+n)(9+n)(11+n)$	$120 + 43n$
79380	2123957	-3063060	$n(9+n)^2$	$100 + 39n$
83160	251561	-360360	$n(3+n)(11+n)$	$48 + 31n$
87318	-8491859	12252240	$(1+n)(9+n)(11+n)$	$120 + 43n$
97020	1999321	-2882880	$(2+n)(7+n)(11+n)$	$96 + 39n$
132300	3997907	-5765760	$(2+n)(9+n)^2$	$100 + 39n$
198450	-1227581	1774080	$(3+n)(9+n)^2$	$100 + 39n$
216216	852707	-1225224	$n(3+n)(15+n)$	$64 + 39n$
360360	263111	-360360	$n(1+n)(13+n)$	$28 + 31n$
630630	-3990557	5765760	$(3+n)(7+n)(13+n)$	$112 + 43n$
918918	-3383801	4900896	$(1+n)(3+n)(17+n)$	$72 + 43n$
1746360	2475007	-3548160	$(4+n)(9+n)(11+n)$	$120 + 43n$
46558512	33464927	-46558512	$n(1+n)(19+n)$	$40 + 43n$

Table 6. Examples of convergence to $\frac{a_0}{a_1 + a_2 \log 2}$. The entry in blue is the one reported by the Ramanujan Project.

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
-80281600	-10675439	9459450	$(2+n)(3+n)(14+n)$	$45 + 35n$
-15728640	392683	-727650	$(4+n)^2(12+n)$	$65 + 35n$
-13107200	-263867	-28350	$(4+n)(5+n)(10+n)$	$55 + 31n$
-10485760	-93699	-103950	$(4+n)(5+n)(12+n)$	$65 + 35n$
-7372800	-884203	727650	$(2+n)(3+n)(12+n)$	$39 + 31n$
-6291456	149419	-257250	$(4+n)(8+n)^2$	$81 + 35n$
-5242880	-86807	9450	$(5+n)(6+n)(10+n)$	$77 + 35n$
-3932160	-116317	3150	$(4+n)(5+n)(8+n)$	$45 + 27n$
-3276800	-158859	22050	$(2+n)(5+n)(10+n)$	$33 + 27n$
-2621440	-48609	-1050	$(5+n)(6+n)(8+n)$	$63 + 31n$
-2359296	-168445	103950	$(2+n)(4+n)(12+n)$	$39 + 31n$
-491520	50593	-66150	$(3+n)(4+n)(10+n)$	$55 + 31n$
-327680	-21271	9450	$(2+n)(4+n)(10+n)$	$33 + 27n$
-230400	-1909	-22050	$(2+n)^2(5+n)$	$9 + 11n$
-196608	-184547	198450	$(3+n)(6+n)(10+n)$	$77 + 35n$
-163840	-4981	-22050	$n(5+n)(8+n)$	$9 + 19n$
-131072	-2951	-630	$(4+n)^2(8+n)$	$45 + 27n$
-122880	-13079	9450	$(2+n)(3+n)(10+n)$	$33 + 27n$
-61440	-2467	-3150	$(2+n)(4+n)(5+n)$	$15 + 15n$
-61440	791	-9450	$n(5+n)(6+n)$	$7 + 15n$
-51200	2839	-9450	$n(4+n)(5+n)$	$5 + 11n$
-49152	-1919	90	$(4+n)^2(6+n)$	$35 + 23n$
-36864	-2693	-450	$(2+n)(4+n)(6+n)$	$21 + 19n$
-18432	-419	-3150	$n(4+n)(6+n)$	$7 + 15n$
-18432	-419	-3150	$n(3+n)(8+n)$	$9 + 19n$
-8192	-487	-54	$(4+n)^3$	$25 + 19n$
-3072	121	-630	$n(4+n)^2$	$5 + 11n$
-2048	-129	-90	$(2+n)(4+n)^2$	$15 + 15n$
-2048	-43	-30	$(3+n)(4+n)(6+n)$	$35 + 23n$
-768	-77	-18	$(2+n)(3+n)(4+n)$	$15 + 15n$
-288	31	-90	$n(2+n)(3+n)$	$3 + 7n$

