True symmetric formulations for fluid-structure interaction in bounded domains. Finite element results

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True Symmetric Variational Formulations for Fluid–structure Interaction in Bounded Domains – Finite Element Results

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Summary It is proposed to establish, in a systematic way, symmetric variational formulations for the problem of transient and modal analysis of bounded coupled fluid–structure linear systems, taking into account gravity and compressibility effects. The discretization of such variational principles by finite element procedures leads therefore to convenient symmetric matrix systems.

This question of symmetry arises from the choice of *displacement* type for the structure and of stress (*pressure*) type for the fluid. If those types of variables are simultaneously used, then modified complementary type variational procedures must be introduced in order to exhibit an adequate fluid–structure coupling.

This can be done by using two new variables, namely the free surface displacement (for gravity effects) and the displacement potential of the fluid (for compressibility effects), together with the displacement field in the structure and the pressure field in the fluid. In those cases, generalized added mass operators – independent of time and circular frequency – are introduced. A link with boundary integral equation is established.

Numerical finite element results are presented in three cases:

—incompressible hydroelastic modal analysis of a liquid propelled launch vehicle,
—elasto-acoustic modal analysis of an incompressible structure containing a compressible gas,
—elastic cylinder partially filled with liquid under gravity effects (comparison with experiments).

10.1 INTRODUCTION

It is proposed to investigate the problem of transient and modal linear analysis of bounded coupled fluid–structure systems. Those systems are constituted by an elastic structure partially or completely filled with a compressible fluid, taking into account possible gravity and/or capillary free surface effects. Such problems arise in aerospace, nuclear, mechanical, naval, and civil engineering areas [1–3]. Let us mention some typical examples. The prediction of the so-called Pogo instabilities occurring on launch vehicle configurations (launchers, the Space Shuttle) requires very complete and very accurate mathematical models in the calculation of the propellant tanks' hydroelastic modes. The response of liquid-filled nuclear reactors and dams to a seismic environment is also of prime importance. Supplementary gravity effects are fundamental in the linear dynamics of wing tip tanks of aircraft. Capillary effects arise in satellite tanks. Acoustoelasticity studies are important for the dynamics of solid propellant boosters and for the prediction of interior noise in aircraft, in automobile passenger compartments, etc.

In all the above-mentioned problems, the physically coupled fluid–structure system may be considered as weakly dissipative for the range of frequencies of interest. Therefore, the computation of the state parameters of the associated conservative system is of prime importance, i.e. a system in which there is no damping in the structure and no viscosity in the fluid. It should be stressed that for bounded media, compressibility and gravity-free surface effects for the fluid *are not* a source of damping, as is the case for unbounded media (radiation phenomena) [3–5].

Among the numerical methods which could be applied to these problems, the finite element method is a particularly efficient one in so far as one can use the standard algorithms for the resolution of a *symmetric matrix system* which usually exist in modular finite element codes. The symmetry of the matrices is a property connected with the existence of symmetric variational formulations of conservative linear systems of continuous media [6, 7].

A boundary integral method has been already developed for the computation of the vibration modes of an elastic tank partially filled with an incompressible liquid [8] and has recently been applied for a case having very simple geometry [9]. We must stress here that the extension and the application of boundary integral procedures to the case of a compressible fluid (acoustoelasticity problems) is very complicated due to the fact that it is *impossible* to obtain, for instance for modal analysis, an eigenvalue problem of the type $[A-\omega^2 B]\times\{X\}=\{0\}$, where ω denotes the circular frequency, and $\{X\}$ the state variables in the structure and on the fluid–structure boundary, *even* if A and B are *non-symmetric*. For transient response analysis, this means that higher time order terms rather than the second time derivative of $\{X\}$ are present. In order to obtain a linear type of eigenvalue problem with respect to ω^2, one

must absolutely keep the interior fluid state variable present in the final equations of the problem. This fact cancels the major advantage of boundary integral methods for this type of problem.

That is why, for general dynamic fluid–structure problems in *bounded domains*, we preferred the finite element method, or, if possible, Ritz-type geometric and physical substructure procedures, in order to have general modular tools for complex geometries and involving complex physical phenomena. In fact, this idea implies that one must, as a first step, exhibit *symmetric* variational formulations as the system is linear and conservative. When, for instance, for harmonic vibrations, a Ritz-type procedure may be employed, the basis vector which spans the space where the solution belongs can be computed by using methods other than the finite element method, provided that they are more efficient, i.e. making use as far as possible of standard efficient algorithms. Each subsystem being also conservative, the method should take advantage of the fact that symmetric variational principles always exist.

As in the analysis of linear conservative fluid–structure systems the fluid is irrotational, a scalar variable can be used, namely the pressure field in the fluid which represents a dual (stress) type of unknown field. When, in conjunction with this dual variable, a primal variable (the displacement field) is used for the elastic structure, the search of *symmetric* variational formulations is not a *standard theoretical task*. From a study of the literature, it seems that a solution of this problem has been very difficult to handle. This is the reason why many non-symmetric matrix systems are exhibited [10, 11], together with some matrix manipulations in order to symmetrize these systems through some physical approximations [12–14], before applying direct finite element procedures [15] or modal techniques [13]. More recently, a symmetrization procedure [16] has been carried out, but this converts the conservative non-symmetric formulation into one which is 'damped' and symmetric and therefore more costly for bounded domains. For aerospace engineering problems, we have succeeded in solving fluid–structure interaction problems through *original symmetric variational* formulations [17–23]. In the present paper, which constitutes the detailed version of reference [22], we do not intend to discuss the effects of gravity on the wet boundaries of the elastic structure [17, 24] as they are generally negligible with respect to other energy contributions. Some of the methods proposed in references [17–22] have been adopted recently and applied to nuclear engineering problems [25].

Alternatively, the choice of primal-type variables both for the structure and the fluid leads to straightforward symmetric formulations, but difficulties arise due to the fact that the discretization of the kinematic admissible space introduces numerical drawbacks connected with the existence of an infinite-dimensional space of zero eigenvalues. The corresponding formulation, which has been established ([19], Appendix 1) for acoustoelasticity problems, has

been applied for the same type of situations in reference [26], which incidentally contains a misleading interpretation of the symmetrization procedures of references [17–19] when using pressure-type variables for the fluid. Let us mention a penalty procedure to avoid the zero eigenvalue discretized space [27].

We present below a *systematic approach* to the general linear transient or modal fluid–structure problem in order to state *symmetric variational formulations.* The main problem arises from the choice of unknown fields of displacement or stress types for the structure and for the fluid. If those types of variables are used simultaneously, then *modified* complementary-type variational procedures must be generally introduced in order to exhibit an appropriate coupling. The various fluid–structure symmetric variational formulations, which are considered here and which lead to symmetric matrix systems, can be applied through a direct finite element analysis or a substructuring type of analysis [18–22].

Numerical results are presented for three cases: (1) incompressible hydroelastic vibrations of a liquid-propelled launch vehicle; (2) acoustoelastic vibrations of an incompressible axisymmetric elastic structure containing a compressible gas; (3) a three-dimensional system constituted by an elastic structure containing a liquid, taking into account gravity effects, together with a comparison with experimental results.

10.2 GENERAL LOCAL EQUATIONS OF THE BOUNDARY VALUE PROBLEM

10.2.1 Notation

Let us introduce the following notation (see Figure 10.1):

ω	circular frequency
$\Omega_s, \Omega_l, \Omega_g$	domain occupied by the structure, the liquid, and the gas at equilibrium
Σ_l (resp. Σ_g)	liquid (resp. gas)–structure interface at equilibrium
Γ	liquid–gas interface at equilibrium (or usual free surface if the effect of the gas is negligible)
R_1, R_2	principal curvature radii of Γ
$\mathbf{n}_s, \mathbf{n}_l, \mathbf{n}_g$	unitary external normal to $\Omega_s, \Omega_l, \Omega_g$
g	gravity field constant
ρ_s, ρ_l, ρ_g	mass density of the structure, the liquid, and the gas at equilibrium
c_l (resp. c_g)	sound velocity in the liquid (resp. the gas)
τ	surface tension constant
$\mathbf{U}_s, \mathbf{U}_l, \mathbf{U}_g$	displacement field of the structure, the liquid, and the gas material points, measured from equilibrium
η	normal displacement of liquid–gas interface (or of the free surface), measured from equilibrium
p_l (resp. p_g)	unsteady pressure field in the liquid (resp. the gas)
φ_l (resp. φ_g)	unsteady displacement potential field in the liquid (resp. the gas)

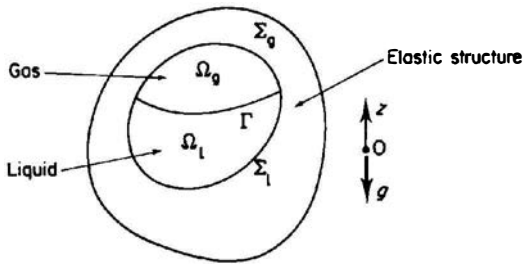

Figure 10.1 General coupled liquid–gas–structure system in an external gravity field environment.

σ Cauchy stress tensor, function of U_s through the constitutive law of hyper-elastic medium
ε deformation tensor
Tr trace symbol of an operator

10.2.2 Physical and geometrical hypothesis

As already mentioned in the introduction, we are dealing with a conservative dynamic linear fluid–structure bounded system. Therefore the fluid (or respectively the structure) is considered as inviscid (or respectively as elastic). We shall not include, for sake of brevity, pre-stress effects [28, 29] due to an internal gas (through a quasi-static approximation) and to the liquid's weight for pressurized tanks. As already mentioned in the introduction, small gravity effects on the fluid–structure interface are neglected [24].

