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The Balkans Continued Fraction

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Abstract. In a previous escapade we gave a collection of continued fractions involving Catalan's constant. This paper provides more general formulae governing those continued fractions. Having distinguished different cases associated to regions in the plan, we nickname those continued fractions "The Balkans" as they divide into areas which are related but still different in nature.

Because we do not provide formal proofs of those machine-constructed formulae we do not claim them to be theorems. Still, each and every proposed formula was extensively tested numerically.

1 Introduction

In a previous escapade [12] we gave a collection of continued fractions involving Catalan's constant. This paper provides more general formulae governing those continued fractions. Having distinguished different cases associated to regions in the plan, we nickname those continued fractions "The Balkans" as they divide into areas which are related but still different in nature.

Because we do not provide formal proofs of those machine-constructed formulae we do not claim them to be theorems. Still, each and every proposed formula was extensively tested numerically.

All the programs included in this article are *self-contained*, i.e. any code snippet can be run independently of the others to fully illustrate the encoded formula. This renders the code longer but has the great advantage of allowing the reader to run and modify each snippet directly by just cutting and pasting it into Mathematica without requiring any other module¹. The code was compacted for the sake of concision but loading it into Mathematica's editor re-indents it automatically.

2 Notations

We denote by $n!!$ the semifactorial of, i.e. the product of all the integers from 1 up to n having the same parity as n :

¹ Each snippet ends by a `ClearAll["Global`*"]`; command whose purpose is to make Mathematica "forget" all passed history.

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k) = n(n-2)(n-4) \cdots$$

Because in all the following we will only apply semifactorials to odd numbers, this can be simplified as:

$$n!! = \prod_{k=1}^{\frac{n+1}{2}} (2k-1) = n(n-2)(n-4) \cdots 3 \cdot 1$$

We denote by Catalan's constant by $G = 0.91596559 \dots$ and let C_n be the n -th Catalan number:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)! n!} = \prod_{k=2}^n \frac{n+k}{k} \quad \text{for } n \geq 0$$

The first Catalan numbers are:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

3 The target

We define for odd j and $\kappa, c \in \mathbb{N}$ the following quantity nicknamed “*The Balkans continued fraction*”:

$$Q_{j,\kappa,c} = j(2-j+2\kappa) + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(c+n)(j+n-1)(1-j+2\kappa+n)}{j(2-j+2\kappa) + (3+4\kappa)n + 3n^2} \right)$$

The question asked is that of finding a general process allowing to compute $Q_{j,\kappa,c}$ while resorting as *little as possible* to numerical simulations or integer relation algorithms. The reason for this is that while integer relation algorithms allow us to “magically” discover relations, they do not provide *general information* about the underlying structure of the constants found. Why? Because we're in it for the thrill of discovery, not just the magic of shortcuts.

3.1 Why this formula in particular?

The Ramanujan Machine Project [14,5], lists a few continued fractions involving G detected in 2020. We do not know why the project did not resort to (rather basic) integer relation algorithms to discover more relations. We hence we decided to play detective and launched a code that found a few hundreds of continued fractions involving G in a few intensive calculations days. All continued fractions were of the form:

$$\epsilon + \prod_{n=1}^{\infty} \left(\frac{-2n(n+\tau)(n+\eta)(n+\mu)}{\epsilon + \delta n + 3n^2} \right) = \frac{a_0}{a_1 + a_2 G} \text{ where } a_0, a_1, a_2 \in \mathbb{Z}$$

We performed two natural tests on the coefficient vectors $(\delta, \epsilon, \tau, \eta, \mu)$: a PCA to determine if the coefficients can be expressed as linear combinations of less than 5 variables and a Hough transform to detect affine relations in the dataset.

PCA revealed that, when projected on $(\delta, \tau, \eta, \mu)$, nearly all data was governed by three linear dimensions². We hence understood that we were facing a linear behavior in a large region (Balkans) plus some sporadic cases (see Remark 13). This was also confirmed by the Hough transform that detected several parallel plans in the 3D-space. We thus decided to focus our efforts on the main plan in the 3D-space and understand it.

3.2 How formulae were reverse-engineered

Now comes the real adventure. The process that allowed us to reverse-engineer the formulae given in this paper is interesting by its own right. A quick look at many examples of the three quantities a_0, a_1, a_2 forming the fractions:

$$Q_{j,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \text{ where } a_0, a_1, a_2 \in \mathbb{Z}$$

showed that the a_i s are products of small prime factors and a few large prime factors. This suggested that a_i s were initially³ of some form:

$$a_i = \text{expression}(j, \kappa, c) = \frac{\prod_{i=0}^{u-1} \phi_i(j, \kappa, c)}{\prod_{i=0}^{v-1} \phi_i(j, \kappa, c)}$$

where the ϕ_i are functions such as $(an+b)!$, 2^{an+b} , $(an+b)!!$, C_{an+b} , Pochhammer symbols of linear combinations of the parameters j, κ, c etc. and a few unknown “mixing” functions causing the appearance of the large prime factors, e.g. polynomials or recurrence relations.

Fortunately, integer relation algorithms allow us to collect many instances of such forms for diverse j, κ, c values. Hence the problem at hand consists in identifying which ϕ_i s are compatible with the cancellations due to the division. If a given ϕ is present in the expression then it is reasonably assumed that when tried for many j, κ, c the new expression:

$$\text{expression}_1(j, \kappa, c) = \frac{\text{expression}(j, \kappa, c)}{\phi(j, \kappa, c)}$$

or

$$\text{expression}_1(j, \kappa, c) = \text{expression}(j, \kappa, c) \cdot \phi(j, \kappa, c)$$

² To which we gave the names j, κ, c .

³ i.e., before simplification intervenes.

will feature less small factors and hence stand-out as an outlier.

The process can hence be repeated with proper backtracking until all the combinatorial ϕ_i s were peeled-off. Then it remains to detect what the remaining “mixing” functions are which is done by monitoring the average growth rate of those surviving constants to emit hypotheses on the type of recurrence relations (or polynomials) at hand or resorting to a variety of integer sequence recognition tools to identify the hidden culprits.

- We started our exploration with the simplest case of **Bosnia & Herzegovina** where $a_2 = 0$.
- Having reverse-engineered **Bosnia & Herzegovina** we moved to **Croatia** for which $a_2 = 0$ as well.
- The conjectured similarity between those two \mathbb{Q} regions guided our software to the formula for **Montenegro** which is somewhat simpler than **Kosovo** and **Serbia** given that **Montenegro** corresponds to $j = 1$.
- Having inferred **Montenegro** we moved on to **Kosovo** whose symmetry⁴ with **Serbia** was quickly noted.

This work demonstrates the interest of statistical classifiers such as Maximum Likelihood Estimation (MLE) and Support Vector Machines (SVM) in mathematical exploration.

As will be shown, this “gradient descent” method proved itself very well, although it required a few thousands of computation hours on a very powerful cluster.

We estimate that 80% of the discovery effort was done by the machine. The remaining 20% being human “piloting” that, we are convinced, is already at the reach of today’s most powerful LLMs, such as Gemini.

It appears much better to read the coming sections first to understand what prey we are stalking and then refer to Section 11 describing the hunting process.

4 **Kosovo, Serbia, Croatia, Montenegro, Bosnia & Herzegovina**

As we will see, we distinguish five cases that we call **Kosovo**, **Serbia**, **Croatia**, **Montenegro** and **Bosnia & Herzegovina** after the regions of interest in the (j, κ) -space shown in Figure 1. In each region c runs over the integers.

A first restriction of our study will be to focus on odd j that produce $Q_{j,\kappa,c}$ values involving G . Note that:

$$Q_{j,c,\kappa} = \begin{cases} \frac{a_0}{a_1 + a_2 G} & \text{if } j \text{ is odd} \\ \frac{a_0}{a_1 + a_2 \log 2} & \text{if } j \text{ is even} \end{cases}$$

⁴ This symmetry simply comes from the fact that the equation $j(2 - 2j + 2\kappa) = x(2 - 2x + 2\kappa)$ has the two solutions $x = j$ and $x = \kappa - j + 1$.

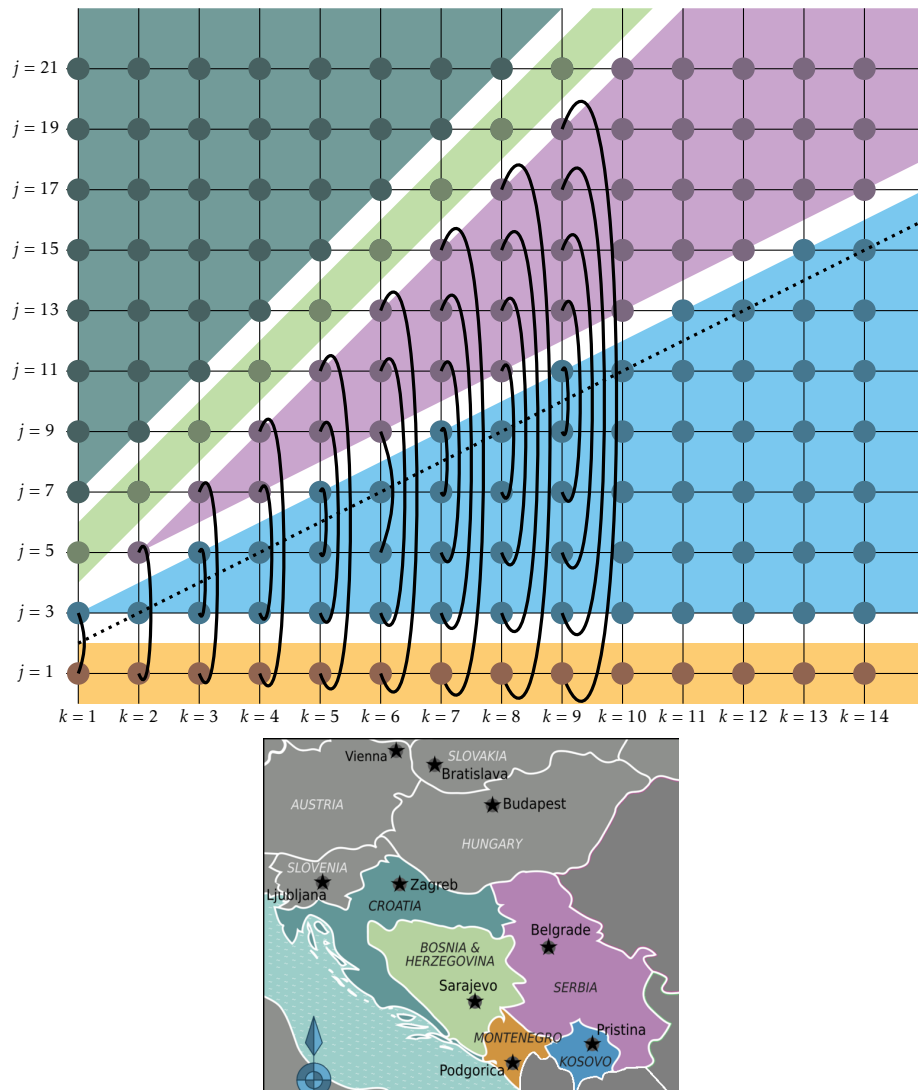


Fig. 1. The Five j,k Areas. The meaning of the arches connecting Kosovo, Serbia, and Montenegro will be clarified later.

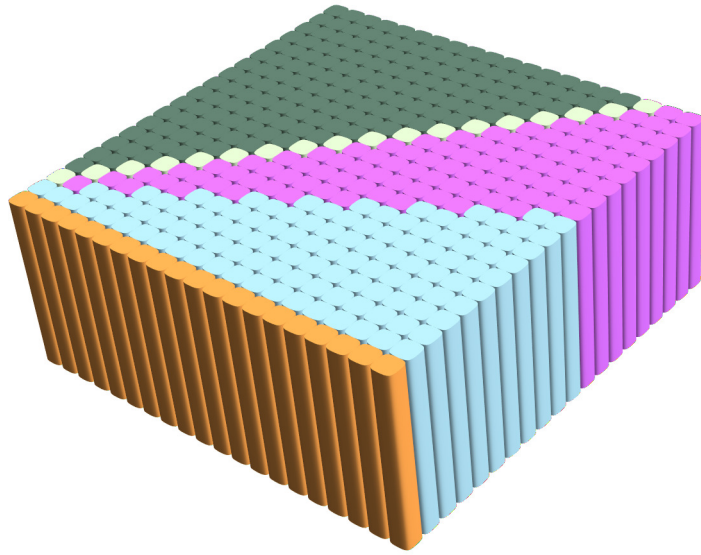


Fig. 2. The areas of Figure 1, with the dimension c added. Each point in space corresponds to a $Q_{j,\kappa,c}$ value.

Area	Domain	$Q_{j,\kappa,c}$ is in
Croatia	$j \geq 2\kappa + 5$	\mathbb{Q}
Bosnia & Herzegovina	$j = 2\kappa + 3$	\mathbb{Q}
Serbia	$2\kappa + 1 \geq j \geq \kappa + 3$	\mathbb{R}
Kosovo	$\kappa + 2 \geq j \geq 3$	\mathbb{R}
Montenegro	$j = 1$	\mathbb{R}

Table 1. Areas. We list fractions involving G as being in \mathbb{R} although it is currently unknown if $G \in \mathbb{Q}$ (this is a major open question).

Note as well that j, κ, c can be negative. In the examples given in the appendix we adopt the notation:

$$\frac{a_0}{a_1 + a_2 \cdot \text{constant}} = T(0) + \mathcal{K}_{n=1}^{\infty} \left(\frac{P(n)}{T(n)} \right)$$

This paper does not treat the even and negative j, κ, c cases (a follow-up paper dealing with those cases being underway).

5 The Montenegro Conjecture

We start with the first j, κ space called **Montenegro**. **Montenegro** corresponds to the case $j = 1$. In other words:

$$Q_{1,\kappa,c} = 2\kappa + 1 + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2(n+2\kappa)(n+c)}{3n^2 + (3+4\kappa)n + 2\kappa + 1} \right)$$

Define the functions:

$$\Delta_{\kappa,c}(\alpha, \beta) = \begin{cases} \alpha + \beta c & \text{if } c < 2 \\ -2c(2c-1)(2(c-\kappa)-1)^2 \Delta_{\kappa,c-2}(\alpha, \beta) & \text{if } c \geq 2 \\ + (8c^2 + (2-8\kappa)c - 2\kappa + 1) \Delta_{\kappa,c-1}(\alpha, \beta) & \end{cases}$$

$$\Gamma_{\kappa,c}(\alpha, \beta) = (2c-1)!!^2 G + \Delta_{\kappa,c-1}(\alpha, \beta) \cdot \prod_{i=0}^{\kappa-1} (2(c-i)-1)$$

$$\delta_{\kappa} = \frac{4^{\kappa-1}}{(2\kappa-1)C_{\kappa-1}} \quad \text{and} \quad \rho_{\kappa} = \frac{\delta_{\kappa}(-1)^{\kappa}(1-2\kappa)}{(2\kappa)!(2\kappa-3)!!}$$

$$\alpha_{\kappa} = \rho_{\kappa} \Delta_{1,\kappa-1}(1, -2) \quad \text{and} \quad \beta_{\kappa} = -\rho_{\kappa} (2\kappa-3)^2 \Delta_{2,\kappa-1}(1, 12) - \alpha_{\kappa}$$

$$\text{Then } \forall \kappa, c \in \mathbb{N}^2, \quad Q_{1,\kappa,c} = \frac{\delta_{\kappa}(2c)!}{\Gamma_{\kappa,c}(\alpha_{\kappa}, \beta_{\kappa})}$$

$Q_{1,\kappa,c}$ is hence an explicitly computable fraction of the form:

$$Q_{1,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \quad \text{where } a_0, a_1, a_2 \in \mathbb{Z}$$

The code snippet testing this formula over the square $1 \leq \kappa, c \leq 14$ is entitled "1. Montenegro".

In summery in **Montenegro** we do not need to resort to any integer relation algorithms to compute $Q_{1,\kappa,c}$ for all κ, c values.

Remark 1. Note that, as described here, the complexity of $\Delta_{\kappa,c}$ is exponential in c , however, using classical Fibonacci memoization, this complexity can be reduced to $O(c \log c)$ thereby resulting in a very efficient algorithm for computing $Q_{1,\kappa,c}$.

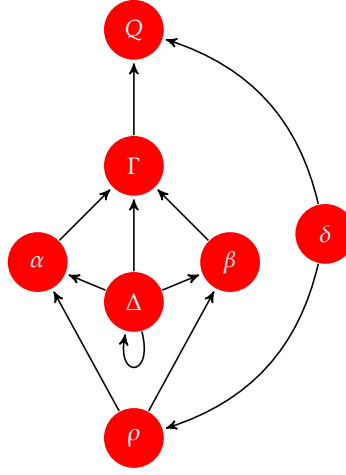


Fig. 3. Functional dependency between the functions computing $Q_{1,\kappa,c}$

6 The Bosnia & Herzegovina Conjecture

Bosnia & Herzegovina corresponds to the line $2\kappa = j - 3$, that is:

$$Q_{j, \frac{j-3}{2}, c} = -j + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(n-2)(n+c)(n+j-1)}{3n^2 + (2j-3)n - j} \right)$$

We start by defining:

$$\alpha_j = (-2)^{\frac{3j-11}{2}} \quad \text{and} \quad \beta_j = \alpha_j(15 - 4j)$$

And let:

$$\Delta_{j,c}(\alpha_j, \beta_j) = \begin{cases} \alpha_j + \beta_j c & \text{if } c < 2 \\ -2c(2c-j)(2c+1)(2c-j+2)\Delta_{j,c-2}(\alpha_j, \beta_j) \\ \quad + (8c^2 + (14-4j)c - 3(j-2))\Delta_{j,c-1}(\alpha_j, \beta_j) & \text{if } c \geq 2 \end{cases}$$

$$g(j, c) = (2c)! 2^{\frac{3j-13}{2}} \prod_{i=1}^{\frac{j-1}{2}} (2c-2i+1)(j-2-2i)$$

$$h(j, c) = (2c-j+2) \prod_{i=1}^{\frac{j-5}{2}} (2c-2i-1)^2$$

Then:

$$Q_{j, \frac{j-3}{2}, c} = \frac{g(j, c)}{\Delta_{j, c-1}(\alpha_j, \beta_j) \cdot h(j, c)} \in \mathbb{Q}$$

Hence, in **Bosnia & Herzegovina** as well we do not need to resort to any integer relation algorithms to compute $Q_{j, \frac{j-3}{2}, c}$ which is an explicitly computable fraction in \mathbb{Q} .