Table 7. Examples of convergence to $\frac{a_0}{a_1+a_2G}$

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
192	13	18	$(2+n)^2(3+n)$	$9+11n$
384	1	90	$n(3+n)(4+n)$	$5+11n$
2304	389	450	$n(3+n)(6+n)$	$7+15n$
3072	179	-18	$(3+n)(4+n)^2$	$25+19n$
4608	133	450	$(2+n)^2(4+n)$	$9+11n$
4608	383	-90	$(2+n)(3+n)(6+n)$	$21+19n$
11520	-1373	3150	$n(2+n)(4+n)$	$3+7n$
12288	973	-750	$(3+n)(6+n)^2$	$49+27n$
12288	1145	-630	$(2+n)(3+n)(8+n)$	$27+23n$
16384	-543	1050	$(3+n)(4+n)(8+n)$	$45+27n$
81920	3983	1350	$(4+n)^2(5+n)$	$25+19n$
89600	-10891	22050	$n(2+n)(5+n)$	$3+7n$
98304	2263	150	$(4+n)(6+n)^2$	$49+27n$
98304	35389	-36750	$(3+n)(6+n)(8+n)$	$63+31n$
122880	6563	3150	$(2+n)(5+n)(6+n)$	$21+19n$
147456	21365	22050	$n(4+n)(8+n)$	$9+19n$
163840	6789	450	$(4+n)(5+n)(6+n)$	$35+23n$
262144	710401	-771750	$(3+n)(8+n)^2$	$81+35n$
294912	18013	-3150	$(2+n)(4+n)(8+n)$	$27+23n$
524288	27787	-22050	$(4+n)(6+n)(10+n)$	$77+35n$
786432	19099	-5250	$(4+n)(6+n)(8+n)$	$63+31n$
983040	25979	-450	$(5+n)(6+n)^2$	$49+27n$
1310720	1723	28350	$(4+n)^2(10+n)$	$55+31n$
2949120	168821	22050	$(2+n)(5+n)(8+n)$	$27+23n$
9830400	-1833409	2182950	$(3+n)(4+n)(12+n)$	$65+35n$
12582912	184025	-7350	$(5+n)(8+n)^2$	$81+35n$
39321600	1965547	-727650	$(2+n)(5+n)(12+n)$	$39+31n$
165150720	12969199	-9459450	$(2+n)(4+n)(14+n)$	$45+35n$
330301440	17687791	-9459450	$(2+n)(5+n)(14+n)$	$45+35n$

Table 8. Examples of convergence to $\frac{a_0}{a_1+a_2G}$

c	a_0	a_1	a_2
3	2	1	0
4	1	1	-1
5	-1	-3	4
6	-2	-17	24
7	-3	-67	96
8	12	667	-960
9	5	666	-960
10	30	9319	-13440
11	-105	-74537	107520
12	-280	-447187	645120
13	126	447173	-645120
14	63	491884	-709632
15	231	3935051	-5677056
16	2772	102311095	-147603456
17	-1287	-102310996	147603456
18	-6006	-1023109531	1476034560
19	45045	16369749493	-23616552960
20	720720	556571437717	-802962800640

Table 9. Examples of convergence to $\frac{a_0}{a_1+a_2 \log 2}$ for R_c

j	α_j	β_j
3	-1	4
5	19	234
7	5818/3	254456/3
9	667115	60003486
11	467946090	71121907440
13	554143204110	127451285438100
15	994115449382940	322092692148962160
17	2516347061651130075	1092094185270706446150
19	8546069024090201027250	4785798287838257081935200
21	37508692924557081882027450	26331102038134635548392485900
23	206659254109760483703789089700	177726957997323983116663150902000
25	1396637676485497608584841260027550	1444123356588023432320434243315206700
27	11361110319787394788017568214856502500	13905999029609441333101619589964946580000
29	109509742351999832489255793094925601037500	156598931559029451368898717824937174831465000
31	1234320809247763942235545044494798498436195000	2039097976865181167119056627863149102390546140000
33	16085205915675471439195309128783843538512283666875	30401039180587356456007967587920548312623820610393750
35	239989379884263177615577263747245812249369757283461250	514537230471714428505965482811829861362838523445500920000

