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# ALGEBRAIC $K$ -THEORY OF THE TWO-PERIODIC FIRST MORAVA $K$ -THEORY

HALDUN ÖZGÜR BAYINDIR

**ABSTRACT.** Using the root adjunction formalism developed in an earlier work and logarithmic THH, we obtain a simplified computation of  $T(2)_* K(ku)$  for  $p > 3$ . Our computational methods also provide  $T(2)_* K(ku/p)$ , where  $ku/p$  is the 2-periodic Morava  $K$ -theory spectrum of height 1.

## 1. INTRODUCTION

One of the central problems in homotopy theory is the computation of the algebraic  $K$ -theory of the sphere spectrum,  $K(\mathbb{S})$ . This is due to the fact that  $K(\mathbb{S})$  contains the smooth Whitehead spectrum of the point as a summand which approximates the concordance spaces of highly connected compact smooth manifolds c.f. [WJR13]. A program initiated by Waldhausen [Wal84] and later carried forward by Ausoni and Rognes [AR02] aims at studying  $K(\mathbb{S})$  via étale descent through  $K(E_n)$  where  $E_n$  is the Morava  $E$ -theory spectrum of height  $n$ . This turns our attention to the computation of  $K(E_n)$ .

Motivated by this plan, Ausoni and Rognes compute  $V(1)_* K(\ell_p)$  in [AR02] for  $p > 3$ . Here,  $\ell_p$  is the Adams summand of the connective cover  $ku_p$  of the  $p$ -completed complex  $K$ -theory spectrum  $KU_p \simeq E_1$ . Later, Ausoni improves this to a computation of the  $V(1)$ -homotopy of  $K(ku_p)$  [Aus10]. Another interest in  $K(ku)$  stems from the fact that it classifies virtual 2-vector bundles, a 2-categorical analogue of ordinary complex vector bundles [BDRR11].

As an outcome of his computations, Ausoni observes that the relationship between  $V(1)_* K(\ell_p)$  and  $V(1)_* K(ku_p)$  through the map  $V(1)_* K(\ell_p) \rightarrow V(1)_* K(ku_p)$  resembles a height 2 analogue of  $K_*(\mathbb{Z}_p; \mathbb{Z}/p) \rightarrow K_*(\mathbb{Z}_p[\zeta_p]; \mathbb{Z}/p)$  for the cyclotomic extension  $\mathbb{Z} \rightarrow \mathbb{Z}_p[\zeta_p]$  where  $\zeta_p$  is a primitive  $p$ th root of unity; the computation of the former is due to Hesselholt and Madsen [HM03, Theorem D]. For instance,  $K(\mathbb{Z}_p[\zeta_p]; \mathbb{Z}/p)$  is essentially given by adjoining a  $p - 1$ -root to  $v_1$  in  $K(\mathbb{Z}_p; \mathbb{Z}/p)$ . On the other hand, Ausoni proves the following for  $\ell_p \rightarrow ku_p$ .

**Theorem 1.1** ([Aus10], Theorem 7.18). *Let  $p > 3$  be a prime. There is an isomorphism of graded abelian groups:*

$$T(2)_* K(ku) \cong T(2)_* K(\ell)[b]/(b^{p-1} + v_2).$$

where  $|b| = 2p + 2$ .

*Remark 1.2.* Indeed, this isomorphism can be improved to that of  $\mathbb{F}_p[b]$ -algebras. We discuss this in Remark 7.20.

*Remark 1.3.* Since  $T(2)_* K(\ell)$  is known due to Ausoni and Rognes [AR02, Theorem 0.3], the theorem above provides an explicit description of  $T(2)_* K(ku)$ .

Following this comparison, Ausoni, the author and Moulinos construct a root adjunction method for ring spectra and study the algebraic  $K$ -theory, THH and logarithmic THH of ring spectra obtained via root adjunction [ABM22]. Let  $A$  be an  $E_1$ -ring spectrum and let  $a \in \pi_{mk}A$ . Under suitable hypothesis, this construction provides another  $E_1$ -ring  $A(\sqrt[m]{a})$  for which the homotopy ring of  $A(\sqrt[m]{a})$  is precisely given by a root adjunction:

$$\pi_*A(\sqrt[m]{a}) \cong \pi_*A[z]/(z^m - a).$$

Furthermore,  $A(\sqrt[m]{a})$  is an  $E_1$ -algebra in  $\text{Fun}(\mathbb{Z}/(m)^{\text{ds}}, \text{Sp})$  equipped with the Day convolution symmetric monoidal structure; we say  $A(\sqrt[m]{a})$  is an  $m$ -graded  $E_1$ -ring. Roughly speaking, this structure may be considered as a splitting  $A(\sqrt[m]{a}) \simeq \bigvee_{i \in \mathbb{Z}/m} A(\sqrt[m]{a})_i$ , which we call the weight grading on  $A(\sqrt[m]{a})$ , for which the multiplication on  $A(\sqrt[m]{a})$  is given by maps respecting this grading over  $\mathbb{Z}/m$ :

$$A(\sqrt[m]{a})_i \wedge A(\sqrt[m]{a})_j \rightarrow A(\sqrt[m]{a})_{i+j}.$$

Furthermore, we have  $A(\sqrt[m]{a})_i = \Sigma^{ik}A$  for  $0 \leq i < m$ . This results in a canonical splitting of  $\text{THH}(A(\sqrt[m]{a}))$  into a coproduct of  $m$ -cofactors as an  $S^1$ -equivariant spectrum. It follows by [ABM22, Theorem 1.9] that at the level of algebraic  $K$ -theory, the map

$$K(A) \rightarrow K(A(\sqrt[m]{a}))$$

is the inclusion of a wedge summand whenever  $A$  is  $p$ -local and  $p \nmid m$ .

The authors prove in [ABM22] that there is an equivalence of  $E_1$ -rings  $ku_p \simeq \ell_p(\sqrt[p]{v_1})$ . This equips  $ku_p$  with the structure of a  $p - 1$ -graded  $E_1$ -ring through  $ku_p \simeq \bigvee_{0 \leq i < p-1} \Sigma^{2i} \ell_p$  and one obtains that  $\text{THH}(ku_p)$  and the logarithmic THH of  $ku_p$  in the sense of [ABM22] admit  $S^1$ -equivariant splittings into  $p - 1$  summands. In this work, our first objective is to obtain a simplified computation of  $T(2)_*K(ku)$ , i.e. a simplified proof of Theorem 1.1, by showing that these splittings carry over to  $\text{TC}(ku_p)$  in a way that provides the graded abelian group  $T(2)_*K(ku)$  as a  $p - 1$ -fold coproduct of shifted copies of  $T(2)_*K(\ell)$  as given in Theorem 1.1. For this, we work with the logarithmic THH (in the sense of Rognes [Rog09]) of  $ku_p$  computed in [RSS18].

In [Aus10, Sections 3 and 4], Ausoni constructs what he calls the higher bott element  $b \in V(1)_{2p+2}K(ku_p)$  and identifies the image of this element in  $V(1)_*\text{THH}(ku_p)$  under the trace map. These are the only results that we take from Ausoni's work as input. In particular, our computation avoids the low dimensional computations and the infinite spectral sequence argument of [Aus10, Sections 5, 6 and 7]. We provide an outline of our computation in Section 2 below.

*Remark 1.4.* Currently, Christian Ausoni, the author, Tommy Lundemo and Steffen Sagave are working on generalizing the methods of this work and [ABM22] to obtain a higher height analogue of Theorem 1.1 that relates the algebraic  $K$ -theory of  $E_n$  to that of the truncated Brown-Peterson spectrum  $BP\langle n \rangle$ .

In a later work [AR12], Ausoni and Rognes compute  $V(1)_*K(\ell/p)$  where  $\ell/p$  is the connective Morava  $K$ -theory spectrum  $k(1)$  of height one. Our computational approach to  $T(2)_*K(ku_p)$  naturally provides a computation of  $T(2)_*K(ku/p)$ ; see Section 2 for an outline. Namely, we obtain the first computation of  $T(2)_*K(ku/p)$ .

Note that,  $ku/p$  is also called the connective 2-periodic Morava  $K$ -theory of height one.

**Theorem 1.5** (Theorem 7.21). *Let  $p > 3$  be a prime. There is an isomorphism of graded abelian groups:*

$$T(2)_* K(ku/p) \cong T(2)_* K(\ell/p) \otimes_{\mathbb{F}_p[v_2]} \mathbb{F}_p[b]$$

with  $|b| = 2p + 2$  and in the tensor product above, we take  $v_2 = -b^{p-1}$ .

Together with [AR12, Theorem 1.1], the theorem above provides a complete description of  $T(2)_* K(ku/p)$ . In particular, similar to Theorem 1.1,  $T(2)_* K(ku/p)$  is given by a  $p - 1$ -fold coproduct of shifted copies of  $T(2)_* K(\ell/p)$ .

Indeed, Ausoni and Rognes compute  $V(1)_* K(\ell/p)$  with the goal of investigating how localization and Galois descent techniques that have been used for (local) number rings or fields can be applied in studying the algebraic  $K$ -theory of ring-spectra, in particular  $\ell_p$  and  $ku_p$ . For this, they define  $K(ff(\ell_p))$ , what they call the algebraic  $K$ -theory of the fraction field of  $\ell_p$ , as the cofiber of the transfer map  $K(L/p) \rightarrow K(L_p)$ , i.e. there is a cofiber sequence

$$K(L/p) \rightarrow K(L_p) \rightarrow K(ff(\ell_p)).$$

Note that  $K(ff(\ell_p))$  is not claimed to be the algebraic  $K$ -theory of an  $E_1$ -ring. Ausoni and Rognes continue their discussion in [AR09] where they state a conjectural formula [AR09, Section 3]:

$$(1.6) \quad T(2)_* K(ff(ku_p)) \cong T(2)_* K(ff(\ell_p)) \otimes_{\mathbb{F}_p[v_2]} \mathbb{F}_p[b],$$

here,  $K(ff(ku_p))$  is defined to be the cofiber of the transfer map below.

$$K(KU/p) \rightarrow K(KU_p) \rightarrow K(ff(ku_p))$$

In Theorem 7.23, we verify the conjectural formula in (1.6) under a suitable hypothesis.

**Notation 1.7.** For  $p > 3$ , we let  $V(1)$  denote the spectrum given by  $\mathbb{S}/(p, v_1)$ . Due to [Oka84], this is a homotopy commutative ring spectrum. Inverting the self map  $v_2$  of  $V(1)$ , we obtain  $T(2) := V(1)[v_2^{\pm 1}]$ . Since  $L_{T(2)}V(1) \simeq T(2)$  [Hov97, Section 1.5], we obtain that  $T(2)$  is also a homotopy commutative ring spectrum and  $V(1) \rightarrow T(2)$  is a map of a homotopy commutative ring spectra.

We let  $\text{CycSp}$  denote the  $\infty$ -category of cyclotomic spectra as in [AMMN22, Definition 2.1]; this is a slight variation of what is called  $p$ -cyclotomic spectra in [NS18]. In particular, an object of  $\text{CycSp}$  is an  $S^1$ -equivariant spectrum  $E$  with an  $S^1$ -equivariant map  $E \rightarrow E^{tC_p}$ .

For a given small symmetric monoidal  $\infty$ -category  $\mathcal{C}$  and a presentably symmetric monoidal  $\infty$ -category  $\mathcal{D}$ , we let  $\text{Fun}(\mathcal{C}, \mathcal{D})$  denote the corresponding functor  $\infty$ -category equipped with the symmetric monoidal structure given by Day convolution [Gla16, Day70]. For a simplicial set  $K$ , we let  $\mathcal{D}^K$  denote the symmetric monoidal  $\infty$ -category given by the simplicial set of maps from  $K$  to  $\mathcal{D}$  equipped with the levelwise symmetric monoidal structure [Lur16, Remark 2.1.3.4].