The corresponding code snippet is "2. Bosnia". We conjecture that **Bosnia & Herzegovina** is actually a particular case of **Croatia** which values are also in \mathbb{Q} .

7 Roadmap

We are now ready to describe the general scheme that will govern the rest of this paper.

7.1 Areas governed by one running variable

As we have just seen, **Montenegro** and **Bosnia & Herzegovina** follow similar behaviors. This pattern revolves around the "magic" values α, β which are formally known for **Montenegro** and **Bosnia & Herzegovina**. Both regions are lines parameterized by a single running variable (κ for **Montenegro** and j for **Bosnia & Herzegovina**).

7.2 The c -level master formula for all Balkans except **Montenegro**

Except **Montenegro** (whose case was settled) all other areas obey a common c -level master formula that we will provide below. Computing $Q_{j, \kappa, c}$ for any c using this c -level master formula requires knowledge of two rational parameters: $\alpha_{j, \kappa}$ and $\beta_{j, \kappa}$.

The cornerstone of the rest of this paper is thus the quest for $\alpha_{j, \kappa}$ and $\beta_{j, \kappa}$ for all Balkan areas except **Montenegro**.

$\alpha_{j, \kappa}, \beta_{j, \kappa}$ can always be inferred by computing numerically⁵ Q_{j, κ, c_1} and Q_{j, κ, c_2} for two values $c_1 \neq c_2$ and solving a system of two equations in the unknowns $\alpha_{j, \kappa}, \beta_{j, \kappa}$.

When $\alpha_{j, \kappa}, \beta_{j, \kappa}$ are found $Q_{j, \kappa, c}$ can be computed for any other $c \notin \{c_1, c_2\}$.

Using this process requires resorting twice to integer relation algorithms such as LLL [11], HJLS [10], PSOS [1,7] or PSLQ [8]. This works perfectly in practice but is not entirely satisfactory because the process has a "blind spot" which is the integer relation oracle. We would like to avoid such blind spots as much as possible and dispose of a fully algebraic process for computing each $Q_{j, \kappa, c}$. The ideal situation being, of course, a direct algebraic computation of $Q_{j, \kappa, c}$ from the data j, κ, c alone.

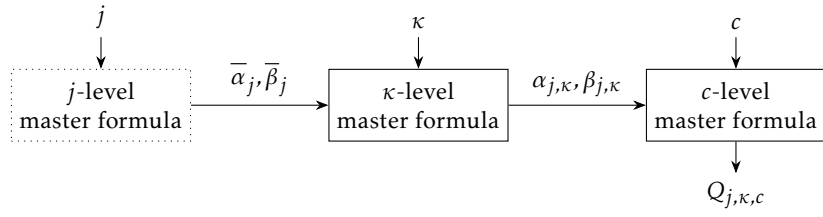


Fig. 4. j , κ and c -level master formulae. j -level master formulae (dotted) are only conjectured to exist. To date we have processes for inferring c -level and κ -level master formulae.

While we do not achieve this goal completely, we grandly progress towards it by minimizing a lot the call to the integer relation oracle.

The snippet "3. Northern Balkans" illustrates the process with all areas except [Montenegro](#). The snippet validates the c -level master formula on the 3D volume $3 \leq j \leq 13$, $1 \leq \kappa \leq 6$, $1 \leq c \leq 7$ (Figure 16) stretching over parts of [Croatia](#), [Bosnia & Herzegovina](#), [Serbia](#) and [Kosovo](#). In each case the program derives the corresponding $\alpha_{j,\kappa}, \beta_{j,\kappa}$ and allows the computing of $Q_{j,\kappa,c}$ for any c . The formally computed results are then successfully compared to numerical ones.

As we will later see, for [Serbia](#) and [Kosovo](#) we have more powerful master formulae operating at the κ -level.

The c -level master formula⁶ is defined as follows, assuming that we are somehow given the magic constants $\alpha_{j,\kappa}, \beta_{j,\kappa}$.

Let:

$$\Delta_{j,\kappa,c}(\alpha_{j,\kappa}, \beta_{j,\kappa}) = \begin{cases} \alpha_{j,\kappa} + \beta_{j,\kappa}c & \text{if } c < 2 \\ -2c(2c-j)(2c-2\kappa+j-2)(2c-2\kappa-1)\Delta_{j,\kappa,c-2}(\alpha_{j,\kappa}, \beta_{j,\kappa}) & \text{if } c \geq 2 \\ \quad + (8c^2 + (2-8\kappa)c + (j-2)(2\kappa-j))\Delta_{j,\kappa,c-1}(\alpha_{j,\kappa}, \beta_{j,\kappa}) & \end{cases}$$

$$f_{j,\kappa,c} = C_{\frac{j-3}{2}} C_{\kappa-1} (j-2)(2\kappa-1)(2c-1)!!^2 \prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1)(\kappa-i+1)$$

$$g_{j,\kappa,c} = (2c)! 2^{\frac{j+4\kappa-7}{2}} \prod_{i=1}^{\frac{j-1}{2}} (2c-2i+1)(2\kappa-2i+1)$$

$$h_{j,\kappa,c} = \prod_{i=0}^{\frac{j-3}{2}} (2c-2i-1) \prod_{i=0}^{\kappa-1} (2c-2i-1)$$

⁵ i.e. using an integer relation algorithm.

⁶ valid for all areas except [Montenegro](#) whose case was anyhow previously settled.

Then:

$$Q_{j,\kappa,c} = \frac{g(j, \kappa, c)}{\Delta_{j,\kappa,c-1}(\alpha_{j,\kappa}, \beta_{j,\kappa}) \cdot h(j, \kappa, c) + f(j, \kappa, c) \cdot G}$$

Remark 2. For **Croatia** $f_{j,\kappa,c} = 0$. This happens automatically given the definition of $f_{j,\kappa,c}$.

7.3 Symmetries

Montenegro, **Kosovo** and **Serbia** feature internal symmetries illustrated by the arches in Figures 1 where connected values are identical. We will also provide a formula connecting their α, β values over the plan. This means that any formula applicable to **Kosovo** will also settle the case of **Serbia** (given that **Montenegro** is settled and given that **Montenegro** provides the values for the upper border line of **Serbia**). We will come to that later on.

8 The Kosovo Conjecture

Kosovo is very specific in that it has a κ -level master formula. This means that for each j , knowledge of four upper-level rational constants⁷:

$$\{\bar{\alpha}_j, \bar{\beta}_j\} = \left\{ \left\{ \alpha_j^\alpha, \alpha_j^\beta \right\}, \left\{ \beta_j^\alpha, \beta_j^\beta \right\} \right\}$$

allows to generate $\alpha_{j,\kappa}, \beta_{j,\kappa}$ for all κ values along a j -line within **Kosovo**.

Finding $\bar{\alpha}_j, \bar{\beta}_j$ for a given j requires solving a system of equations using four known $\alpha_{j,\kappa_1}, \beta_{j,\kappa_1}, \alpha_{j,\kappa_2}, \beta_{j,\kappa_2}$ for some $\kappa_1 \neq \kappa_2$. To determine the two $\alpha_{j,\kappa_i}, \beta_{j,\kappa_i}$ pairs⁸ we can solve two systems of equations (each in two unknowns) using any $Q_{j,\kappa_1,c_1}, Q_{j,\kappa_1,c_2}, Q_{j,\kappa_2,c_3}$ and Q_{j,κ_2,c_4} . Note that there is no opposition to take $c_1 = c_3$ and $c_2 = c_4$. This process is illustrated in Figure 5.

8.1 The Kosovo κ -level master formula

The κ -level master formula for **Kosovo** allows to infer $\alpha_{j,\kappa}, \beta_{j,\kappa}$ from $\bar{\alpha}_j, \bar{\beta}_j$ for a fixed j and a variable κ .

In other words, the **Kosovo** κ -level master formula generates for a fixed j and for a variable κ the data $\alpha_{j,\kappa}, \beta_{j,\kappa}$ necessary to operate the general c -level master formula given in subsection 7.2.

Recall that the c -level master formula of subsection 7.2 generates for a fixed j, κ and a variable c a formal expression of the continued fraction $Q_{j,\kappa,c}$, given the auxiliary input $\alpha_{j,\kappa}, \beta_{j,\kappa}$.

⁷ This somewhat unusual notation is used to note the “ α of α ”, the “ β of α ” etc, as the same type of formula is applied at both c and κ levels.

⁸ for $i \in \{1, 2\}$.

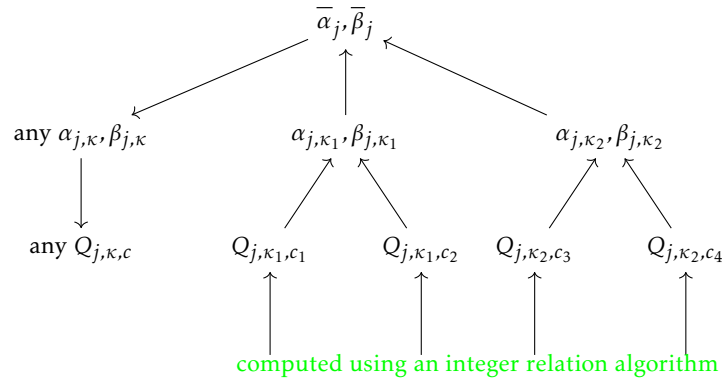


Fig. 5. κ -level resolution process for Kosovo.

Define:

$$\pi(j, \kappa) = \prod_{i=0}^{\frac{j-3}{2}} (\kappa - i)(2\kappa - 2i - 1)^2$$

$$\ell(n, j, \kappa) = \frac{(-1)^{\kappa+1} (2\kappa)!^2}{\kappa! 2^{3\kappa-2} (2\kappa - j)(2\kappa - 1)(n((2\kappa - j - 2)(3 - 2\kappa) - 1) + 1) \cdot \pi(j, \kappa)}$$

$$\eta(n, j, \kappa) = (2\kappa + 2j - 9 - 2n)(2\kappa + j - 8 - 2n)(-2\kappa + 5 - j)(2\kappa + j - 6)$$

$$\phi(n, j, \kappa) = 8\kappa^2 + \kappa(10j - 48 - 8n) + 3j^2 - (28 + 4n)j + 68 + 18n$$

$$\bar{\Delta}_{n,j,\kappa}(\alpha, \beta) = \begin{cases} \alpha + \beta\kappa & \text{if } \kappa < 2 \\ \eta(n, j, \kappa) \cdot \bar{\Delta}_{n,j,\kappa-2}(\alpha, \beta) + \phi(n, j, \kappa) \cdot \bar{\Delta}_{n,j,\kappa-1}(\alpha, \beta) & \text{if } \kappa \geq 2 \end{cases}$$

We assume that we are given the four constants:

$$\{\bar{\alpha}_j, \bar{\beta}_j\} = \left\{ \left\{ \alpha_j^\alpha, \alpha_j^\beta \right\}, \left\{ \beta_j^\alpha, \beta_j^\beta \right\} \right\} \in \mathbb{Q}^4$$

Then:

$$\alpha_{j,\kappa} = \frac{\bar{\Delta}_{0,j,\kappa-j+2}(\alpha_j^\alpha, \alpha_j^\beta)}{\ell(0, j, \kappa)} \quad \text{and} \quad \beta_{j,\kappa} = \frac{\bar{\Delta}_{1,j,\kappa-j+2}(\beta_j^\alpha, \beta_j^\beta)}{\ell(1, j, \kappa)} - \alpha_{j,\kappa}$$

The process is illustrated by the code snippet "4. Kosovo".

The computation of the “magic” lists of constants $\{\bar{\alpha}_j, \bar{\beta}_j\}$ hard-coded in the snippet "4. Kosovo" is done by resorting to integer relation resolution in snippet "5. resolution".

Remark 3. It is interesting to note that thanks to the inner symmetry within [Kosovo](#), it is possible to determine $\bar{\alpha}_{j+2}, \bar{\beta}_{j+2}$ if $\bar{\alpha}_j, \bar{\beta}_j, \bar{\alpha}_{j+4}, \bar{\beta}_{j+4}$ are known. Taking as an example $j = 5$, Figure 1 shows that $Q_{5,3,c} = Q_{3,3,c}$. Hence knowledge of $\bar{\alpha}_3, \bar{\beta}_3$ will be used to compute $Q_{3,3,c}$ which is identical to $Q_{5,3,c}$.

Note that $Q_{5,5,c} = Q_{7,5,c}$. Hence knowledge of $\bar{\alpha}_7, \bar{\beta}_7$ will be used to compute $Q_{7,5,c}$ which is identical to $Q_{5,5,c}$. Finally, having in hand $Q_{5,3,c}, Q_{5,5,c}$ we can solve a system in two unknowns and determine $\bar{\alpha}_5, \bar{\beta}_5$. Unfortunately this process has an information flow that only operates in “sandwich mode” allowing to determine $\bar{\alpha}_{j+2}, \bar{\beta}_{j+2}$ from $\bar{\alpha}_j, \bar{\beta}_j, \bar{\alpha}_{j+4}, \bar{\beta}_{j+4}$. This information flow cannot be reversed into an “escalator” allowing to ascend to level $j + 4$ from levels j and $j + 2$. The situation is illustrated in Figure 6 where $x \rightarrow y$ denotes the relation “ y is computable from x ”.

Remark 4. It is also possible to directly infer the $\alpha_{j,\kappa}, \beta_{j,\kappa}$ through symmetry. Note that the relation below also works for negative j, κ values.

Define:

$$\zeta(j, u) = \frac{1}{2^u} \prod_{i=0}^{u-1} (2 + 2i - j)$$

$$\tau_{j,u} = \begin{cases} \frac{\zeta(j, u)}{|\zeta(j, u)|} \cdot (2j - 2u - 3)!!(2u - j)!!(-2)^u & \text{if } 2u > j - 1 \\ \zeta(j, u) \cdot (-4)^u & \text{otherwise} \end{cases}$$

Then:

$$\frac{\alpha_{j,j-u-1}}{\alpha_{j-2u,j-u-1}} = \frac{\beta_{j,j-u-1}}{\beta_{j-2u,j-u-1}} = \tau_{j,u}$$

In particular $\tau_{j,j-1} = \frac{1}{2^{j-4}}$. The code is snippet "6. symmetry".

9 The Croatia Conjecture

We suspect that [Bosnia & Herzegovina](#) is a particular border case of [Croatia](#).

Our automated software detected the following [stunning](#) behavior providing a κ -level formula for [Croatia](#).