j	$\bar{\alpha}_j$	$\bar{\beta}_j$
3	-1/3	-14/3
5	-17	-8
7	-758	-27820
9	-302117	-23010044
11	-1091480994/5	-146282046156/5
13	-262476468810	-54596049230880
15	-475443072646380	-141682352738003640
17	-1211573031414907725	-489475664504671450500
19	-4135193781750207709650	-2175112041708995560914300
21	-18218507239728799899288030	-12097088912487772715204794320
23	-100680148628028059378172563700	-82356361704096372069838207986600
25	-682078864161239229949889893754850	-673893917980353010760236819146271800
27	-5559692282317104149119150499246482500	-6527078739785105011529098668023829975000
29	-53681246247288656939970174534335708392500	-73865394837022289182570623863734010339760000
31	-605939306349175124039948450466713432304279000	-965867254322525126192328035746702493817188406000
33	-7906287653442943862409767973858652552097126548125	-14452693830499521315903006473321900713406050369972500
35	-118090012323922699712409299094969252935070156336941250	-245391045609131483699190960852072336578359955374403917500

Table 10. The first $\bar{\alpha}_j = \{\alpha_j, \alpha_j\}$ values and the first $\bar{\beta}_j = \{\beta_j, \beta_j\}$ values.

A Mathematica Code

```

1 (* code snippet "1. Montenegro" *)
2 NumQ[k_, c_] := 2k+1+ContinuedFractionK[-2n^2(n+2k)(n+c), 3n^2+(3+4k)n+2k+1, {n, 1, 5000}];
3 vD[a_, b_, k_, c_] := If[c < 2, a+b*c, -2c(2c-1)(2(c-k)-1)^2vD[a, b, k, c-2]+(8c^2+(2-8k)c-2k+1)vD[a, b, k, c-1]];
4 vG[a_, b_, k_, c_] := ((2c-1)!!)^2Catalan+Product[(2(c-j)-1), {j, 0, k-1}]*vD[a, b, k, c-1];
5 vd[k_] := 4^(k-1)/(2k-1)/CatalanNumber[k-1]; vr[k_] := vd[k](-1)^(k)(1-2k)/((2k)!(2k-3)!!);
6 va[k_] := vr[k]vD[1, -2, 1, k-1]; vb[k_] := -vr[k](2k-3)^2vD[1, 12, 2, k-1]-va[k];
7 QFor[k_, c_] := vd[k](2c)!/vG[va[k], vb[k], k, c];
8 Print[Union[Flatten[Table[N[QFor[k, c], 200] == N[NumQ[k, c], 200], {c, 1, 14}, {k, 1, 14}]]][[1]]]; ClearAll["Global'"];