We are concerned by small motion amplitudes around an equilibrium natural state at rest. Therefore the fluid may be considered as irrotational.

10.2.3 General local equations of the boundary value problem

10.2.3.1 *In the structure*

The dynamic equilibrium equation is given by

$$\operatorname{div}\sigma-\rho_s\frac{\partial^2\mathbf{U}_s}{\partial t^2}=0 \quad \text{in} \quad \Omega_s. \tag{10.1}$$

The stress–strain elastic law is given by

$$\sigma=A\varepsilon \quad \text{in} \quad \Omega_s. \tag{10.2}$$

Strain–displacement relationships are described by

$$\varepsilon = \tfrac{1}{2}[\text{grad}\,\mathbf{U}_s + (\text{grad}\,U_s)^T]. \tag{10.3}$$

10.2.3.2 *In the fluid*

Let us consider, for instance, the general case of a stratified fluid medium constituted by a heavy liquid and a gas (Figure 10.1). The liquid–gas interface is at equilibrium under an external gravity field (including capillary effects in the case of a low gravity environment).

The dynamic equations in the fluid are a particular case of equation (10.1) with a spherical stress tensor (which takes a value opposite to that of the pressure). For the liquid,

$$-\text{grad}\,p_l - \rho_l \frac{\partial^2 \mathbf{U}_l}{\partial t^2} = 0 \quad \text{in} \quad \Omega_l. \tag{10.4}$$

For the gas,

$$-\text{grad}\,p_g - \rho_g \frac{\partial^2 \mathbf{U}_g}{\partial t^2} = 0 \quad \text{in} \quad \Omega_g. \tag{10.5}$$

The equation corresponding to (10.2) and (10.3), i.e. the relationship between the pressure and the displacement fields, is obtained through the classical barotropic law and continuity equation. For the liquid,

$$p_l = -\rho_l c_l^2\,\text{div}\,\mathbf{U}_l \quad \text{in} \quad \Omega_l. \tag{10.6}$$

For the gas,

$$p_g = -\rho_g c_g^2\,\text{div}\,\mathbf{U}_g \quad \text{in} \quad \Omega_g. \tag{10.7}$$

Furthermore, $\text{curl}\,\mathbf{U}_l = 0$ and $\text{curl}\,\mathbf{U}_g = 0$.

10.2.3.3 *Liquid–gas coupling equations*

The liquid–gas interface which is at equilibrium under the external gravity field may be considered as a surface medium which has its own constitutive law, namely

$$p_l - p_g = \rho_l g \eta \quad \text{on} \quad \Gamma. \tag{10.8}$$

Remark 1 In the case of a low external gravity field, one must add the operator $\mathcal{L}(\eta)$ corresponding to a membrane theory-type behaviour of the interface [20, 29, 30]:

$$\mathcal{L}(\eta) = \tau\left[-\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)\eta + \text{div}\,\text{grad}\,\eta\right], \tag{10.9}$$

div and grad being considered as surface operators in (10.9).

The constitutive law of the interface medium Γ becomes

$$p_l - p_g = \rho_l g \cos(\mathbf{n}_l, z)\eta + \mathscr{L}(\eta). \tag{10.10}$$

In this case, one must add an appropriate boundary condition on the boundary $\partial\Gamma$ of Γ which expresses the constant angle condition [1, 30]:

$$\frac{\partial \eta}{\partial \nu} = \alpha\eta \quad \text{on} \quad \partial\Gamma. \tag{10.11}$$

(α is a constant depending of the fluid–structure interface characteristics and ν is the normal to $\partial\Gamma$ in the tangent plane to Γ at the contact line between Γ and the tank.)

Special attention must be paid in order to couple the preceding theory of capillarity with a structure relevant to the three-dimensional elastic theory. This last problem is analogous to the classical problem of the matching of two media, one described by three-dimensional elasticity theory and the other by second gradient theory (plate, shell) [29]. We shall not describe this particular point in the present analysis.

Remark 2 We have, for the sake of simplicity, considered here only the case where one of the two fluids is heavy; that is why only the contribution $\rho_l g\eta$ is present.

The coupling equation (10.8) (or (10.10) and (10.11)) must be completed by the kinematic coupling equations:

$$\eta = \mathbf{U}_l \cdot \mathbf{n}_l \quad \text{on} \quad \Gamma, \tag{10.12}$$

$$\eta = -\mathbf{U}_g \cdot \mathbf{n}_g \quad \text{on} \quad \Gamma. \tag{10.13}$$

10.2.3.4 *Fluid–structure coupling equations*

Dynamic equilibrium equations on the interface are given by

$$\sigma\mathbf{n}_s = p_l\mathbf{n}_l \quad \text{on} \quad \Sigma_l, \tag{10.14}$$

$$\sigma\mathbf{n}_s = p_g\mathbf{n}_g \quad \text{on} \quad \Sigma_g. \tag{10.15}$$

Kinematic equations on the interface (inviscid fluid) are given by

$$\mathbf{U}_s \cdot \mathbf{n}_s = \mathbf{U}_l \cdot \mathbf{n}_s \quad \text{on} \quad \Sigma_l, \tag{10.16}$$

$$\mathbf{U}_s \cdot \mathbf{n}_s = \mathbf{U}_g \cdot \mathbf{n}_s \quad \text{on} \quad \Sigma_g. \tag{10.17}$$

10.2.3.5 *Prescribed boundary and initial conditions*

Let us denote by S the boundary $\partial\Omega - (\Sigma_l \cup \Sigma_g)$ of the structure. The system is subjected to prescribed forces acting, for example, on a part S_1 of S:

$$\sigma\mathbf{n}_s = F(M, t) \quad \text{on} \quad S_1. \tag{10.18}$$

The system is, for example; clamped on S_2 ($S = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$):

$$\mathbf{U}_s(M, t) = 0, \quad \text{on} \quad S_2, \qquad \forall t, \tag{10.19}$$

Remark 3 If S_2 is free, no supplementary difficulties arise, the corresponding adjustment is left to the reader.

One must finally add the appropriate initial Cauchy conditions. If the state field variables are of displacement types $\mathbf{U}_s$, $\mathbf{U}_l$, $\mathbf{U}_g$, and η, we have $\mathbf{U}_s(M, 0)$, $(\partial \mathbf{U}_s / \partial t)(M, 0)$ (with the respective subscripts l and g) given; $\eta(M, 0)$ is also given.

10.2.4 Dynamic variational formulation – Generalities

Without entering into mathematical details [31–33], let us recall that, as is usual in the variational treatment of linear dynamic systems, the time t is considered as a parameter.

A function $f(M, t)$ is defined as the mapping $M \to f(M, t)$, using the same notation f for convenience; then we consider $f: t \to f(t)$.

Let V be the product space of the admissible spaces in which belong each state variable field of the boundary value problem. V is independent of t and corresponds to the classical mathematical spaces introduced for the functional analysis studies of linear problems of the mechanics of continuous media.

Let $X(t)$ be an element of V.

The dynamic variational formulation of a classical linear dynamic conservative system is settled as follows. Find a function $t \to X(t)$ of $[0, t'] \to V$ such that

$$a(\ddot{X}(t), \delta X) + b(X(t), \delta X) = (F(t), \delta X) \qquad \forall \delta X \in V, \tag{10.20}$$

plus initial conditions. The double dot, i.e. $\ddot{X}$, denotes the second derivative with respect to t and δX, independent of t, is the usual test function. (The notation δX is useful for the correlation with virtual principles concepts of mechanics [6, 7].) $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are two bilinear forms of $V \times V$. The existence and uniqueness of a solution $X(t)$ of (10.20) is closely connected with the properties of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$.

In particular, $a(\cdot, \cdot)$ usually defines a scalar product on an appropriate space and is therefore *symmetric*, positive definite. $b(\cdot, \cdot)$ must also be *symmetric* and must satisfy mathematical inequality properties such as coercivity. If, for example, X is a displacement field, then $a(\cdot, \cdot)$ corresponds to a so-called kinetic bilinear operator on $V \times V$, connected to the kinetic energy of the linear conservative system as $a(\cdot, \cdot)$ is symmetric; similarly, $b(\cdot, \cdot)$ corresponds to a so-called potential bilinear operator on $V \times V$, connected to the potential energy of the linear conservative system as $b(\cdot, \cdot)$ is symmetric.

If we are dealing with *free harmonic vibrations*, then (10.20) becomes an eigenvalue problem involving the same operators $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. The problem is then settled as follows. Find $\omega^2 \in \mathbb{R}^2$ and $X \in V$ such that

$$-\omega^2 a(X, \delta X) + b(X, \delta X) = 0 \qquad \forall \delta X \in V. \tag{10.21}$$

With the same symmetry and mathematical property of the operators (compactness), (10.21) expresses the stationarity of the following Rayleigh quotient:

$$R(X) = \frac{b(X, X)}{a(X, X)}, \qquad X \in V, \tag{10.22}$$

or equivalently:

$$\delta[-\omega^2 a(X, X) + b(X, X)] = 0 \quad \text{on} \quad V, \qquad \forall \delta X \in V. \tag{10.23}$$

This leads to the classical min or min-max properties of eigenvalues [34, 35] together with the usual algebraic properties: a infinite numerable sequence of eigenvalues; to each eigenvalue is associated a finite-dimensional subspace (expected for the infinite eigenvalue); the direct sum of these subspaces constitutes the admissible space V.

