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Explicit Formulae for Some Polynomial Continued Fractions

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Abstract. In a previous article we gave a collection of continued fractions involving Catalan's constant. In this note we provide the general formula governing those continued fractions. We also show that a single formula involving $\log 2$ discovered by the Ramanujan machine is part of a large family of which we provide examples.

1 Introduction

In a previous article [2] we gave a collection of continued fractions involving Catalan's constant. In this note we provide the general formula governing those continued fractions.

We also show that a single formula involving $\log 2$ discovered by the Ramanujan Machine Project [1, 3, 4] is part of a large family of which we provide examples.

2 Notations

We denote by $n!!$ the semifactorial of, i.e. the product of all the integers from 1 up to n having the same parity as n :

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k) = n(n-2)(n-4)\cdots$$

Because in all the following we will only apply semifactorials to odd numbers, this can be simplified as:

$$n!! = \prod_{k=1}^{\frac{n+1}{2}} (2k-1) = n(n-2)(n-4)\cdots 3 \cdot 1$$

We denote by Catalan's constant by $G = 0.91596559\dots$ and let C_n be the n -th Catalan number:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)! n!} = \prod_{k=2}^n \frac{n+k}{k} \quad \text{for } n \geq 0$$

The first Catalan numbers are:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

3 The Explicit Formulae

Consider:

$$Q_{c,\kappa} = 2\kappa + 1 + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2(n+2\kappa)(n+c)}{3n^2 + (3+4\kappa)n + 2\kappa + 1} \right)$$

Define the functions:

$$\Delta_{c,\kappa}(\alpha, \beta) = \begin{cases} \alpha + \beta c & \text{if } c < 2 \\ -2c(2c-1)(2(c-\kappa)-1)^2 \Delta_{c-2,\kappa} & \text{if } c \geq 2 \\ + (8c^2 + (2-8\kappa)c - 2\kappa + 1) \Delta_{c-1,\kappa} & \end{cases}$$

$$\Gamma_{c,\kappa}(\alpha, \beta) = (2c-1)!!^2 G + \Delta_{c-1,\kappa}(\alpha, \beta) \cdot \prod_{i=0}^{\kappa} (2(c-i)-1)$$

$$\delta_{\kappa} = \frac{4^{\kappa-1}}{(2\kappa-1)C_{\kappa-1}} \quad \text{and} \quad \rho_{\kappa} = \frac{\delta_{\kappa}(-1)^k(1-2k)}{(2k)!(2k-3)!!}$$

$$\alpha_{\kappa} = \rho_{\kappa} \Delta_{k-1,1}(1, -2) \quad \text{and} \quad \beta_{\kappa} = -\rho_{\kappa} (2k-3)^2 \Delta_{k-1,2}(1, 12) - \alpha_{\kappa}$$

$$\text{Then } \forall \kappa, c \in \mathbb{N}^2, \quad Q_{c,\kappa} = \frac{\delta_{\kappa}(2c)!}{\Gamma_{c,\kappa}(\alpha_{\kappa}, \beta_{\kappa})}$$

$Q_{c,\kappa}$ is hence an explicitly computable fraction of the form:

$$Q_{c,\kappa} = \frac{a_0}{a_1 + a_2 G} \quad \text{where } a_0, a_1, a_2 \in \mathbb{Z}$$

```

1 QNum[c_, k_] :=
2 2 k + 1 +
3 ContinuedFractionK[-2 n^2 (n + 2 k) (n + c), 3 n^2 + (3 + 4 k) n + 2 k +
4 1, {n, 1, 2000}];
5 Del[a_, b_, c_, k_] := If[c < 2, a + b c,
6 -2 c (2 c - 1) (2 (c - k) - 1)^2 Del[a, b, c - 2,
7 k] + (8 c^2 + (2 - 8 k) c - 2 k + 1) Del[a, b, c - 1, k]];
8
9 Gam[a_, b_, c_, k_] := ((2 c - 1)!!)^2 Catalan +
10 Product[(2 (c - j) - 1), {j, 0, k - 1}] * Del[a, b, c - 1, k];
11
12 d[k_] := 4^(k - 1) / (2 k - 1) / CatalanNumber[k - 1];
13
14 Rho[k_] := d[k] (-1)^(k) (1 - 2 k) / ((2 k)! (2 k - 3)!!);

```

```

15
16 Alp[k_] := Rho[k] Del[1, -2, k - 1, 1];
17
18 Bet[k_] := - Rho[k] (2 k - 3)^2 Del[1, 12, k - 1, 2] - Alp[k];
19
20 QFor[c_, k_] := d[k] (2 c)!/Gam[Alp[k], Bet[k], c, k];
21
22 Union[Flatten[Table[N[QFor[c, k], 200] == N[QNum[c, k], 200], {c, 1, 20}, {
    k, 1, 20}]]]