```

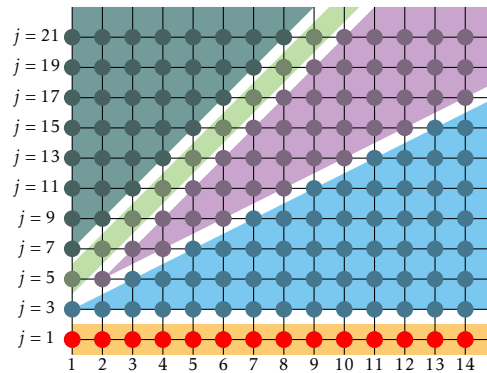


Fig. 8. The points on which **Montenegro** is tested for $1 \leq c \leq 14$ by the snippet "1. Montenegro" are shown in **red**.

```

1 (* code snippet "2. Bosnia" *)
2 NumQ[j_, c_] := -j + ContinuedFractionK[-2n(n-2)(n+c)(n+j-1), 3n^2+(2j-3)n-j, {n, 1, 5000}];
3 vD[j_, c_] := If[c < 2, (-2)^((3j-11)/2)(1+(15-4j)c), -2c(2c-j)(2c+1)(2c-j+2)vD[j, c-2] + (8c^2+(14-4j)c-3(j-2))vD[j, c-1]];
4 g[j_, c_] := Product[(2c-2i+1)(j-2-2i), {i, 1, (j-1)/2}](2c)! 2^((3j-13)/2);
5 h[j_, c_] := (2c-j+2) Product[(2c-2i-1)^2, {i, 0, (j-5)/2}];
6 Q[j_, c_] := Simplify[g[j, c]/(vD[j, c-1]*h[j, c])];
7 Print[Union[Flatten[Table[N[NumQ[j, c]] == Q[j, c], 200], {j, 5, 13, 2}, {c, 1, 14}]]][[1]]; ClearAll["Global'*"];

```

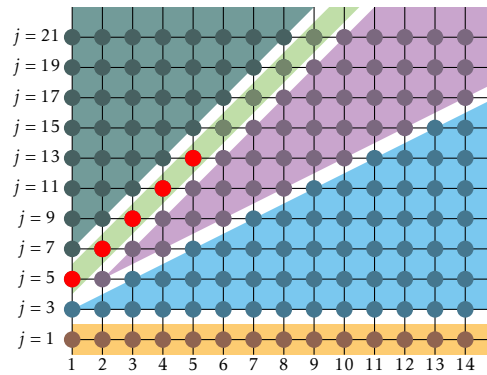


Fig. 9. The points on which Bosnia & Herzegovina is tested for $1 \leq c \leq 14$ by the snippet "2. Bosnia" are shown in red.

```

1 (* code snippet "3. Northern Balkans" *)
2 NumQ[j_, k_, c_] := j(2-j+2k) + ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n), 3n^2+(3+4k)n+j(2-j+2k), {n, 1, 15000}];
3 vD[ab_, j_, k_, c_] := If[c < 2, ab[[1]] + ab[[2]]c, -2c(2c-j)(2c-2k+j-2)(2c-2k-1)vD[ab, j, k, c-2] + (8c^2+(2-8k)c+(j-2)(2k-j))vD[ab, j, k, c-1]];

```



```

4 f[j_,k_,c_]:=Product[(2c-2k+2i-1)(k-i+1),{i,1,(j-1)/2}]CatalanNumber[(j-3)/2](j-2)(2k-1)*(2c-1)!!^2*CatalanNumber[k-1];
5 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
6 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
7 Q[j_,k_,c_,ab_]:=Simplify[g[j,k,c]/(vD[ab,j,k,c-1]h[j,k,c]+f[j,k,c]Catalan)];
8 Ratio:=Function[{j,k,c},r=N[NumQ[j,k,c],2000];v=FindIntegerNullVector[{1,r,N[Catalan*r,2000]}];-v[[1]]/v[[2]]];
9 GenAB[j_]:=Table[{a,b}/.Solve[Table[{a,b}.{1,c-1]==g[j,k,c]/Ratio[j,k,c]/h[j,k,c],{c,1,2}],{a,b}][[1]],{k,1,6}];
10 result={};For[j=3,j<=13,AB=GenAB[j];AppendTo[result,Union[Flatten[Table[N[Q[j,k,c,AB[[k]]]==NumQ[j,k,c],20],{k,1,6},{c
,1,7}]]]];j+=2];Print[Union[Flatten[result]][[1]]];