Remark 4 We should notice that, starting from (10.20), and using Laplace transformation, we obtain a spectral problem (10.21). *The connection between transient and harmonic formulations is straightforward.*

We must notice here that if the symmetry properties *are not satisfied*, all the preceding considerations are invalid; in particular the existence theorem in the transient case remains to be proved. Moreover, from the *physical point of view*, a linear conservative system which therefore possesses quadratic operators connected to energy concepts must lead to symmetric variational formulations.

We stress the fact that if, as mentioned in Section 10.1, many non-symmetric formulations arose from the linear conservative dynamic fluid–structure problem, the situation is due to an unappropriate choice of state variable fields. This will be seen in the next sections when *primal and dual* types of variables are introduced.

10.3 VARIATIONAL FORMULATIONS IN THE CASE OF RIGID, MOTIONLESS STRUCTURES

10.3.1 Introduction

In what follows we shall discuss primal (displacement) and various dual (pressure) variational formulations in the case of a rigid, motionless structure

in order to prepare the tools which will be needed in Section 10.4 for the elastic situation.

In order, when dealing with dual field variables, to retrieve the usual physical situation of linear dynamic conservative system, i.e. with potential and kinetic bilinear operators corresponding to equations (10.20) or (10.21), we must introduce, in the general case, a so-called indirect dual–primal symmetric variational formulation. The word primal is employed because the liquid–gas interface (or the free surface) is described by a primal type field, namely the normal displacement η of the interface.

For the sake of simplicity, we shall not consider capillary effects, but we shall show at the end of this section how the variational formulations may be modified accordingly. Moreover, we shall consider the standard modal situation, the forced transient case being *easily recovered* by replacing $-\omega^2 X$ by $\ddot{X}(t)$ and by introducing as a second member the linear form corresponding to the prescribed forces.

By using the same notation for the state variable field and the corresponding modal amplitude, the equations of Section 10.2.3 may be given as follows: *in the liquid*,

$$-\text{grad}\, p_l + \rho_l \omega^2 \mathbf{U}_l = 0 \quad \text{in} \quad \Omega_l, \tag{10.24}$$

$$p_l = -\rho_l c_l^2 \,\text{div}\, \mathbf{U}_l \quad \text{in} \quad \Omega_l; \tag{10.25}$$

in the gas,

$$-\text{grad}\, p_g + \rho_g \omega^2 \mathbf{U}_g = 0 \quad \text{in} \quad \Omega_g, \tag{10.26}$$

$$p_g = -\rho_g c_g^2 \,\text{div}\, \mathbf{U}_g \quad \text{in} \quad \Omega_g. \tag{10.27}$$

We can easily see that, as ρ_l and ρ_g are constant in Ω_l and Ω_g, (10.24) and (10.26) lead to the results $\text{curl}\, \mathbf{U}_l = 0$ and $\text{curl}\, \mathbf{U}_g = 0$. Furthermore, this property comes from the fact that, together with the barotropic behaviour of the field, linearized motions around an equilibrium state at rest is considered here.

The liquid–gas coupling equations may be written as follows:

$$p_l - p_g = \rho_l g \eta \quad \text{on} \quad \Gamma, \tag{10.28}$$

$$\eta = \mathbf{U}_l \cdot \mathbf{n}_l \quad \text{on} \quad \Gamma, \tag{10.29}$$

$$\eta = -\mathbf{U}_g \cdot \mathbf{n}_g \quad \text{on} \quad \Gamma. \tag{10.30}$$

The *fluid-structure equations* are given by

$$\mathbf{U}_l \cdot \mathbf{n}_l = 0 \quad \text{on} \quad \Sigma_l, \tag{10.31}$$

$$\mathbf{U}_g \cdot \mathbf{n}_g = 0 \quad \text{on} \quad \Sigma_g. \tag{10.32}$$

10.3.2 Primal (displacement) symmetric variational formulation

Let us consider the case in which three media are coupled together, each one being described by displacement-type variables $\mathbf{U}_l$, η, and $\mathbf{U}_g$.

The interface medium can be viewed as a medium characterized by the constitutive law (10.28) leading to a potential energy without a kinetic energy due to an absence of mass (membrane or spring-type forces). The variational formulation – obviously symmetric – is consequently straightforward to establish.

If V_1 denotes the appropriate admissible space, i.e.

$$V_1 = \{\eta, \mathbf{U}_l, \mathbf{U}_g | \mathbf{U}_l \cdot \mathbf{n}_l = 0|_{\Sigma_l}, \ \mathbf{U}_g \cdot \mathbf{n}_g = 0|_{\Sigma_g},$$
$$\eta = \mathbf{U}_l \cdot \mathbf{n}_l|_\Gamma, \ \eta = -\mathbf{U}_g \cdot \mathbf{n}_g|_\Gamma\}, \tag{10.33}$$

then we may find the triplet $(\eta, \mathbf{U}_l, \mathbf{U}_g) \in V_1$, and $\omega^2 \in \mathbb{R}^+$, such that

$$\int_{\Omega_l} \rho_l c_l^2 \operatorname{div} \mathbf{U}_l \operatorname{div} \delta\mathbf{U}_l + \int_\Gamma \rho_l g \eta \, \delta\eta$$
$$+ \int_{\Omega_g} \rho_g c_g^2 \operatorname{div} \mathbf{U}_g \operatorname{div} \delta\mathbf{U}_g$$
$$- \omega^2 \left[\int_{\Omega_l} \rho_l \mathbf{U}_l \cdot \delta\mathbf{U}_l + \int_{\Omega_g} \rho_g \mathbf{U}_g \cdot \delta\mathbf{U}_g \right] = 0 \qquad \forall (\delta\eta, \delta\mathbf{U}_l, \delta\mathbf{U}_g) \in V_1. \tag{10.34}$$

Remark 5 From a mathematical point of view, $\mathbf{U}_l$ (respectively $\mathbf{U}_g$) belongs to $H(\operatorname{div}; \Omega_l)$ (respectively $H(\operatorname{div}; \Omega_g)$, a space of square-summable functions with their divergence also square-summable. η is only square-summable on Γ.

Remark 6 From the discretization point of view, independently of the fact that the matrices are banded, one must deal with the existence of an infinite vector space corresponding to a zero eigenvalue and leading to spurious rotational modes (see reference [26], which exhibits numerical difficulties in a simplified case without gravity and with one acoustic medium, and the penalty method of reference [27]).

Remark 7 If the liquid is incompressible ($\operatorname{div} \mathbf{U}_l = 0$ and $c_l \to \infty$), the constraint

$$\operatorname{div} \mathbf{U}_l = 0 \tag{10.35}$$

must be added in V_1. The discretization of such a constraint needs particular regularization-penalty techniques [36–38].

Remark 8 If gravity effects are negligible by comparison with other energy contributions, one has to cancel the $\int_\Gamma \rho_l g \eta \, \delta\eta$ contribution (by setting $g = 0$ or using a constraint procedure existing in standard codes).

Remark 9 In order to take into account capillary effects, one has to replace the potential bilinear operator $\int_\Gamma \rho_l g \eta\, \delta\eta$ by the following, which is easily derived from (10.10) and (10.11) after integrating by parts and applying surface–Green formulae:

$$\int_\Gamma \rho_l g \cos(\mathbf{n}_l, z)\eta\, \delta\eta + \tau \int_\Gamma \operatorname{grad} \eta \cdot \operatorname{grad} \delta\eta - \tau \int_\Gamma \alpha\eta\, \delta\eta. \quad (10.36)$$

From a mathematical point of view, η and the surface gradient of η must be square-summable on Γ in the distribution sense.

Remark 10 By using a dual variable for the fluid, the difficulties exhibited in Remark 6 can be circumvented. Using the scalar product in the sense of kinetic energy, $\mathbf{U}$ must be orthogonal to the zero-divergence space:

$$\int_\Omega \mathbf{U} \cdot \delta\mathbf{U} = 0,\ \forall \delta\mathbf{U} \in \{\delta\mathbf{U} | \operatorname{div} \delta\mathbf{U}\} \Rightarrow \mathbf{U} = \operatorname{grad} \varphi, \qquad \operatorname{curl} \mathbf{U} = 0.$$

10.3.3 Dual variational formulations

In Section 10.3.2 the displacement equation for the fluid was obtained by using, for example, equation (10.25) in equation (10.24). In order to express (10.24) in terms of p_l only, one has, after dividing equation (10.24) by ρ_l, to apply the divergence operator to (10.24) and to take account of (10.25).

In the liquid we obtain

$$\operatorname{div}\left(\frac{1}{\rho_l} \operatorname{grad} p_l\right) + \frac{\omega^2}{\rho_l c_l^2} p_l = 0 \quad \text{in} \quad \Omega_l. \quad (10.37)$$

In the gas

$$\operatorname{div}\left(\frac{1}{\rho_g} \operatorname{grad} p_g\right) + \frac{\omega^2}{\rho_g c_g^2} p_g = 0 \quad \text{in} \quad \Omega_g. \quad (10.38)$$

The *liquid–gas coupling equation* is given by

$$p_l - p_g = \rho_l g \eta \quad \text{on} \quad \Gamma. \quad (10.39)$$

From (10.24) and (10.29), we find

$$\eta = \frac{1}{\rho_l \omega^2} \frac{\partial p_l}{\partial \mathbf{n}_l} \quad \text{on} \quad \Gamma. \quad (10.40)$$

From (10.26) and (10.30),

$$\eta = -\frac{1}{\rho_g \omega^2} \frac{\partial p_g}{\partial \mathbf{n}_g} \quad \text{on} \quad \Gamma. \quad (10.41)$$

The fluid–structure equations on Σ_l and Σ_g are

$$\frac{\partial p_l}{\partial \mathbf{n}_l} = 0 \quad \text{on} \quad \Sigma_l, \tag{10.42}$$

$$\frac{\partial p_g}{\partial \mathbf{n}_g} = 0 \quad \text{on} \quad \Sigma_g. \tag{10.43}$$

We have to couple three media, two of them being described by dual variables – namely the pressure field and the interface medium – by a *primal* displacement scalar variable.