For  $m \geq 0$ , we when we mention the abelian group  $\mathbb{Z}/m$  as a symmetric monoidal monoidal  $\infty$ -category, we mean the corresponding discrete symmetric monoidal  $\infty$ -category. This is often denoted by  $(\mathbb{Z}/m)^{\text{ds}}$  in the literature.

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## 2. OUTLINE

Here, we provide an outline of the proofs of Theorems 1.1 and 1.5. In Section 4, we use the Sp-linear Fourier transform developed in [CSY21] to show that the action of the  $p$ -adic Adams operations on  $ku_p$  equips  $ku_p$  with the structure of a  $p - 1$ -graded  $E_\infty$ -ring, i.e. an  $E_\infty$ -algebra in  $\text{Fun}(\mathbb{Z}/(p - 1), \text{Sp})$ , compatible with the splitting

$$ku_p \simeq \bigvee_{0 \leq i < p-1} \Sigma^{2i} \ell_p.$$

Furthermore, we show that the resulting  $p - 1$ -graded  $E_1$ -ring structure on  $ku_p$  agrees with that provided by the root adjunction methods of [ABM22], i.e. that provided by the equivalence  $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$ .

In Section 5, we discuss the resulting grading on  $\text{THH}(ku_p)$  and  $\text{TC}(ku_p)$  with further details provided in Appendix A where we show that  $\text{TC}(-)$  is a lax symmetric monoidal functor from the  $\infty$ -category of  $p - 1$ -graded  $E_1$ -rings to the  $\infty$ -category of  $p - 1$ -graded spectra. We deduce that  $\text{THH}(ku_p)$  is an  $E_\infty$ -algebra in  $p - 1$ -graded cyclotomic spectra and that  $\text{TC}(ku_p)$  and  $\text{TC}^-(ku_p)$  are  $p - 1$ -graded  $E_\infty$ -rings.

Since  $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$ , it follows from the results of [ABM22] that

$$\text{TC}(ku_p)_0 \simeq \text{TC}(\ell_p) \quad \text{and} \quad \text{TC}^-(ku_p)_0 \simeq \text{TC}^-(\ell_p).$$

Therefore, to obtain Theorem 1.1, it suffices to show that there is a unit

$$b \in T(2)_{2p+2} \text{TC}(ku_p)$$

of weight 1. For this, we use the element  $b \in V(1)_{2p+2} \text{K}(ku_p)$  constructed in [Aus10, Section 3].

We show in Section 6 that logarithmic THH of  $ku_p$  (as in [RSS18]) also admits an  $S^1$ -equivariant splitting compatible with that on  $\text{THH}(ku_p)$ . After this, we obtain that  $b$  represents a unit in  $T(2)_* \text{TC}^-(ku_p)$  by using the logarithmic THH computations of Rognes, Sagave and Schlichtkrull [RSS18], see Section 7. From this, it follows easily that  $b$  is indeed a unit of weight 1 in  $T(2)_* \text{TC}(ku_p)$ . This provides  $T(2)_* \text{TC}(ku_p)$ , i.e. Theorem 1.1, since

$$T(2)_* \text{TC}(ku_p)_0 \cong T(2)_* \text{TC}(\ell_p)$$

and that  $T(2)_* \text{TC}(ku_p)$  is periodic in the weight direction due to the unit  $b$  of weight 1 in  $T(2)_* \text{TC}(ku_p)$ .

To compute  $T(2)_* \text{K}(ku/p)$ , i.e. to prove Theorem 1.5, we construct  $ku/p$  as an algebra over  $ku_p$  in the  $\infty$ -category of  $p - 1$ -graded spectra in Section 4, i.e.  $ku/p$  is a  $p - 1$ -graded  $ku_p$ -algebra. Furthermore, we show that

$$ku/p \simeq \ell/p(\sqrt[p-1]{v_1})$$

as  $p - 1$ -graded  $E_1$ -rings. As a result, we obtain that  $\mathrm{TC}(ku/p)$  is a  $p - 1$ -graded  $\mathrm{TC}(ku_p)$ -module (i.e. a module over  $\mathrm{TC}(ku_p)$  in  $\mathrm{Fun}(\mathbb{Z}/(p - 1), \mathrm{Sp})$ ) and that

$$\mathrm{TC}(ku/p)_0 \simeq \mathrm{TC}(\ell/p).$$

After this, Theorem 1.5 follows by noting that  $T(2)_* \mathrm{TC}(ku_p)$  contains a unit of weight 1 and therefore, every  $p - 1$ -graded module over it, such as  $T(2)_* \mathrm{TC}(ku/p)$ , is periodic in the weight direction, see Section 7.

### 3. GRADED RING SPECTRA

Here, we set our conventions for graded objects in a presentably symmetric monoidal stable  $\infty$ -category  $\mathcal{C}$ . We start by noting that there is an equivalence of  $\infty$ -categories

$$\mathrm{Fun}(\mathbb{Z}/m, \mathcal{C}) \simeq \prod_{i \in \mathbb{Z}/m} \mathcal{C}.$$

We call an object  $C$  of  $\mathrm{Fun}(\mathbb{Z}/m, \mathcal{C})$  an  $m$ -graded object of  $\mathcal{C}$  and let  $C_i$  denote  $C(i)$ . If  $C$  is an  $E_k$ -algebra in  $\mathrm{Fun}(\mathbb{Z}/m, \mathcal{C})$ , we say  $C$  is an  $m$ -graded  $E_k$ -algebra in  $\mathcal{C}$ . If  $C'$  is an  $E_{k-1}$   $C$ -algebra in  $\mathrm{Fun}(\mathbb{Z}/m, \mathcal{C})$ , we say  $C'$  is an  $m$ -graded  $C$ -algebra. For an  $M \in \mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$ , we say  $M$  is an  $m$ -graded spectrum and an  $E_k$ -algebra in  $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$  is called an  $m$ -graded  $E_k$ -ring. For  $m = 0$ , we drop  $m$  and talk about graded spectra, graded  $E_k$ -rings and so on.

The map  $\mathbb{Z}/m \rightarrow 0$  provides a symmetric monoidal left adjoint functor

$$D: \mathrm{Fun}(\mathbb{Z}/m, \mathcal{C}) \rightarrow \mathrm{Fun}(0, \mathcal{C}) \simeq \mathcal{C}$$

given by left Kan extension [Nik16, Corollary 3.8]. We call  $D(C)$  the underlying object of  $C$  and this is given by the formula

$$D(M) \simeq \prod_{i \in \mathbb{Z}/m} M_i.$$

We often omit  $D$  in our notation.

We say an  $m$ -graded object  $C$  of  $\mathcal{C}$  is concentrated in weight 0 if  $C_i \simeq 0$  whenever  $i \neq 0$ . The inclusion  $0 \rightarrow \mathbb{Z}/m$  provides another adjunction:

$$\mathcal{C} \simeq \mathrm{Fun}(0, \mathcal{C}) \begin{array}{c} \xrightarrow{F_0} \\ \xleftarrow{G_0} \end{array} \mathrm{Fun}(\mathbb{Z}/m, \mathcal{C})$$

where the left adjoint  $F_0$  is symmetric monoidal and given by left Kan extension and  $G_0$  is given by restriction, i.e.  $G_0(C) = C_0$ . For  $C \in \mathcal{C}$ ,  $F_0(C)$  provides  $C$  as an  $m$ -graded object concentrated in weight 0. We often omit  $F_0$  and for a given  $C \in \mathrm{Fun}(\mathbb{Z}/m, \mathcal{C})$ , we denote the  $m$ -graded object  $F_0(C_0)$  by  $C_0$ .

For an  $m$ -graded  $E_k$ -ring  $A$ , the counit of this adjunction provides a map  $A_0 \rightarrow A$  of  $m$ -graded  $E_k$ -rings. If  $A$  is concentrated in weight 0, the counit of  $F_0 \dashv G_0$  provides an equivalence

$$(3.1) \quad F_0 G_0(A) \simeq A.$$

#### 4. COMPLEX $K$ -THEORY SPECTRUM AS A $p - 1$ -GRADED $E_\infty$ -RING

Here, we use the results of [CSY21] to obtain a  $p - 1$ -graded  $E_\infty$ -ring structure on  $ku_p$ . Furthermore, we show that the resulting  $p - 1$ -graded  $E_1$ -ring structure on  $ku_p$  agrees with that provided by the root adjunction methods of [ABM22].

Similarly, we construct a 2-periodic  $p - 1$ -graded Morava  $K$ -theory spectrum of height 1, i.e.  $ku_p/p$ , as a  $p - 1$ -graded  $E_\infty$   $ku_p$ -algebra.

**4.1. Complex  $K$ -theory spectrum.** Recall that  $L_p \rightarrow KU_p$  is a Galois extension with Galois group  $\Delta := \mathbb{Z}/(p - 1)$  in the sense of [Rog08, Section 5.5.4]. Taking connective covers, one obtains that  $ku_p$  is a  $\Delta$ -equivariant  $E_\infty$   $\ell_p$ -algebra and there is an equivalence:

$$\ell_p \simeq ku_p^{h\Delta}.$$

We consider  $\Delta$  as the cyclic subgroup  $\mathbb{Z}/(p - 1)$  of  $\mathbb{Z}_p^\times$ . Let  $\delta$  denote a generator of  $\Delta$  and  $\alpha$  denote the corresponding element in  $\mathbb{Z}_p$ . On  $\pi_*ku_p \cong \mathbb{Z}_p[u_2]$ , we have  $\pi_*(\delta)(u_2^i) = \alpha^i u_2^i$ . Note that since  $|\Delta| = p - 1$  and since  $ku_p$  is  $p$ -local, the homotopy fixed points above can be computed by taking fixed points at the level of homotopy groups.

The  $p - 1$ -graded  $E_\infty$   $\ell_p$ -algebra structure on  $ku_p$  is a consequence of the Fourier transform developed in [CSY21, Section 3]. Due to [CSY21, Corollary 3.9],  $\ell_p$  admits a primitive  $p - 1$ -root of unity in the sense of [CSY21, Definition 3.3] which we can choose to be  $\alpha \in \mathbb{Z}_p \cong \pi_0\ell_p$  above. Let  $\Delta^*$  denote the Pontryagin dual

$$\Delta^* := \text{hom}(\Delta, \mathbb{Z}/(p - 1))$$

of  $\Delta$  for which we have  $\Delta^* \cong \mathbb{Z}/(p - 1)$ . In this situation, [CSY21, Proposition 3.13] provides a symmetric monoidal functor:

$$(4.1) \quad \mathfrak{F}: \text{LMod}_{\ell_p}^{B\Delta} \rightarrow \text{Fun}(\mathbb{Z}/(p - 1), \text{LMod}_{\ell_p})$$

from the  $\infty$ -category of  $\Delta$ -equivariant  $\ell_p$ -modules to the  $\infty$ -category of  $p - 1$ -graded  $\ell_p$ -modules. Indeed, this functor is an equivalence of  $\infty$ -categories. For the  $E_\infty$ -algebra  $ku_p$  in  $\text{LMod}_{\ell_p}^{B\Delta}$ , we will show that  $\mathfrak{F}(ku_p)$  provides the desired  $p - 1$ -graded  $E_\infty$   $\ell_p$ -algebra structure on  $ku_p$ .