```

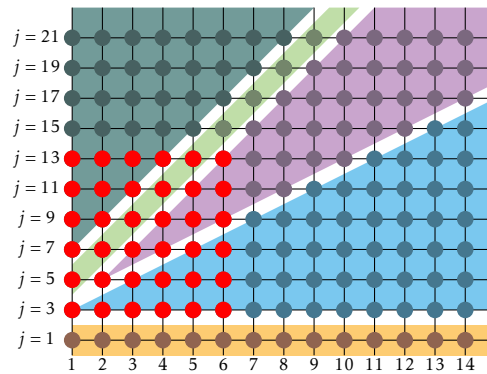


Fig. 10. The points on which the c -level master formula is tested for $1 \leq c \leq 7$ by the snippet "3. Northern Balkans" are shown in red. Note that the Montenegro line does not obey that same c -level formula but was validated by snippet "1. Montenegro".

```

1 (* code snippet "4. Kosovo" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,15000}];

```

```

3 vD:=Function[{ab,j,k,c},If[c<2,ab[[1]]+ab[[2]]c,-2c(2c-j)(2c-2k+j-2)(2c-2k-1)vD[ab,j,k,c-2]+(8c^2+(2-8k)c+(j-2)(2k-j))
   vD[ab,j,k,c-1]]];
4 f[j_,k_,c_]:=Product[(2c-2k+2i-1)(k-i+1),{i,1,(j-1)/2}]CatalanNumber[(j-3)/2](j-2)(2k-1)*(2c-1)!!^2*CatalanNumber[k-1];
5 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
6 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
7 Q[j_,k_,c_,ab_]:=Simplify[g[j,k,c]/(vD[ab,j,k,c-1]*h[j,k,c]+f[j,k,c]Catalan)];
8 l[n_,j_,k_]:=(-1)^(k+1)(2k)!^2/k!/2^(3*k-2)/Product[(k-i)(2k-2i-1)^2,{i,0,(j-3)/2}]/(2k-j)/(2k-1)/(n((2k-j-2)(3-2k)-1)
   +1);
9 kD:=Function[{n,ab,j,k},If[k<2,ab[[1]]+ab[[2]]k,(2k+2j-9-2n)(2k+j-8-2n)(-2k+5-j)(2k+j-6)*kD[n,ab,j,k-2]+(8k^2+k(10j
   -48-8n)+(3j^2-(28+4n)j+68+18n))kD[n,ab,j,k-1]]];
10 Descend:=Function[{j,k},abh[0]={{-1,4},{19,234},{5818/3,254456/3},{667115,60003486},{467946090,71121907440},
11 {554143204110,127451285438100},{994115449382940,322092692148962160},{2516347061651130075,1092094185270706446150},
12 {8546069024090201027250,4785798287838257081935200},{37508692924557081882027450,26331102038134635548392485900},
13 {206659254109760483703789089700,177726957997323983116663150902000},
14 {1396637676485497608584841260027550,1444123356588023432320434243315206700},
15 {11361110319787394788017568214856502500,13905999029609441333101619589964946580000},
16 {109509742351999832489255793094925601037500,156598931559029451368898717824937174831465000},
17 {1234320809247763942235545044494798498436195000,2039097976865181167119056627863149102390546140000},
18 {16085205915675471439195309128783843538512283666875,30401039180587356456007967587920548312623820610393750},
19 {239989379884263177615577263747245812249369757283461250,514537230471714428505965482811829861362838523445500920000}};
20 abh[1]={{-1/3,-14/3},{-17,-8},{-758,-27820},{-302117,-23010044},{-1091480994/5,-146282046156/5},
21 {-262476468810,-54596049230880},{-475443072646380,-141682352738003640},{-1211573031414907725,-489475664504671450500},
22 {-4135193781750207709650,-2175112041708995560914300},{-18218507239728799899288030,-12097088912487772715204794320},
23 {-100680148628028059378172563700,-82356361704096372069838207986600},
24 {-682078864161239229949889893754850,-673893917980353010760236819146271800},
25 {-5559692282317104149119150499246482500,-6527078739785105011529098668023829975000},
26 {-53681246247288656939970174534335708392500,-73865394837022289182570623863734010339760000},
27 {-605939306349175124039948450466713432304279000,-965867254322525126192328035746702493817188406000},
28 {-7906287653442943862409767973858652552097126548125,-14452693830499521315903006473321900713406050369972500},
29 {-118090012323922699712409299094969252935070156336941250,-245391045609131483699190960852072336578359955374403917500}};
30 ab=Table[kD[u,abh[u]][(j-1)/2],j,k-j+2]/l[u,j,k],{u,0,1}];{ab[[1]],ab[[2]]-ab[[1]]}];