Let:

$$\mu_{i,j} = -(-2)^{\frac{3j-11-4i}{2}} \prod_{q=1}^i (j - 2q - 2)$$

For every $i = 0, 1, 2, \dots$ there exist two polynomials in j , denoted $\psi_1(i, j)$ and $\psi_2(i, j)$ such that for $j \geq 2i + 5$ we have:

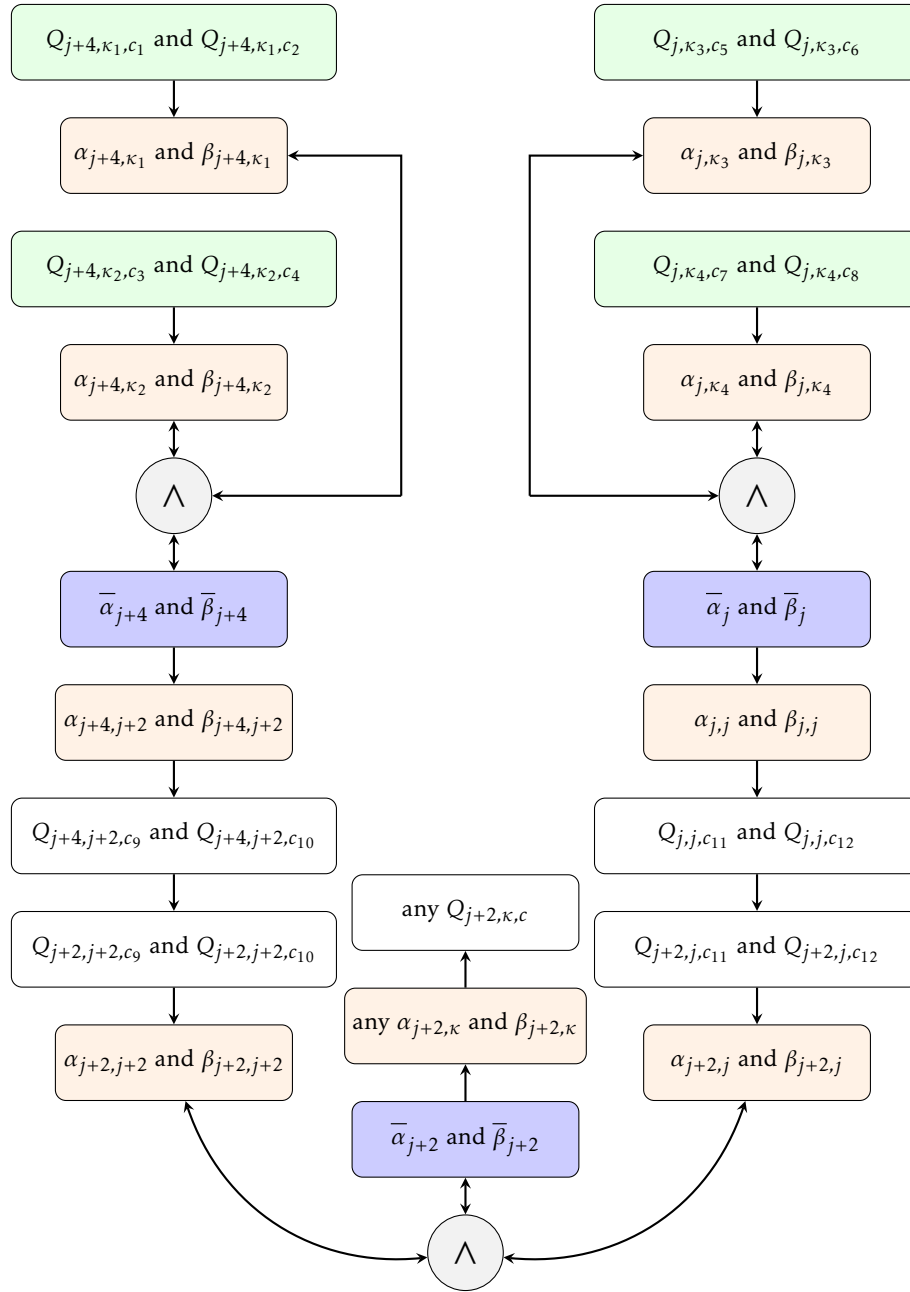


Fig. 6. Inferring $\bar{\alpha}_{j+2}, \bar{\beta}_{j+2}$ from $\bar{\alpha}_j, \bar{\beta}_j$ and $\bar{\alpha}_{j+4}, \bar{\beta}_{j+4}$. The $Q_{j, \kappa, c}$ in green boxes are determined using integer relation algorithms.

$$\alpha_{j, \frac{j-2i-3}{2}} = \frac{\psi_1(i, j)}{\mu(i, j)} \quad \text{and} \quad \beta_{j, \frac{j-2i-3}{2}} = \frac{\psi_2(i, j)}{\mu(i, j)}$$

Tables 2 and 3 provide the first values of the polynomials ψ_1, ψ_2 .
See code snippet "7. Croatia".

i	$\psi_1(i, j)$
0	-1
1	$14 - j$
2	$-464 + 58j - 3j^2$
3	$27936 - 4692j + 432j^2 - 15j^3$
4	$-2659968 + 542256j - 67836j^2 + 4260j^3 - 105j^4$
5	$367568640 - 86278560j + 13203480j^2 - 1139700j^3 + 51450j^4 - 945j^5$

Table 2. $\psi_1(i, j)$ for $0 \leq i \leq 5$.

i	$\psi_2(i, j)$
0	$-15 + 4j$
1	$306 - 95j + 4j^2$
2	$-13360 + 4646j - 357j^2 + 12j^3$
3	$999648 - 379692j + 40368j^2 - 2457j^3 + 60j^4$
4	$-113885568 + 46449360j - 6124164j^2 + 513228j^3 - 22935j^4 + 420j^5$
5	$18333538560 - 7933530720j + 1224286440j^2 - 126833100j^3 + 7864950j^4 - 266175j^5 + 3780j^6$

Table 3. $\psi_2(i, j)$ for $0 \leq i \leq 5$.

Remark 5.

The leading coefficients of $\psi_1(i, j)$ (i.e. 1, 1, 3, 15, 105, 945, ...) are $(2i-1)!!$ whereas the leading coefficients of $\psi_2(i, j)$ (i.e. 4, 4, 12, 60, 420, 3780, ...) are $4(2i-1)!!$.

Remark 6. The ψ polynomials can always be written under a nested form, e.g.:

$$\begin{aligned} \psi_1(j, 6) = & -14487726825 - (104826150 + (452605725 \\ & + (121200300 + (13697775 + (640710 \\ & + 10395 \cdot (j - 27))(j - 25))(j - 23))(j - 21))(j - 19))(j - 17) \end{aligned}$$

$$\begin{aligned} \psi_2(j, 6) = & 3198013886925 + (145296572850 + (5207427225 \\ & + (4353102000 + (877052475 + (78210090 + (3023055 \\ & + 41580 \cdot (j - 29))(j - 27))(j - 25))(j - 23))(j - 21))(j - 19))(j - 17) \end{aligned}$$

Remark 7. The GCD between the u -th coefficient of $\psi_1(i, j)$ and the u -th coefficient of $\psi_2(i, j)$ is always smooth as illustrated in the code snippet "8. coefficients". Our code could not reverse-engineer those coefficients. This might come from the fact that the coefficients involve too many ϕ_i or (more likely) that a specific ϕ_i , absent in our catalog, intervenes in the coefficients.

10 Balkans Knowledge Summary

In summary:

- For **Kosovo** and **Croatia** we have κ -level formulae⁹.
- Any $Q_{j,\kappa,c}$ value in **Montenegro** is algebraically computable.
- Any $Q_{j,\kappa,c}$ value in **Bosnia & Herzegovina** is algebraically computable.
- **Serbia** is fully determined by our knowledge of **Montenegro** and **Kosovo**.

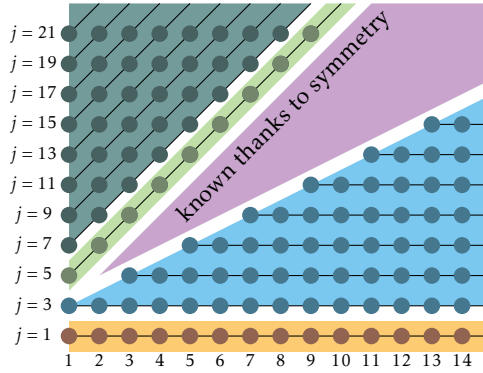


Fig. 7. The axes along which the κ -level master formulae operate in each area. We hence see that a finite amount of “magic” information in one dimension allows to algebraically compute $Q_{j,\kappa,c}$ over the two remaining dimensions.

We conjecture that there exists (possibly very complex) j -level master formulae for all regions allowing to algebraically access any $Q_{j,\kappa,c}$ value. Another challenge, on which the authors are currently working, is characterizing the case of even j values (that yield formulae involving $\log 2$) and negative j, κ, c values.

As a motivational example let us unveil the simple $\log 2$ example $Q_{2,\kappa,0}$.

In this case:

⁹ This means that oracle knowledge of $\bar{\alpha}_j, \bar{\beta}_j$ is required to generate $Q_{j,\kappa,c}$ for all κ, c . $\bar{\alpha}_j, \bar{\beta}_j$ can be inferred from any four $Q_{j,\kappa_1,c_1}, Q_{j,\kappa_1,c_2}, Q_{j,\kappa_2,c_3}, Q_{j,\kappa_2,c_3}$ values.

Area	j -level formula	κ -level formula	c -level formula
Kosovo+Serbia	unknown	✓	✓
Croatia	unknown	✓	✓
Montenegro	not needed	✓	✓
Bosnia & Herzegovina	not needed	✓	✓

Table 4. Knowledge Summary

$$Q_{2,\kappa,0} = \frac{(-1)^{\kappa+1} a_\kappa}{-b_\kappa + a_\kappa \log 2}$$

Where a_κ is the LCM of the list of κ integers starting with κ (in Mathematica: `Table[Apply[LCM, Table[i, {i, \kappa, 2\kappa-1}]], {\kappa, 1, 100}]`) while the b_κ are the numerators of the coefficients in the power series for $-\log(1+x)\log(1-x)$.

Recall that the coefficients in the expansion of $\log(1+x)\log(1-x)$ are given by

$$\frac{1}{2} \binom{2n}{n} \int_{x=0}^1 (x(1-x))^{n-1} \log(x) dx$$

and

$$\log(1+x)\log(1-x) = \frac{1}{2} \int_{z=0}^1 \frac{\log(z)}{z(1-z)} \left(\frac{1}{\sqrt{1-4x^2z(1-z)}} - 1 \right) dz$$

Evidently, providing formal proofs of the formulae provided in this paper is a challenge by its own right. The Conservative Matrix Field approach of [6] is one promising direction to investigate. To date, attempts to code automated substitution-simplification-induction proofs were unsuccessful. Yet another interesting question is that of *reversal*: Given a $Q_{j,\kappa,c}$ (i.e. a_0, a_1, a_2) find j, κ, c ¹⁰.

The main open questions remaining are very simple to formulate:

Open Question 1

Find a way to calculate directly the constants: $\{\bar{\alpha}_j, \bar{\beta}_j\} = \left\{ \left\{ \alpha_j^\alpha, \alpha_j^\beta \right\}, \left\{ \beta_j^\alpha, \beta_j^\beta \right\} \right\}$.

Open Question 2

Find a way to calculate directly the polynomials $\psi_1(i, j)$ and $\psi_2(i, j)$.

Open Question 3

Prove the formulae given in this paper.

¹⁰ Reversal is not always possible over Montenegro because $\forall \kappa, Q_{1,\kappa,1} = Q_{1,1,\kappa}$.

The following observations are hints that may serve in future quests:

Remark 8.

$$Q_{1,0,c} = \frac{(2c)!}{2(2c-1)!!^2 G - \Delta_{c-1,0}}$$

where:

$$\Delta_{1,0,c} = \begin{cases} 1 + 10c & \text{if } c < 2 \\ 2c(1-2c)^3 \Delta_{1,0,c-2} + (8c^2 + 2c + 1) \Delta_{1,0,c-1} & \text{if } c \geq 2 \end{cases}$$

or under an equivalent more compact form:

$$\Delta_{1,0,c} = \begin{cases} 1 & \text{if } c = 0 \\ (2c)! + (2c+1)^2 \Delta_{1,0,c-1} & \text{if } c > 0 \end{cases}$$

and even [4]:

$$\Delta_{1,0,c-1} = (2c)! \left(\frac{2G\binom{2c}{c}}{4^c} - \int_0^\infty \frac{t}{\cosh^{2c+1}(t)} dt \right)$$

Remark 9.

$$\begin{aligned} \lim_{c \rightarrow \infty} Q_{j,\kappa,c+1} - Q_{j,\kappa,c} &= 2 \\ \lim_{\kappa \rightarrow \infty} Q_{j,\kappa+1,c} - Q_{j,\kappa,c} &= 2j \\ \lim_{\kappa \rightarrow \infty} Q_{j+1,j+2r+1,c} - Q_{j,j+2r+1,c} &= 4r + 1 \end{aligned}$$

Remark 10. Denoting:

$$Q_{j,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \text{ where } a_0, a_1, a_2 \in \mathbb{Z}$$

The following formula (code snippet "9. ratio") is valid all over areas:

$$\rho_{j,\kappa,c} = \prod_{i=1}^{\frac{j-1}{2}} \frac{(2c-2\kappa+2i-1)(\kappa-i+1)}{(2c-2i+1)(2\kappa-2i+1)} \text{ and } \varepsilon_{j,\kappa} = 2\kappa + \frac{j-7}{2} + \left\lfloor \frac{1}{j} \right\rfloor$$

$$\frac{a_0}{a_2} = \frac{(2c)! \cdot 2^{\varepsilon_{j,\kappa}}}{(2c-1)!!^2 \cdot C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa-1) \cdot (j-2) \cdot \rho_{j,\kappa,c}}$$

Where $\rho_{1,\kappa,c} = 1$ by definition.

Remark 11. Let $x \in \mathbb{Q}$, denote by $p(x)$ the highest power of two dividing the numerator of x . We have:

$$p(\alpha_j) = p(\beta_j) = p(C_{\frac{j-3}{2}})$$

Remark 12. Although possibly unrelated, we note that low-degree continued fractions involving $\log 2$ can be also obtained with lower degree polynomials, e.g. (See code snippet "10. log2-a"):

$$\frac{2}{L\left(\frac{1}{2}, 1, c-1\right)} = \frac{2}{\sum_{n=0}^{\infty} \frac{e^{\pi i n}}{(n+c-1)}} = \frac{1}{2^{c-2} \log(2) - \sum_{j=1}^{c-2} \frac{2^{c-j-2}}{j}} = c + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2}{3n+c} \right)$$

We get a similar behavior for:

$$R_c = c + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n^2 - 2n}{3n+c} \right)$$

where $2^{c-4} \cdot (c-3) \cdot a_0 = a_2$ and for which we provide numerical examples in Table 12.

See code snippet "11. log2-b".

Remark 13. We also discovered other continued fractions involving G outside the Balkans. This suggests the existence of a more general formula encompassing both the Balkans and those other territories (called "Inostranstvo").

We denote those relations:

$$Q'_{\delta, \epsilon, \tau, \eta, \mu} = \epsilon + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(n+\tau)(n+\eta)(n+\mu)}{\epsilon + \delta n + 3n^2} \right) = \frac{a_0}{a_1 + a_2 G} \text{ where } a_0, a_1, a_2 \in \mathbb{Z}$$

Luckily, given that we have two coefficients¹¹ in the denominator of the continued fractions, we can compare in one plot the coefficients ϵ and δ of the Balkans and of Inostranstvo in one figure (Figure 8). Similarly we can visualize in 3D the τ, η, μ of both regions (Figure 9). The alignments of red points show that there is clearly another structured family hiding out beyond the Balkans.

PCA and automated matching revealed that ϵ, δ are dependent on τ, η, μ and:

$$Q'_{\tau, \eta, \mu} = x + 2(\tau + \eta + 1)i + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(n+\tau)(n+\eta)(n+2i+\mu)}{x + 2(\tau + \eta + 1)i + (2(\tau + \eta + \mu + 2i) + 3)n + 3n^2} \right)$$

Where $x = (1 + \eta)(1 + \mu) + \tau(1 + \eta + \mu)$.

This formula works when all variables¹² have identical parity, i.e.:

$$\tau \equiv \eta \equiv \mu \pmod{2}$$

¹¹ We exclude the 3 of $3n^2$ in the denominator, which is common to both the Balkans and to Inostranstvo.

¹² (code snippet "14. Inostranstvo")

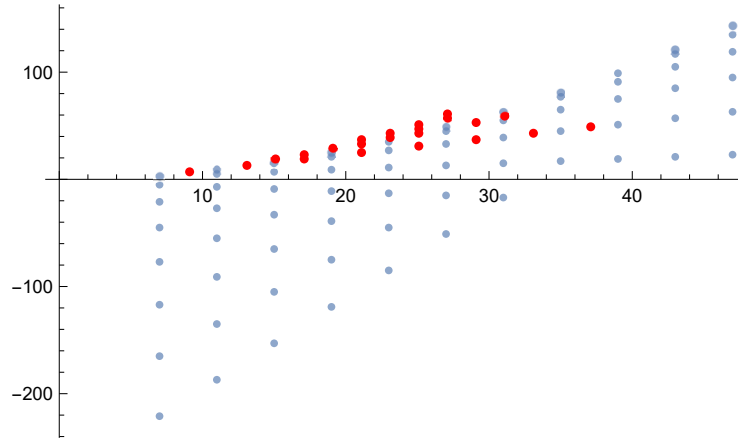


Fig. 8. ϵ, δ for the Balkans (in blue) and Inostranstvo (in red).

For instance:

$$Q'_i = 7 + 6i + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(n+1)^2(n+2i+1)}{7+6i+(9+4i)n+3n^2} \right) \text{ for } i = 0, 1 \dots$$

Whose formal expression (code snippet "12. Inostranstvo1") turns out to be:

$$Q'_i = \frac{(2i+1)!}{\Delta'_i - 2G(2i+1)!!^2}$$

$$\Delta'_i = \begin{cases} 2+15i & \text{if } i < 2 \\ 2(2i-1)^3(1-i) \cdot \Delta'_{i-2} + (8i^2-2i+3) \cdot \Delta'_{i-1} & \text{if } i \geq 2 \end{cases}$$

or¹³:

$$Q''_i = 23 + 10i + \mathcal{K}_{n=1}^{\infty} \left(\frac{-2n(n+1)(n+3)(n+2i+3)}{23+10i+(17+4i)n+3n^2} \right) \text{ for } i = 0, 1 \dots$$

We did not investigate further the various Inostranstvo families but conjecture that they share the same behaviors as the Balkans.