First, we describe  $\mathfrak{F}(ku_p)$  as a  $p - 1$ -graded  $\ell_p$ -module. Indeed,  $\mathfrak{F}$  provides the underlying eigenspectrum decomposition as described in [CSY21, Remark 3.14]. Namely, by [CSY21, Definition 3.12] we have

$$\mathfrak{F}(ku_p)_i \simeq (\ell_p(-\varphi_i) \wedge_{\ell_p} ku_p)^{h\Delta}$$

where  $\ell_p(-\varphi_i)$  is given by  $\ell_p$  as an  $\ell_p$ -module but  $\delta$  acts through multiplication by  $\alpha^{-i}$  on  $\pi_*(\ell_p(-\varphi_i))$  [CSY21, Definition 3.10]; here,  $\varphi_i$  is the map  $\mathbb{Z}/(p - 1) \rightarrow \mathbb{Z}/(p - 1)$  that multiplies by  $i$ . Again, homotopy fixed points may be computed by taking fixed points at the level of homotopy groups and one observes that

$$\pi_*(\mathfrak{F}(ku_p)_i) \cong \pi_*((\ell_p(-\varphi_i) \wedge_{\ell_p} ku_p)^{h\Delta})$$

is precisely given by the eigenspace corresponding to  $\alpha^i$  in  $\pi_*ku_p$ . This eigenspace is  $\pi_*(\Sigma^{2i}\ell_p) \subseteq \pi_*ku_p$ . In particular,  $\pi_*(\mathfrak{F}(ku_p)_i)$  is free of rank 1 as a  $\pi_*\ell_p$ -module and therefore, we obtain equivalences of  $\ell_p$ -modules

$$\mathfrak{F}(ku_p)_i \simeq \Sigma^{2i}\ell_p.$$

Therefore, the underlying  $\ell_p$ -module of the  $p - 1$ -graded  $\ell_p$ -module  $\mathfrak{F}(ku_p)$  is given by

$$(4.2) \quad D(\mathfrak{F}(ku_p)) \simeq \bigvee_{0 \leq i < p-1} \Sigma^{2i} \ell_p \simeq ku_p$$

as desired. The following proposition identifies the underlying  $E_\infty \ell_p$ -algebra of  $\mathfrak{F}(ku_p)$  with  $ku_p$ .

**Proposition 4.3.** *There is an equivalence of  $E_\infty \ell_p$ -algebras*

$$D(\mathfrak{F}(ku_p)) \simeq ku_p.$$

*Proof.* Using the strong monoidality of  $\mathfrak{F}$ , one observes that there is an isomorphism  $\pi_*(D(\mathfrak{F}(ku_p))) \cong \pi_*ku_p$  of  $\pi_*\ell_p$ -algebras. Inverting  $v_1 \in \pi_*\ell_p$ , we obtain the following commuting diagram of  $E_\infty$ -rings.

$$(4.4) \quad \begin{array}{ccc} \ell_p & \longrightarrow & D(\mathfrak{F}(ku_p)) \\ \downarrow & & \downarrow \\ L_p & \longrightarrow & D(\mathfrak{F}(ku_p))[v_1^{-1}] \end{array}$$

There is an isomorphism of  $\pi_*L_p$ -algebras

$$\pi_*(D(\mathfrak{F}(ku_p))[v_1^{-1}]) \cong \pi_*KU_p.$$

Since  $\pi_*L_p \rightarrow \pi_*KU_p$  is an étale map of Dirac rings [HP22, Example 4.32], we deduce by [HP22, Theorem 1.10] that the isomorphism above lifts to an equivalence of  $E_\infty L_p$ -algebras

$$D(\mathfrak{F}(ku_p))[v_1^{-1}] \simeq KU_p.$$

Alternatively, this also follows by [BR07, Proposition 2.2.3].

Since the right hand vertical arrow in Diagram (4.4) is a connective cover, the universal property of connective covers (in  $E_\infty \ell_p$ -algebras) provide an equivalence

$$D(\mathfrak{F}(ku_p)) \simeq ku_p$$

of  $E_\infty \ell_p$ -algebras. □

**Theorem 4.5.** *The  $E_\infty \ell_p$ -algebra  $ku_p$  admits the structure of a  $p - 1$ -graded  $E_\infty \ell_p$ -algebra such that*

$$(ku_p)_i \simeq \Sigma^{2i} \ell_p.$$

*Remark 4.6.* From this point, when we mention  $ku_p$  as a  $p - 1$ -graded  $E_\infty \ell_p$ -algebra, we mean  $\mathfrak{F}(ku_p)$ .

*Remark 4.7.* We would like to thank Tommy Lundemo for pointing out that it should also be possible to construct a  $p - 1$ -graded  $E_\infty$ -algebra structure on  $ku_{(p)}$  using [Sag14, Proposition 4.15] which states that  $ku_{(p)}$  can be obtained from  $\ell$  via base change through the polynomial like  $E_\infty$ -algebras of [Sag14, Construction 4.2].



**4.2. Adjoining roots to ring spectra.** Here, we summarize the root adjunction method developed in [ABM22, Construction 4.6].

Let  $k > 0$  be even and let  $\mathbb{S}[z_k]$  denote the free  $E_1$ -ring spectrum on  $\mathbb{S}^k$  (this is denoted by  $\mathbb{S}[\sigma_k]$  in [ABM22]). Taking  $z_k$  to be of weight 1,  $\mathbb{S}[z_k]$  admits the structure of a graded  $E_2$ -algebra [ABM22, Construction 3.3]. By left Kan extending  $\mathbb{S}[z_k]$  through  $\mathbb{Z} \rightarrow \mathbb{Z}/m$ , one obtains an  $m$ -graded  $E_2$ -ring that we also call  $\mathbb{S}[z_k]$  with  $z_k$  in weight 1.

Let  $A$  be an  $E_1$   $\mathbb{S}[z_{mk}]$ -algebra where  $z_{mk}$  acts through  $a \in \pi_{mk}A$ . Using  $F_0$ , we obtain a map of  $m$ -graded  $E_2$ -rings concentrated in weight 0:

$$\mathbb{S}[z_{mk}] \rightarrow A.$$

Furthermore, [ABM22, Proposition 3.9] provides a map  $\mathbb{S}[z_{mk}] \rightarrow \mathbb{S}[z_k]$  of  $m$ -graded  $E_2$ -rings carrying the weight 0 class  $z_{mk}$  to  $z_k^m$  in homotopy. Finally, the  $m$ -graded  $E_1$ -ring  $A(\sqrt[m]{a})$  is defined via the following relative smash product in  $m$ -graded spectra:

$$(4.8) \quad A(\sqrt[m]{a}) := A \wedge_{\mathbb{S}[z_{mk}]} \mathbb{S}[z_k].$$

This comes equipped with a map  $A \rightarrow A(\sqrt[m]{a})$  of  $m$ -graded  $E_1$ -rings given by the counit of the adjunction  $F_0 \dashv G_0$ .

It follows by the Künneth spectral sequence that at the level of homotopy rings, one obtains precisely the desired root adjunction:

$$(4.9) \quad \pi_*(A(\sqrt[m]{a})) \cong \pi_*(A)[y]/(y^m - a).$$

The class  $y$  above comes from  $z_k \in \pi_*\mathbb{S}[z_k]$  and therefore it is of weight 1. Furthermore,  $\pi_*A \subseteq \pi_*(A(\sqrt[m]{a}))$  is the subring of weight 0 elements.

**Lemma 4.10.** *Let  $k \geq 0$  be even. The  $m$ -graded  $E_2$ -ring obtained from the graded  $E_2$ -ring  $\mathbb{S}[z_k]$  by left Kan extending through  $\mathbb{Z} \rightarrow \mathbb{Z}/m$  admits an even cell decomposition.*

*Proof.* This follows as in the proof of [ABM22, Lemma 3.6]. The  $E_2$ -ring  $\mathbb{S}[z_k]$  admits an even cell decomposition [ABM22, Proposition 3.4], i.e. it is given by a filtered colimit of graded  $E_2$ -rings starting with the free graded  $E_2$ -algebra on  $\mathbb{S}^k$  and the later stages given by attaching an even  $E_2$ -cell to the former. Note that left Kan extension through  $\mathbb{Z} \rightarrow \mathbb{Z}/m$  is left adjoint and symmetric monoidal. Therefore, it preserves free algebras, even cell attachments and filtered colimits. This provides the  $m$ -graded  $E_2$ -ring  $\mathbb{S}[z_k]$  with an even cell decomposition.  $\square$

**4.3. Complex  $K$ -theory spectrum via root adjunction.** Here, we show that the  $p-1$ -graded  $E_1$   $\ell_p$ -algebra structure on  $ku_p$  provided by Proposition 4.3 agrees with that obtained by adjoining a root to  $v_1 \in \pi_*\ell_p$ .

Let  $\mathbb{S}[z_2]$  be the  $p-1$ -graded  $E_2$ -algebra with  $z_2$  in weight 1 as mentioned earlier. Since  $\pi_*\ell_p$  is concentrated in even degrees, Lemma 4.10 provides a  $p-1$ -graded  $E_2$ -ring map

$$(4.11) \quad \mathbb{S}[z_2] \rightarrow ku_p$$

that carries  $z_2$  to  $u_2$  in homotopy. The aforementioned  $p-1$ -graded  $E_2$ -map  $\mathbb{S}[z_{2(p-1)}] \rightarrow \mathbb{S}[z_2]$  carrying the weight 0 class  $z_{2(p-1)}$  to  $z_2^{p-1}$  induces the second equivalence below.

$$\mathbb{S}[z_{2(p-1)}] \simeq F_0G_0(\mathbb{S}[z_{2(p-1)}]) \simeq F_0G_0(\mathbb{S}[z_2])$$

The first equivalence follows by (3.1) and these are equivalences of  $p - 1$ -graded  $E_2$ -rings.

Similarly, the  $p - 1$ -graded  $E_\infty$ -ring map  $\ell_p \rightarrow ku_p$  (with  $\ell_p$  concentrated in weight 0), provides an equivalence  $\ell_p \simeq F_0G_0(ku_p)$  of  $p - 1$ -graded  $E_\infty$ -rings concentrated in weight 0. We obtain the following commuting diagram of  $p - 1$ -graded  $E_2$ -rings by applying the natural transformation  $F_0G_0 \rightarrow id$  to (4.11) and using the last two equivalences we mentioned above.

$$(4.12) \quad \begin{array}{ccc} \mathbb{S}[z_{2(p-1)}] & \longrightarrow & \ell_p \\ \downarrow & & \downarrow \\ \mathbb{S}[z_2] & \longrightarrow & ku_p \end{array}$$

In particular, the map  $\ell_p \rightarrow ku_p$  is a map of  $p - 1$ -graded  $\mathbb{S}[z_{2(p-1)}]$ -algebras. The extension/restriction of scalars adjunction induced by the left hand vertical map provides a map

$$\ell_p \wedge_{\mathbb{S}[z_{2(p-1)}]} \mathbb{S}[z_2] \xrightarrow{\cong} ku_p$$

of  $p - 1$ -graded  $E_1$   $\mathbb{S}[z_2]$ -algebras. Note that the left hand side above is a form of  $\ell_p(\sqrt[p-1]{v_1})$  as in (4.8). Considering (4.9), one observes that the map above is an equivalence as desired. This proves the following.

**Proposition 4.13.** *Let  $ku_p$  denote a  $p - 1$ -graded  $E_1$ -ring provided by Theorem 4.5. Then there is an equivalence of  $p - 1$ -graded  $E_1$ -rings*

$$ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$$

for the form of  $\ell_p(\sqrt[p-1]{v_1})$  constructed above.

**4.4. Two periodic Morava K-theory as a  $p - 1$ -graded  $ku_p$ -algebra.** Using the even cell decomposition of  $\mathbb{S}[z_0]$ , we obtain a map  $\mathbb{S}[z_0] \rightarrow \ell_p$  of  $E_2$ -rings that carries  $z_0$  to  $p$  in homotopy. Through this, we define the connective first Morava  $K$ -theory  $k(1) \simeq \ell/p$  as an  $E_1$   $\ell_p$ -algebra as follows:

$$\ell/p := \mathbb{S} \wedge_{\mathbb{S}[z_0]} \ell_p.$$

Here, we make use of the  $E_\infty$  map  $\mathbb{S}[z_0] \rightarrow \mathbb{S}$  sending  $z_0$  to 0; this is the weight 0-Postnikov truncation of  $\mathbb{S}[z_0]$  ([HW20, Lemma B.0.6]). Using  $F_0$ , we consider  $\ell/p$  as a  $p - 1$ -graded  $\ell_p$ -algebra concentrated in weight 0.