10.3.3.1 *Dual–Primal direct symmetric variational formulation*

The corresponding dual–primal symmetric variational formulation can be easily obtained.

If V_2 denotes the appropriate admissible space, i.e.

$$V_2 = \{(\eta, p_l, p_g)/\text{mathematical usual vector spaces}\}, \tag{10.44}$$

then we must find the triplet $\{\eta, p_l, p_g\} \in V_2$, $\omega^2 \in \mathbb{R}^+$ such that

$$\begin{aligned} &\int_{\Omega_l} \frac{1}{\rho_l} \operatorname{grad} p_l \cdot \operatorname{grad} \delta p_l + \int_{\Omega_g} \frac{1}{\rho_g} \operatorname{grad} p_g \cdot \operatorname{grad} \delta p_g \\ &\quad - \omega^2 \left[\int_{\Omega_l} \frac{1}{\rho_l c_l^2} p_l \, \delta p_l + \int_{\Omega_g} \frac{1}{\rho_g c_g^2} p_g \, \delta p_g \right. \\ &\quad - \int_\Gamma \rho_l g \eta \, \delta\eta + \int_\Gamma p_l \, \delta\eta + \int_\Gamma \delta p_l \eta \\ &\quad \left. - \int_\Gamma p_g \, \delta\eta - \int_\Gamma \delta p_g \eta \right] = 0 \qquad \forall \{\delta\eta, \delta p_l, \delta p_g\} \in V_2. \end{aligned} \tag{10.45}$$

Remark 11 From the mathematical point of view, $\eta \in L^2(\Gamma) p_l (\text{resp. } p_g) \in H^1(\Omega_l)$ (resp. $H^1(\Omega_g)$).

All the operators involved in (10.45) are *symmetric*, therefore, instead of (10.45), we can (see Section 10.24) find the triplet $\{\eta, p_l, p_g\} \in V_2$, $\omega^2 \in \mathbb{R}^+$ such that

$$\begin{aligned} \delta \left\{ \int_{\Omega_l} \frac{1}{2\rho_l} |\operatorname{grad} p_l|^2 + \int_{\Omega_g} \frac{1}{2\rho_g} |\operatorname{grad} p_g|^2 - \omega^2 \left[\int_{\Omega_l} \frac{1}{2\rho_l c_l^2} p_l^2 + \int_{\Omega_g} \frac{1}{2\rho_g c_g^2} p_g^2 \right. \right. \\ \left. \left. - \frac{1}{2} \int_\Gamma \rho_l g \eta^2 + \int_\Gamma (p_l - p_g) \eta \right] \right\} = 0 \qquad \forall \{\delta\eta, \delta p_l, \delta p_g\} \in V_2. \end{aligned} \tag{10.46}$$

We must stress here the fact that all the potential energy contributions of the fluid, namely

$$\frac{1}{2}\int_{\Omega_l}\frac{p_l^2}{\rho_l c_l^2} \quad \text{and} \quad \frac{1}{2}\int_{\Omega_g}\frac{p_g^2}{\rho_g c_g^2},$$

are actually factors of ω^2. This is not the usual situation.

Let us return to Section 10.2.4 and let us partially split a linear dynamic conservative system into two subsystems labelled by subscripts 1 and 2.

If, separately, subsystem 1 is described through primal variable X_1 and subsystem 2 is described through a dual variable X_2, we have just seen (and this was readily observed from the method by which we obtained equation (10.37) from (10.24) and (10.25), which inverts the position of the kinetic and potential terms) that we have

$$-\omega^2 a_1(X_1, \delta X_1) + b_1(X_1, \delta X_1) = 0 \quad \text{in the domain (1)}, \tag{10.47}$$

$$-\omega^2 b_2(X_2, \delta X_2) + a_2(X_2, \delta X_2) = 0 \quad \text{in the domain (2)}, \tag{10.48}$$

where $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ are the kinetic bilinear forms and $b_1(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$ are the potential bilinear forms.

Obviously these two subsystems cannot be matched by following the *classical physical rule of adding the potential contributions of each susbsystem* and *the corresponding kinetic ones, through appropriate coupling terms*, because $a_2(\cdot,\cdot)$ does not possess the factor ω^2.

One has to make an exception if one of the two subsystems has only a kinetic *or* potential linear contribution, because there could be an artificial premultiplication by ω^2 ($\omega \neq 0$). This was the case in the preceding example because the liquid–gas interface has only one type of energy contribution: the potential one. That is why the matching was possible, but we have to notice that the potential energy contribution of the interface acts as a 'mass' contribution.

Remark 12 When we say that there is no possible matching it must be understood that no matching is possible in order to obtain symmetric bilinear forms, as it would be the case if one would add $a_1(\cdot,\cdot)$ to $a_2(\cdot,\cdot)$, as they are separately symmetric.

Remark 13 Without entering into details, the interface variable η could have been eliminated through the equation (10.39), with $p_l \neq p_g$ on Γ:

$$\eta = (p_l - p_g)/\rho_l g, \tag{10.49}$$

and using (10.49) in (10.40) and (10.41). In fact, we have three representations

of the free surface energy: *the primal one,*

$$\frac{1}{2}\int_{\Gamma} \rho_l g \eta^2; \tag{10.50}$$

the primal–dual one,

$$-\frac{1}{2}\int_{\Gamma} \rho_l g \eta^2 + \int_{\Gamma} (p_l - p_g)\eta, \tag{10.51}$$

which is the Legendre–Fenchel [7] transform of (10.50); and *the dual one,*

$$\frac{1}{2}\int_{\Gamma} \frac{(p_l - p_g)^2}{\rho_l g}. \tag{10.52}$$

Remark 14 If capillary effects are taken into account, then in the formulation (10.45), the contribution $\int_{\Gamma} \rho_l g \eta\, \delta\eta$ must be replaced using the relation (10.36). Of course, contrary to the situation in Remark 13 above, η cannot be eliminated when using (10.10). The capillary constitutive law involves too complex a partial differential operator to allow us to exhibit a dual energy of the liquid–gas interface.

10.3.3.2 *Indirect dual–primal symmetric variational formulation. Introduction of a supplementary new state variable: the displacement field potential*

Taking into account the results of the analysis in Section 10.3.3.1, one can guess that the symmetric primal dual formulations cannot match, except for very particular cases, with a primal displacement formulation of an elastic structure, in order to obtain symmetric bilinear operators.

(a) *Local equations of the problem* In order to recover (10.46), the right physical places of the kinetic and potential energies, we introduce an auxiliary field variable: the potential displacement φ_l (respectively φ_g) in the liquid (respectively the gas).

We write (24) as follows:

$$-\text{grad}\,(p_l/\rho_l\omega^2) + \mathbf{U}_l = 0 \quad \text{in} \quad \Omega_l, \tag{10.53}$$

with

$$p_l = \rho_l \omega^2 \varphi_l \quad \text{in} \quad \Omega_l, \tag{10.54}$$

$$\mathbf{U}_l = \text{grad}\,\varphi_l \quad \text{in} \quad \Omega_l. \tag{10.55}$$

The same transformation holds for the gas.

The equations introduced at the beginning of Section 3.3 are then given a follows: *in the liquid,*

$$\Delta\varphi_l + \frac{p_l}{\rho_l c_l^2} = 0 \quad \text{in} \quad \Omega_l, \tag{10.56}$$

$$p_l = \rho_l \omega^2 \varphi_l \quad \text{in} \quad \Omega_l; \tag{10.57}$$

in the gas,

$$\Delta\varphi_g + \frac{p_g}{\rho_g c_g^2} = 0 \quad \text{in} \quad \Omega_g, \tag{10.58}$$

$$p_g = \rho_g \omega^2 \varphi_g \quad \text{in} \quad \Omega_g; \tag{10.59}$$

the liquid–gas coupling equations are

$$\rho_l \omega^2 \varphi_l - \rho_g \omega^2 \varphi_g = \rho_l g \eta \quad \text{on} \quad \Gamma, \tag{10.60}$$

$$\frac{\partial \varphi_l}{\partial \mathbf{n}_l} = \eta \quad \text{on} \quad \Gamma, \tag{10.61}$$

$$\frac{\partial \varphi_g}{\partial \mathbf{n}_g} = -\eta \quad \text{on} \quad \Gamma; \tag{10.62}$$

and the *fluid–structure equations* on Σ_l and Σ_g are

$$\frac{\partial \varphi_l}{\partial \mathbf{n}_l} = 0 \quad \text{on} \quad \Sigma_l, \tag{10.63}$$

$$\frac{\partial \varphi_g}{\partial \mathbf{n}_g} = 0 \quad \text{on} \quad \Sigma_g. \tag{10.64}$$

We shall present two equivalent symmetric variational formulations of the above equations. The first one is of mixed type and involves the five fields (φ_l, φ_g, p_l, p_g, and η). The second one, which is equivalent to the first one, involves only three fields, because we shall show that φ_l (respectively φ_g) is an appropriate function of p_l (respectively p_g) and η through generalized 'added mass operators' concepts.