Remark 14. A frustrating exploration that did not succeed can be re-attempted by interested readers. The reasons for which this approach failed can be any combination of the following:

¹³ for which

$$\frac{a_0}{a_2} = \frac{(2j+5)!}{(2j+4)(2j+5)!!^2}$$

(code snippet "13. Inostranstvo2")

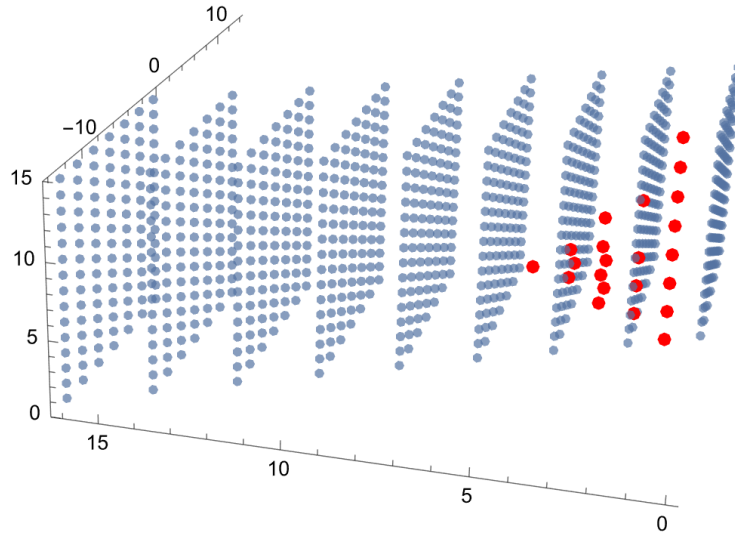


Fig. 9. τ, η, μ for the Balkans (in blue) and Inostranstvo (in red).

- The hypothesis below is wrong.
- We did not explore deep enough.
- A bug in our overly complex code.
- The unknowns are in \mathbb{Q} and not in \mathbb{Z} .

We wish to test the hypothesis that α_j^α has as a factor some function Δ_j'' of the form:

$$\Delta_j'' = \begin{cases} \alpha + \beta j & \text{if } j < 2 \\ P(j) \cdot \Delta'_{j-2} + R(j) \cdot \Delta'_{j-1} & \text{if } j \geq 2 \end{cases}$$

Where α, β are unknown integers and P, Q are unknown polynomials with moderate size integer coefficients.

We note that $\alpha_2^\alpha \bmod 2909 = \alpha_3^\alpha \bmod 5801 = 0$.

Set some integer exploration bounds B_0, B_1, B_2 .

Run a search over $\alpha, \beta \in [-B_0, B_0]$ and for each candidate pair perform the following test:

List all the integer pairs $P_i, R_i \in [-B_1, B_1]$ for which:

$$\tilde{\Delta}_2'' = 0 \bmod 2909 = 0$$

where:

$$\tilde{\Delta}_j'' = \begin{cases} \alpha + \beta j & \text{if } j < 2 \\ P_i \cdot \Delta'_{j-2} + R_i \cdot \Delta'_{j-1} & \text{if } j \geq 2 \end{cases}$$

List all polynomials $P(x), R(x)$ having integer coefficients $p_i, r_i \in [-B_2, B_2]$ such that: $P(2) = P_i$ and $R(2) = R_i$.

For each candidate check if $\Delta_3'' \bmod 5801 = 0$. Note that the search on all the coefficients of P, R is not required as the free coefficients can be determined as a function of all others. This task does not necessarily require exhaustive search and can be tackled, for instance, by integer programming.

The reason to be of this experiment is that the magnitude of α_j^α is compatible with that of Δ_j'' when, as usual, $\deg(P(x)) = 4$ and $\deg(R(x)) = 2$.

Figure 10 shows the relative \log_2 -scale growth of the functions $\alpha_j^\alpha, \alpha_j^\beta$ and Δ_j'' where:

$$\{\alpha, \beta, P(x), R(x)\} = \{-1, 20, -x^4, x^2\}$$

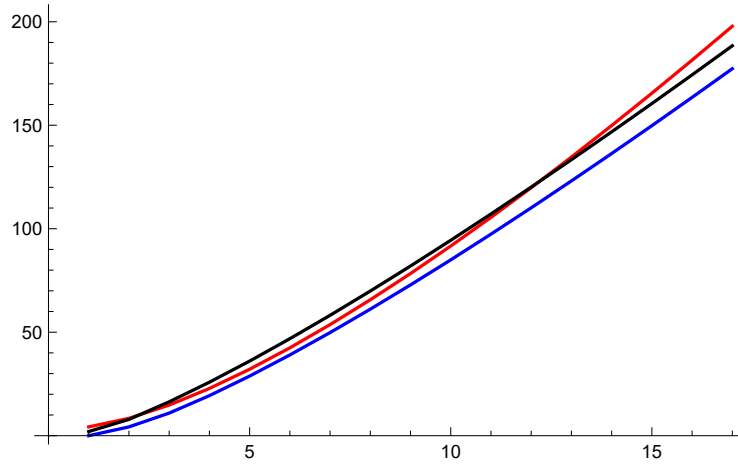


Fig. 10. α_j^α (blue), α_j^β (black) and Δ_j'' (red). Absolute value, \log_2 -scale.

This comforted the hypothesis¹⁴ that α_j^α and α_j^β are Δ -cores multiplied or divided by some moderate-magnitude combinatorial ϕ_i s.

Unfortunately, except fishing some artifacts, that fail from $j = 5$ and on, we did not spot any valid candidate.

Alternatively, one can attempt a faster but more complex analytic approach consisting in using each of the coefficients p_i, r_i to absorb one divisibility constraint, i.e.:

$$\Delta_j''(\alpha, \beta, P(x), R(x)) \bmod \rho_i = 0 \text{ where (for instance):}$$

¹⁴ That we (unsuccessfully) tried to confirm.

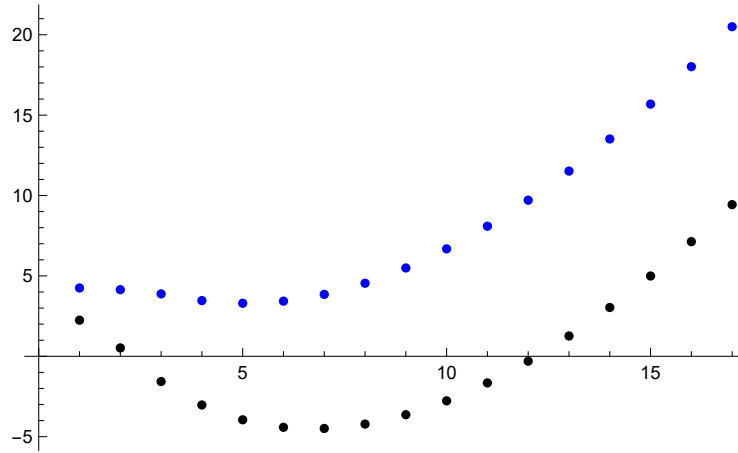


Fig. 11. red minus blue and red minus black for Figure 10.

$$\{\rho_2, \rho_3, \dots\} = \{2909, 5801, 1093, 103192403, 5522863607683, \dots\}$$

We did not attempt this approach given the prohibitive complexity of coding it.

Remark 15. The Δ functions appearing in this paper are particular cases of “Generalized Fibonacci Polynomials” (GFBs) studied by various authors, e.g. [9]. GFBs have numerous properties that might shed light on the open questions listed *supra*. We did not investigate this further.

Yet another route consists in considering the targets as algebraic equations and attempting on them an algebraic sieving approach such as [3]. This was not investigated this process given its theoretical and logistical complexity.

Finally, the similarity between the infinite sums given in [13] and the continued fractions investigated in this paper may reveal connections allowing to prove our conjectures. While analyzing [13] we noted a probable misprint in the formulae given for y_3 and y_5 (bottom of page 9 of [13]). We hence conducted our own experiments and discovered the relations described in Table 5 where:

$$w_2c + w_3 + w_1 \sum_{n=1}^{\infty} (-1)^{n+1} \prod_{i=1}^7 (2n + 2i - 3 + \epsilon)^{-e_i} = 0$$

11 The Gradient Descent Process

The Gradient Descent process that generated the formulae will be presented in steps. We sample execution at critical points using the example $j = 11, \kappa = 6$ and $40 \leq c \leq 47$ to explain the automated exploration process.

ϵ	c	e_1, e_2, \dots, e_7	w_1, w_2, w_3
0	π	0, 0, 2, 2, 2, 2, 0	57153600, -33075, 103904
0	π	0, 1, 1, 2, 2, 1, 1	69854400, -24255, 76192
0	π	0, 1, 2, 1, 1, 2, 1	62868960, -14553, 45712
0	π	0, 2, 2, 1, 2, 2, 0	-65318400, -4725, 14848
0	π	0, 2, 2, 2, 2, 0, 0	-6350400, -3675, 11552
0	π	1, 1, 2, 2, 1, 1, 0	-907200, -315, 992
0	π	1, 2, 1, 1, 2, 1, 0	-635040, -147, 464
0	π	1, 2, 2, 2, 2, 1, 0	16934400, -245, 768
0	π	2, 1, 1, 1, 1, 2, 0	-59535000, -6615, 21292
0	π	2, 2, 0, 0, 2, 1, 1	-93139200, -2695, 9344
0	π	2, 2, 0, 0, 2, 2, 0	-52920000, -2695, 9056
0	π	2, 2, 1, 2, 2, 0, 0	-50803200, 3675, -11264
0	π	2, 2, 2, 2, 0, 0, 0	129600, -75, 224
0	G	0, 1, 2, 2, 2, 1, 0	-50803200, 66150, -60577
0	G	0, 3, 2, 3, 0, 0, 0	-3456000, -6750, 6197
0	G	1, 2, 2, 2, 1, 0, 0	-2419200, -3150, 2909
0	G	1, 3, 0, 3, 1, 0, 0	8064000, 8750, -8109
0	G	2, 2, 0, 2, 2, 0, 0	-33868800, -22050, 21131
0	G	3, 2, 3, 0, 0, 0, 0	-27648, 54, -25
1	$\log 2$	0, 0, 0, 2, 2, 2, 2	-33177600, -38400, 26617
1	$\log 2$	0, 0, 2, 2, 2, 2, 0	1382400, -1600, 1109
1	$\log 2$	0, 1, 1, 2, 2, 1, 1	11059200, -7680, 5323
1	$\log 2$	0, 1, 2, 1, 1, 2, 1	-22118400, 10240, -7097
1	$\log 2$	0, 2, 2, 0, 0, 2, 2	17280000, -1760, 1219
1	$\log 2$	0, 2, 2, 2, 2, 0, 0	442368, 512, -355
1	$\log 2$	1, 1, 2, 0, 0, 2, 2	-88473600, 5120, -3539
1	$\log 2$	1, 1, 2, 2, 1, 1, 0	-230400, -160, 111
1	$\log 2$	1, 2, 1, 1, 2, 1, 0	-22118400, -10240, 7109
1	$\log 2$	2, 2, 0, 0, 2, 1, 1	-88473600, -5120, 3627
1	$\log 2$	2, 2, 0, 0, 2, 2, 0	69120000, 7040, -4951
1	$\log 2$	2, 2, 1, 2, 2, 0, 0	7077888, -1024, 707
1	$\log 2$	2, 2, 2, 2, 0, 0, 0	-13824, 16, -11

Table 5. Relations for π , G and $\log 2$. See code snippet "15. Series".

11.1 The starting point

We first recall our notations:

$$Q_{j,\kappa,c} = \frac{a_0}{a_1 + a_2 G} \text{ where } a_0, a_1, a_2 \in \mathbb{Z}$$

We have by now “seen the end of the movie”, and we know that for all areas:

$$n_0(j, \kappa, c) = \frac{a_0}{a_2} = \frac{(2c)! \cdot 2^{\varepsilon_{j,\kappa}}}{(2c-1)!!^2 \cdot C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa-1) \cdot (j-2) \cdot \rho_{j,\kappa,c}}$$

Where:

$$\rho_{j,\kappa,c} = \prod_{i=1}^{\frac{j-1}{2}} \frac{(2c-2\kappa+2i-1)(\kappa-i+1)}{(2c-2i+1)(2\kappa-2i+1)} \text{ and } \varepsilon_{j,\kappa} = 2\kappa + \frac{j-7}{2} + \left\lfloor \frac{1}{j} \right\rfloor$$

We started our journey by manually inspecting $n_0(j, \kappa, c)$ for several j, κ, c values, noting that $n_0(j, \kappa, c)$ is always very smooth.

This suggests that $n_0(j, \kappa, c)$ is the product of basic combinatorial functions such as factorials, binomials, semifactorials, Catalan numbers, Pochhammer symbols etc.

But the question is – of course – **which functions?**

We now know that $n_0(j, \kappa, c)$ is an exotic zoo containing the following animals:

$$\phi_0 = (2c)!, \quad \phi_1 = (2c-1)!!, \quad \phi_2 = (2c-1)!!, \quad \phi_3 = C_{\kappa-1}, \quad \phi_4 = C_{\frac{j-3}{2}}$$

$$\phi_5 = (2\kappa-1), \quad \phi_6 = (j-2), \quad \phi_7 = 2^{2\kappa}, \quad \phi_8 = 2^{\frac{j-7}{2}}, \quad \phi_9 = 2^{\lfloor \frac{1}{j} \rfloor}$$

$$\phi_{10} = \prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1), \quad \phi_{11} = \prod_{i=1}^{\frac{j-1}{2}} (\kappa-i+1), \quad \phi_{12} = \prod_{i=1}^{\frac{j-1}{2}} (2c-2i+1)$$

$$\phi_{13} = \prod_{i=1}^{\frac{j-1}{2}} (2\kappa-2i+1)$$

$$n_0(j, \kappa, c) = \frac{\phi_0 \cdot \phi_7 \cdot \phi_8 \cdot \phi_9 \cdot \phi_{12} \cdot \phi_{13}}{\phi_1 \cdot \phi_2 \cdot \phi_3 \cdot \phi_4 \cdot \phi_5 \cdot \phi_6 \cdot \phi_{10} \cdot \phi_{11}}$$

However, at start, we have no idea what the ϕ_i s were nor do we know how many ϕ_i s are there.

Remark 16. The basic functions in our catalog are not independent as some multiplicatively generate others. e.g., Catalan numbers, binomials, multinomials and Pochhammer symbols are all products of factorials. Adding to the catalog 2^x we reach semifactorials etc. The code can hence *successfully* follow different paths for a given target $n_0(j, \kappa, c)$. While those functional dependencies do not impact the final result, they do impact complexity: e.g., using Catalan numbers *reduces the depth of search*¹⁵ but *increases its width*.

11.2 Mutating functions

The algorithm performs a gradient descent on $n_i(j, \kappa, c)$, using an LLM to guide the descent. Catalog functions are not used in “bare metal” mode. They appear with specific linear combinations of $j, \kappa, c, 1$. To capture those combinations let:

$$\sigma(\bar{u}) = u_0 j + u_1 \kappa + u_2 c + u_3 \quad \text{where } \bar{u} = \{u_0, u_1, u_2, u_3\}$$

In other words¹⁶:

$$\begin{aligned} \phi_0 &= \sigma(0, 0, 2, 0)!, \quad \phi_1 = \phi_2 = \sigma(0, 0, 2, -1)!!, \quad \phi_3 = C_{\sigma(0,1,0,-1)} \\ \phi_4 &= C_{\sigma(\frac{1}{2},0,0,-\frac{3}{2})}, \quad \phi_5 = \sigma(0, 2, 0, -1), \quad \phi_6 = \sigma(1, 0, 0, -2), \quad \phi_7 = 2^{\sigma(0,2,0,0)} \\ \phi_8 &= 2^{\sigma(\frac{1}{2},0,0,-\frac{7}{2})}, \quad \phi_{10} = \prod_{i=1}^{\sigma(\frac{1}{2},0,0,-\frac{1}{2})} (\sigma(0, -2, 2, -1) + 2i) \\ \phi_{11} &= \prod_{i=1}^{\sigma(\frac{1}{2},0,0,-\frac{1}{2})} (\sigma(0, 1, 0, 1) - i), \quad \phi_{12} = \prod_{i=1}^{\sigma(\frac{1}{2},0,0,-\frac{1}{2})} (\sigma(2, 0, 0, 1) - 2i) \\ \phi_{13} &= \prod_{i=1}^{\sigma(\frac{1}{2},0,0,-\frac{1}{2})} (\sigma(0, 2, 0, 1) - 2i) \end{aligned}$$

11.3 What information do we have?

The integer relations oracle¹⁷ provides tens of thousands of $n_0(j, \kappa, c)$ instances. We record these in a database B where each entry has the form:

$$B_i = \{j_i, \kappa_i, c_i, t_i\} = \{j_i, \kappa_i, c_i, n_0(j_i, \kappa_i, c_i)\}$$

The code needs to infer the \bar{u} s that intervene in $n_0(j, \kappa, c)$ from those numerous numerical examples.

¹⁵ When a Catalan number is identified three factorial identifications are avoided.

¹⁶ $\phi_9 = 2^{\lfloor \frac{1}{j} \rfloor}$ was not in the catalog and was added manually to unify two families of general formulae discovered by two runs.

¹⁷ In our case LLL.