We define the connective two periodic first Morava  $K$ -theory  $ku/p$  as a  $p - 1$ -graded  $ku_p$ -algebra as follows.

$$ku/p := \ell/p \wedge_{\ell_p} ku_p$$

**Proposition 4.14.** *There is an equivalence of  $p - 1$ -graded  $E_1$ -rings*

$$ku/p \simeq \ell/p(\sqrt[p-1]{v_n})$$

for some form of  $\ell/p(\sqrt[p-1]{v_n})$ .

*Proof.* There is a map of  $p - 1$ -graded  $\ell_p$ -algebras

$$\ell/p \rightarrow ku/p := \ell/p \wedge_{\ell_p} ku_p.$$

The target carries a  $p - 1$ -graded  $ku_p$ -algebra structure compatible with its  $p - 1$ -graded  $\ell_p$ -algebra structure. Forgetting through Diagram (4.12), this is a map of

$p - 1$ -graded  $\mathbb{S}[z_{2(p-1)}]$ -algebras where the target carries a compatible  $p - 1$ -graded  $\mathbb{S}[z_2]$ -algebra structure. Extending scalars, we obtain a map of  $p - 1$ -graded  $E_1$ -rings:

$$\ell/p \wedge_{\mathbb{S}[z_{2(p-1)}]} \mathbb{S}[z_2] \xrightarrow{\cong} ku/p,$$

which can easily be shown to be an equivalence.  $\square$

## 5. GRADED THH, TC<sup>-</sup> AND TC

Let  $X$  be a  $p$ -local  $m$ -graded  $E_k$ -ring for  $m > 0$ . In [AMMN22, Appendix A], the authors prove that in this situation,  $\mathrm{THH}(X)$  is an  $m$ -graded  $E_{k-1}$ -algebra in  $\mathrm{Sp}^{BS^1}$ . In particular,  $\mathrm{THH}(X)$  admits an  $S^1$ -equivariant splitting into a coproduct of  $m$ -cofactors. Since the homotopy fixed points functor and the Tate construction commute with finite coproducts, this splits  $\mathrm{THH}(X)^{hS^1}$  and  $(\mathrm{THH}(X)^{tC_p})^{hS^1}$  as well. However, these splittings may not carry over to  $\mathrm{TC}(X)$  in general since the canonical map is given by maps  $\mathrm{THH}(X)_i^{hS^1} \rightarrow (\mathrm{THH}(X)_i^{tC_p})^{hS^1}$  that preserve the  $m$ -grading whereas the Frobenius map is given by maps

$$\mathrm{THH}(X)_i \rightarrow \mathrm{THH}(X)_{pi}^{tC_p}.$$

In particular, the fiber sequence defining  $\mathrm{TC}(X)$

$$\mathrm{TC}(X) \rightarrow \bigvee_{i \in \mathbb{Z}/m} \mathrm{THH}(X)_i^{hS^1} \xrightarrow{\varphi_p^{hS^1} - \mathrm{can}} \bigvee_{i \in \mathbb{Z}/m} (\mathrm{THH}(X)_i^{tC_p})^{hS^1}$$

may not split. On the other hand, if  $p = 1$  in  $\mathbb{Z}/m$ , the Frobenius map also respects the  $m$ -grading. This results in a splitting of the fiber sequence defining  $\mathrm{TC}(X)$  and hence a splitting of  $\mathrm{TC}(X)$  into  $m$ -factors. Since  $p = 1$  in  $\mathbb{Z}/(p - 1)$ , this applies to our examples. In this section, we make this precise and deduce that  $\mathrm{TC}(ku_p)$  is a  $p - 1$ -graded  $E_\infty$ -ring and that  $\mathrm{TC}(ku/p)$  is a  $p - 1$ -graded  $\mathrm{TC}(ku_p)$ -module.

For a given spectrum  $F$ , we let  $F^{\mathrm{triv}}$  denote the cyclotomic spectrum with trivial  $S^1$ -action and the Frobenius map given by the composite  $F \rightarrow F^{hC_p} \rightarrow F^{tC_p}$  where the first map comes from the fact that  $F$  has the trivial action and the second map is the canonical map.

**Definition 5.1.** Since  $\mathrm{CycSp}$  is a stable and presentably symmetric monoidal  $\infty$ -category, it follows by Shipley's theorem that there is a unique cocontinuous symmetric monoidal functor

$$(-)^{\mathrm{triv}}: \mathrm{Sp} \rightarrow \mathrm{CycSp}$$

given by the trivial cyclotomic structure described above.

The right adjoint to  $(-)^{\mathrm{triv}}$  is the lax symmetric monoidal functor

$$\mathrm{TC}: \mathrm{CycSp} \rightarrow \mathrm{Sp}$$

given by

$$\mathrm{TC}(-) \simeq \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, -).$$

For the rest of this section, assume that  $m$  is a positive integer such that  $p = 1$  in  $\mathbb{Z}/m$ . Using the results of [AMMN22, Appendix A] we prove in Appendix A below that there is a symmetric monoidal functor

$$\mathrm{Alg}_{E_1}(\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})) \xrightarrow{\mathrm{THH}} \mathrm{Fun}(\mathbb{Z}/m, \mathrm{CycSp}).$$

Furthermore, it follows by [Nik16, Corollary 3.7] that the levelwise application of TC provides a lax symmetric monoidal functor:

$$\mathrm{TC}: \mathrm{Fun}(\mathbb{Z}/m, \mathrm{CycSp}) \rightarrow \mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp}),$$

that we also call TC. In Appendix A, we prove that the following diagram of lax symmetric monoidal functors commutes.

$$(5.2) \quad \begin{array}{ccccc} \mathrm{Alg}_{E_1}(\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})) & \xrightarrow{\mathrm{THH}} & \mathrm{Fun}(\mathbb{Z}/m, \mathrm{CycSp}) & \xrightarrow{\mathrm{TC}} & \mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Alg}_{E_1}(\mathrm{Sp}) & \xrightarrow{\mathrm{THH}} & \mathrm{CycSp} & \xrightarrow{\mathrm{TC}} & \mathrm{Sp} \end{array}$$

The vertical maps above are given by left Kan extension along  $\mathbb{Z}/m \rightarrow 0$ , i.e. they provide the underlying objects.

*Remark 5.3.* The composite  $\mathrm{TC} \circ \mathrm{THH}$  at the bottom row above may not in general give the correct result since we only consider one prime in  $\mathrm{CycSp}$ . However, this is not an issue since we only work with  $p$ -complete objects in our applications.

**Construction 5.4.** Since  $ku_p$  is a  $p-1$ -graded  $E_\infty$ -ring, we obtain that  $\mathrm{THH}(ku_p)$  is a  $p-1$ -graded  $E_\infty$ -algebra in cyclotomic spectra and that  $\mathrm{TC}(ku_p)$  is a  $p-1$ -graded  $E_\infty$ -ring.

Furthermore, in Section 4.4, we defined  $ku/p$  as a  $p-1$ -graded  $ku_p$ -algebra. In particular, this implies that  $ku/p$  is a right module over  $ku_p$  in the  $\infty$ -category of  $p-1$ -graded  $E_1$ -rings, see [ABM22, Construction 4.11]. Therefore,  $\mathrm{THH}(ku/p)$  is a right  $\mathrm{THH}(ku_p)$ -module in the  $\infty$ -category of  $p-1$ -graded cyclotomic spectra and that  $\mathrm{TC}(ku/p)$  is a right  $\mathrm{TC}(ku_p)$ -module in the  $\infty$ -category of  $p-1$ -graded spectra.

*Remark 5.5.* Furthermore, the levelwise application of the symmetric monoidal functor  $\mathrm{CycSp} \rightarrow \mathrm{Sp}^{BS^1}$  that forgets the Frobenius map shows that  $\mathrm{THH}(X)$  is an  $m$ -graded  $E_{k-1}$ -algebra in  $\mathrm{Sp}^{BS^1}$  whenever  $X$  is an  $m$ -graded  $E_k$ -ring. In particular,  $\mathrm{THH}(X)^{hS^1}$  and  $(\mathrm{THH}(X)^{tC_p})^{hS^1}$  also admit the structures of  $m$ -graded  $E_{k-1}$ -algebras.

**5.1. Weight zero splitting of THH for root adjunctions.** In Section 4.3, we show that the  $p-1$ -graded  $E_1$ -ring structure on  $ku_p$  agrees with that given by the root adjunction method of [ABM22]. The reason we do this is so that we can make use of Theorem 4.17 of [ABM22] which states that for a  $p$ -local  $A$ ,  $\mathrm{THH}(A) \rightarrow \mathrm{THH}(A(\sqrt[p]{a}))_0$  is an equivalence whenever  $p \nmid m$ . Furthermore, this equivalence carries over to topological cyclic homology due to [ABM22, Theorem 5.5]. We obtain the following.

**Proposition 5.6.** *The canonical maps*

$$\mathrm{THH}(\ell_p) \xrightarrow{\cong} \mathrm{THH}(ku_p)_0 \quad \text{and} \quad \mathrm{TC}(\ell_p) \xrightarrow{\cong} \mathrm{TC}(ku_p)_0$$

*are equivalences.*

**Proposition 5.7.** *The canonical maps*

$$\mathrm{THH}(\ell/p) \xrightarrow{\cong} \mathrm{THH}(ku/p)_0 \quad \text{and} \quad \mathrm{TC}(\ell/p) \xrightarrow{\cong} \mathrm{TC}(ku/p)_0$$

*are equivalences.*

## 6. LOGARITHMIC THH OF THE COMPLEX $K$ -THEORY SPECTRUM

Here, we use the  $p-1$ -grading on  $\mathrm{THH}(ku_p)$  to obtain a splitting of the logarithmic THH of  $ku_p$  as an  $S^1$ -equivariant spectrum. We identify the resulting splitting at the level of  $V(1)$ -homotopy by using the logarithmic THH computations of Rognes, Sagave and Schlichtkrull in [RSS18]. For the rest of this section, let  $p > 3$ .

*Remark 6.1.* For the rest, we consider  $V(1) \rightarrow T(2)$  as a map of commutative monoids in the homotopy category of  $p-1$ -graded cyclotomic spectra with the trivial cyclotomic structure concentrated in weight 0 (using  $F_0 \circ (-)^{triv}$ ).

*Remark 6.2.* In the following, we move freely between  $\mathrm{THH}(ku_p)$  and  $\mathrm{THH}(ku_p)$  since ultimately, we are interested in the  $V(1)$ -homotopy of these objects for which we have an equivalence  $V(1) \wedge \mathrm{THH}(ku_p) \simeq V(1) \wedge \mathrm{THH}(ku_p)$ . Similarly, we move freely between  $V(1) \wedge \mathrm{THH}(\ell)$  and  $V(1) \wedge \mathrm{THH}(\ell_p)$ .

Let  $\mathrm{THH}(ku_p | u_2)$  denote the logarithmic THH of  $ku_p$  with respect to the Bott class  $u_2 \in \pi_*(ku_p) \cong \mathbb{Z}_{(p)}[u_2]$  in the sense of [RSS18]. In [RSS18], this is denoted by  $\mathrm{THH}(ku_p, D(u))$ . This is an  $S^1$ -equivariant  $E_\infty$ -algebra and there is a cofiber sequence of  $S^1$ -equivariant spectra:

$$(6.3) \quad \mathrm{THH}(ku_p) \rightarrow \mathrm{THH}(ku_p | u_2) \rightarrow \Sigma \mathrm{THH}(\mathbb{Z}_{(p)}),$$

where the first map is a map of  $E_\infty$ -algebras in  $S^1$ -equivariant spectra, see the discussion after [RSS15, Definition 4.6].

Here, our goal is to prove the following proposition where  $\mathrm{THH}(\ell | v_1)$  denotes the logarithmic THH of  $\ell$  with respect to the class  $v_1 \in \pi_*\ell$  as defined in [RSS18] where it is denoted by  $\mathrm{THH}(\ell, D(v))$ .