```

```

31 Q[j_,k_,c_,ab_]:=Simplify[g[j,k,c]/(vd[ab,j,k,c-1]*h[j,k,c]+f[j,k,c]Catalan)];
32 Print[Union[Flatten[Table[N[NumQ[j,k,c]==Q[j,k,c,Descend[j,k]],200],{j,3,11,2},{k,j-2,10},{c,1,7}]]][[1]]];ClearAll["
Global'*"];

```

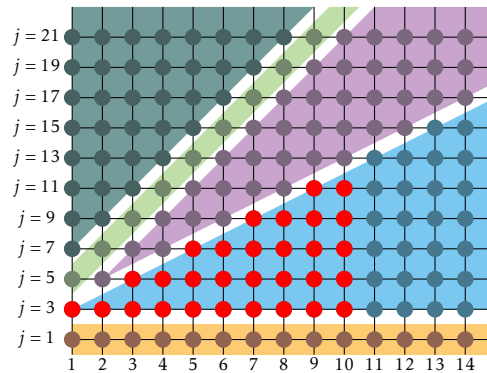


Fig. 11. The points on which Kosovo is tested for $1 \leq c \leq 7$ by the snippet "4. Kosovo" are shown in red.

```

1 (* code snippet "5. resolution" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,10000}];
3 f[j_,k_,c_]:=Product[(2c-2k+2i-1)(k-i+1),{i,1,(j-1)/2}]CatalanNumber[(j-3)/2](j-2)(2k-1)*(2c-1)!!^2*CatalanNumber[k-1];
4 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
5 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
6 l:=Function[{n,j,k},(-1)^(k+1)(2k)!^2/k!/2^(3k-2)/Product[(k-i)(2k-2i-1)^2,{i,0,(j-3)/2}]/(2k-j)/(2k-1)/(n((2k-j-2)(3-2
k)-1)+1)];

```

```

7 AB:=Function[{j,k},For[c=1,c<=2,r=N[NumQ[j,k,c],2000]];v=FindIntegerNullVector[{1,r,N[Catalan*r,2000]};d[c]=(-g[j,k,c](
      v[[3]]Catalan+v[[2]])/v[[1]]-f[j,k,c]Catalan)/h[j,k,c];c++;{d[1],d[2]-d[1]};ABlists:=Function[lim,z[1]=z[0]={};
8 For[j=3,j<=lim,For[w=0,w<=1,e[w]=Table[AB[j,k].{1,w}l[w,j,k],{k,j-2,j-1}];
9 ab[w]={a,b}/.Solve[Table[a+b(u-1)==e[w][[u]],{u,1,2}],{a,b}];AppendTo[z[w],ab[w][[1]]];w++;j+=2];{z[1],z[0]};
10 Print[ABlists[15]];ClearAll["Global'"];

```

```

1 (* code snippet "6. symmetry" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,15000}];
3 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
4 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
5 zeta[j_,u_]:=Product[2+2i-j,{i,0,u-1]}/2^u;Ratio:=Function[{j,k,c},r=N[NumQ[j,k,c],2000];
6 v=FindIntegerNullVector[{1,r,N[Catalan*r,2000]};-v[[1]]/v[[2]]];
7 GenAB[j_,k_]:=({a,b}/.Solve[Table[{a,b}.{1,c-1]==g[j,k,c]/Ratio[j,k,c]/h[j,k,c],{c,1,2}],{a,b}][[1]]);
8 tau[j_,u_]:=If[u>(j-1)/2,Sign[zeta[j,u]](2j-2u-3)!!(2u-j)!!(-2)^u,zeta[j,u](-4)^u];
9 For[j=3,j<=13,Print[MatrixForm[Table[{{j,j-u-1},{j-2u,j-u-1},{(GenAB[j,j-u-1]/GenAB[j-2u,j-u-1])/tau[j,u]}=={1,1}},{u,1,j
      +3}]]];j+=2];ClearAll["Global'"];