11.4 A first step

Assume that the code is discovering the ϕ_i s and their \bar{u}_s one by one. The code is at some step ω at which it has already discovered the ϕ_i s shown in red in the formulae:

$$n_0(j, \kappa, c) = \frac{a_0}{a_2} = \frac{(2c)! \cdot 2^{\epsilon_{j,\kappa}}}{(2c-1)!!^2 \cdot C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa-1) \cdot (j-2) \cdot \rho_{j,\kappa,c}}$$

And:

$$\rho_{j,\kappa,c} = \frac{\prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1)(\kappa-i+1)}{\prod_{i=1}^{\frac{j-1}{2}} (2c-2i+1)(2\kappa-2i+1)}$$

Divide the t_i of each record B_i by the red components evaluated at j_i, κ_i, c_i . Update the target to:

$$n_\omega(j, \kappa, c) = (2c-1)!!^2 \cdot C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa-1) \cdot (j-2) \cdot \prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1)(\kappa-i+1)$$

A successfully mutated function, e.g. $\phi_1 = (2c-1)!! = \sigma(0, 0, 2, -1)!!$ stands-out because:

$$n_\omega(j, \kappa, c) \bmod \sigma(0, 0, 2, -1)!! = 0 \quad \text{for all } j, \kappa, c \text{ values}$$

Evidently, this criterion generates false positives. For instance:

$$n_\omega(j, \kappa, c) \bmod \sigma(0, 0, 2, -1)!! = 0 \Rightarrow n_\omega(j, \kappa, c) \bmod \sigma(0, 0, 2, -3)!! = 0$$

Remark 17. False positives come in two flavors: “False false positives” and “True false positives”. $\sigma(0, 0, 2, -3)!!$ is a “false false positive”. Detecting $\sigma(0, 0, 2, -3)!!$ is useful as it decreases the target. In the example above, instead of peeling-off $(2c-1)!!$ in one round, a first round will peel-off $(2c-3)!!$ and leave an extra $(2c-1)$ to some subsequent round. By opposition a “True false positive” is a candidate appearing as a factor of the target by the sole effect of chance over the available dataset.

To visualize efficiently execution we introduce *Backgammon diagrams*.

11.5 Backgammon diagrams

The gradient descent monitoring tool is called “Backgammon diagrams” because of its visual similarity to a backgammon board (Figure 12).

In a Backgammon diagram the x -axis shows a search performed over a coordinate u_s within an interval $u_s \in \{u_{\text{start}}, \dots, u_{\text{end}}\}$.

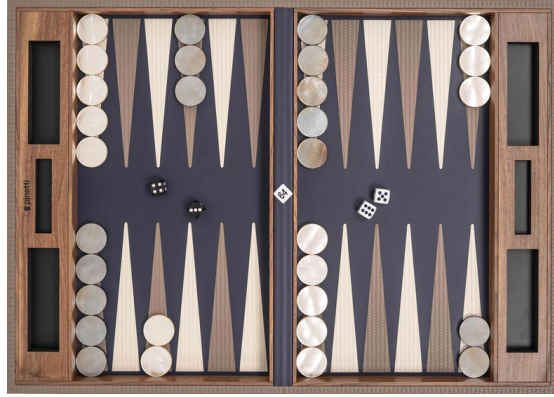


Fig. 12. A typical Backgammon board.

In all the following, consider that all \bar{u} coordinates other than s were fixed to correct values denoted by \checkmark .

The draughts represent experiments with fixed j, κ values and different c values. In the example $j = 11, \kappa = 6$ and $40 \leq c \leq 47$. Each draught color corresponds to a different c value ($\bullet, \circ, \blacktriangle, \triangle, \blacksquare, \square, \blacklozenge, \lozenge$). The legend is not repeated to save space. Draughts were slightly lifted to avoid covering each other.

If for a given c value:

$$n_{\omega}(j, \kappa, c) \bmod \phi_i(\sigma(\checkmark, \dots, \checkmark, u_s, \checkmark, \dots, \checkmark)) = 0$$

then the draught of c 's color is lowered to the bottom of diagram, else it is raised to the top. Hence, a glance at the diagram shows which u_s values are compatible with the target $n_{\omega}(j, \kappa, c)$ for each c .

Two additional features enhance reading: a red line showing the correct answer¹⁸ and little red triangles (\blacktriangle) denoting u_s values for which tests succeeded for all c values.

This view is grandly simplified with respect to reality. In the code j, κ vary as well (resulting in multi-dimensional and hence unvisualizable diagrams) and several u_s s are simultaneously tried at each round.

11.6 A worked-out example

We want to disassemble:

$$n_{\omega}(j, \kappa, c) = (2c - 1)!!^2 \cdot C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa - 1) \cdot (j - 2) \cdot \prod_{i=1}^{\frac{j-1}{2}} (2c - 2\kappa + 2i - 1)(\kappa - i + 1)$$

¹⁸ This is, of course, not provided by the software.

step	$\omega + 1$
target	$n_\omega(j, \kappa, c)$
candidate	$(uc - 1)!!$
new target	$n_{\omega+1}(j, \kappa, c) = n_\omega(j, \kappa, c)/(2c - 1)!!$
database update	$\forall i \text{ do } t_i = t_i/(2c_i - 1)!!$
remarks	We have one solution which is $u = 2$.

step	illustrating option 1: $\omega + 2$
target	$n_{\omega+1}(j, \kappa, c)$
candidate	illustrating option 1: $(uc - 1)!!$
new target	$n_{\omega+2}(j, \kappa, c) = n_{\omega+1}(j, \kappa, c)/(2c - 1)!!$
database update	$\forall i \text{ do } t_i = t_i/(2c_i - 1)!!$
remarks	We have one solution which is $u = 2$.

step	illustrating option 2: $\omega + 2$
target	$n_{\omega+1}(j, \kappa, c)$
candidate	illustrating option 2: $(2c + u)!!$
new target	$n_{\omega+2}(j, \kappa, c) = n_{\omega+1}(j, \kappa, c)/(2c - 1)!!$
database update	$\forall i \text{ do } t_i = t_i/(2c_i - 1)!!$
remarks	4 solutions added to backtracking list.

step	illustrating option 1: $\omega + 3$
target	$n_{\omega+2}(j, \kappa, c)$
candidate	illustrating option 1: $(uc - 1)!!$
new target	None, wrong guess.
database update	None, wrong guess.
remarks	No solutions: repeat step $\omega + 3$ with another candidate.

step	illustrating option 2: $\omega + 3$
target	$n_{\omega+2}(j, \kappa, c)$
candidate	illustrating option 2: $(u\kappa - 1)$
new target	$n_{\omega+3}(j, \kappa, c) = n_{\omega+2}(j, \kappa, c)/(2\kappa - 1)$
database update	$\forall i \text{ do } t_i = t_i/(2\kappa_i - 1)$
remarks	7 solutions added to backtracking list.

step	illustrating option 3: $\omega + 3$
target	$n_{\omega+2}(j, \kappa, c)$
candidate	illustrating option 3: $(2\kappa + u)$
new target	$n_{\omega+3}(j, \kappa, c) = n_{\omega+2}(j, \kappa, c)/(2\kappa - 1)$
database update	$\forall i \text{ do } t_i = t_i/(2\kappa_i - 1)$
remarks	14 solutions added to backtracking list.

step	illustrating option 1: $\omega + 4$
target	$n_{\omega+3}(j, \kappa, c)$
candidate	illustrating option 1: $(j + u)$
new target	$n_{\omega+4}(j, \kappa, c) = n_{\omega+3}(j, \kappa, c)/(j - 2)$
database update	$\forall i \text{ do } t_i = t_i/(j_i - 2)$
remarks	13 solutions added to backtracking list.

step	illustrating option 2: $\omega + 4$
target	$n_{\omega+3}(j, \kappa, c)$
candidate	illustrating option 2: $(uj - 2)$
new target	$n_{\omega+4}(j, \kappa, c) = n_{\omega+3}(j, \kappa, c)/(j - 2)$
database update	$\forall i \text{ do } t_i = t_i/(j_i - 2)$
remarks	9 solutions added to backtracking list.

step	illustrating option 1: $\omega + 5$
target	$n_{\omega+4}(j, \kappa, c)$
candidate	illustrating option 1: $C_{\kappa+u}$
new target	$n_{\omega+5}(j, \kappa, c) = n_{\omega+4}(j, \kappa, c)/C_{\kappa-1}$
database update	$\forall i \text{ do } t_i = t_i/C_{\kappa_i-1}$
remarks	7 solutions added to backtracking list.

step	illustrating option 2: $\omega + 5$
target	$n_{\omega+4}(j, \kappa, c)$
candidate	illustrating option 2: $C_{u\kappa-1}$
new target	$n_{\omega+5}(j, \kappa, c) = n_{\omega+4}(j, \kappa, c)/C_{\kappa-1}$
database update	$\forall i \text{ do } t_i = t_i/C_{\kappa_i-1}$
remarks	2 solutions added to backtracking list.

step	illustrating option 1: $\omega + 6$
target	$n_{\omega+5}(j, \kappa, c)$
candidate	illustrating option 1: $\prod_{x=1}^{\frac{j-1}{2}} (2c + uk + 2x - 1)$
new target	$n_{\omega+6}(j, \kappa, c) = n_{\omega+5}(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (2c - 2k + 2x - 1)$
database update	$\forall i \text{ do } t_i = t_i / \prod_{x=1}^{\frac{j-1}{2}} (2c_i - 2k_i + 2x - 1)$
remarks	We have one solution which is $u = -2$.

step	illustrating option 2: $\omega + 6$
target	$n_{\omega+5}(j, \kappa, c)$
candidate	illustrating option 2: $\prod_{x=1}^{\frac{j-1}{2}} (2c - 2k + 2x + u)$
new target	$n_{\omega+6}(j, \kappa, c) = n_{\omega+5}(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (2c - 2k + 2x - 1)$
database update	$\forall i \text{ do } t_i = t_i / \prod_{x=1}^{\frac{j-1}{2}} (2c_i - 2k_i + 2x - 1)$
remarks	We have one solution which is $u = -1$.

step	illustrating option 3: $\omega + 6$
target	$n_{\omega+5}(j, \kappa, c)$
candidate	illustrating option 3: $\prod_{x=1}^{\frac{j-1}{2}} (uc - 2k + 2x - 1)$
new target	$n_{\omega+6}(j, \kappa, c) = n_{\omega+5}(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (uc - 2k + 2x - 1)$
database update	$\forall i \text{ do } t_i = t_i / \prod_{x=1}^{\frac{j-1}{2}} (uc_i - 2k_i + 2x - 1)$
remarks	We have one solution which is $u = 2$.

step	illustrating option 4: $\omega + 6$
target	$n_{\omega+5}(j, \kappa, c)$
candidate	illustrating option 4: $\prod_{x=1}^{\frac{2uj+j-1}{2}} (2c - 2\kappa + 2x - 1)$
new target	$n_{\omega+6}(j, \kappa, c) = n_{\omega+5}(j, \kappa, c) / \prod_{x=1}^{\frac{2uj+j-1}{2}} (2c - 2\kappa + 2x - 1)$
database update	$\forall i \text{ do } t_i = t_i / \prod_{x=1}^{\frac{2uj_i+j_i-1}{2}} (2c_i - 2\kappa_i + 2x - 1)$
remarks	We have one solution which is $u = 0$.

step	$\omega + 7$
target	$n_{\omega+6}(j, \kappa, c)$
candidate	$C_{\frac{j+2u+1}{2}}$
new target	$n_{\omega+7}(j, \kappa, c) = n_{\omega+6}(j, \kappa, c) / C_{\frac{j-3}{2}}$
database update	$\forall i \text{ do } t_i = t_i / C_{\frac{j_i-3}{2}}$
remarks	3 solutions added to backtracking list.

step	$\omega + 8$
target	$n_{\omega+7}(j, \kappa, c)$
candidate	$\prod_{x=1}^{\frac{j-1}{2}} (\kappa - x + u)$
new target	$n_{\omega+8}(j, \kappa, c) = n_{\omega+7}(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (\kappa - x + 1)$
database update	$\forall i \text{ do } t_i = t_i / \prod_{x=1}^{\frac{j_i-1}{2}} (\kappa_i - x + 1)$
remarks	4 solutions added to backtracking list.

step	$\omega + 8$
target	$n_{\omega+7}(j, \kappa, c)$
candidate	$\prod_{x=1}^{\frac{j-1}{2}} (\kappa + ux + 1)$
new target	$n_{\omega+8}(j, \kappa, c) = n_{\omega+7}(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (\kappa - x + 1)$
database update	$\forall i \text{ do } t_i = t_i / \prod_{x=1}^{\frac{j_i-1}{2}} (\kappa_i - x + 1)$
remarks	2 solutions added to backtracking list.

At this point one of the backtracking branches gives the constant function 1. $n_{\omega}(j, \kappa, c)$ was hence disassembled.

11.7 The Decimator

A very significant speed-up is achieved by a software module called “the Decimator”. The Decimator restricts the \vec{u} space by removing u_i affine combinations incompatible with the target.

Consider the target:

$$n_{\omega+2}(j, \kappa, c) = C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa-1) \cdot (j-2) \cdot \prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1)(\kappa-i+1)$$

Create the database B given in Table 6.

i	j_i	κ_i	c_i	$t_i = n_{\omega+2}(j_i, \kappa_i, c_i)$
1	11	6	40	86562004597992000
2	11	6	41	99107222655672000
3	11	6	42	113065986409992000
4	11	6	43	128554477699032000
5	11	6	44	145695074725569600
6	11	6	45	164616513001617600
7	11	6	46	185454046292961600
8	11	6	47	208349607563697600

Table 6. The database B

Assume that we want to test a candidate $y(j, \kappa, c) = u_0j + u_1\kappa + u_2c + u_3$. We are typically interested in exploring each u_i over a small interval, e.g. $[-8, 8]$. Start by creating a list \mathcal{L} of all $(2 \times 8 + 1)^4$ \vec{u} -values.

For each $\{u_0, u_1, u_2, u_3\}$ value, if a t_i is not divisible by at least one $y(j_i, \kappa_i, c_i)$ then remove $\{u_0, u_1, u_2, u_3\}$ from \mathcal{L} .

Table 7 gives the number of \vec{u} s eliminated from $[-8, 8]^4$ by each c value.

c	40	41	42	43	44	45	46	47
eliminated \vec{u}s	51721	54371	55635	56161	52771	51203	51609	46383

Table 7. Number of eliminated \vec{u} combinations per c tried.

When we merge all the forbidden \vec{u} s, removing duplicates, we get a collection of 78567 combinations to skip. We are hence left with $(2 \times 8 + 1)^4 - 78567 = 4954$ survivors to test, i.e. 5.9%.

The above example is restricted to eight c toy-values. In reality we perform calculations on $\simeq 10^5$ combinations of (j, κ, c) . This reduces drastically

the search space¹⁹. Note however that as c increases the percentage of newly removed \bar{u} values per round decreases.

Note that, in practice even an exploration in $[-3, 3]^4$ would have sufficed²⁰. For $\vec{u} \in [-3, 3]^4$ we get 332 survivors out of 2401 (14%).

For $[-8, 8]^4$ and $40 \leq c \leq 47$, decimating for $y(j, \kappa, c) = C_{u_0j+u_1\kappa+u_2c+u_3}$ removes 83233 candidates, leaving only 288 possibilities (0.35%).

Decimating over $[-5, 5]^5$ and $20 \leq c \leq 60$ for the candidate:

$$y(j, \kappa, c) = \prod_{i=1}^{\frac{i-1}{2}} (u_0j + u_1\kappa + u_2c + u_3 + u_4i)$$

results in 1496 survivors out of 161051 (0.93%).

Counter-intuitively, the more “complex”²¹ the candidate is the more efficiently it is decimated. We hence select the candidates in decreasing complexity order. Measuring the complexity of mathematical formulas can be a subjective task, as it depends on various factors such as the number of terms, the presence of functions, exponents, and variables, as well as the overall structure. There’s no definitive metric to quantify formula complexity. The following criteria can nonetheless be used to “measure” complexity and hence get rid of ϕ_i s as fast as possible:

Counting Elements: We can count the number of distinct elements in each candidate, such as variables, constants, operators, and functions. For example, the candidate $\kappa - 1$ has two elements while in others we may have multiple variables, exponents, a product symbol, and a summation, which increases its complexity.

Nesting and Hierarchy: Analyze the nesting of operations and functions within the candidates. A candidate with multiple levels of nesting or hierarchy can be regarded as more complex.

Mathematical Operations: Consider the types of mathematical operations present in the candidates. More complex operations, such as exponentiation and summation, contribute to higher complexity compared to simpler operations like addition or multiplication.

Function Complexity: If the candidate includes functions, their complexity should be taken into account. For instance, a Catalan number can add complexity compared to linear or constant functions.

Symbolic Representation: Represent each candidate in a symbolic format, such as a parse tree or abstract syntax tree. Compare the depth and branching of the trees as a rough measure of complexity.

Information Theory: Explore concepts from information theory, such as Kolmogorov complexity or algorithmic information theory, to quantify the amount of information needed to describe each candidate. This approach can be quite theoretical and may not provide a practical measure for all cases.

¹⁹ e.g., exploring for $20 \leq c \leq 60$ reduces the survivors’ pool to 3702 (4.4%).

²⁰ The 7 in ϕ_8 would have been decimated in two rounds.

²¹ i.e., closer to a random oracle.

We resorted to a much more brutal approach [2]. Using the API²², we asked GPT-4 to compare candidates pairwise and obtained a subjective comparison of each pair and translated the ternary results²³ to a directed graph. We performed a random walk of 10^6 steps on the graph and counted the number of times each candidate was visited. The most visited candidate was considered as the “most complex”. We then removed this candidate from the graph and started over again.

As a final note, we remark that some ϕ_i s are “process killers”. This is the case of candidates such as $(2c)!$. Because, in essence, $(2c)!$ contains just about any number of interest, it silences the modular tests. We hence start by launching the process with $\bar{u}!$ and start the backtracking afresh for each $n_0(j, \kappa, c)/\bar{u}!$.

11.8 Ascending to descend

As underlined in [15], some descents require intermediate ascents. The process described so far advances only in the case of a monotonous descent. In other words, if the initial ϕ_i s were wrongly chosen the process will not converge.