**Proposition 6.4.** *There is an equivalence of  $S^1$ -equivariant spectra:*

$$V(1) \wedge \mathrm{THH}(ku_p | u_2) \simeq V(1) \wedge \mathrm{THH}(\ell | v_1) \vee \left( \bigvee_{i \in \mathbb{Z}/(p-1) | i \neq 0} V(1) \wedge \mathrm{THH}(ku_p)_i \right),$$

given by the coproduct of the map  $V(1) \wedge \mathrm{THH}(\ell | v_1) \rightarrow V(1) \wedge \mathrm{THH}(ku_p | u_2)$  with the composite:

$$\bigvee_{i \in \mathbb{Z}/(p-1) | i \neq 0} V(1) \wedge \mathrm{THH}(ku_p)_i \rightarrow V(1) \wedge \mathrm{THH}(ku_p) \rightarrow V(1) \wedge \mathrm{THH}(ku_p | u_2),$$

where the first map is given by the inclusion of the given summands of the  $p-1$ -graded spectrum  $\mathrm{THH}(ku_p)$  and the second one is the canonical one.

*Remark 6.5.* Since  $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$ , this is an immediate consequence of the results of [ABM22, Section 6] if we assume that the definition of logarithmic THH in [ABM22] agrees with that used in [RSS18]. This compatibility result is not available at the moment, and therefore, we will not assume it. On the other hand, Devalapurkar and Moulinos prove this compatibility result in their upcoming work.

*Proof.* Due to [RSS18, Theorem 4.4], there is a map of homotopy cofiber sequences of  $S^1$ -equivariant spectra:

$$\begin{array}{ccccc} V(1) \wedge \mathrm{THH}(\mathbb{Z}_{(p)}) & \longrightarrow & V(1) \wedge \mathrm{THH}(\ell) & \longrightarrow & V(1) \wedge \mathrm{THH}(\ell \mid v_1) \\ \downarrow & & \downarrow & & \downarrow \\ V(1) \wedge \mathrm{THH}(\mathbb{Z}_{(p)}) & \longrightarrow & V(1) \wedge \mathrm{THH}(ku_p) & \longrightarrow & V(1) \wedge \mathrm{THH}(ku_p \mid u_2) \end{array}$$

as mentioned in [RSS18, Equation (8.1)]. Here, the left hand vertical map is an equivalence. Therefore, the bottom left horizontal map factors as

$$V(1) \wedge \mathrm{THH}(\mathbb{Z}_{(p)}) \rightarrow V(1) \wedge \mathrm{THH}(\ell) \rightarrow V(1) \wedge \mathrm{THH}(ku_p) \simeq \bigvee_{i \in \mathbb{Z}/(p-1)} V(1) \wedge \mathrm{THH}(ku_p)_i.$$

The second map above is the inclusion of the weight 0 summand due to Proposition 5.6. In particular, the cofiber sequence given by the bottom row splits through the splitting of  $\mathrm{THH}(ku_p)$ . Namely, this cofiber sequence is given by a coproduct of the cofiber sequence given by the top row and the cofiber sequence

$$* \rightarrow \bigvee_{i \in \mathbb{Z}/(p-1) \mid i \neq 0} V(1) \wedge \mathrm{THH}(ku_p)_i \xrightarrow{\simeq} \bigvee_{i \in \mathbb{Z}/(p-1) \mid i \neq 0} V(1) \wedge \mathrm{THH}(ku_p)_i.$$

This identifies the cofiber, i.e.  $V(1) \wedge \mathrm{THH}(ku_p \mid u_2)$  as stated in the proposition.  $\square$

We will identify the homotopy groups of the summands of  $V(1) \wedge \mathrm{THH}(ku_p \mid u_2)$  given by the splitting above. For this, we start by recalling the computations of  $V(1)_* \mathrm{THH}(\ell \mid v_1)$  and  $V(1)_* \mathrm{THH}(ku_p \mid u_2)$  from [RSS18]. For what follows,  $E(x, y)$ ,  $P(x)$  and  $P_k(x)$  denote the exterior algebra over  $\mathbb{F}_p$  in two variables, the polynomial algebra  $\mathbb{F}_p[x]$  and the truncated polynomial algebra  $\mathbb{F}_p[x]/x^k$  respectively.

**Theorem 6.6.** [RSS18, Theorems 7.3 and 8.1] *There are ring isomorphisms:*

$$\begin{aligned} V(1)_* \mathrm{THH}(\ell \mid v_1) &\cong E(\lambda_1, d \log v_1) \otimes P(\kappa_1) \\ V(1)_* \mathrm{THH}(ku_p \mid u_2) &\cong P_{p-1}(u_2) \otimes E(\lambda_1, d \log u_2) \otimes P(\kappa_1) \end{aligned}$$

where  $|\lambda_1| = 2p - 1$ ,  $|\kappa_1| = 2p$ ,  $|d \log v_1| = |d \log u_2| = 1$  and  $|u_2| = 2$ . Furthermore, the map

$$V(1)_* \mathrm{THH}(\ell \mid v_1) \rightarrow V(1)_* \mathrm{THH}(ku_p \mid u_2)$$

is given by the ring map that carries  $d \log v_1$  to  $-d \log u_2$ ,  $\lambda_1$  to  $\lambda_1$  and  $\kappa_1$  to  $\kappa_1$ .

Recall that there is an action of the group  $\Delta := \mathbb{Z}/(p-1)$  on  $ku_p$  through Adams operations. Let  $\delta \in \Delta$  be a chosen generator and we choose a  $\beta \in \mathbb{F}_p^\times$  such that  $\pi_*(\mathbb{S}/p \wedge \delta)(u_2) = \beta u_2$ ; here,

$$\pi_*(\mathbb{S}/p \wedge \delta): \pi_*(\mathbb{S}/p \wedge ku_p) \rightarrow \pi_*(\mathbb{S}/p \wedge ku_p) \cong \mathbb{F}_p[u_2].$$

A given  $x \in V(1)_* \mathrm{THH}(ku_p)$  is said to have  $\delta$ -weight  $i \in \mathbb{Z}/(p-1)$  if the automorphism of  $V(1)_* \mathrm{THH}(ku_p)$  induced by  $\delta$  carries  $x$  to  $\beta^i x$  [Aus05, Definition 8.2]; the  $\delta$ -weights of the generators of  $V(1)_* \mathrm{THH}(ku_p)$  are given in [Aus05, Proposition 10.1]. One defines  $\delta$ -weight in a similar way for  $V(1)_* K(ku_p)$ ,  $V(1)_* \mathrm{TC}(ku_p)$  etc.

It follows by [ABM22, Proposition 8.2] that  $V(1)_* \mathrm{THH}(ku_p)_i$  is precisely given by the classes of  $\delta$ -weight  $i$  in  $V(1)_* \mathrm{THH}(ku_p)$ . In other words,  $\delta$ -weight and our weight gradings agree for  $V(1)_* \mathrm{THH}(ku_p)$ .

**Proposition 6.7.** *For  $0 < i < p - 1$ , the image of the inclusion*

$$\psi_i: V(1)_* \mathrm{THH}(ku_p)_i \rightarrow V(1)_* \mathrm{THH}(ku_{(p)} \mid u_2)$$

is given by:

$$\{u_2^i\} \otimes E(\lambda_1, d \log u_2) \otimes P(\kappa_1).$$

Here, the maps  $\psi_i$  are given by Proposition 6.4.

*Proof.* It follows from Theorem 6.6 that the image of the inclusion

$$\psi_0: V(1)_* \mathrm{THH}(\ell \mid v_1) \rightarrow V(1)_* \mathrm{THH}(ku_{(p)} \mid u_2)$$

is given by

$$V_0 := \{1\} \otimes E(\lambda_1, d \log u_2) \otimes P(\kappa_1);$$

we say  $V_0 = \mathrm{im} \psi_0$ . Also, let

$$V_i = \{u_2^i\} \otimes E(\lambda_1, d \log u_2) \otimes P(\kappa_1).$$

It follows by inspection on [RSS18, Theorem 8.5] that every  $\mathbb{F}_p$ -module generator of  $V_i$  given above gets hit by an element of  $\delta$ -weight  $i$  under the map

$$V(1)_* \mathrm{THH}(ku_{(p)}) \rightarrow V(1)_* \mathrm{THH}(ku_{(p)} \mid u_2)^1.$$

Since  $\delta$ -weight  $i$  elements of  $V(1)_* \mathrm{THH}(ku_p)$  correspond to the  $\mathbb{F}_p$ -submodule

$$V(1)_* \mathrm{THH}(ku_p)_i \subseteq V(1)_* \mathrm{THH}(ku_p),$$

we deduce that  $V_i \subseteq \mathrm{im} \psi_i$  for every  $i$ . Since

$$\bigoplus_{i \in \mathbb{Z}/p-1} V_i \cong V(1)_* \mathrm{THH}(ku_{(p)} \mid u_2) \cong \bigoplus_{i \in \mathbb{Z}/p-1} \mathrm{im} \psi_i,$$

and since all the vector spaces involved are finite dimensional at each homotopy degree, we deduce that  $V_i = \mathrm{im} \psi_i$  as desired. Note that the second isomorphism above follows by Proposition 6.4. □

## 7. TOPOLOGICAL CYCLIC HOMOLOGY OF COMPLEX $K$ -THEORY

Let  $p > 3$  for the rest of this section. Here, we compute  $T(2)_* \mathrm{K}(ku_p)$  and  $T(2)_* \mathrm{K}(ku/p)$ .

*Remark 7.1.* Since  $V(1)$  is a finite spectrum,  $V(1) \wedge -$  commutes with all constructions involving colimits and limits. For instance, it commutes with homotopy fixed points and one has  $\mathrm{TC}(V(1) \wedge E) \simeq V(1) \wedge \mathrm{TC}(E)$  for every cyclotomic spectrum  $E$ .

---

<sup>1</sup>For this, note that the classes  $a_i$ ,  $b_i$  and  $u_2$  are of  $\delta$ -weight 1 and the classes  $\lambda_1$  and  $\mu_2$  are of  $\delta$ -weight 0 in  $V(1)_* \mathrm{THH}(ku_p)$ .

**7.1. Higher Bott element.** In [Aus10, Section 3], Ausoni constructs a non-trivial class  $b \in V(1)_{2p+2} K(ku_p)$  of  $\delta$ -weight 1, that he calls the higher Bott element, by considering the units of  $ku_p$ . Namely,  $b$  is constructed using a map

$$K(\mathbb{Z}, 2) \rightarrow GL_1(ku_p)$$

and it originates from  $K(2)_* K(\mathbb{Z}, 3)$  which is known due to Ravenel-Wilson [RW80].

Let  $b \in V(1)_{2p+2} TC(ku_p)$  also denote the image of this class under the map  $V(1)_* K(ku_p) \rightarrow V(1)_* TC(ku_p)$ ; this is also a non-trivial class due to the following.

**Proposition 7.2.** *The classes  $b$  mentioned above satisfies the following properties.*

- (1) *The map  $V(1)_* TC(ku_p) \rightarrow V(1)_* THH(ku_p)$  carries  $b$  to a  $\delta$ -weight 1 class denoted as  $b_1$  in [Aus05], see [Aus10, Lemma 4.4]. Since  $b_1$  is of  $\delta$ -weight 1, we have*

$$b_1 \in V(1)_* THH(ku_p)_1.$$

- (2) *The map  $V(1)_* THH(ku_p) \rightarrow V(1)_* THH(ku_p | u_2)$  carries  $b_1$  to  $u_2 \kappa_1$  [RSS18, Theorem 8.5].*
- (3) *In  $V(1)_* K(ku_p)$ , we have  $b(b^{p-1} + v_2) = 0$  [Aus10, Proposition 2.7].*

We prove the following.