```

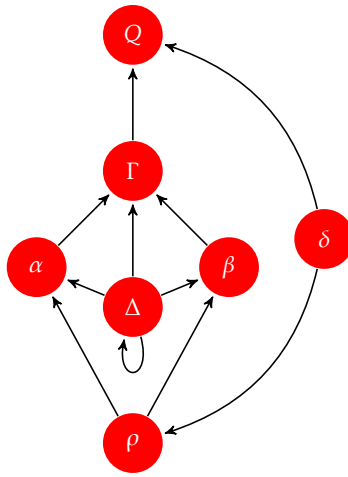


Fig. 1. Functional dependency between the functions computing $Q_{c,k}$

Note that, as described, the complexity of $\Delta_{c,k}$ is exponential in c , however, using classical Fibonacci memoization, this complexity can be reduced to $O(c)$ thereby resulting in a very efficient algorithm for computing $Q_{c,k}$.

4 Low Degree Continued Fractions Involving $\log 2$

Continued fractions involving $\log 2$ can be also obtained with lower degree polynomials, e.g.:

$$\frac{2}{L\left(\frac{1}{2}, 1, c-1\right)} = \frac{2}{\sum_{n=0}^{\infty} \frac{e^{\pi i n}}{(n+c-1)}} = \frac{1}{2^{c-2} \log(2) - \sum_{j=1}^{c-2} \frac{2^{c-j-2}}{j}} = c + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2}{3n+c} \right)$$

```

1 Q[c_] := c + ContinuedFractionK[-2 n^2, 3 n + c, {n, 1, 2000}];
2 For[c = 2, c <= 25,
3   r = N[Q[c], 80];
4   v = FindIntegerNullVector[{1, r, N[Log[2] r, 80]}];
5   Print[{b, c,
6     Simplify[
7       2/LerchPhi[1/2, 1, -1 + c] == -v[[1]]/(v[[2]] + Log[2] v[[3]]) ==
8       1/(Log[2] 2^(c - 2) - Sum[2^(c - j - 2)/j, {j, c - 2}])]];
9   c++];

```

We get a similar behavior for:

$$R_c = c + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2 - 2n}{3n + c} \right)$$

where $2^{c-4} \cdot (c-3) \cdot a_0 = a_2$ and for which we provide numerical examples in Table 5.

```

1 R[c_] := c + ContinuedFractionK[-2 n^2 - 2 n, 3 n + c, {n, 1, 2000}];
2 For[c = 3, c <= 20,
3   r = N[R[c], 200];
4   v = FindIntegerNullVector[{1, r, N[Log[2] r, 200]}];
5   Print[{c,
6     2^(c - 4) * (c - 3) * v[[1]] ==
7     v[[3]], -v[[1]]/(v[[2]] + Log[2] v[[3]])}];
8   c++];

```

5 Further Generalizations

$Q_{c,k} = Q_{1,c,k}$ is a particular case $j = 1$ of a more general continued fraction:

$$Q_{j,c,k} = j(2 - j + 2\kappa) + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(c+n)(j+n-1)(1-j+2\kappa+n)}{j(2-j+2\kappa) + (3+4\kappa)n + 3n^2} \right)$$

When $Q_{j,c,k}$ is not a rational, $Q_{j,c,k}$ converges to:

$$Q_{j,c,k} = \begin{cases} \frac{a_0}{a_1 + a_2 G} & \text{if } j \text{ is odd} \\ \frac{a_0}{a_1 + a_2 \log 2} & \text{if } j \text{ is even} \end{cases}$$

In the examples given in the appendix we adopt the notation:

$$\frac{a_0}{a_1 + a_2 \cdot \text{constant}} = T(0) + \mathcal{K}_{n=1}^{\infty} \left(\frac{P(n)}{T(n)} \right)$$