We thus need a process allowing temporary ascents to get out of bowls. To that end we use a modified version of Broyden-Fletcher-Goldfarb-Shanno’s (BFGS) algorithm. This requires a more refined measure of the penalty/profit of each move which cannot just be the Boolean “ x divides y ”.

Denote by p_i the i -th prime.

We introduce a measure called “brittleness”²⁴ denoted $\Xi(n)$.

$$\text{Let: } n = \prod_{i=0}^{a-1} p_i^{m_i} \in \mathbb{Q} \text{ be a simplified fraction. Then } \Xi(n) = \sum_{i=0}^{a-1} |m_i|$$

In other words, $\Xi(n)$ counts, with repetition, the number of distinct factors appearing in the numerator or in the denominator of n .

The following example illustrates the evolution of brittleness (y -axis) during the peeling-off process.

Dotted lines \Rightarrow Target’s brittleness at the concerned j, κ, c points.

Draughts \Rightarrow Raised to $\Xi(\text{target/candidate})$ for different u values.

As before, \blacktriangle denotes us at which all draughts are lower than their same-color dotted lines. The thin red vertical line is the correct answer.

As an example start with the target:

$$n_1(j, \kappa, c) = C_{\kappa-1} \cdot C_{\frac{j-3}{2}} \cdot (2\kappa-1) \cdot (j-2) \cdot \prod_{i=1}^{\frac{j-1}{2}} (2c-2\kappa+2i-1)(\kappa-i+1)$$

²² endpoint <https://api.openai.com/v1/engines/davinci-codex/completions>

²³ “ A is more complex than B ”: $A \rightarrow B$, “ B is more complex than A ”: $B \rightarrow A$ or “unsure”: no edge.

²⁴ Brittleness is a generalization the prime Ω function to \mathbb{Q} : $\Xi(n) = \Omega(\text{numerator}(n)) + \Omega(\text{denominator}(n))$.

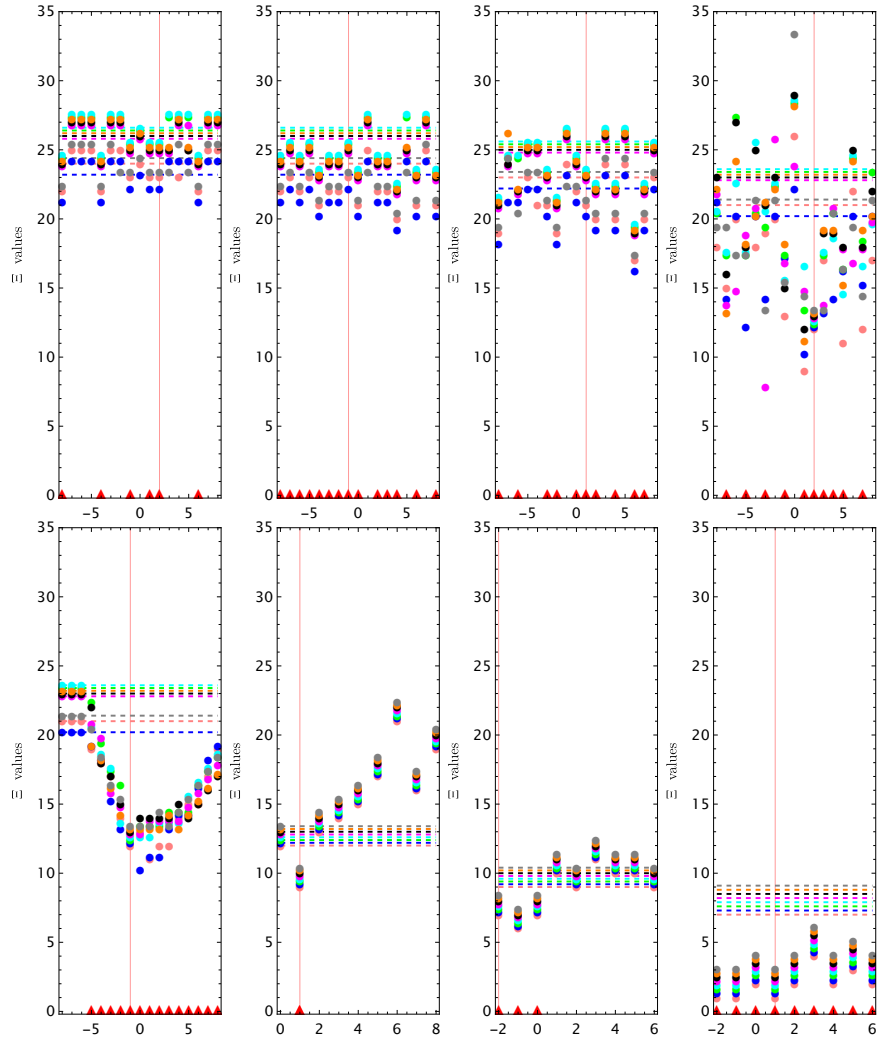


Fig. 13. Steps are ordered from left→right and up→down, i.e. $\frac{1}{3}, \frac{1'}{4}, \frac{2}{5}, \frac{3}{6}$. Note the steady decrease of Ξ levels. Steps 1 and 1' are two division options occurring at the same step and so are 3 and 3'.

Figure 13 represents the following steps:

step	1 of Figure 13
target	$n_0(j, \kappa, c)$
candidate	illustrating first option: $u\kappa - 1$
new target	$n_1(j, \kappa, c) = n_0(j, \kappa, c)/(2\kappa - 1)$

step	1' of Figure 13
target	$n_0(j, \kappa, c)$
candidate	illustrating second option: $2\kappa + u$
new target	$n_1(j, \kappa, c) = n_0(j, \kappa, c)/(2\kappa - 1)$

step	2 of Figure 13
target	$n_1(j, \kappa, c)$
candidate	$uj - 2$
new target	$n_2(j, \kappa, c) = n_1(j, \kappa, c)/(j - 2)$

step	3 of Figure 13
target	$n_2(j, \kappa, c)$
candidate	illustrating first option: $\prod_{x=1}^{\frac{j-1}{2}} (2c - 2k + ux - 1)$
new target	$n_3(j, \kappa, c) = n_2(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (2c - 2k + 2x - 1)$

step	3' of Figure 13
target	$n_2(j, \kappa, c)$
candidate	illustrating second option: $\prod_{x=1}^{\frac{j+2u+1}{2}} (2c - 2k + 2x - 1)$
new target	$n_3(j, \kappa, c) = n_2(j, \kappa, c) / \prod_{x=1}^{\frac{j-1}{2}} (2c - 2k + 2x - 1)$

step	4 of Figure 13
target	$n_3(j, \kappa, c)$
candidate	$C_{u\kappa-1}$
new target	$n_4(j, \kappa, c) = n_3(j, \kappa, c)/C_{\kappa-1}$

step	5 of Figure 13
target	$n_4(j, \kappa, c)$
candidate	$C_{\frac{j+2u+1}{2}}$
new target	$n_5(j, \kappa, c) = n_4(j, \kappa, c)/C_{\frac{j-3}{2}}$

step	6 of Figure 13
target	$n_5(j, \kappa, c)$
candidate	$\prod_{x=1}^{\frac{j-1}{2}} (k - x + u)$
new target	None: Processed finished.

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
-7351344	-32375839	46558512	$n(3+n)(17+n)$	$72+43n$
-2450448	-1768477	2450448	$n(1+n)(17+n)$	$36+39n$
-1081080	-16147379	23279256	$n(5+n)(15+n)$	$96+43n$
-793800	-232217	322560	$(4+n)(9+n)^2$	$100+39n$
-504504	-16140515	23279256	$n(7+n)(13+n)$	$112+43n$
-436590	7989199	-11531520	$(3+n)(9+n)(11+n)$	$120+43n$
-180180	-1001393	1441440	$(2+n)(5+n)(13+n)$	$84+39n$
-174636	-8069449	11639628	$n(9+n)(11+n)$	$120+43n$
-72072	-850133	1225224	$n(5+n)(13+n)$	$84+39n$
-72072	-251099	360360	$n(3+n)(13+n)$	$56+35n$
-72072	-52279	72072	$n(1+n)(15+n)$	$32+35n$
-45045	124048	-180180	$(1+n)(3+n)(13+n)$	$56+35n$
-28028	-999391	1441440	$(2+n)(7+n)(13+n)$	$112+43n$
-27720	-20417	27720	$n(1+n)(11+n)$	$24+27n$
-22050	27649	-40320	$(3+n)(7+n)(9+n)$	$80+35n$
-19305	424423	-612612	$(1+n)(5+n)(15+n)$	$96+43n$
-17640	-250007	360360	$n(7+n)(9+n)$	$80+35n$
-14700	-153907	221760	$(2+n)(7+n)(9+n)$	$80+35n$
-10395	124741	-180180	$(1+n)(5+n)(11+n)$	$72+35n$
-9450	76691	-110880	$(1+n)(5+n)(9+n)$	$60+31n$
-7350	-62563	90090	$n(7+n)^2$	$64+31n$
-7007	424566	-612612	$(1+n)(7+n)(13+n)$	$112+43n$
-6300	-14087	20160	$(2+n)(5+n)(9+n)$	$60+31n$
-6006	99839	-144144	$(1+n)(5+n)(13+n)$	$84+39n$
-4900	-21043	30240	$(2+n)(7+n)^2$	$64+31n$
-2520	-1879	2520	$n(1+n)(9+n)$	$20+23n$
-2450	28781	-41580	$(1+n)(7+n)^2$	$64+31n$
-1225	367	-560	$(3+n)(7+n)^2$	$64+31n$
-525	-2413	3465	$n(5+n)(7+n)$	$48+27n$
-450	1151	-1680	$(1+n)(5+n)^2$	$36+23n$
-350	1739	-2520	$(1+n)(5+n)(7+n)$	$48+27n$
-180	-299	420	$n(3+n)(5+n)$	$24+19n$
-105	142	-210	$(1+n)(3+n)(7+n)$	$32+23n$
-18	7	-12	$(1+n)(3+n)^2$	$16+15n$
-6	-5	6	$n(1+n)(3+n)$	$8+11n$

Table 8. Examples of convergence to $\frac{a_0}{a_1+a_2 \log 2}$

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
1	1	-1	$n(1+n)^2$	$4+7n$
9	11	-15	$n(3+n)^2$	$16+15n$
50	147	-210	$n(5+n)^2$	$36+23n$
60	47	-60	$n(1+n)(5+n)$	$12+15n$
90	-79	120	$(1+n)(3+n)(5+n)$	$24+19n$
420	319	-420	$n(1+n)(7+n)$	$16+19n$
420	887	-1260	$n(3+n)(7+n)$	$32+23n$
900	361	-480	$(2+n)(5+n)^2$	$36+23n$
1890	-3443	5040	$(1+n)(3+n)(9+n)$	$40+27n$
2100	2377	-3360	$(2+n)(5+n)(7+n)$	$48+27n$
5544	50035	-72072	$n(5+n)(11+n)$	$72+35n$
6468	-249713	360360	$(1+n)(7+n)(11+n)$	$96+39n$
7560	19409	-27720	$n(3+n)(9+n)$	$40+27n$
8316	-19031	27720	$(1+n)(3+n)(11+n)$	$48+31n$
15444	-49705	72072	$(1+n)(3+n)(15+n)$	$64+39n$
22050	-499279	720720	$(1+n)(7+n)(9+n)$	$80+35n$
24255	-76586	110880	$(3+n)(7+n)(11+n)$	$96+39n$
25740	200107	-288288	$(2+n)(5+n)(15+n)$	$96+43n$
37800	250427	-360360	$n(5+n)(9+n)$	$60+31n$
38808	849671	-1225224	$n(7+n)(11+n)$	$96+39n$
39690	-1997851	2882880	$(1+n)(9+n)^2$	$100+39n$
41580	154327	-221760	$(2+n)(5+n)(11+n)$	$72+35n$
58212	3997025	-5765760	$(2+n)(9+n)(11+n)$	$120+43n$
79380	2123957	-3063060	$n(9+n)^2$	$100+39n$
83160	251561	-360360	$n(3+n)(11+n)$	$48+31n$
87318	-8491859	12252240	$(1+n)(9+n)(11+n)$	$120+43n$
97020	1999321	-2882880	$(2+n)(7+n)(11+n)$	$96+39n$
132300	3997907	-5765760	$(2+n)(9+n)^2$	$100+39n$
198450	-1227581	1774080	$(3+n)(9+n)^2$	$100+39n$
216216	852707	-1225224	$n(3+n)(15+n)$	$64+39n$
360360	263111	-360360	$n(1+n)(13+n)$	$28+31n$
630630	-3990557	5765760	$(3+n)(7+n)(13+n)$	$112+43n$
918918	-3383801	4900896	$(1+n)(3+n)(17+n)$	$72+43n$
1746360	2475007	-3548160	$(4+n)(9+n)(11+n)$	$120+43n$
46558512	33464927	-46558512	$n(1+n)(19+n)$	$40+43n$

Table 9. Examples of convergence to $\frac{a_0}{a_1+a_2 \log 2}$. The entry in blue is the one reported by the Ramanujan Project.

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
-80281600	-10675439	9459450	$(2+n)(3+n)(14+n)$	$45 + 35n$
-15728640	392683	-727650	$(4+n)^2(12+n)$	$65 + 35n$
-13107200	-263867	-28350	$(4+n)(5+n)(10+n)$	$55 + 31n$
-10485760	-93699	-103950	$(4+n)(5+n)(12+n)$	$65 + 35n$
-7372800	-884203	727650	$(2+n)(3+n)(12+n)$	$39 + 31n$
-6291456	149419	-257250	$(4+n)(8+n)^2$	$81 + 35n$
-5242880	-86807	9450	$(5+n)(6+n)(10+n)$	$77 + 35n$
-3932160	-116317	3150	$(4+n)(5+n)(8+n)$	$45 + 27n$
-3276800	-158859	22050	$(2+n)(5+n)(10+n)$	$33 + 27n$
-2621440	-48609	-1050	$(5+n)(6+n)(8+n)$	$63 + 31n$
-2359296	-168445	103950	$(2+n)(4+n)(12+n)$	$39 + 31n$
-491520	50593	-66150	$(3+n)(4+n)(10+n)$	$55 + 31n$
-327680	-21271	9450	$(2+n)(4+n)(10+n)$	$33 + 27n$
-230400	-1909	-22050	$(2+n)^2(5+n)$	$9 + 11n$
-196608	-184547	198450	$(3+n)(6+n)(10+n)$	$77 + 35n$
-163840	-4981	-22050	$n(5+n)(8+n)$	$9 + 19n$
-131072	-2951	-630	$(4+n)^2(8+n)$	$45 + 27n$
-122880	-13079	9450	$(2+n)(3+n)(10+n)$	$33 + 27n$
-61440	-2467	-3150	$(2+n)(4+n)(5+n)$	$15 + 15n$
-61440	791	-9450	$n(5+n)(6+n)$	$7 + 15n$
-51200	2839	-9450	$n(4+n)(5+n)$	$5 + 11n$
-49152	-1919	90	$(4+n)^2(6+n)$	$35 + 23n$
-36864	-2693	-450	$(2+n)(4+n)(6+n)$	$21 + 19n$
-18432	-419	-3150	$n(4+n)(6+n)$	$7 + 15n$
-18432	-419	-3150	$n(3+n)(8+n)$	$9 + 19n$
-8192	-487	-54	$(4+n)^3$	$25 + 19n$
-3072	121	-630	$n(4+n)^2$	$5 + 11n$
-2048	-129	-90	$(2+n)(4+n)^2$	$15 + 15n$
-2048	-43	-30	$(3+n)(4+n)(6+n)$	$35 + 23n$
-768	-77	-18	$(2+n)(3+n)(4+n)$	$15 + 15n$
-288	31	-90	$n(2+n)(3+n)$	$3 + 7n$

Table 10. Examples of convergence to $\frac{a_0}{a_1+a_2G}$

a_0	a_1	a_2	$P(n)/(-2n)$	$T(n) - 3n^2$
192	13	18	$(2+n)^2(3+n)$	$9+11n$
384	1	90	$n(3+n)(4+n)$	$5+11n$
2304	389	450	$n(3+n)(6+n)$	$7+15n$
3072	179	-18	$(3+n)(4+n)^2$	$25+19n$
4608	133	450	$(2+n)^2(4+n)$	$9+11n$
4608	383	-90	$(2+n)(3+n)(6+n)$	$21+19n$
11520	-1373	3150	$n(2+n)(4+n)$	$3+7n$
12288	973	-750	$(3+n)(6+n)^2$	$49+27n$
12288	1145	-630	$(2+n)(3+n)(8+n)$	$27+23n$
16384	-543	1050	$(3+n)(4+n)(8+n)$	$45+27n$
81920	3983	1350	$(4+n)^2(5+n)$	$25+19n$
89600	-10891	22050	$n(2+n)(5+n)$	$3+7n$
98304	2263	150	$(4+n)(6+n)^2$	$49+27n$
98304	35389	-36750	$(3+n)(6+n)(8+n)$	$63+31n$
122880	6563	3150	$(2+n)(5+n)(6+n)$	$21+19n$
147456	21365	22050	$n(4+n)(8+n)$	$9+19n$
163840	6789	450	$(4+n)(5+n)(6+n)$	$35+23n$
262144	710401	-771750	$(3+n)(8+n)^2$	$81+35n$
294912	18013	-3150	$(2+n)(4+n)(8+n)$	$27+23n$
524288	27787	-22050	$(4+n)(6+n)(10+n)$	$77+35n$
786432	19099	-5250	$(4+n)(6+n)(8+n)$	$63+31n$
983040	25979	-450	$(5+n)(6+n)^2$	$49+27n$
1310720	1723	28350	$(4+n)^2(10+n)$	$55+31n$
2949120	168821	22050	$(2+n)(5+n)(8+n)$	$27+23n$
9830400	-1833409	2182950	$(3+n)(4+n)(12+n)$	$65+35n$
12582912	184025	-7350	$(5+n)(8+n)^2$	$81+35n$
39321600	1965547	-727650	$(2+n)(5+n)(12+n)$	$39+31n$
165150720	12969199	-9459450	$(2+n)(4+n)(14+n)$	$45+35n$
330301440	17687791	-9459450	$(2+n)(5+n)(14+n)$	$45+35n$