**Proposition 7.3.** *The higher Bott element  $b \in V(1)_* TC(ku_p)$  is a homogeneous element of weight 1 in the  $p-1$ -grading. In other words,  $b \in V(1)_* TC(ku_p)_1$ . Similarly, the corresponding element  $b \in V(1)_* THH(ku_p)^{hS^1}$  is also of weight 1.*

*Proof.* Indeed, we show that all the elements in  $V(1)_* TC(THH(ku_p)_i)$  are of  $\delta$ -weight  $i$ . Since  $\Delta = \mathbb{Z}/(p-1)$  is an abelian group, the map  $\delta: ku_p \rightarrow ku_p$  induced by the chosen generator  $\delta \in \Delta$  is a map of  $E_\infty$ -algebras in the  $\infty$ -category of  $\Delta$ -equivariant  $\ell_p$ -modules (not just a map of  $E_\infty$   $\ell_p$ -algebras). Therefore, using  $\mathfrak{F}$  in (4.1),  $\delta: ku_p \rightarrow ku_p$  can be considered as a map of  $p-1$ -graded  $E_\infty$   $\ell_p$ -algebras.

As a result, the induced map  $THH(\delta)$  is a map of  $p-1$ -graded cyclotomic objects. In particular, it preserves weight at the level of  $TC$ ,  $TC^-$  and  $TP$ . Recall that each

$$x \in V(1)_* THH(ku_p)_i$$

is of  $\delta$ -weight  $i$ . Therefore, the map induced by  $\delta$  at the level of the homotopy fixed point spectral sequence for  $V(1)_* THH(ku_p)_i^{hS^1}$  is given by multiplication by  $\beta^i \in \mathbb{F}_p^\times$ . Since  $V(1)_* THH(ku_p)_i$  is finite at each degree, this spectral sequence is strongly convergent [Boa99, Theorem 7.1]. We deduce that every class in  $V(1)_* THH(ku_p)_i^{hS^1}$  with defined  $\delta$ -weight have  $\delta$ -weight  $i$ . On the other hand,  $V(1)_* THH(\delta)_i^{hS^1}$  is diagonalizable (since its  $p-1$ st power is identity), i.e.  $V(1)_* THH(ku_p)_i^{hS^1}$  have a basis for which  $\delta$ -weight is defined for each basis element. Therefore, we deduce that all the classes in  $V(1)_* THH(ku_p)_i^{hS^1}$  are of  $\delta$ -weight  $i$ . The same argument shows that every class in  $V(1)_* THH(ku_p)_i^{tS^1}$  is of  $\delta$ -weight  $i$ .

The fiber sequence defining  $TC$  also shows that each class in  $V(1)_* TC(THH(ku_p)_i)$  either have  $\delta$ -weight  $i$  or have undefined  $\delta$ -weight, but since this action is again diagonalizable, we deduce that every class in  $V(1)_* TC(THH(ku_p)_i)$  is of  $\delta$ -weight  $i$ . Since  $b$  is of  $\delta$ -weight 1, the result follows.  $\square$



**7.2. Topological cyclic homology.** As mentioned earlier, we need to show that  $b \in T(2)_* \mathrm{TC}(ku_p)$  is a unit. For this, we construct multiplication by  $b$  as a self map of the cyclotomic spectrum  $V(1) \wedge \mathrm{THH}(ku_p)$  and show that it induces a self equivalence of  $T(2) \wedge \mathrm{TC}(ku_p)$ . We first show that  $b$  provides a unit in  $T(2)_* \mathrm{THH}(ku_p)^{hS^1}$  by comparing it with the corresponding multiplication in  $T(2)_* \mathrm{THH}(ku_p | u_2)^{hS^1}$ .

**Construction 7.4.** We start with the map  $\mathbb{S}^{2p+2} \rightarrow \mathrm{TC}(V(1) \wedge \mathrm{THH}(ku_p))$  representing  $b$ . Using the adjunction  $(-)^{\mathrm{triv}} \dashv \mathrm{TC}$  mentioned in Definition 5.1, one obtains a map of cyclotomic spectra

$$b_1: \Sigma^{2p+2} \mathbb{S}^{\mathrm{triv}} \rightarrow V(1) \wedge \mathrm{THH}(ku_p)$$

representing the class  $b_1$ . We define

$$m_b: \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_p) \rightarrow V(1) \wedge \mathrm{THH}(ku_p)$$

as the following composite map of cyclotomic spectra.

$$\begin{aligned} m_b: \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_p) &\simeq V(1) \wedge \mathrm{THH}(ku_p) \wedge \Sigma^{2p+2} \mathbb{S}^{\mathrm{triv}} \xrightarrow{id \wedge b_1} \\ &V(1) \wedge \mathrm{THH}(ku_p) \wedge V(1) \wedge \mathrm{THH}(ku_p) \rightarrow V(1) \wedge \mathrm{THH}(ku_p) \end{aligned}$$

Here,  $id$  denotes the identity map of  $V(1) \wedge \mathrm{THH}(ku_p)$  and the second map above is given by the multiplication maps of  $\mathrm{THH}(ku_p)$  and  $V(1)$ .

We construct a similar map for logarithmic  $\mathrm{THH}$  of  $ku_{(p)}$  which is compatible with the one constructed above.

**Construction 7.5.** The first map below is the underlying  $S^1$ -equivariant map of the map  $b_1$  in Construction 7.4; the second map is the usual one.

$$u_2 \kappa_1: \Sigma^{2p+2} \mathbb{S}^{\mathrm{triv}} \xrightarrow{b_1} V(1) \wedge \mathrm{THH}(ku_p) \rightarrow V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2)$$

This composite is a map of  $S^1$ -equivariant spectra. Furthermore, it represents  $u_2 \kappa_1$  in homotopy due to Proposition 7.2. As in Construction 7.4, we define an  $S^1$ -equivariant map:

$$m_{u_2 \kappa_1}: \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \rightarrow V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2),$$

through the following composite.

$$\begin{aligned} m_{u_2 \kappa_1}: \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) &\simeq V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \wedge \Sigma^{2p+2} \mathbb{S}^{\mathrm{triv}} \xrightarrow{id \wedge u_2 \kappa_1} \\ &V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \wedge V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \rightarrow V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \end{aligned}$$

Since  $V(1) \wedge \mathrm{THH}(ku_p) \rightarrow V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2)$  is a map of monoids in the homotopy category of  $S^1$ -equivariant spectra, the following canonical diagram of  $S^1$ -equivariant spectra commutes up to homotopy.

$$(7.6) \quad \begin{array}{ccc} \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_p) & \longrightarrow & \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \\ \downarrow m_b & & \downarrow m_{u_2 \kappa_1} \\ V(1) \wedge \mathrm{THH}(ku_p) & \longrightarrow & V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \end{array}$$

**Proposition 7.7.** *The map  $\mathrm{THH}(\ell) \rightarrow \mathrm{THH}(\ell | v_1)$  induces an equivalence*

$$L_{T(2)} \mathrm{THH}(\ell)^{hS^1} \xrightarrow{\simeq} L_{T(2)} \mathrm{THH}(\ell | v_1)^{hS^1}.$$

*Proof.* There is an  $E_\infty$ -map  $K(\mathbb{Z}_{(p)}) \rightarrow \mathrm{THH}(\mathbb{Z}_{(p)})^{hS^1}$  and we have  $L_{T(2)} K(\mathbb{Z}_{(p)}) \simeq 0$  due to [LMMT20, Purity Theorem]. This implies that

$$L_{T(2)} \mathrm{THH}(\mathbb{Z}_{(p)})^{hS^1} \simeq 0.$$

Since the cofiber of the map  $\mathrm{THH}(\ell)^{hS^1} \rightarrow \mathrm{THH}(\ell | v_1)^{hS^1}$  is given by  $\Sigma \mathrm{THH}(\mathbb{Z}_{(p)})^{hS^1}$ , this provides the desired result.  $\square$

*Remark 7.8.* As mentioned earlier, the map  $V(1) \rightarrow T(2)$  is given by the  $T(2)$ -localization

$$V(1) \rightarrow L_{T(2)} V(1) \simeq T(2).$$

For a given spectrum  $E$ ,  $T(2) \wedge E$  is a homotopy  $T(2)$ -module and therefore,  $T(2) \wedge E$  is  $T(2)$ -local. Furthermore,  $V(1) \wedge E \rightarrow T(2) \wedge E$  is a  $T(2)$ -equivalence as  $V(1) \rightarrow T(2)$  is. Therefore,  $V(1) \wedge E \rightarrow T(2) \wedge E$  is given by the  $T(2)$ -localization:

$$V(1) \wedge E \rightarrow L_{T(2)}(V(1) \wedge E) \simeq T(2) \wedge E.$$

**Proposition 7.9.** *For the composite  $S^1$ -equivariant map:*

$$\begin{aligned} f: \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(\ell | v_1) &\rightarrow \Sigma^{2p+2} V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \xrightarrow{m_{u_2 \kappa_1}} \\ &V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2) \rightarrow V(1) \wedge \mathrm{THH}(ku_p)_1, \end{aligned}$$

$L_{T(2)}(f^{hS^1})$  is an equivalence. Here, the first and the last maps are those provided by Proposition 6.4; indeed, the last map above is the projection to the factor of  $V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2)$  corresponding to  $1 \in \mathbb{Z}/(p-1)$ .

*Proof.* Due to Theorem 6.6 and Proposition 6.7,  $\pi_* f$  can be given by the composite map

$$\begin{aligned} \pi_* f: E(\lambda_1, d \log v_1) \otimes P(\kappa_1) &\rightarrow \{1\} \otimes E(\lambda_1, d \log u_2) \otimes P(\kappa_1) \\ &\xrightarrow{u_2 \kappa_1} \{u_2\} \otimes E(\lambda_1, d \log u_2) \otimes P(\kappa_1) \end{aligned}$$

where the first map above sends  $d \log v_1$  to  $-d \log u_2$  and fixes the other generators and the second map multiplies by  $u_2 \kappa_1$ . The first map is an isomorphism and the second map above is an isomorphism in sufficiently large degrees. We deduce that  $\pi_* f$  is an isomorphism in sufficiently large degrees.

In particular, the cofiber of  $f$ , lets call it  $C$ , is bounded from above in homotopy. Therefore,  $C^{hS^1}$  is also bounded from above in homotopy since  $(-)^{hS^1}$  preserves co-connectivity. In particular,  $L_{T(2)}(C^{hS^1}) \simeq 0$ . This means that  $L_{T(2)}(f^{hS^1})$  is an equivalence as desired.  $\square$

*Remark 7.10.* In the construction of the map  $f$  above, if we used  $V(1) \wedge \mathrm{THH}(ku_p)$  (together with its weight splitting and  $m_b$ ) instead of  $V(1) \wedge \mathrm{THH}(ku_{(p)} | u_2)$ , the proof above would fail to go through. This is because the cofiber of  $f$  would not be bounded from above. This is precisely the reason why we use logarithmic THH for our computations.

The following is the non-logarithmic analogue of the the proposition above.

**Proposition 7.11.** *For the composite  $S^1$ -equivariant map:*

$$g: \Sigma^{2p+2}V(1) \wedge \mathrm{THH}(ku_p)_0 \rightarrow \Sigma^{2p+2}V(1) \wedge \mathrm{THH}(ku_p) \xrightarrow{m_b} \\ V(1) \wedge \mathrm{THH}(ku_p) \rightarrow V(1) \wedge \mathrm{THH}(ku_p)_1,$$

$L_{T(2)}(g^{hS^1})$  is an equivalence. Here, the first and the last maps are given by the  $p-1$ -grading on  $\mathrm{THH}(ku_p)$ .

*Proof.* For this, we consider the following (up to homotopy) commuting diagram of  $S^1$ -equivariant spectra.

$$\begin{array}{ccc} \Sigma^{2p+2}V(1) \wedge \mathrm{THH}(ku_p)_0 & \longrightarrow & \Sigma^{2p+2}V(1) \wedge \mathrm{THH}(\ell \mid v_1) \\ \downarrow & & \downarrow \\ \Sigma^{2p+2}V(1) \wedge \mathrm{THH}(ku_p) & \longrightarrow & \Sigma^{2p+2}V(1) \wedge \mathrm{THH}(ku_{(p)} \mid u_2) \\ \downarrow m_b & & \downarrow m_{u_2\kappa_1} \\ V(1) \wedge \mathrm{THH}(ku_p) & \longrightarrow & V(1) \wedge \mathrm{THH}(ku_{(p)} \mid u_2) \\ \downarrow & & \downarrow \\ V(1) \wedge \mathrm{THH}(ku_p)_1 & \xrightarrow{id} & V(1) \wedge \mathrm{THH}(ku_p)_1 \end{array}$$

$g$  (left curved arrow),  $f$  (right curved arrow)

The sequence of vertical maps on the left hand side is the composite defining  $g$  and the sequence of vertical maps on the right hand side is the composite defining the map  $f$  in Proposition 7.9. The upper horizontal map is given by the passage from  $\mathrm{THH}$  to log  $\mathrm{THH}$  by noting  $\mathrm{THH}(ku_p)_0 \simeq \mathrm{THH}(\ell_p)$  (see Proposition 5.6). The lower horizontal map is the identity map. The inner square above is given by Diagram (7.6) which commutes up to homotopy. By the definition of the rest of the maps, one observes that the diagram above commutes.