(b) *Mixed symmetric variational formulations* The test function procedure applied to equations (10.56) (with $\delta\varphi_l$), (10.58) (with $\delta\varphi_g$), (10.57) (with δp_l), (10.59) (with δp_g), and (10.60) (with $\delta\eta$) leads to five symmetric field following variational formulation.

If V_3 denotes the appropriate admissible space, i.e.

$$V_3 = \{\eta, p_l, p_g, \varphi_l, \varphi_g | \text{mathematical usual vector spaces}\}, \tag{10.65}$$

then we have the problem of finding $(\eta, p_l, p_g, \varphi_l, \varphi_g) \in V_3$, $\omega^2 \in \mathbb{R}^+$ such that

$$\int_{\Omega_l} \frac{p_l \delta p_l}{\rho_l c_l^2} + \int_{\Omega_g} \frac{p_g \delta p_g}{\rho_g c_g^2} + \int_\Gamma \rho_l g \eta\, \delta\eta - \omega^2 \Bigg[- \int_{\Omega_l} \rho_l \operatorname{grad} \varphi_l \cdot \operatorname{grad} \delta\varphi_l$$
$$+ \int_\Gamma \rho_l\, \delta\varphi_l \eta + \int_\Gamma \rho_l \varphi_l\, \delta\eta + \int_{\Omega_l} \frac{\varphi_l\, \delta p_l}{c_l^2} + \int_{\Omega_l} \frac{\delta\varphi_l p_l}{c_l^2}$$
$$- \int_{\Omega_g} \rho_g \operatorname{grad} \varphi_g \cdot \operatorname{grad} \delta\varphi_g + \int_\Gamma \rho_g\, \delta\varphi_g \eta + \int_\Gamma \rho_g \varphi_g\, \delta\eta$$
$$+ \int_{\Omega_g} \frac{\varphi_g\, \delta p_g}{c_g^2} + \int_{\Omega_g} \frac{\delta\varphi_g p_g}{c_g^2} \Bigg] = 0 \qquad \forall (\delta\eta, \delta p_l, \delta p_g, \delta\varphi_l, \delta\varphi_g) \in V_3. \qquad (10.66)$$

To give this problem in a more convenient manner, as all operators are symmetric, we need to find $(\eta, p_l, p_g, \varphi_l, \varphi_g) \in V_3$, $\omega^2 \in \mathbb{R}^+$ such that

$$\delta \Bigg\{ \int_{\Omega_l} \frac{p_l^2}{2\rho_l c_l^2} + \int_{\Omega_g} \frac{p_g^2}{2\rho_g c_g^2} + \frac{1}{2} \int_\Gamma \rho_l g \eta^2$$
$$- \omega^2 \Bigg[\left(-\frac{1}{2} \int_{\Omega_l} \rho_l |\operatorname{grad} \varphi_l|^2 + \int_{\Omega_l} \rho_l \varphi_l \eta + \int_{\Omega_l} \frac{p_l \varphi_l}{c_l^2} \right)$$
$$+ \left(-\frac{1}{2} \int_{\Omega_g} \rho_g |\operatorname{grad} \varphi_g|^2 + \int_{\Omega_g} \rho_g \varphi_g \eta + \int_{\Omega_g} \frac{p_g \varphi_g}{c_g^2} \right) \Bigg] \Bigg\} = 0$$
$$\forall (\delta\eta, \delta p_l, \delta p_g, \delta\varphi_l, \delta\varphi_g) \in V_3. \qquad (10.67)$$

Remark 15 From the mathematical point of view, $\eta \in L^2(\Gamma)$, p_l(respectively p_g) $\in L^2(\Omega_l)$(respectively $L^2(\Omega_g)$), φ_l(respectively φ_g) $\in H^1(\Omega_l)$(respectively $H^1(\Omega_g)$).

Remark 16 $p_l \neq p_g$, $\varphi_l \neq \varphi_g$ on Γ. If no gravity effects are taken into account, then $p_l = p_g$, but $\varphi_l \neq \varphi_g$ on Γ.

Remark 17 In (10.67) all the potential and kinetic energies of the system are in the right physical places. For example

$$\frac{1}{2} \int_{\Omega_l} \frac{p_l^2}{\rho_l c_l^2} + \frac{1}{2} \int_{\Omega_g} \frac{p_g^2}{\rho_g c_g^2} + \frac{1}{2} \int_\Gamma \rho_l g \eta^2$$

is the sum of the three potential energies of the three media. We shall see in the next section that the quantity

$$-\frac{1}{2} \int_{\Omega_l} \rho_l |\operatorname{grad} \varphi_l|^2 + \int_{\Omega_l} \rho_l \varphi_l \eta + \int_{\Omega_l} \frac{1}{c_l^2} p_l \varphi_l$$

represents the kinetic energy of the liquid and implies an *added mass concept.*

Remark 18 If capillary effects are taken into account, one must replace the potential energy $\frac{1}{2}\int_\Gamma \rho_l g \eta^2$ using the relation (10.36), and therefore $\eta \in L^2(\Gamma)$ and grad $\eta \in (L^2(\Gamma))^2$.

Remark 19 *Discretization by the finite element method* The various symmetric operators involved in the symmetric formulation (10.66) leads to the following eigenvalue problem:

$$\left\{\begin{bmatrix} A & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \omega^2 \begin{bmatrix} 0 & 0 & 0 & C_1 & C_2 \\ 0 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & E_1 & E_2 \\ C_1 & D_1^T & E_1 & -F_l & 0 \\ C_2 & D_2^T & E_2 & 0 & -F_g \end{bmatrix}\right\} \begin{bmatrix} \eta \\ p_l \\ p_g \\ \varphi_l \\ \varphi_g \end{bmatrix} = 0. \tag{10.68}$$

The matrices F_l are singular, of rank $N-1$; N is the number of degrees of freedom in φ_l (the same analysis holds for F_g). It is easy to see that all the φ_l (respectively all the φ_g) except one can be eliminated as a function of η, p_l (respectively of η, p_g) using a *static standard condensation procedure applied to the mass supermatrix.* The stiffness matrix remains unchanged but the resulting mass matrix can be interpreted as an added mass matrix operator. If gravity effects are neglected and if there is only one acoustic liquid medium with a free surface, then one must impose the usual condition on the free surface Γ:

$$\varphi_l = 0 \quad \text{on} \quad \Gamma. \tag{10.69}$$

The matrix F_l is not singular and all the φ_l variables can be eliminated as a function of p_l, η (the same for F_g).

(c) *Three field* (η, p_l, p_g) *symmetric variational formulation through generalized added mass operators* We shall define two auxiliary static problems, one for the liquid and one for the gas. We shall go into details for the liquid; the transposition to the gas being a question of notation.

Let us consider the system given by (10.56), (10.61), and (10.63):

$$\left.\begin{aligned} \Delta\varphi_l + \frac{p_l}{\rho_l c_l^2} = 0 \quad &\text{in} \quad \Omega_l, \\ \frac{\partial \varphi_l}{\partial n_l} = \eta \quad &\text{on} \quad \Gamma, \\ \frac{\partial \varphi_l}{\partial n_l} = 0 \quad &\text{on} \quad \Sigma. \end{aligned}\right\} \tag{10.68}$$

We define φ_l as the solution of this *static problem*, where η and p_l are supposed temporarily known.

We need an existence condition of this Newman problem:

$$\int_{\Omega_l} \frac{p_l}{\rho_l c_l^2} + \int_\Gamma \eta = 0, \tag{10.69}$$

together with the uniqueness condition

$$\int_{\Omega_l} \varphi_l = 0. \tag{10.70}$$

The mapping $(\eta, p_l) \to \varphi_l$ is linear.

We can now define the kinetic energy of the liquid as a quadratic function of η and p_l provided that they satisfy (10.69), and this represents the generalized added mass concept through the following extremal property (which gives the definition of the added mass operator M_A^l):

$$\begin{aligned}\frac{1}{2}\int_{\Omega_l} \rho_l |\mathbf{U}_l|^2 &= \frac{1}{2}\int_{\Omega_l} \rho_l |\text{grad}\ \varphi_l|^2 \\ &= \max_{\varphi_l}\left(-\frac{1}{2}\int_{\Omega_l} \rho_l |\text{grad}\ \varphi_l|^2 + \int_\Gamma \rho_l \varphi_l \eta + \int_{\Omega_l} \frac{p_l \varphi_l}{c_l^2}\right) \\ &= M_A^l\left(\begin{bmatrix} p_l \\ \eta \end{bmatrix}, \begin{bmatrix} p_l \\ \eta \end{bmatrix}\right), \qquad \varphi_l \in \left\{\varphi_l \,\middle|\, \int_{\Omega_l} \varphi_l = 0\right\}. \end{aligned} \tag{10.71}$$

The proof is the same as in the theory of elasticity, when we search the value of the total potential energy for the solution [7, 39].

We define in the same manner the gas added mass, which represents the kinetic energy of the gas expressed as a quadratic function of p_g and η through the extremal property

$$M_A^g\left(\begin{bmatrix} p_g \\ \eta \end{bmatrix}, \begin{bmatrix} p_g \\ \eta \end{bmatrix}\right) = \max_{\varphi_g}\left(-\frac{1}{2}\int_{\Omega_g} \rho_g |\text{grad}\ \varphi_g|^2 + \int_\Gamma \rho_g \varphi_g \eta + \int_{\Omega_g} \frac{p_g \varphi_g}{c_g^2}\right) \tag{10.72}$$

under the existence condition

$$\int_{\Omega_g} \frac{p_g}{\rho_g c_g^2} + \int_\Gamma \eta = 0, \tag{10.73}$$

$$\varphi_g \in \left\{\varphi_g \,\middle|\, \int_{\Omega_g} \varphi_g = 0\right\}. \tag{10.74}$$

If we look at the formulation (10.67), as commented on in Remark 17, we said that the quantity

$$-\frac{1}{2}\int_{\Omega_l} \rho_l |\text{grad}\ \varphi_l|^2 + \int_{\Omega_l} \rho_l \varphi_l \eta + \int_{\Omega_l} \frac{p_l \varphi_l}{c_l^2}$$

represents the kinetic energy of liquid. The proof has been given by the added mass operator concept defined above.