```

```

1 (* code snippet "7. Croatia" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,2000}];
3 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
4 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
5 Ratio:=Function[{j,k,c},r=N[NumQ[j,k,c],4000];v=FindIntegerNullVector[{1,r,N[Catalan*r,4000]};-v[[1]]/v[[2]]];
6 GenAB[j_,k_]:=({a,b}/.Solve[Table[{a,b}.{1,c-1]==g[j,k,c]/Ratio[j,k,c]/h[j,k,c],{c,1,2}],{a,b}][[1]]);
7 psi1[i_,j_]:={-1,14-j,-464+58j-3j^2,27936-4692j+432j^2-15j^3,-2659968+542256j-67836j^2+4260j^3-105j
      ^4,367568640-86278560j+13203480j^2-1139700j^3+51450j^4-945j^5}[[i]];
8 psi2[i_,j_]:={4j-15,306-95j+4j^2,-13360+4646j-357j^2+12j^3,999648-379692j+40368j^2-2457j^3+60j^4,-113885568+
9 46449360j-6124164j^2+513228j^3-22935j^4+420j^5,18333538560-7933530720j+1224286440j^2-126833100j^3+7864950j^4-266175j
      ^5+3780j^6}[[i]];
10 mu[i_,j_]:=-Product[j-2q-2,{q,1,i}]/(-2)^((3j-11-4i)/2);
11 Print[Union[Table[Union[Table[GenAB[j,(j-3)/2-i]=={psi1[1+i,j],psi2[1+i,j]}/mu[i,j],{j,5+2i,37,2}][[1]],{i
      ,0,5}][[1]]];ClearAll["Global'"];

```

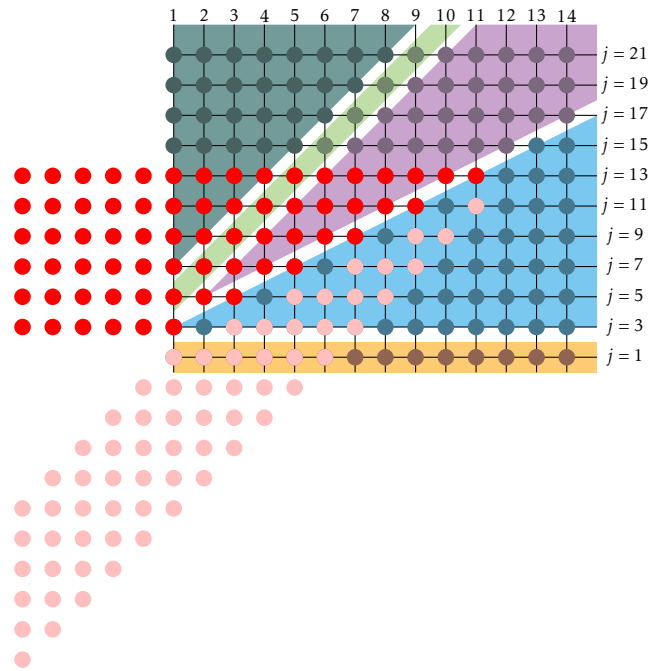


Fig. 12. The points on which the Kosovo-Serbia symmetry is tested at the $(\alpha_{j,\kappa}, \beta_{j,\kappa})$ -level by the snippet "6. symmetry". Tested points are shown in red and their symmetrical correspondants in pink.