Due to Proposition 7.9,  $L_{T(2)}(f^{hS^1})$  is an equivalence. Furthermore, the top horizontal map is also an equivalence after applying  $L_{T(2)}(-^{hS^1})$  due to Propositions 7.7 and 5.6. Since the lower horizontal map is also an equivalence, we deduce that  $L_{T(2)}(g^{hS^1})$  is an equivalence as desired.  $\square$

For the rest, we also let  $b \in V(1)_* \mathrm{THH}(ku_p)^{hS^1}$  denote the image of the higher Bott element  $b \in V(1)_* \mathrm{K}(ku_p)$  under the trace map  $V(1)_* \mathrm{K}(ku_p) \rightarrow V(1)_* \mathrm{THH}(ku_p)^{hS^1}$ .

**Corollary 7.12.** *After restricting and corestricting, multiplication by  $b$  provides an isomorphism*

$$\cdot b: T(2)_* \Sigma^{2p+2} \mathrm{THH}(ku_p)_0^{hS^1} \xrightarrow{\cong} T(2)_* \mathrm{THH}(ku_p)_1^{hS^1}$$

between the sets of weight 0 and weight 1 classes in  $T(2)_* \mathrm{THH}(ku_p)^{hS^1}$ .

*Proof.* By Remark 7.8 and the lax monoidal structure of the fixed points functor  $-^{hS^1}$ , this map is given by  $\pi_* L_{T(2)}(g^{hS^1})$  which is an isomorphism due to Proposition 7.11.  $\square$

**Corollary 7.13.** *In  $T(2)_* \mathrm{THH}(ku_p)^{hS^1}$ , we have  $b^{p-1} = -v_2$ . In particular,  $b \in T(2)_* \mathrm{THH}(ku_p)^{hS^1}$  is a unit.*

*Proof.* By Proposition 7.2, we have  $b(b^{p-1} + v_2) = 0$  in  $T(2)_* \mathrm{K}(ku_p)$ . Using the ring map  $T(2)_* \mathrm{K}(ku_p) \rightarrow T(2)_* \mathrm{THH}(ku_p)^{hS^1}$ , we obtain that

$$(7.14) \quad b(b^{p-1} + v_2) = 0$$

in  $T(2)_* \mathrm{THH}(ku_p)^{hS^1}$ .

Due to Proposition 7.3,  $b$  is of weight 1 in  $V(1)_* \mathrm{THH}(ku_p)^{hS^1}$ . In particular,  $b^{p-1} + v_2$  is of weight 0 as  $v_2$  is of weight 0. However, multiplication by  $b$  does not annihilate any non-trivial weight 0 classes in  $T(2)_* \mathrm{THH}(ku_p)^{hS^1}$  due to Corollary 7.12. This, together with (7.14) implies that  $b^{p-1} + v_2 = 0$  in  $T(2)_* \mathrm{THH}(ku_p)^{hS^1}$  as desired.  $\square$

We are going to use the following two propositions to deduce that  $L_{T(2)} \mathrm{TC}(m_b)$  is an equivalence; i.e. that  $b$  is a unit in  $T(2)_* \mathrm{TC}(ku_p)$ .

**Proposition 7.15.** *The map*

$$L_{T(2)}(m_b^{hS^1}): \Sigma^{2p+2}T(2) \wedge \mathrm{THH}(ku_p)^{hS^1} \xrightarrow{\simeq} T(2) \wedge \mathrm{THH}(ku_p)^{hS^1}$$

*is an equivalence.*

*Proof.* Using the lax structure of the homotopy fixed points functor  $(-)^{hS^1}$  and Remark 7.8, one observes that the map  $\pi_* L_{T(2)}(m_b^{hS^1})$  is precisely the map

$$T(2)_* \mathrm{THH}(ku_p)^{hS^1} \rightarrow T(2)_* \mathrm{THH}(ku_p)^{hS^1}$$

given by multiplication by  $b$ . This is an isomorphism due to Corollary 7.13.  $\square$

**Proposition 7.16.** *The map  $L_{T(2)}(m_b^{tS^1})$  is an equivalence.*

*Proof.* Since  $m_b$  is an  $S^1$ -equivariant map, we have the following commuting diagram given by the canonical natural transformation in [NS18, Corollary I.4.3].

$$\begin{array}{ccc} \Sigma^{2p+2}T(2) \wedge \mathrm{THH}(ku_p)^{hS^1} & \xrightarrow[\simeq]{can} & \Sigma^{2p+2}T(2) \wedge \mathrm{THH}(ku_p)^{tS^1} \\ \simeq \downarrow L_{T(2)}(m_b^{hS^1}) & & \downarrow L_{T(2)}(m_b^{tS^1}) \\ T(2) \wedge \mathrm{THH}(ku_p)^{hS^1} & \xrightarrow[\simeq]{can} & T(2) \wedge \mathrm{THH}(ku_p)^{tS^1} \end{array}$$

The maps *can* above are equivalences since their fibers are given by

$$T(2) \wedge \Sigma \mathrm{THH}(ku_p)_{hS^1} \simeq 0$$

due to [NS18, Corollary I.4.3]. The left hand vertical map is an equivalence due to Proposition 7.15; and therefore, the right hand vertical map is also an equivalence.  $\square$

**Proposition 7.17.** *The higher bott element  $b \in T(2)_{2p+2} \mathrm{TC}(ku_p)$  is a unit.*

*Proof.* Recall that  $m_b$  is a map of cyclotomic spectra by construction. Furthermore, the map

$$\pi_* L_{T(2)} \mathrm{TC}(m_b): T(2)_* \Sigma^{2p+2} \mathrm{TC}(ku_p) \rightarrow T(2)_* \mathrm{TC}(ku_p)$$

is given by multiplication by  $b$ . Therefore, it is sufficient to show that  $L_{T(2)} \mathrm{TC}(m_b)$  is an equivalence. Since  $m_b$  is cyclotomic, this induces a map of fiber sequences as follows, see [NS18, Lemma II.4.2].

$$\begin{array}{ccc}
\Sigma^{2p+2}T(2) \wedge \mathrm{TC}(ku_p) & \xrightarrow{L_{T(2)} \mathrm{TC}(m_b)} & T(2) \wedge \mathrm{TC}(ku_p) \\
\downarrow & & \downarrow \\
\Sigma^{2p+2}T(2) \wedge \mathrm{THH}(ku_p)^{hS^1} & \xrightarrow[\simeq]{L_{T(2)}(m_b^{hS^1})} & T(2) \wedge \mathrm{THH}(ku_p)^{hS^1} \\
\downarrow \varphi_p^{hS^1} \text{-can} & & \downarrow \varphi_p^{hS^1} \text{-can} \\
\Sigma^{2p+2}T(2) \wedge \mathrm{THH}(ku_p)^{tS^1} & \xrightarrow[\simeq]{L_{T(2)}(m_b^{tS^1})} & T(2) \wedge \mathrm{THH}(ku_p)^{tS^1}
\end{array}$$

The middle and the bottom horizontal maps are equivalences due to Propositions 7.15 and 7.16. Since this is a map of fiber sequences, we deduce that the top horizontal map is an equivalence as desired.  $\square$

**Theorem 7.18** (Theorem 1.1). *Let  $p > 3$  be a prime. There is an isomorphism of graded abelian groups:*

$$T(2)_* \mathrm{K}(ku) \cong T(2)_* \mathrm{K}(\ell)[b]/(b^{p-1} + v_2),$$

where  $|b| = 2p + 2$ .

*Proof.* Due to [LMMT20, Purity Theorem] and the Dundas-Goodwillie-McCarthy theorem, we have

$$T(2)_* \mathrm{K}(ku) \cong T(2)_* \mathrm{TC}(ku_p) \quad \text{and} \quad T(2)_* \mathrm{K}(\ell) \cong T(2)_* \mathrm{TC}(\ell_p).$$

Therefore, it is sufficient to prove the corresponding statement:

$$T(2)_* \mathrm{TC}(ku_p) \cong T(2)_* \mathrm{TC}(\ell_p)[b]/(b^{p-1} + v_2),$$

at the level of topological cyclic homology. Since  $T(2)_* \mathrm{TC}(ku_p)$  is a  $p - 1$ -graded ring with a unit in weight 1 (Propositions 7.3 and 7.17), it is periodic in its weight direction. In other words, multiplication by  $b^i$  provides an isomorphism

$$\cdot b^i: T(2)_* \Sigma^{(2p+2)i} \mathrm{TC}(ku_p)_0 \xrightarrow{\cong} T(2)_* \mathrm{TC}(ku_p)_i$$

for each  $0 < i < p - 1$ . Furthermore,  $T(2)_* \mathrm{TC}(ku_p)_0 \cong T(2)_* \mathrm{TC}(\ell_p)$  due to Proposition 5.6. This proves the desired isomorphism.  $\square$

*Remark 7.19.* Blumberg, Gepner and Tabuada prove that there is a lax symmetric monoidal transformation from algebraic  $K$ -theory to topological cyclic homology [BGT14] given by the cyclotomic trace. On the other hand, we defined the lax symmetric monoidal structure on  $\mathrm{TC}(-)$  using cyclotomic spectra as in [NS18] whereas the older definition of topological cyclic homology is used in [BGT14]. On connective  $E_1$ -rings, the two definitions of topological cyclic homology provide the same spectrum [NS18]. From this, one obtains a map of spectra  $\mathrm{K}(-) \rightarrow \mathrm{TC}(-)$  on connective  $E_1$ -rings for  $\mathrm{TC}(-)$  as in Definition 5.1. To our knowledge, a lax symmetric monoidal comparison of the two definitions of topological cyclic homology is not currently available in the literature. Therefore, we do not assume the existence of a lax symmetric monoidal transformation  $\mathrm{K}(-) \rightarrow \mathrm{TC}(-)$  unless we explicitly state otherwise. Note that we only make this assumption in Remarks 7.20 and 7.22 and Theorem 7.23.

On the other hand, it is highly expected that the agreement of the two definitions of TC can be improved to that of lax symmetric monoidal functors and it should be possible to obtain a lax symmetric monoidal transformation  $K(-) \rightarrow \text{TC}(-)$  for TC as in Definition 5.1.

*Remark 7.20.* If we assume that the trace map  $K(-) \rightarrow \text{TC}(-)$  is lax symmetric monoidal (see Remark 7.19), then we have a  $T(2)$ -equivalence  $K(ku) \rightarrow \text{TC}(ku_p)$  of  $E_\infty$ -rings. This shows that  $b$  is a unit in  $T(2)_* K(ku)$ . Since  $b(b^{p-1} + v_2) = 0$  in  $T(2)_* K(ku)$  (Proposition 7.2), one obtains that  $b^{p-1} = -v_2$  in  $T(2)_* K(ku)$ . In particular, the isomorphism in Theorem 1.1 improves to an isomorphism of graded rings.

**7.3. Algebraic K-theory of the 2-periodic Morava K-theory.** At this point, Theorem 1.5 follows easily from our previous arguments.