Table 11. Examples of convergence to $\frac{a_0}{a_1+a_2G}$

c	a_0	a_1	a_2
3	2	1	0
4	1	1	-1
5	-1	-3	4
6	-2	-17	24
7	-3	-67	96
8	12	667	-960
9	5	666	-960
10	30	9319	-13440
11	-105	-74537	107520
12	-280	-447187	645120
13	126	447173	-645120
14	63	491884	-709632
15	231	3935051	-5677056
16	2772	102311095	-147603456
17	-1287	-102310996	147603456
18	-6006	-1023109531	1476034560
19	45045	16369749493	-23616552960
20	720720	556571437717	-802962800640

Table 12. Examples of convergence to $\frac{a_0}{a_1+a_2 \log 2}$ for R_c

j	α_j	β_j
3	-1	4
5	19	234
7	5818/3	254456/3
9	667115	60003486
11	467946090	71121907440
13	554143204110	127451285438100
15	994115449382940	322092692148962160
17	2516347061651130075	1092094185270706446150
19	8546069024090201027250	4785798287838257081935200
21	37508692924557081882027450	26331102038134635548392485900
23	206659254109760483703789089700	177726957997323983116663150902000
25	1396637676485497608584841260027550	1444123356588023432320434243315206700
27	11361110319787394788017568214856502500	13905999029609441333101619589964946580000
29	109509742351999832489255793094925601037500	156598931559029451368898717824937174831465000
31	1234320809247763942235545044494798498436195000	2039097976865181167119056627863149102390546140000
33	16085205915675471439195309128783843538512283666875	30401039180587356456007967587920548312623820610393750
35	239989379884263177615577263747245812249369757283461250	514537230471714428505965482811829861362838523445500920000

j	$\bar{\alpha}_j$	$\bar{\beta}_j$
3	-1/3	-14/3
5	-17	-8
7	-758	-27820
9	-302117	-23010044
11	-1091480994/5	-146282046156/5
13	-262476468810	-54596049230880
15	-475443072646380	-141682352738003640
17	-1211573031414907725	-489475664504671450500
19	-4135193781750207709650	-2175112041708995560914300
21	-18218507239728799899288030	-12097088912487772715204794320
23	-100680148628028059378172563700	-82356361704096372069838207986600
25	-682078864161239229949889893754850	-673893917980353010760236819146271800
27	-5559692282317104149119150499246482500	-652707873978510501152909868023829975000
29	-53681246247288656939970174534335708392500	-73865394837022289182570623863734010339760000
31	-605939306349175124039948450466713432304279000	-965867254322525126192328035746702493817188406000
33	-7906287653442943862409767973858652552097126548125	-14452693830499521315903006473321900713406050369972500
35	-118090012323922699712409299094969252935070156336941250	-2453910456091314836991909680852072336578359955374403917500

Table 13. The first $\bar{\alpha}_j = \left\{ \alpha_j, \beta_j \right\}$ values and the first $\bar{\beta}_j = \left\{ \alpha_j, \beta_j \right\}$ values.

A Mathematica Code

```

1 (* code snippet "1. Montenegro" *)
2 NumQ[k_, c_] := 2k+1+ContinuedFractionK[-2n^2(n+2k)(n+c), 3n^2+(3+4k)n+2k+1, {n, 1, 5000}];
3 vD[a_, b_, k_, c_] := If[c < 2, a+b*c, -2c(2c-1)(2(c-k)-1)^2vD[a, b, k, c-2]+(8c^2+(2-8k)c-2k+1)vD[a, b, k, c-1]];
4 vG[a_, b_, k_, c_] := ((2c-1)!!)^2Catalan+Product[(2(c-j)-1), {j, 0, k-1}]*vD[a, b, k, c-1];
5 vd[k_] := 4^(k-1)/(2k-1)/CatalanNumber[k-1]; vr[k_] := vd[k](-1)^(k)(1-2k)/((2k)!(2k-3)!!);
6 va[k_] := vr[k]vD[1, -2, 1, k-1]; vb[k_] := -vr[k](2k-3)^2vD[1, 12, 2, k-1]-va[k];
7 QFor[k_, c_] := vd[k](2c)!/vG[va[k], vb[k], k, c];
8 Print[Union[Flatten[Table[N[QFor[k, c], 200]==N[NumQ[k, c], 200], {c, 1, 14}, {k, 1, 14}]]][[1]]]; ClearAll["Global'"];

```

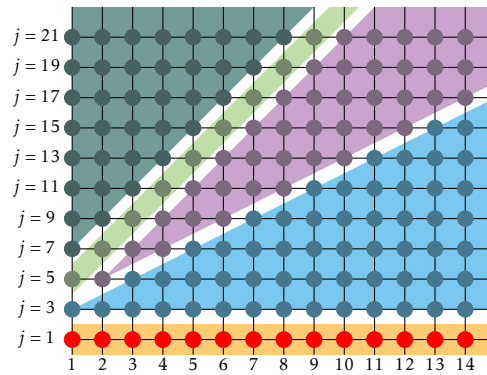


Fig. 14. The points on which **Montenegro** is tested for $1 \leq c \leq 14$ by the snippet "1. Montenegro" are shown in **red**.

```

1 (* code snippet "2. Bosnia" *)
2 NumQ[j_,c_]:= -j+ContinuedFractionK[-2n(n-2)(n+c)(n+j-1),3n^2+(2j-3)n-j,{n,1,5000}];
3 vD[j_,c_]:=If[c<2,(-2)^((3j-11)/2)(1+(15-4j)c),-2c(2c-j)(2c+1)(2c-j+2)vD[j,c-2]+(8c^2+(14-4j)c-3(j-2))vD[j,c-1]];
4 g[j_,c_]:=Product[(2c-2i+1)(j-2-2i),{i,1,(j-1)/2}](2c)!2^((3j-13)/2);
5 h[j_,c_]:= (2c-j+2)Product[(2c-2i-1)^2,{i,0,(j-5)/2}];
6 Q[j_,c_]:=Simplify[g[j,c]/(vD[j,c-1]*h[j,c])];
7 Print[Union[Flatten[Table[N[NumQ[j,c]==Q[j,c],200],{j,5,13,2},{c,1,14}]]][[1]]];ClearAll["Global'"];

```

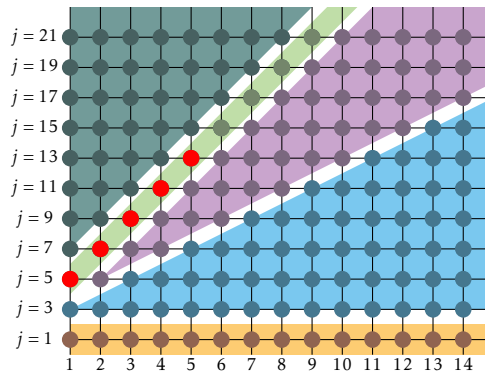


Fig. 15. The points on which *Bosnia & Herzegovina* is tested for $1 \leq c \leq 14$ by the snippet "2. Bosnia" are shown in red.

```

1 (* code snippet "3. Northern Balkans"*)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,15000}];
3 vD[ab_,j_,k_,c_]:=If[c<2,ab[[1]]+ab[[2]]c,-2c(2c-j)(2c-2k+j-2)(2c-2k-1)vD[ab,j,k,c-2]+(8c^2+(2-8k)c+(j-2)(2k-j))vD[ab,j,k,c-1]];

```



```

4 f[j_,k_,c_]:=Product[(2c-2k+2i-1)(k-i+1),{i,1,(j-1)/2}]CatalanNumber[(j-3)/2](j-2)(2k-1)*(2c-1)!!^2*CatalanNumber[k-1];
5 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
6 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
7 Q[j_,k_,c_,ab_]:=Simplify[g[j,k,c]/(vD[ab,j,k,c-1]h[j,k,c]+f[j,k,c]Catalan)];
8 Ratio:=Function[{j,k,c},r=N[NumQ[j,k,c],2000];v=FindIntegerNullVector[{1,r,N[Catalan*r,2000]};-v[[1]]/v[[2]]];
9 GenAB[j_]:=Table[{a,b}/.Solve[Table[{a,b}.{1,c-1]==g[j,k,c]/Ratio[j,k,c]/h[j,k,c],{c,1,2}],{a,b}][[1]],{k,1,6}];
10 result={};For[j=-7,j<=13,If[j==1,j=3];AB=GenAB[j];AppendTo[result,Union[Flatten[Table[N[Q[j,k,c,AB[[k]]]==NumQ[j,k,c
],20],{k,1,6},{c,1,7}]]]];j+=2];Print[Union[Flatten[result]][[1]]];ClearAll["Global`*"];

```

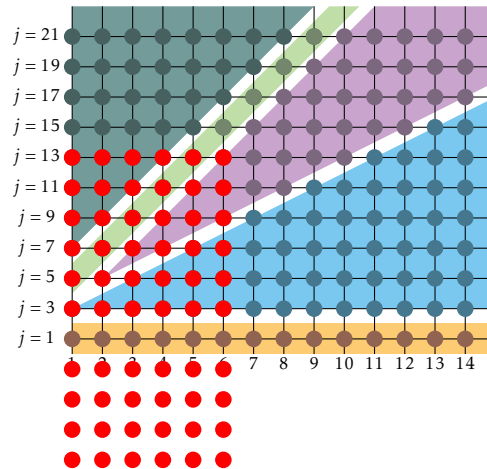


Fig. 16. The points on which the c -level master formula is tested for $1 \leq c \leq 7$ by the snippet "3. Northern Balkans" are shown in red. Note that the Montenegro line does not obey that same c -level formula but was validated by snippet "1. Montenegro". Note that the formula is also tested for negative j values (not shown as red dots in the diagram).

```

1 (* code snippet "4. Kosovo" *)
2 NumQ[j_, k_, c_] := j(2-j+2k) + ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n), 3n^2+(3+4k)n+j(2-j+2k), {n, 1, 15000}];
3 vD := Function[{ab, j, k, c}, If[c < 2, ab[[1]] + ab[[2]]c, -2c(2c-j)(2c-2k+j-2)(2c-2k-1)vD[ab, j, k, c-2] + (8c^2+(2-8k)c+(j-2)(2k-j))
   vD[ab, j, k, c-1]];
4 f[j_, k_, c_] := Product[(2c-2k+2i-1)(k-i+1), {i, 1, (j-1)/2}] CatalanNumber[(j-3)/2] (j-2)(2k-1) * (2c-1)!!^2 * CatalanNumber[k-1];
5 g[j_, k_, c_] := Product[(2c-2i+1)(2k-2i+1), {i, 1, (j-1)/2}] (2c)! 2^(2k+(j-7)/2);
6 h[j_, k_, c_] := Product[2c-2i-1, {i, 0, (j-3)/2}] Product[2c-2i-1, {i, 0, k-1}];
7 Q[j_, k_, c_, ab_] := Simplify[g[j, k, c] / (vD[ab, j, k, c-1] * h[j, k, c] + f[j, k, c] Catalan)];
8 l[n_, j_, k_] := (-1)^(k+1) (2k)!^2 / k! / 2^(3*k-2) / Product[(k-i)(2k-2i-1)^2, {i, 0, (j-3)/2}] / (2k-j) / (2k-1) / (n((2k-j-2)(3-2k)-1)
   + 1);
9 kD := Function[{n, ab, j, k}, If[k < 2, ab[[1]] + ab[[2]]k, (2k+2j-9-2n)(2k+j-8-2n)(-2k+5-j)(2k+j-6) * kD[n, ab, j, k-2] + (8k^2+k(10j
   -48-8n) + (3j^2-(28+4n)j+68+18n))kD[n, ab, j, k-1]];
10 Descend := Function[{j, k}, abh[0] = {{-1, 4}, {19, 234}, {5818/3, 254456/3}, {667115, 60003486}, {467946090, 71121907440},
11 {554143204110, 127451285438100}, {994115449382940, 322092692148962160}, {2516347061651130075, 1092094185270706446150},
12 {8546069024090201027250, 4785798287838257081935200}, {37508692924557081882027450, 26331102038134635548392485900},
13 {206659254109760483703789089700, 177726957997323983116663150902000},
14 {1396637676485497608584841260027550, 1444123356588023432320434243315206700},
15 {11361110319787394788017568214856502500, 13905999029609441333101619589964946580000},
16 {109509742351999832489255793094925601037500, 156598931559029451368898717824937174831465000},
17 {1234320809247763942235545044494798498436195000, 2039097976865181167119056627863149102390546140000},
18 {16085205915675471439195309128783843538512283666875, 30401039180587356456007967587920548312623820610393750},
19 {239989379884263177615577263747245812249369757283461250, 514537230471714428505965482811829861362838523445500920000}};
20 abh[1] = {{-1/3, -14/3}, {-17, -8}, {-758, -27820}, {-302117, -23010044}, {-1091480994/5, -146282046156/5},
21 {-262476468810, -54596049230880}, {-475443072646380, -141682352738003640}, {-1211573031414907725, -489475664504671450500},
22 {-4135193781750207709650, -2175112041708995560914300}, {-18218507239728799899288030, -12097088912487772715204794320},
23 {-100680148628028059378172563700, -82356361704096372069838207986600},
24 {-682078864161239229949889893754850, -673893917980353010760236819146271800},
25 {-5559692282317104149119150499246482500, -6527078739785105011529098668023829975000},
26 {-53681246247288656939970174534335708392500, -73865394837022289182570623863734010339760000},
27 {-605939306349175124039948450466713432304279000, -965867254322525126192328035746702493817188406000},

```

```

28 {-7906287653442943862409767973858652552097126548125,-14452693830499521315903006473321900713406050369972500},
29 {-118090012323922699712409299094969252935070156336941250,-245391045609131483699190960852072336578359955374403917500}};
30 ab=Table[kD[u,abh[u][[(j-1)/2]],j,k-j+2]/L[u,j,k],{u,0,1}];{ab[[1]],ab[[2]]-ab[[1]]};
31 Q[j_,k_,c_,ab_]:=Simplify[g[j,k,c]/(vD[ab,j,k,c-1]*h[j,k,c]+f[j,k,c]Catalan)];
32 Print[Union[Flatten[Table[N[NumQ[j,k,c]==Q[j,k,c,Descend[j,k]],200],{j,3,11,2},{k,j-2,10},{c,1,7}]]][[1]]];ClearAll["
Global'*"];

```

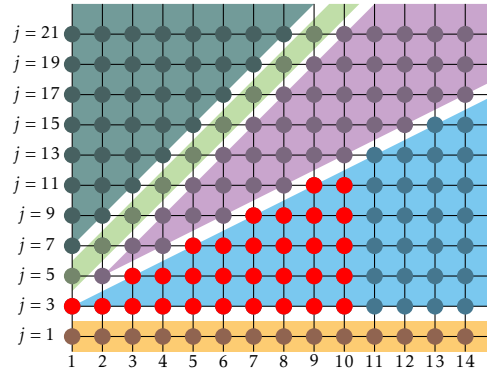


Fig. 17. The points on which *Kosovo* is tested for $1 \leq c \leq 7$ by the snippet "4. Kosovo" are shown in red.

```

1 (* code snippet "5. resolution" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,10000}];
3 f[j_,k_,c_]:=Product[(2c-2k+2i-1)(k-i+1),{i,1,(j-1)/2}]CatalanNumber[(j-3)/2](j-2)(2k-1)*(2c-1)!!^2*CatalanNumber[k-1];
4 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
5 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];

```

```

6 l:=Function[{n,j,k},(-1)^(k+1)(2k)!^2/k!/2^(3k-2)/Product[(k-i)(2k-2i-1)^2,{i,0,(j-3)/2}]/(2k-j)/(2k-1)/(n((2k-j-2)(3-2
k)-1)+1)];
7 AB:=Function[{j,k},For[c=1,c<=2,r=N[NumQ[j,k,c],2000];v=FindIntegerNullVector[{1,r,N[Catalan*r,2000]}];d[c]=(-g[j,k,c](
v[[3]]Catalan+v[[2]])/v[[1]]-f[j,k,c]Catalan)/h[j,k,c];c++;{d[1],d[2]-d[1]};ABlists:=Function[lim,z[1]=z[0]={};
8 For[j=3,j<=lim,For[w=0,w<=1,e[w]=Table[AB[j,k].{1,w}l[w,j,k],{k,j-2,j-1}];
9 ab[w]={a,b}/.Solve[Table[a+b(u-1)==e[w][[u]],{u,1,2}],{a,b}];AppendTo[z[w],ab[w][[1]]];w++;j+=2];{z[1],z[0]};
10 Print[ABlists[15]];ClearAll["Global' *"];