If V_4 denotes the appropriate admissible space, i.e.

$$V_4 = \left\{ p_l, p_g, \eta \;\middle|\; \int_{\Omega_l} \frac{p_l}{\rho_l c_l^2} + \int_\Gamma \eta = 0, \int_{\Omega_l} \frac{p_g}{\rho_g c_g^2} + \int_\Gamma \eta = 0 \right\}, \tag{10.75}$$

then the three fields symmetric variational formulation is given by finding $(p_l, p_g, \eta) \in V_4$, $\omega^2 \in \mathbb{R}^+$ such that

$$\delta \left\{ \int_{\Omega_l} \frac{p_l^2}{2\rho_l c_l^2} + \int_{\Omega_g} \frac{p_g^2}{2\rho_g c_g^2} + \frac{1}{2} \int_\Gamma \rho_l g \eta^2 - \omega^2 \left[M_A^l \left(\begin{bmatrix} p_l \\ \eta \end{bmatrix}, \begin{bmatrix} p_l \\ \eta \end{bmatrix} \right) + M_A^g \left(\begin{bmatrix} p_g \\ \eta \end{bmatrix}, \begin{bmatrix} p_g \\ \eta \end{bmatrix} \right) \right] \right\} = 0$$

$$\forall (\delta p_l, \delta p_g, \delta \eta) \in V_4. \tag{10.76}$$

A way of discretizing M_A^l and M_A^g by finite element procedures is precisely to form the corresponding part of the mass matrix in (10.68) and to use a standard static condensation method which gives the discretized added mass operators. M_A^l and M_A^g operators are added through an assembly procedure on η.

10.4 VARIATIONAL FORMULATIONS IN THE CASE OF AN ELASTIC STRUCTURE

10.4.1 Introduction

If the structure is described by a primal displacement field, the only way to get a symmetric variational fluid–structure formulation (see Sections 10.2.4 and 10.3.3.1) is to use the indirect dual–primal variational formulation of Section 10.3.3.2. It is, as already shown, the only one which exhibits the potential and kinetic energies of the fluid system at proper places for a symmetric *coupling* with a displacement representation of the structure.

We stress here that if a dynamic dual formulation is used for the structure, then the direct dual fluid formulation represents the proper one (Section 10.3.3.1).

10.4.2 Local equations of the problem

We shall write these equations taking account of the results of Section 10.3.3.2 (as usual in the modal situation). *In the structure,* taking account of (10.1)–(10.3),

$$\operatorname{div} \sigma(\mathbf{U}_s) + \rho_s \omega^2 \mathbf{U}_s = 0 \quad \text{in} \quad \Omega_s. \tag{10.77}$$

In the liquid, we have the system (10.68):

$$\left.\begin{aligned} \Delta\varphi_l + \frac{p_l}{\rho_l c_l^2} = 0 \quad &\text{in} \quad \Omega_l, \\ \frac{\partial \varphi_l}{\partial \mathbf{n}_l} = \eta \quad &\text{on} \quad \Gamma, \\ \frac{\partial \varphi_l}{\partial \mathbf{n}_l} = \mathbf{U}_s \cdot \mathbf{n}_l \quad &\text{on} \quad \Sigma_l. \end{aligned}\right\} \tag{10.78}$$

In the gas,

$$\left.\begin{aligned} \Delta\varphi_g + \frac{p_g}{\rho_g c_g^2} = 0 \quad &\text{in} \quad \Omega_g, \\ \frac{\partial \varphi_g}{\partial \mathbf{n}_g} = -\eta \quad &\text{on} \quad \Gamma, \\ \frac{\partial \varphi_g}{\partial \mathbf{n}_g} = \mathbf{U}_s \cdot \mathbf{n}_g \quad &\text{on} \quad \Sigma_g. \end{aligned}\right\} \tag{10.79}$$

Liquid–gas dynamic coupling is given by

$$\rho_l \omega^2 \varphi_l - \rho_g \omega^2 \varphi_g = \rho_l g \eta \quad \text{on} \quad \Gamma. \tag{10.80}$$

Fluid–structure coupling is given by

$$\sigma(\mathbf{U}_s)\mathbf{n}_s = \rho_l \omega^2 \varphi_l \mathbf{n}_l \quad \text{on} \quad \Sigma_l \tag{10.81}$$

$$\sigma(\mathbf{U}_s)\mathbf{n}_s = \rho_g \omega^2 \varphi_g \mathbf{n}_g \quad \text{on} \quad \Sigma_g \tag{10.82}$$

$$\mathbf{U}_s \cdot \mathbf{n}_s = \frac{\partial \varphi_l}{\partial \mathbf{n}_s} \quad \text{on} \quad \Sigma_l \tag{10.83}$$

$$\mathbf{U}_s \cdot \mathbf{n}_s = \frac{\partial \varphi_g}{\partial \mathbf{n}_s} \quad \text{on} \quad \Sigma_g. \tag{10.84}$$

10.4.3 Symmetric variational formulation of the fluid–structure system

Taking into account the results of Section 10.3.3.1 concerning the coupling of two systems described by primal and dual variables, we have the four following systems: the structure (primal field), the gas (dual field), the liquid (dual field), the interface (primal field).

In order to match the systems properly, keeping the displacement primal field for the structure, as seen before, the triple gas–interface–liquid must use the indirect dual–primal (for this triple) symmetric variational formulation of Section 10.3.3.2. It can be observed that all the formulations introduced in the rigid case are symmetric, but only one of them matches the primal

formulation of the fluid; a similar situation would arise if, for instance, one fluid were described by a primal field and the other by a dual field.

As previously (see Sections 10.3.3.2(b) and (c)), we can easily derive a six-field variational formulation and a four-field equivalent variational formulation through added mass operators.

10.4.3.1 *Mixed symmetric variational formulation*

The formulation is derived in a manner similar to that given in Section 10.3.3.2(b) by taking into account the equations in the structure (i.e. (10.77)) multiplying by the test function $\delta \mathbf{U}_s$, and, after integrating by parts, using equations (10.81) and (10.82), and modifying accordingly the formulation (10.67) in order to take into account the following boundary conditions ((10.83) and (10.84)):

$$\frac{\partial \varphi_l}{\partial \mathbf{n}_s} = \mathbf{U}_s \cdot \mathbf{n}_s \quad \text{on} \quad \Sigma_l,$$

$$\frac{\partial \varphi_g}{\partial \mathbf{n}_g} = \mathbf{U}_g \cdot \mathbf{n}_g \quad \text{on} \quad \Sigma_g.$$

We can see that (10.67) is easily transformed into a new functional by adding the potential and kinetic contributions of the elastic structure, and by adding two contributions in the fluid in order to take into account the preceding boundary conditions.

If V_5 denotes the appropriate admissible space, i.e. $V_5 = \{\eta, p_l, p_g, \varphi_l, \varphi_g, \mathbf{U}_s | \text{plus mathematical appropriate spaces, plus constraints on } \mathbf{U}_s \text{ if any}\}$ then the problem is to find $(\eta, p_l, p_g, \varphi_l, \varphi_g, \mathbf{U}_s) \in V_5$, $\omega^2 \in \mathbb{R}^+$ such that

$$\delta \left\{ \frac{1}{2} \int_{\Omega_s} \mathrm{Tr}\,(\sigma(\mathbf{U}_s)\mathscr{E}(\mathbf{U}_s)) + \int_{\Omega_l} \frac{p_l^2}{2\rho_l c_l} + \int_{\Omega_g} \frac{p_g^2}{2\rho_g c_g^2} + \frac{1}{2} \int_{\Gamma} \rho_l g \eta^2 \right.$$
$$- \omega^2 \left[\left(-\frac{1}{2} \int_{\Omega_l} \rho_l |\mathrm{grad}\, \varphi_l|^2 + \int_{\Omega_l} \rho_l \varphi_l \eta + \int_{\Omega_l} \frac{p_l \varphi_l}{c^2} + \int_{\Sigma_l} \rho_l \varphi_l \mathbf{U}_s \cdot \mathbf{n}_l \right) \right.$$
$$+ \left(-\frac{1}{2} \int_{\Omega_g} \rho_g |\mathrm{grad}\, \varphi_g|^2 + \int_{\Omega_g} \rho_g \varphi_g \eta + \int_{\Omega_g} \frac{p_g \varphi_g}{c^2} + \int_{\Sigma_l} \rho_g \varphi_g \mathbf{U}_s \cdot \mathbf{n}_g \right)$$
$$\left. \left. + \frac{1}{2} \int_{\Omega_s} \rho_s |\mathbf{U}_s|^2 \right] \right\} = 0 \qquad \forall (\delta\eta, \delta p_l, \delta p_g, \delta\varphi_l, \delta\varphi_g, \delta\mathbf{U}_s) \in V_5. \tag{10.85}$$

Remark 20 From the mathematical point of view $\mathbf{U}_s \in (H^1(\Omega_s))^3$.