For odd j , we have (whenever there is no division by zero):

$$\frac{a_0}{a_2} = \frac{(2c)! \cdot 2^{\lfloor 1/j \rfloor + 2k} \cdot \prod_{i=1}^{\frac{j-1}{2}} \frac{(4c - 4i + 2) \cdot (2k - 2i + 1)}{(k - i + 1) \cdot (2c + 2i - 2k - 1)}}{8 \cdot C_{\frac{j-3}{2}} \cdot (j-2) \cdot (2c-1)! \cdot (2k-1) \cdot C_{k-1}}$$

```

1 Num := Function[{j, c, k},
2   j (2 - j + 2 k) +
3   ContinuedFractionK[-2 n (c + n) (j + n - 1) (1 - j + 2 k + n),
4   3 n^2 + (3 + 4 k) n + j (2 - j + 2 k), {n, 1, 100000}]];
5
6 For[j = 1, j <= 7,
7   For[k = 1, k <= 7,
8     For[c = 1, c <= 7,
9       r = N[Num[j, c, k], 3000];
10      v = FindIntegerNullVector[{1, r, N[Catalan r, 3000]}];
11      q = {-v[[1]}/
12          v[[3]] == ((2 c)! 2^(2 k + (j - 7)/2 + Floor[1/j])/
13                  CatalanNumber[k - 1]/(2 k - 1)/(2 c - 1)!^2/(j - 2)/
14                  Product[(2 c - 2 k + 2 i -
15                          1) (k - i + 1)/(2 c - 2 i + 1)/(2 k - 2 i + 1), {i,
16                          1, (j - 1)/2}]/CatalanNumber[(j - 3)/2])};
17      Print[{j, k, c, q}];
18      c++;
19      k++;
20      j += 2]

```

References

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3. G. Raayoni, S. Gottlieb, Y. Manor, G. Pisha, Y. Harris, U. Mendlovic, D. Haviv, Y. Hadad, and I. Kaminer. Generating Conjectures on Fundamental Constants with the Ramanujan Machine. *Nature*, 590(7844):67–73, Feb 2021.
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a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
-7351344	-32375839	46558512	$n(3+n)(17+n)$	$72 + 43n$
-2450448	-1768477	2450448	$n(1+n)(17+n)$	$36 + 39n$
-1081080	-16147379	23279256	$n(5+n)(15+n)$	$96 + 43n$
-793800	-232217	322560	$(4+n)(9+n)^2$	$100 + 39n$
-504504	-16140515	23279256	$n(7+n)(13+n)$	$112 + 43n$
-436590	7989199	-11531520	$(3+n)(9+n)(11+n)$	$120 + 43n$
-180180	-1001393	1441440	$(2+n)(5+n)(13+n)$	$84 + 39n$
-174636	-8069449	11639628	$n(9+n)(11+n)$	$120 + 43n$
-72072	-850133	1225224	$n(5+n)(13+n)$	$84 + 39n$
-72072	-251099	360360	$n(3+n)(13+n)$	$56 + 35n$
-72072	-52279	72072	$n(1+n)(15+n)$	$32 + 35n$
-45045	124048	-180180	$(1+n)(3+n)(13+n)$	$56 + 35n$
-28028	-999391	1441440	$(2+n)(7+n)(13+n)$	$112 + 43n$
-27720	-20417	27720	$n(1+n)(11+n)$	$24 + 27n$
-22050	27649	-40320	$(3+n)(7+n)(9+n)$	$80 + 35n$
-19305	424423	-612612	$(1+n)(5+n)(15+n)$	$96 + 43n$
-17640	-250007	360360	$n(7+n)(9+n)$	$80 + 35n$
-14700	-153907	221760	$(2+n)(7+n)(9+n)$	$80 + 35n$
-10395	124741	-180180	$(1+n)(5+n)(11+n)$	$72 + 35n$
-9450	76691	-110880	$(1+n)(5+n)(9+n)$	$60 + 31n$
-7350	-62563	90090	$n(7+n)^2$	$64 + 31n$
-7007	424566	-612612	$(1+n)(7+n)(13+n)$	$112 + 43n$
-6300	-14087	20160	$(2+n)(5+n)(9+n)$	$60 + 31n$
-6006	99839	-144144	$(1+n)(5+n)(13+n)$	$84 + 39n$
-4900	-21043	30240	$(2+n)(7+n)^2$	$64 + 31n$
-2520	-1879	2520	$n(1+n)(9+n)$	$20 + 23n$
-2450	28781	-41580	$(1+n)(7+n)^2$	$64 + 31n$
-1225	367	-560	$(3+n)(7+n)^2$	$64 + 31n$
-525	-2413	3465	$n(5+n)(7+n)$	$48 + 27n$
-450	1151	-1680	$(1+n)(5+n)^2$	$36 + 23n$
-350	1739	-2520	$(1+n)(5+n)(7+n)$	$48 + 27n$
-180	-299	420	$n(3+n)(5+n)$	$24 + 19n$
-105	142	-210	$(1+n)(3+n)(7+n)$	$32 + 23n$
-18	7	-12	$(1+n)(3+n)^2$	$16 + 15n$
-6	-5	6	$n(1+n)(3+n)$	$8 + 11n$