```

```

1 (* code snippet "6. symmetry" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,15000}];
3 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
4 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
5 zeta[j_,u_]:=Product[2+2i-j,{i,0,u-1}]/2^u;Ratio:=Function[{j,k,c},r=N[NumQ[j,k,c],2000];
6 v=FindIntegerNullVector[{1,r,N[Catalan*r,2000]}];-v[[1]]/v[[2]]];
7 GenAB[j_,k_]:=({a,b}/.Solve[Table[{a,b}.{1,c-1]==g[j,k,c]/h[j,k,c],{c,1,2}],{a,b}][[1]]);
8 tau[j_,u_]:=If[u>(j-1)/2,Sign[zeta[j,u]](2j-2u-3)!!(2u-j)!!(-2)^u,zeta[j,u](-4)^u];
9 For[j=3,j<=13,Print[MatrixForm[Table[{{j,j-u-1},{j-2u,j-u-1},(GenAB[j,j-u-1]/GenAB[j-2u,j-u-1])/tau[j,u]=={1,1}},{u,1,j
+3}]]];j+=2];ClearAll["Global' *"];

```

```

1 (* code snippet "7. Croatia" *)
2 NumQ[j_,k_,c_]:=j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n),3n^2+(3+4k)n+j(2-j+2k),{n,1,2000}];
3 g[j_,k_,c_]:=Product[(2c-2i+1)(2k-2i+1),{i,1,(j-1)/2}](2c)!2^(2k+(j-7)/2);
4 h[j_,k_,c_]:=Product[2c-2i-1,{i,0,(j-3)/2}]Product[2c-2i-1,{i,0,k-1}];
5 Ratio:=Function[{j,k,c},r=N[NumQ[j,k,c],4000];v=FindIntegerNullVector[{1,r,N[Catalan*r,4000]}];-v[[1]]/v[[2]]];
6 GenAB[j_,k_]:=({a,b}/.Solve[Table[{a,b}.{1,c-1]==g[j,k,c]/Ratio[j,k,c]/h[j,k,c],{c,1,2}],{a,b}][[1]]);
7 psil[i_,j_]:={-1,14-j,-464+58j-3j^2,27936-4692j+432j^2-15j^3,-2659968+542256j-67836j^2+4260j^3-105j
^4,367568640-86278560j+13203480j^2-1139700j^3+51450j^4-945j^5}[[i]];
8 psi2[i_,j_]:={4j-15,306-95j+4j^2,-13360+4646j-357j^2+12j^3,999648-379692j+40368j^2-2457j^3+60j^4,-113885568+
9 46449360j-6124164j^2+513228j^3-22935j^4+420j^5,18333538560-7933530720j+1224286440j^2-126833100j^3+7864950j^4-266175j
^5+3780j^6}[[i]];
10 mu[i_,j_]:=-Product[j-2q-2,{q,1,i}]/(-2)^((3j-11-4i)/2);

```

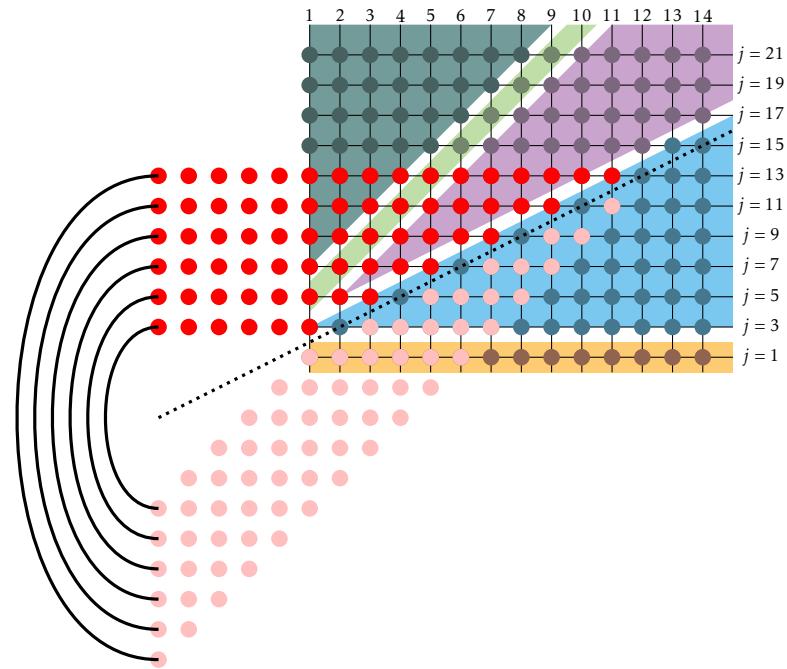


Fig. 18. The points on which the Kosovo-Serbia symmetry is tested at the $(\alpha_{j,\kappa}, \beta_{j,\kappa})$ -level by the snippet "6. symmetry". Tested points are shown in red and their symmetrical correspondents in pink.

```

11 Print[Union[Table[Union[Table[GenAB[j, (j-3)/2-i]=={psi1[1+i, j], psi2[1+i, j]}/mu[i, j], {j, 5+2i, 37, 2}]]][[1]], {i
, 0, 5}]]][[1]]; ClearAll["Global'*"];

```

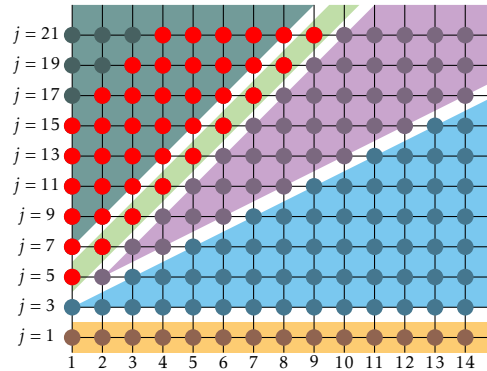


Fig. 19. The points on which `Croatia` is tested. Note that the test is performed directly at the $(\alpha_{j,\kappa}, \beta_{j,\kappa})$ -level. Points tested by the snippet "7. Croatia" are shown in red. Points tested beyond the quadrant are not shown.

```

1 (* code snippet "8. coefficients" *)
2 coef=CoefficientList[{3198013886925+(145296572850+(5207427225+(4353102000+(877052475+(78210090+(3023055+41580(X-29)) (X
  -27)) (X-25)) (X-23)) (X-21)) (X-19)) (X-17), -14487726825-(104826150+(452605725+(121200300+(13697775+(640710+10395(X
  -27)) (X-25)) (X-23)) (X-21)) (X-19)) (X-17)), X];Print[Table[GCD[coef[[1, i]], coef[[2, i]]], {i, 1, 7}]];ClearAll["
  Global`*"];

```

```

1 (* code snippet "9. ratio" *)
2 NumQ[j_, k_, c_] := j(2-j+2k)+ContinuedFractionK[-2n(c+n)(j+n-1)(1-j+2k+n), 3n^2+(3+4k)n+j(2-j+2k), {n, 1, 50000}];
3 For[j=1, j<=7, For[k=Abs[j-2], k<=7, For[c=1, c<=7, r=N[NumQ[j, k, c], 3000]; v=FindIntegerNullVector[{1, r, N[Catalan*r, 3000]}];
4 q=(-v[[1]]/v[[3]])==((2c)!2^(2k+(j-7)/2+Floor[1/j])/CatalanNumber[k-1]/(2k-1)/(2c-1)!!^2/(j-2)/Product[(2c-2k+2i-
5 1)(k-i+1)/(2c-2i+1)/(2k-2i+1), {i, 1, (j-1)/2}]/CatalanNumber[(j-3)/2]);Print[{j, k, c, q}];c++;k++;j+=2];ClearAll["Global
  `*"];

```

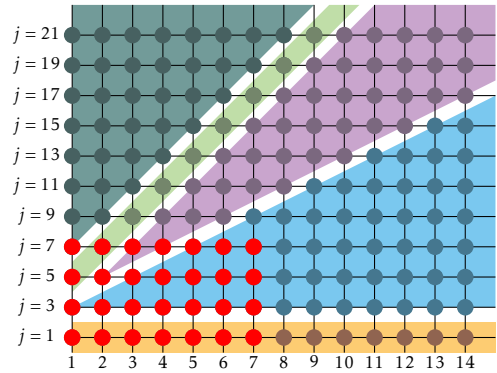


Fig. 20. The points on which the ratio conjecture is tested for $1 \leq c \leq 7$ by the snippet "9. ratio" are shown in red.

```

1 (* code snippet "10. log2-a" *)
2 Q[c_]:=c+ContinuedFractionK[-2n^2,3n+c,{n,1,2000}];
3 For[c=2,c<=25,r=N[Q[c],80];
4 v=FindIntegerNullVector[{1,r,N[Log[2]*r,80]}];
5 Print[{c,Simplify[2/LerchPhi[1/2,1,-1+c]==-v[[1]]/(v[[2]]+Log[2]v[[3]])==1/(Log[2]2^(c-2)-Sum[2^(c-j-2)/j,{j,c-2}]]}];
   c++];ClearAll["Global'"];

```

```

1 (* code snippet "11. log2-b"*)
2 R[c_]:=c+ContinuedFractionK[-2n^2-2n,3n+c,{n,1,2000}];
3 For[c=3,c<=20,r=N[R[c],200];v=FindIntegerNullVector[{1,r,N[Log[2]*r,200]}];
4 Print[{c,2^(c-4)*(c-3)*v[[1]]==v[[3]],-v[[1]]/(v[[2]]+Log[2]v[[3]])};c++];ClearAll["Global'"];

```

```

1 (* code snippet "12. Inostranstvo1" *)

```

```

2 NumQ[j_]:=7+6j+ContinuedFractionK[-2n(n+1)^2(n+2j+1),3n^2+(9+4j)n+(7+6j)},{n,1,10000}];
3 D1:=Function[i,If[i<2,2+15i,2(2i-1)^3(1-i)*D1[i-2]+(8i^2-2i+3)D1[i-1]]];For[j=0,j<=14,r=N[NumQ[j],80];
4 v=FindIntegerNullVector[{1,r,N[Catalan*r,80]};Q=-v[[1]]/(v[[2]]+Catalanv[[3]]);Print[{j,(2*j+1)!/(D1[j]-2Catalan(2*j
+1)!!^2)==Q}];j++;

```

```

1 (* code snippet "13. Inostranstvo2" *)

```

```

2 NumQ[j_]:=23+10j+ContinuedFractionK[-2n(n+1)(n+3)(n+2j+3),3n^2+(17+4j)n+(23+10j)},{n,1,30000}];
3 For[j=0,j<=30,r=N[NumQ[j],180];v=FindIntegerNullVector[{1,r,N[Catalan*r,180]};Print[{j,(2j+5)!/(2j+4)/(2j+5)!!^2==v
[[1]]/v[[3]]}];j++;

```

```

1 (* code snippet "14. Inostranstvo" *)

```

```

2 NumQ:=Function[{i,a,b,c},z=a+b+1;x=ab+(c+1)z;x+2zi+ContinuedFractionK[-2n(n+a)(n+b)(n+2i+c),
3 3n^2+(2z+2c+1+4i)n+(x+2zi)},{n,1,5000}]];lim=6;ToV:=Function[{i,a,b,c},r=N[NumQ[i,a,b,c],80];v=FindIntegerNullVector[{1,
r,N[Catalan*r,80]};
4 Q=-v[[1]]/(v[[2]]+Catalanv[[3]]);Table[ToV[i,a+p,b+p,c+p},{p,0,1},{a,0,lim,2},{i,0,lim},{c,0,lim,2},{b,a,lim,2}]

```

```

1 (* code snippet "15. Series" *)

```

```

2 all={Pi,{0,0,2,2,2,0},{57153600,-33075,103904},0},Pi,{0,1,1,2,2,1,1},{69854400,-24255,76192},0},Pi
, {0,1,2,1,1,2,1},{62868960,-14553,45712},0},Pi,{0,2,2,1,2,2,0},{-65318400,-4725,14848},0},Pi
, {0,2,2,2,0,0},{-6350400,-3675,11552},0},Pi,{1,1,2,2,1,1,0},{-907200,-315,992},0},Pi
, {1,2,1,1,2,1,0},{-635040,-147,464},0},Pi,{1,2,2,2,1,0},{16934400,-245,768},0},Pi
, {2,1,1,1,1,2,0},{-59535000,-6615,21292},0},Pi,{2,2,0,0,2,1,1},{-93139200,-2695,9344},0},Pi
, {2,2,0,0,2,2,0},{-52920000,-2695,9056},0},Pi,{2,2,1,2,2,0,0},{-50803200,3675,-11264},0},Pi
, {2,2,2,0,0,0},{129600,-75,224},0},Catalan,{0,1,2,2,1,0},{-50803200,66150,-60577},0},Catalan
, {0,3,2,3,0,0,0},{-3456000,-6750,6197},0},Catalan,{1,2,2,1,0,0},{-2419200,-3150,2909},0},Catalan
, {1,3,0,3,1,0,0},{8064000,8750,-8109},0},Catalan,{2,2,0,2,2,0,0},{-33868800,-22050,21131},0},Catalan
, {3,2,3,0,0,0,0},{-27648,54,-25},0},Log[2],{0,0,0,2,2,2,2},{-33177600,-38400,26617},1},Log
[2],{0,0,2,2,2,2,0},{1382400,-1600,1109},1},Log[2],{0,1,1,2,2,1,1},{11059200,-7680,5323},1},Log
[2],{0,1,2,1,1,2,1},{-22118400,10240,-7097},1},Log[2],{0,2,2,0,0,2,2},{17280000,-1760,1219},1},Log
[2],{0,2,2,2,2,0,0},{442368,512,-355},1},Log[2],{1,1,2,0,0,2,2},{-88473600,5120,-3539},1},Log

```



```

[2], {1, 1, 2, 2, 1, 1, 0}, {-230400, -160, 111}, 1}, {Log[2], {1, 2, 1, 1, 2, 1, 0}, {-22118400, -10240, 7109}, 1}, {Log
[2], {2, 2, 0, 0, 2, 1, 1}, {-88473600, -5120, 3627}, 1}, {Log[2], {2, 2, 0, 0, 2, 2, 0}, {69120000, 7040, -4951}, 1}, {Log
[2], {2, 2, 1, 2, 2, 0, 0}, {7077888, -1024, 707}, 1}, {Log[2], {2, 2, 2, 2, 0, 0, 0}, {-13824, 16, -11}, 1}};
3 G:=Function[{e, ep}, (-1)^(n+1)/Product[(2n+2i-3+ep)^e[[i]], {i, 1, Length[e]}]]; F:=Function[{e, ep}, Sum[G[e, ep]/.{n->j}, {j
, 1, 100000}]]; Union[Table[{F[all[[i, 2]], all[[i, 4]]], all[[i, 1]], 1}.all[[i, 3]]<10^30, {i, 1, Length[all]}]]

```

B A bit of science-fiction

Note that if we had a κ -level formula for **Serbia** the problem would be settled and we would have a j -level master formula. We stress that we do not have such a formula, hence what follows is only a hypothetical science-fiction blueprint.

We would split the problem into two parts: Computing $Q_{j,\kappa,c}$ for $j < 7$ and for $j \geq 7$. For $j < 7$ we would just use the κ -level formulae given in this paper for $j \in \{1, 3, 5\}$ that we would compute once and for all.

We would then profit of the symmetry to “ascend” from lines 1,3 to any $j \geq 7$ as shown in Figure 22. The rest of the process would follow Figure 5. A more succinct equivalent process is described in Figure 23. We would have hence obtained a polynomial-time process (polynomial in j, κ, c) allowing to compute precisely and formally any $Q_{j,\kappa,c}$.

Unfortunately the κ -level formula for **Kosovo** does not work for **Serbia**!



Fig. 21. The remaining challenge: find a j -level formula so that $Q_{j,\kappa,c}$ *Requiescat in pace*.

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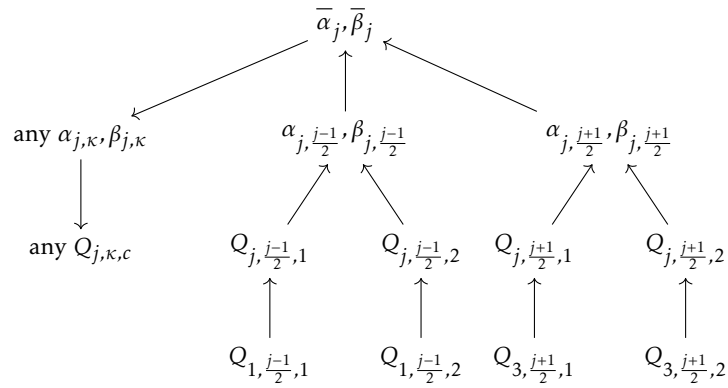


Fig. 22. Blueprint for a hypothetical j -level master formula for $j \geq 7$ (for now science fiction).

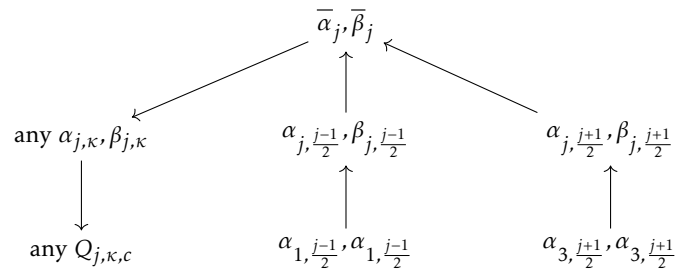


Fig. 23. Blueprint for a direct j -level master formula for $j \geq 7$ using the last remark of Section 8.1 (for now science fiction).

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