Remark 21 In (10.85) all the potential and kinetic energies of the system are in the proper physical places. This equation clearly gives the sum of the

potential energies of the four media. We shall see in the next section that the quantity

$$-\frac{1}{2}\int_{\Omega_l}\rho_l|\text{grad}\,\varphi_l|^2+\int_{\Omega_l}\rho_l\varphi_l\eta+\int_{\Omega_l}\frac{1}{c_l^2}p_l\varphi_l+\int_{\Sigma_l}\rho_l\varphi_l\mathbf{n}_l\cdot\mathbf{U}_s$$

represents the kinetic energy of the fluid (here, for example, the liquid) and implies an added mass concept.

Remark 22 As in the rigid case, φ_l can be eliminated as a function of $\eta, p_l, \mathbf{U}_s\cdot\mathbf{n}_l$.

Remark 23 The discretization by the finite element method is carried out by the same procedure as indicated in Remark 19.

10.4.3.2 *Four-field* $(\eta, p_l, p_g, \mathbf{U}_s)$ *symmetric variational formulation through generalized added mass operators concepts*

The procedure is the same as in Section 10.3.3.2(c). Using (10.78), we define φ_l as the solution of (10.78) considered as a static problem, where $\eta, \mathbf{U}_s\cdot\mathbf{n}_l$ and p_l are supposed to be temporarily known.

We need the existence condition of the following von Neumann problem,

$$\int_{\Omega_l}\frac{p_l}{\rho_l c_l^2}+\int_{\Gamma}\eta+\int_{\Sigma_l}\rho_l\varphi\mathbf{U}_s\cdot\mathbf{n}_l=0, \tag{10.86}$$

together with the uniqueness condition

$$\int_{\Omega_l}\varphi_l=0. \tag{10.87}$$

The added mass operator then satisfies the following extremal property:

$$\begin{aligned}\frac{1}{2}\int_{\Omega_l}\rho_l|\mathbf{U}_l|^2&=\frac{1}{2}\int_{\Omega_l}\rho_l|\text{grad}\,\varphi_l|^2\\&=\max_{\varphi_l}\left(-\frac{1}{2}\int_{\Omega_l}\rho_l|\text{grad}\,\varphi_l|^2+\int_{\Gamma}\rho_l\varphi_l\eta+\int_{\Omega_l}\frac{p_l\varphi_l}{c_l^2}+\int_{\Sigma_l}\rho_l\varphi_l\mathbf{U}_s\cdot\mathbf{n}_l\right)\\&=M_A^l\left(\begin{bmatrix}p_l\\ \eta\\ \mathbf{U}_s\cdot\mathbf{n}_l\end{bmatrix},\begin{bmatrix}p_l\\ \eta\\ \mathbf{U}_s\cdot\mathbf{n}_l\end{bmatrix}\right),\qquad \varphi_l\in\left\{\varphi_l\,\middle|\,\int_{\Omega_l}\varphi_l=0\right\}.\end{aligned} \tag{10.88}$$

If V_6 denotes the following appropriate admissible space, i.e.

$$V_6 = \left\{ p_l, p_g, \eta, \mathbf{U}_s \;\middle|\; \int_{\Omega_l} \frac{p_l}{\rho_l c_l^2} + \int_\Gamma \eta + \int_{\Sigma_l} \rho_l \varphi_l \mathbf{U}_s \cdot \mathbf{n}_l = 0; \right.$$

$$\left. \text{respectively subscript } g \right\},$$

then the four-field symmetric variational formulation is given by finding $(p_l, p_g, \eta, \mathbf{U}_s) \in V_6$, $\omega^2 \in \mathbb{R}^+$, such that

$$\delta \left\{ \int_{\Omega_s} \mathrm{Tr}\,(\sigma(\mathbf{U}_s)\mathscr{E}(\mathbf{U}_s)) + \int_{\Omega_l} \frac{p_l^2}{2\rho_l c_l^2} + \int_{\Omega_g} \frac{p_g^2}{2\rho_g c_g^2} + \frac{1}{2}\int_\Gamma \rho_l g \eta^2 \right.$$

$$\left. - \omega^2 \left[M_A^l \left(\begin{bmatrix} p_l \\ \eta \\ \mathbf{U}_s \cdot \mathbf{n}_l \end{bmatrix}, \begin{bmatrix} p_l \\ \eta \\ \mathbf{U}_s \cdot \mathbf{n}_l \end{bmatrix} \right) + M_A^g \left(\begin{bmatrix} p_g \\ \eta \\ \mathbf{U}_s \cdot \mathbf{n}_g \end{bmatrix}, \begin{bmatrix} p_g \\ \eta \\ \mathbf{U}_s \cdot \mathbf{n}_g \end{bmatrix} \right) \right] = 0$$

$$\forall (\delta p_l, \delta p_g, \delta\eta, \delta \mathbf{U}_s) \in V_6. \qquad (10.89)$$

Remark 24 For transient responses, one has simply to replace

$$-\omega^2 \begin{bmatrix} p_l \\ \eta \\ \mathbf{U}_s \cdot \mathbf{n}_l \end{bmatrix}$$

by

$$\begin{bmatrix} \ddot{p}_l \\ \ddot{\eta} \\ \ddot{\mathbf{U}}_s \cdot \mathbf{n}_l \end{bmatrix},$$

where

$$\begin{bmatrix} \delta p_l \\ \delta\eta \\ \delta \mathbf{U}_s \cdot \mathbf{n}_l \end{bmatrix}$$

is independent of time, in the bilinear forms of (10.89) and to add the linear form corresponding to external forces.

Remark 25 Equations (10.85) and (10.88) are greatly simplified if there is only a liquid (incompressible) without gravity free surface effects: p_l and η do not then appear in the formulation and (10.88) represents the classical added mass operator.

10.5 NUMERICAL RESULTS

In the three following cases, the fluid–structure modules of the ONERA Structure Division research set of codes, ASTRONEF, are used.

10.5.1 Incompressible hydroelastic vibrations of the liquid propelled launch vehicle Ariane (second stage flight)

We present here the axisymmetric modes of the launcher. The added mass operators are computed separately for each tank. The total number of degrees of freedom is 380 (the dimension of the total matrix system before the added mass condensation procedure is 970).

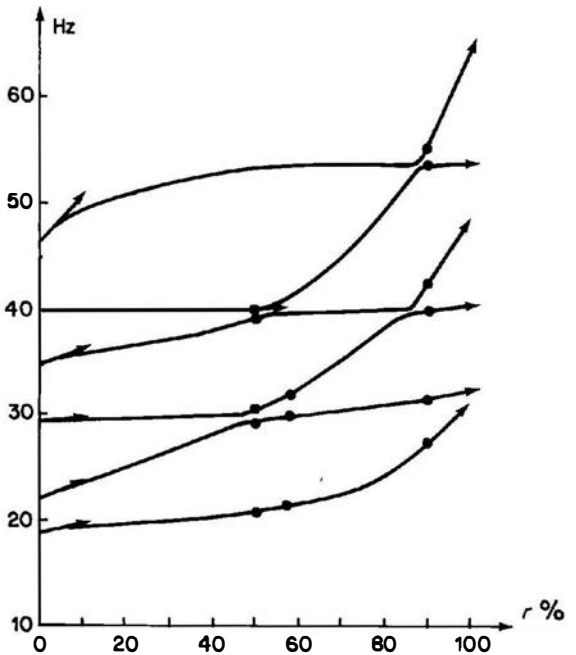

Figure 10.2 Evolution of frequencies as a function of filling ratio ($r = 100\%$ = empty).

Figure 10.2 represents the evolution of the frequencies as a function of the filling ratio.

Figure 10.3 represents, for a half-empty configuration, the third stage being full, the axisymmetric modal shapes of the launcher.