1 (* code snippet "8. coefficients" *)

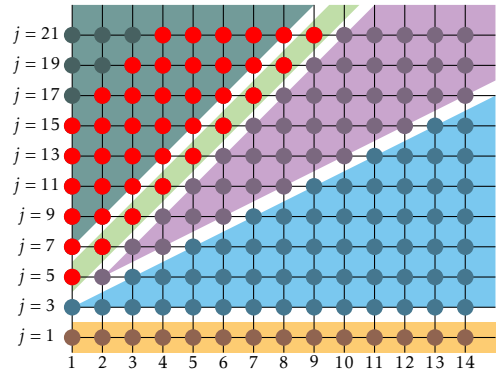


Fig. 13. The points on which `Croatia` is tested. Note that the test is performed directly at the $(\alpha_{j,\kappa}, \beta_{j,\kappa})$ -level. Points tested by the snippet "7. Croatia" are shown in red. Points tested beyond the quadrant are not shown.

```

2 coef=CoefficientList[{3198013886925+(145296572850+(5207427225+(4353102000+(877052475+(78210090+(3023055+41580(X-29)) (X
-27)) (X-25)) (X-23)) (X-21)) (X-19)) (X-17), -14487726825-(104826150+(452605725+(121200300+(13697775+(640710+10395(X
-27)) (X-25)) (X-23)) (X-21)) (X-19)) (X-17)}, X];
3 Print[Table[GCD[coef[[1, i]], coef[[2, i]], {i, 1, 7}]]; ClearAll["Global`*"];

```

```

1 (* code snippet "9. ratio" *)
2 NumQ[j_, k_, c_] := j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n), 3n^2+(3+4k)n+j(2-j+2k), {n, 1, 50000}];
3 For[j=1, j<=7, For[k=Abs[j-2], k<=7, For[c=1, c<=7, r=N[NumQ[j, k, c], 3000]; v=FindIntegerNullVector[{1, r, N[Catalan*r, 3000]}]];
4 q = (-v[[1]]/v[[3]]) == ((2c)!2^(2k+(j-7)/2+Floor[1/j])/CatalanNumber[k-1]/(2k-1)/(2c-1)!!^2/(j-2)/Product[(2c-2k+2i-
5 1)(k-i+1)/(2c-2i+1)/(2k-2i+1), {i, 1, (j-1)/2}]/CatalanNumber[(j-3)/2]);
6 Print[{j, k, c, q}]; c++; k++; j+=2]; ClearAll["Global`*"];

```

```

1 (* code snippet "10. log2-a" *)

```

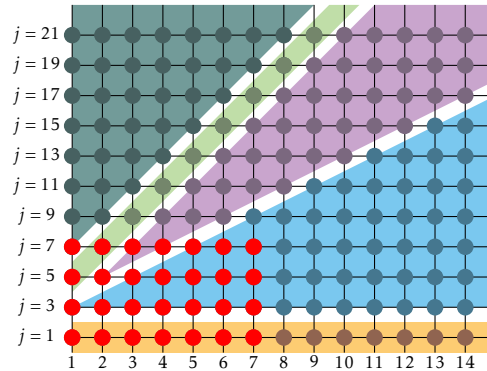


Fig. 14. The points on which the ratio conjecture is tested for $1 \leq c \leq 7$ by the snippet "9. ratio" are shown in red.

```

2 Q[c_]:=c+ContinuedFractionK[-2n^2,3n+c,{n,1,2000}];
3 For[c=2,c<=25,r=N[Q[c],80];
4 v=FindIntegerNullVector[{1,r,N[Log[2]*r,80]}];
5 Print[{c,Simplify[2/LerchPhi[1/2,1,-1+c]==-v[[1]]/(v[[2]]+Log[2]v[[3]])==1/(Log[2]2^(c-2)-Sum[2^(c-j-2)/j,{j,c-2}]]}];
   c++;ClearAll["Global'"];

```

```

1 (* code snippet "11. log2-b"*)
2 R[c_]:=c+ContinuedFractionK[-2n^2-2n,3n+c,{n,1,2000}];
3 For[c=3,c<=20,r=N[R[c],200];v=FindIntegerNullVector[{1,r,N[Log[2]*r,200]}];
4 Print[{c,2^(c-4)*(c-3)*v[[1]]==v[[3]],-v[[1]]/(v[[2]]+Log[2]v[[3]])}];c++;ClearAll["Global'"];

```

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