Table 1. Examples of convergence to $\frac{a_0}{a_1 + a_2 \log 2}$

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
1	1	-1	$n(1+n)^2$	$4+7n$
9	11	-15	$n(3+n)^2$	$16+15n$
50	147	-210	$n(5+n)^2$	$36+23n$
60	47	-60	$n(1+n)(5+n)$	$12+15n$
90	-79	120	$(1+n)(3+n)(5+n)$	$24+19n$
420	319	-420	$n(1+n)(7+n)$	$16+19n$
420	887	-1260	$n(3+n)(7+n)$	$32+23n$
900	361	-480	$(2+n)(5+n)^2$	$36+23n$
1890	-3443	5040	$(1+n)(3+n)(9+n)$	$40+27n$
2100	2377	-3360	$(2+n)(5+n)(7+n)$	$48+27n$
5544	50035	-72072	$n(5+n)(11+n)$	$72+35n$
6468	-249713	360360	$(1+n)(7+n)(11+n)$	$96+39n$
7560	19409	-27720	$n(3+n)(9+n)$	$40+27n$
8316	-19031	27720	$(1+n)(3+n)(11+n)$	$48+31n$
15444	-49705	72072	$(1+n)(3+n)(15+n)$	$64+39n$
22050	-499279	720720	$(1+n)(7+n)(9+n)$	$80+35n$
24255	-76586	110880	$(3+n)(7+n)(11+n)$	$96+39n$
25740	200107	-288288	$(2+n)(5+n)(15+n)$	$96+43n$
37800	250427	-360360	$n(5+n)(9+n)$	$60+31n$
38808	849671	-1225224	$n(7+n)(11+n)$	$96+39n$
39690	-1997851	2882880	$(1+n)(9+n)^2$	$100+39n$
41580	154327	-221760	$(2+n)(5+n)(11+n)$	$72+35n$
58212	3997025	-5765760	$(2+n)(9+n)(11+n)$	$120+43n$
79380	2123957	-3063060	$n(9+n)^2$	$100+39n$
83160	251561	-360360	$n(3+n)(11+n)$	$48+31n$
87318	-8491859	12252240	$(1+n)(9+n)(11+n)$	$120+43n$
97020	1999321	-2882880	$(2+n)(7+n)(11+n)$	$96+39n$
132300	3997907	-5765760	$(2+n)(9+n)^2$	$100+39n$
198450	-1227581	1774080	$(3+n)(9+n)^2$	$100+39n$
216216	852707	-1225224	$n(3+n)(15+n)$	$64+39n$
360360	263111	-360360	$n(1+n)(13+n)$	$28+31n$
630630	-3990557	5765760	$(3+n)(7+n)(13+n)$	$112+43n$
918918	-3383801	4900896	$(1+n)(3+n)(17+n)$	$72+43n$
1746360	2475007	-3548160	$(4+n)(9+n)(11+n)$	$120+43n$
46558512	33464927	-46558512	$n(1+n)(19+n)$	$40+43n$

Table 2. Examples of convergence to $\frac{a_0}{a_1+a_2 \log 2}$. The entry in blue is the one reported by the Ramanujan Project.

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
-80281600	-10675439	9459450	$(2+n)(3+n)(14+n)$	$45 + 35n$
-15728640	392683	-727650	$(4+n)^2(12+n)$	$65 + 35n$
-13107200	-263867	-28350	$(4+n)(5+n)(10+n)$	$55 + 31n$
-10485760	-93699	-103950	$(4+n)(5+n)(12+n)$	$65 + 35n$
-7372800	-884203	727650	$(2+n)(3+n)(12+n)$	$39 + 31n$
-6291456	149419	-257250	$(4+n)(8+n)^2$	$81 + 35n$
-5242880	-86807	9450	$(5+n)(6+n)(10+n)$	$77 + 35n$
-3932160	-116317	3150	$(4+n)(5+n)(8+n)$	$45 + 27n$
-3276800	-158859	22050	$(2+n)(5+n)(10+n)$	$33 + 27n$
-2621440	-48609	-1050	$(5+n)(6+n)(8+n)$	$63 + 31n$
-2359296	-168445	103950	$(2+n)(4+n)(12+n)$	$39 + 31n$
-491520	50593	-66150	$(3+n)(4+n)(10+n)$	$55 + 31n$
-327680	-21271	9450	$(2+n)(4+n)(10+n)$	$33 + 27n$
-230400	-1909	-22050	$(2+n)^2(5+n)$	$9 + 11n$
-196608	-184547	198450	$(3+n)(6+n)(10+n)$	$77 + 35n$
-163840	-4981	-22050	$n(5+n)(8+n)$	$9 + 19n$
-131072	-2951	-630	$(4+n)^2(8+n)$	$45 + 27n$
-122880	-13079	9450	$(2+n)(3+n)(10+n)$	$33 + 27n$
-61440	-2467	-3150	$(2+n)(4+n)(5+n)$	$15 + 15n$
-61440	791	-9450	$n(5+n)(6+n)$	$7 + 15n$
-51200	2839	-9450	$n(4+n)(5+n)$	$5 + 11n$
-49152	-1919	90	$(4+n)^2(6+n)$	$35 + 23n$
-36864	-2693	-450	$(2+n)(4+n)(6+n)$	$21 + 19n$
-18432	-419	-3150	$n(4+n)(6+n)$	$7 + 15n$
-18432	-419	-3150	$n(3+n)(8+n)$	$9 + 19n$
-8192	-487	-54	$(4+n)^3$	$25 + 19n$
-3072	121	-630	$n(4+n)^2$	$5 + 11n$
-2048	-129	-90	$(2+n)(4+n)^2$	$15 + 15n$
-2048	-43	-30	$(3+n)(4+n)(6+n)$	$35 + 23n$
-768	-77	-18	$(2+n)(3+n)(4+n)$	$15 + 15n$
-288	31	-90	$n(2+n)(3+n)$	$3 + 7n$