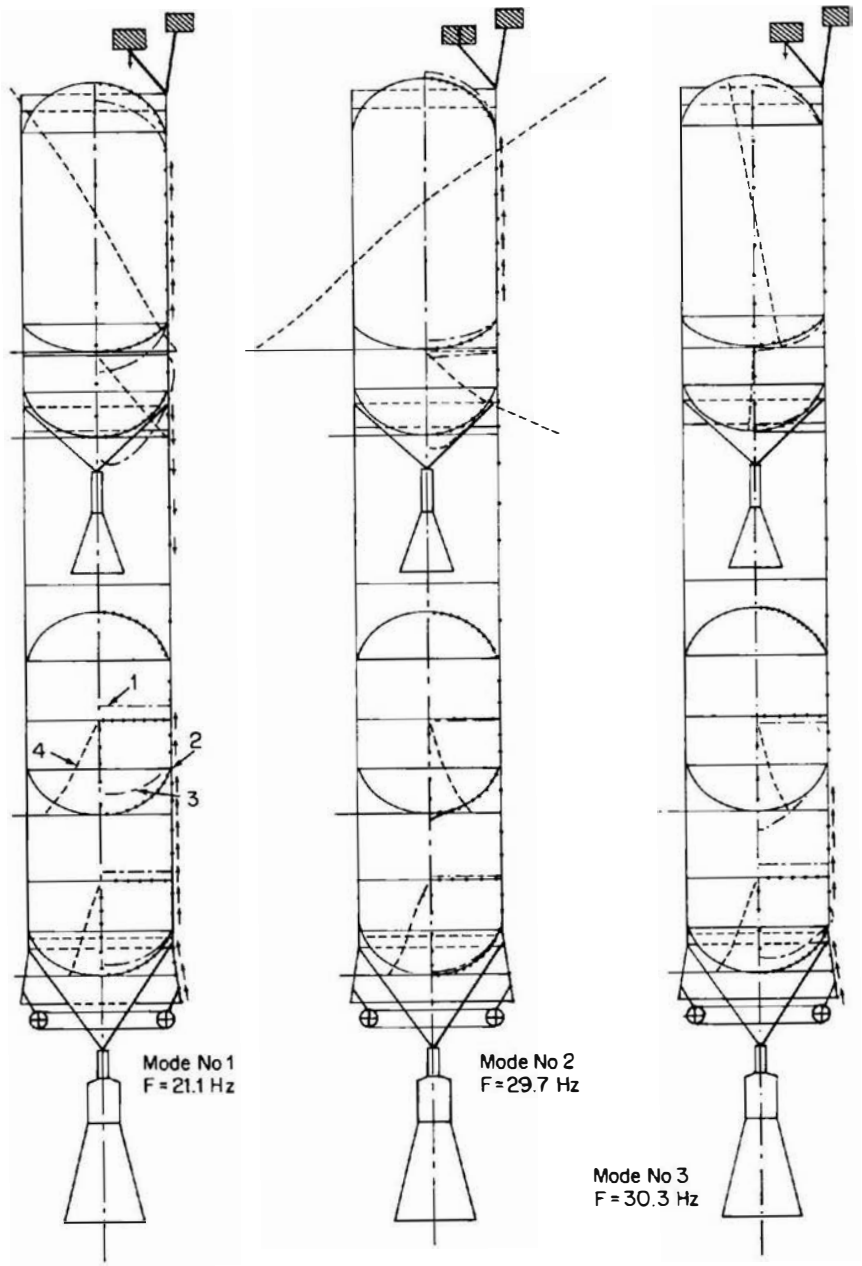

Figure 10.3 Modal shapes of the second stage flight of the launcher Ariane (filling ratio 50%). 1, Free surface displacement; 2, 3, structural displacement; 4, evoulution of the pressure along the axis of revolution.

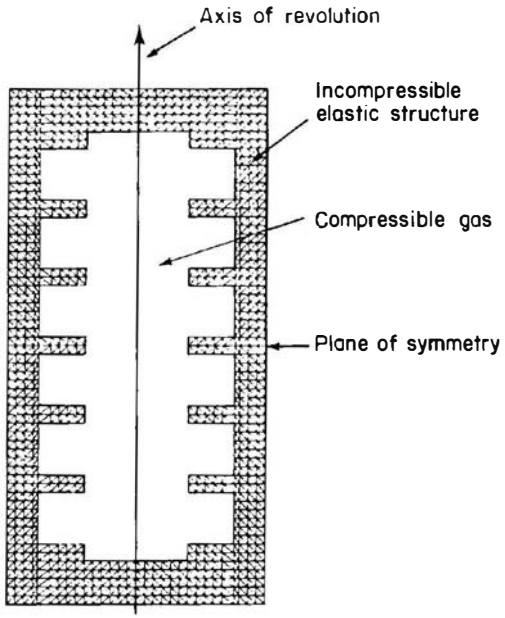

Figure 10.4 Gas–structure system of revolution in a meridian plane.

10.5.2 Incompressible axisymmetric elastic structure containing a compressible gas

The procedure consists of applying the symmetric formulation particularized to the problem of consideration through a dynamic substructure analysis [19] through a Ritz method. The basis functions are computed by a finite element method using the modes in a rigid cavity and the modes of the incompressible elastic structure *in vacuo*. The coupling reveals that the *in vacuo* structural mode takes 30% of the kinetic energy of the closed acoustic mode.

Figures 10.4–10.6 give an illustration of the coupled eigenmodes. The computation of the incompressible *in vacuo* modes of the elastic structure involves special regularization techniques [40]. (In Figure 10.5, the diameter of the structure is 1.5 m and the length is 3 m.)

10.5.3 Elastic cylinder partially filled with liquid under gravity

Figure 10.7 represents the physical system (under transverse excitation). The length of the cylinder is 1.730 m and the diameter is 0.173 m. The main motivation of this study is the problem of the influence of sloshing in using tip tanks on the vibration modes of a wing.

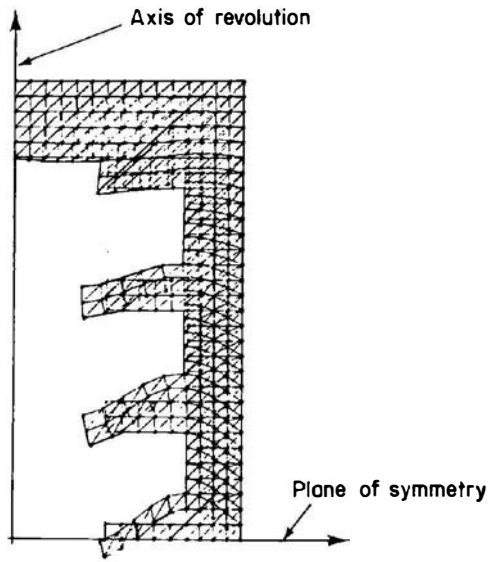

Figure 10.5 Structural finite element mesh and fifth modal shape.

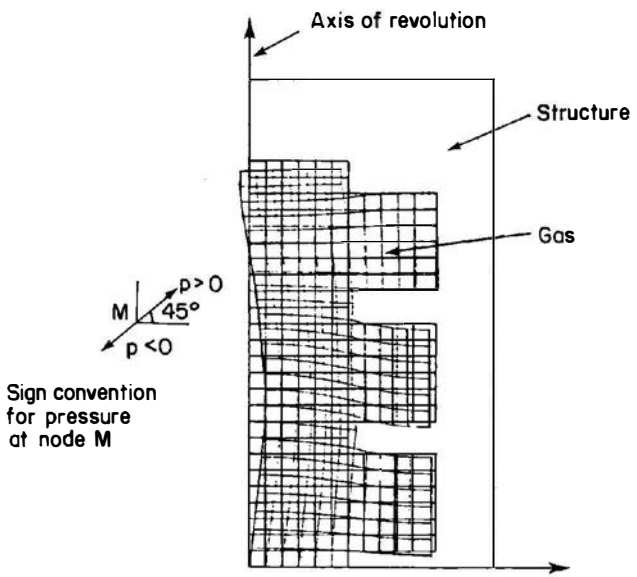

Figure 10.6 Gas finite element mesh and fifth pressure modal shape.

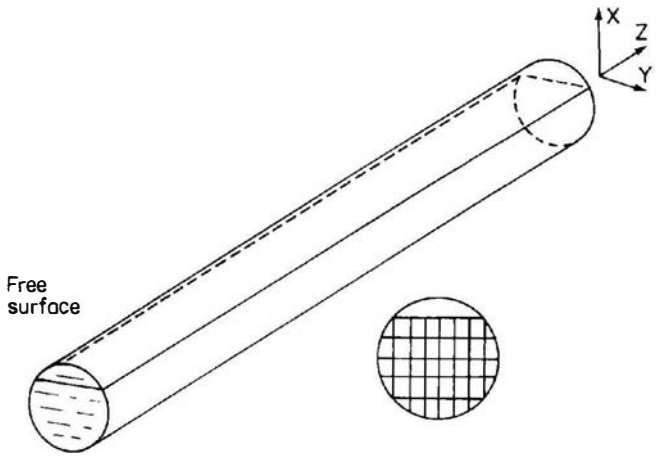

Figure 10.7 Elastic cylinder partially filled with liquid. Cross-section of the three-dimensional isoparametric elements.

Table 10.1 shows the very good accuracy between the experimental data [41] and the numerical results.

10.6 GENERALIZATION: STRUCTURE–STRUCTURE INTERACTION

Variational principles governing the problems of continuum mechanics, i.e. displacement, complementary energy, and mixed principles [6, 7] have not been applied, as far as we know, in vibration and transient responses mechanics as regards the coupling of these various variational principles to one another. But it might be interesting to divide a structure spatially into zones in which one principle would be privileged relative to another, in that it puts into play in each zone the most pertinent variables of the problem (Figure 10.8). This

Table 10.1 Comparison between experimental and numerical frequencies (in hertz) for transverse excitation (rotation around X-axis)

Half-filled cylinder			Three-quarter-filled cylinder		
Numerical	Experimental	Relative error (%)	Numerical	Experimental	Relative error (%)
1.970	1.962	0.4	2.164	2.220	2.5
3.708	3.680	0.8	2.295	—	—
4.948	—	—	4.002	3.900	2.6
12.006	11.76	2.1	4.807	4.890	1.7
			5.692	—	—

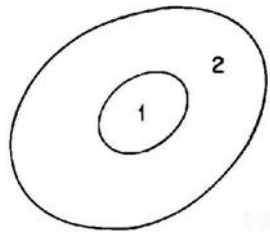

Figure 10.8 Structure split into two subdomains.

could be interesting for problems where the stresses present high gradients. The same problem of symmetrization occurs here as was the case in the fluid–structure problem, and involves a similar type of procedure, i.e. introduction of an auxiliary variable satisfying an auxiliary static problem [42, 43]. One could also obtain the benefit, for transient problems, of the so-called partitioned procedures [5] which have been useful for infinite medium fluid–structure interaction [44].

10.7 CONCLUSION

A systematic approach has been presented for variational symmetric formulations of conservative linear coupled fluid–structure bounded systems. The approach explains clearly why non-symmetric formulations in terms of displacement potential or pressure have been found in the literature. Generalization to problems of elasticity is under development [43]. This makes use of appropriate unknown fields in specified subdomains of a structure. Similar ideas lead to *a posteriori* error estimates [45]. Extension to non-linear problems of dynamic coupled bounded systems is an on-going open research area, for example for non-linear dynamic fluid–structure interaction in bounded media [46, 47].

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