Table 3. Examples of convergence to $\frac{a_0}{a_1+a_2G}$

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
192	13	18	$(2+n)^2(3+n)$	$9+11n$
384	1	90	$n(3+n)(4+n)$	$5+11n$
2304	389	450	$n(3+n)(6+n)$	$7+15n$
3072	179	-18	$(3+n)(4+n)^2$	$25+19n$
4608	133	450	$(2+n)^2(4+n)$	$9+11n$
4608	383	-90	$(2+n)(3+n)(6+n)$	$21+19n$
11520	-1373	3150	$n(2+n)(4+n)$	$3+7n$
12288	973	-750	$(3+n)(6+n)^2$	$49+27n$
12288	1145	-630	$(2+n)(3+n)(8+n)$	$27+23n$
16384	-543	1050	$(3+n)(4+n)(8+n)$	$45+27n$
81920	3983	1350	$(4+n)^2(5+n)$	$25+19n$
89600	-10891	22050	$n(2+n)(5+n)$	$3+7n$
98304	2263	150	$(4+n)(6+n)^2$	$49+27n$
98304	35389	-36750	$(3+n)(6+n)(8+n)$	$63+31n$
122880	6563	3150	$(2+n)(5+n)(6+n)$	$21+19n$
147456	21365	22050	$n(4+n)(8+n)$	$9+19n$
163840	6789	450	$(4+n)(5+n)(6+n)$	$35+23n$
262144	710401	-771750	$(3+n)(8+n)^2$	$81+35n$
294912	18013	-3150	$(2+n)(4+n)(8+n)$	$27+23n$
524288	27787	-22050	$(4+n)(6+n)(10+n)$	$77+35n$
786432	19099	-5250	$(4+n)(6+n)(8+n)$	$63+31n$
983040	25979	-450	$(5+n)(6+n)^2$	$49+27n$
1310720	1723	28350	$(4+n)^2(10+n)$	$55+31n$
2949120	168821	22050	$(2+n)(5+n)(8+n)$	$27+23n$
9830400	-1833409	2182950	$(3+n)(4+n)(12+n)$	$65+35n$
12582912	184025	-7350	$(5+n)(8+n)^2$	$81+35n$
39321600	1965547	-727650	$(2+n)(5+n)(12+n)$	$39+31n$
165150720	12969199	-9459450	$(2+n)(4+n)(14+n)$	$45+35n$
330301440	17687791	-9459450	$(2+n)(5+n)(14+n)$	$45+35n$

Table 4. Examples of convergence to $\frac{a_0}{a_1+a_2G}$

c	a_0	a_1	a_2
3	2	1	0
4	1	1	-1
5	-1	-3	4
6	-2	-17	24
7	-3	-67	96
8	12	667	-960
9	5	666	-960
10	30	9319	-13440
11	-105	-74537	107520
12	-280	-447187	645120
13	126	447173	-645120
14	63	491884	-709632
15	231	3935051	-5677056
16	2772	102311095	-147603456
17	-1287	-102310996	147603456
18	-6006	-1023109531	1476034560
19	45045	16369749493	-23616552960
20	720720	556571437717	-802962800640

Table 5. Examples of convergence to $\frac{a_0}{a_1+a_2 \log 2}$ for R_c