**Theorem 7.21** (Theorem 1.5). *Let  $p > 3$  be a prime. There is an isomorphism of graded abelian groups:*

$$T(2)_* K(ku/p) \cong T(2)_* K(\ell/p) \otimes_{\mathbb{F}_p[v_2]} \mathbb{F}_p[b]$$

with  $|b| = 2p + 2$  and in the tensor product above, we take  $v_2 = -b^{p-1}$ .

*Proof.* As before, it is sufficient to prove the same identity at the level of topological cyclic homology, i.e. we need to show that

$$T(2)_* \text{TC}(ku/p) \cong T(2)_* \text{TC}(\ell/p) \otimes_{\mathbb{F}_p[v_2]} \mathbb{F}_p[b].$$

Recall from Construction 5.4 that  $\text{TC}(ku/p)$  is a module over  $\text{TC}(ku_p)$  in  $p - 1$ -graded spectra. By Proposition 5.7, we have

$$T(2)_* \text{TC}(ku/p)_0 \cong T(2)_* \text{TC}(\ell/p).$$

Furthermore, there is a unit  $b \in T(2)_* \text{TC}(ku_p)$  of weight 1, (Propositions 7.3 and 7.17). Therefore, multiplying by powers of  $b$  induces isomorphisms

$$\cdot b^i : T(2)_* \Sigma^{(2p+2)i} \text{TC}(ku/p)_0 \xrightarrow{\cong} T(2)_* \text{TC}(ku/p)_i$$

for each  $0 < i < p - 1$ . This provides the desired result.  $\square$

*Remark 7.22.* As in Remark 7.20, if we assume that the trace map provides a lax symmetric monoidal transformation  $K(-) \rightarrow \text{TC}(-)$  (see Remark 7.19), then the isomorphism in Theorem 1.5 improves to an isomorphism of  $\mathbb{F}_p[b]$ -modules.

Now we prove Theorem 7.23 verifying the the conjectural formula of Ausoni and Rognes [AR09, Section 3] that we stated in (1.6). For this, note that due to [LMMT20, Purity Theorem],  $T(2) \wedge K(\text{ff}(ku_p))$  and  $T(2) \wedge K(\text{ff}(\ell_p))$  are given by the cofibers of the transfer maps  $T(2) \wedge K(ku/p) \rightarrow T(2) \wedge K(ku_p)$  and  $T(2) \wedge K(\ell/p) \rightarrow T(2) \wedge K(\ell_p)$  respectively, see [AR09, Diagrams 3.1 and 3.10].

**Theorem 7.23.** *Let  $p > 3$  be a prime. Assume that the natural transformation  $K(-) \rightarrow \text{TC}(-)$  on connective  $E_1$ -rings given by the cyclotomic trace is lax symmetric monoidal (see Remark 7.19). There is an isomorphism of  $\mathbb{F}_p[b]$ -modules:*

$$(7.24) \quad T(2)_* K(\text{ff}(ku_p)) \cong T(2)_* K(\text{ff}(\ell_p)) \otimes_{\mathbb{F}_p[v_2]} \mathbb{F}_p[b]$$

where  $v_2 = -b^{p-1}$  and  $|b| = 2p + 2$ .

*Proof.* The trace map  $K(-) \rightarrow TC(-)$  is a  $T(2)$ -equivalence for  $ku_p$ ,  $ku/p$ ,  $\ell_p$  and  $\ell/p$  [LMMT20, Corollary E]. Therefore, we obtain that  $T(2) \wedge K(ku_p)$  is a monoid in the homotopy category of  $p-1$ -graded spectra and  $T(2) \wedge K(ku/p)$  is a left module over  $T(2) \wedge K(ku_p)$  in the homotopy category of  $p-1$ -graded spectra.

Let

$$\tau: T(2) \wedge K(ku/p) \rightarrow T(2) \wedge K(ku_p)$$

denote the map induced by transfer along  $ku_p \rightarrow ku/p$ . Since

$$T(2) \wedge K(ku/p) \simeq \bigvee_{0 \leq i < p-1} T(2) \wedge K(ku/p)_i,$$

it is sufficient to understand the restriction of  $\tau$  to  $T(2) \wedge K(ku/p)_i$  for each  $i$ . For  $i = 0$ , we consider the commuting diagram of  $E_1$ -rings:

$$\begin{array}{ccc} \ell_p & \longrightarrow & \ell/p \\ \downarrow & & \downarrow \\ ku_p & \longrightarrow & ku/p. \end{array}$$

We obtain a commuting diagram of spectra:

$$(7.25) \quad \begin{array}{ccc} K(\ell/p) & \longrightarrow & K(\ell_p) \\ \downarrow & & \downarrow \\ K(ku/p) & \longrightarrow & K(ku_p) \end{array}$$

by using the following equivalence of the corresponding functors induced at the level of module categories:

$$ku_p \wedge_{\ell_p} - \simeq (ku_p \wedge_{\ell_p} \ell/p) \wedge_{\ell/p} - \simeq ku/p \wedge_{\ell/p} -.$$

Let  $\tau': T(2) \wedge K(\ell/p) \rightarrow T(2) \wedge K(\ell)$  denote the map induced by transfer along  $\ell \rightarrow \ell/p$ . Diagram (7.25) provides that the restriction of  $\tau$  to  $T(2) \wedge K(ku/p)_0$  is given by the following map.

$$\begin{aligned} T(2) \wedge K(ku/p)_0 &\simeq T(2) \wedge K(\ell/p) \xrightarrow{\tau'} \\ &T(2) \wedge K(\ell_p) \simeq T(2) \wedge K(ku_p)_0 \rightarrow T(2) \wedge K(ku_p) \end{aligned}$$

Here, the last map is the inclusion of the weight 0-component and the equivalences above are provided by Propositions 5.6 and 5.7. Let  $\tau_0$  denote the map

$$\tau_0: T(2) \wedge K(ku/p)_0 \rightarrow T(2) \wedge K(ku_p)_0$$

in the composite above.

Let  $0 < i < p-1$ . To describe the restriction of  $\tau$  to  $T(2) \wedge K(ku/p)_i$ , we use the fact that  $\tau$  is a map of  $T(2) \wedge K(ku_p)$ -modules in the stable homotopy category, see [AR09, Section 3]. By Propositions 7.3 and 7.17, there is a unit  $b^i \in T(2)_* K(ku_p)$  of weight  $i$ . Omitting the suspension functor, let  $m_1: T(2) \wedge K(ku/p)_0 \rightarrow T(2) \wedge K(ku/p)_i$  and  $m_2: T(2) \wedge K(ku_p)_0 \rightarrow T(2) \wedge K(ku_p)_i$  denote the equivalences given by multiplication with  $b^i \in T(2)_* K(ku_p)$ . Abusing notation, let  $m_1$  and  $m_2$  also denote the respective

endomorphisms of  $T(2) \wedge K(ku/p)$  and  $T(2) \wedge K(ku_p)$ . We have the following up-to-homotopy commuting diagram of spectra.

$$\begin{array}{ccccc}
 & & & T(2) \wedge K(ku_p)_0 & \\
 & & \nearrow \tau_0 & \downarrow & \searrow m_2 \\
 T(2) \wedge K(ku/p)_0 & \longrightarrow & T(2) \wedge K(ku/p) & \xrightarrow{\tau} & T(2) \wedge K(ku_p) & \xrightarrow{\simeq} & T(2) \wedge K(ku_p)_i \\
 \simeq \downarrow m_1 & & \downarrow m_1 & & \downarrow m_2 & & \swarrow \\
 T(2) \wedge K(ku/p)_i & \longrightarrow & T(2) \wedge K(ku/p) & \xrightarrow{\tau} & T(2) \wedge K(ku_p) & & 
 \end{array}$$

Here, the unmarked arrows are the canonical inclusions. The bottom left hand square and the right hand diagram commute since  $T(2) \wedge K(ku/p)$  and  $T(2) \wedge K(ku_p)$  are modules over  $T(2) \wedge K(ku_p)$  in the homotopy category of  $p-1$ -graded spectra. The inner square commutes as  $\tau$  is a map of modules over  $T(2) \wedge K(ku_p)$  in the homotopy category of spectra. The top left diagram commutes due to our previous identification of  $\tau_0$ .

The commuting diagram above shows that the restriction of  $\tau$  to  $T(2) \wedge K(ku/p)_i$  is given by the composite:

$$\begin{aligned}
 T(2) \wedge K(ku/p)_i &\xrightarrow[\simeq]{m_1^{-1}} T(2) \wedge K(ku/p)_0 \xrightarrow{\tau_0} \\
 &T(2) \wedge K(ku_p)_0 \xrightarrow[\simeq]{m_2} T(2) \wedge K(ku_p)_i \rightarrow T(2) \wedge K(ku_p).
 \end{aligned}$$

Letting  $\tau_i: T(2) \wedge K(ku/p)_i \rightarrow T(2) \wedge K(ku_p)_i$  be as in the composite above, we obtain that  $\tau$  is given by

$$\tau \simeq \bigvee_{0 \leq i < p-1} \tau_i$$

and that each  $\tau_i$  is equivalent to  $\tau_0$  up to a suspension. Since  $\tau_0$  is equivalent to  $\tau'$ , this provides the desired splitting of the cofiber of  $\tau$  as a coproduct of shifted copies of  $T(2) \wedge K(ff(\ell_p))$ . This proves (7.24) as an isomorphism of abelian groups. Due to the argument above, the resulting cofactors of  $T(2) \wedge K(ku/p)$  are connected through multiplication by  $b$  and this shows that (7.24) is an isomorphism of  $\mathbb{F}_p[b]$ -modules.  $\square$

## APPENDIX A. GRADED CYCLOTOMIC SPECTRA

For this section, let  $m$  be a positive integer such that  $p = 1$  in  $\mathbb{Z}/m$ . Here, our goal is to construct a symmetric monoidal functor

$$\text{THH}: \text{Alg}_{E_1}(\text{Fun}(\mathbb{Z}/m, \text{Sp})) \rightarrow \text{Fun}(\mathbb{Z}/m, \text{CycSp})$$

and show that the resulting diagram:

$$(5.2) \quad \begin{array}{ccccc}
 \text{Alg}_{E_1}(\text{Fun}(\mathbb{Z}/m, \text{Sp})) & \xrightarrow{\text{THH}} & \text{Fun}(\mathbb{Z}/m, \text{CycSp}) & \xrightarrow{\text{TC}} & \text{Fun}(\mathbb{Z}/m, \text{Sp}) \\
 \downarrow & & \downarrow D & & \downarrow D' \\
 \text{Alg}_{E_1}(\text{Sp}) & \xrightarrow{\text{THH}} & \text{CycSp} & \xrightarrow{\text{TC}} & \text{Sp}
 \end{array}$$



of lax symmetric monoidal functors commutes. Note that this diagram is also stated as Diagram (5.2) in Section 5. Recall that the vertical functors above are given by left Kan extending through  $\mathbb{Z}/m \rightarrow 0$ , i.e. they provide the corresponding underlying objects. Furthermore, the upper right hand horizontal arrow TC is given by levelwise application of TC:  $\text{CycSp} \rightarrow \text{Sp}$ .

We first prove the following proposition which states that the right hand square in Diagram (5.2) commutes.

**Proposition A.1.** *Let  $m > 0$  such that  $p = 1$  in  $\mathbb{Z}/m$ . Then the following diagram:*

$$\begin{array}{ccc} \text{Fun}(\mathbb{Z}/m, \text{CycSp}) & \xrightarrow{\text{TC}} & \text{Fun}(\mathbb{Z}/m, \text{Sp}) \\ \downarrow D & & \downarrow D' \\ \text{CycSp} & \xrightarrow{\text{TC}} & \text{Sp}, \end{array}$$

*of lax symmetric monoidal functors commutes. In other words, the right hand side of Diagram (5.2), commutes.*

*Proof.* Let  $R$  and  $R'$  denote the right adjoints of  $D$  and  $D'$  respectively. The functors  $R$  and  $R'$  are given by restriction along  $\mathbb{Z}/m \rightarrow 0$ . Here, we denote the top horizontal arrow by  $\text{TC}^{\text{level}}$  to distinguish it from the bottom horizontal arrow TC.

First, we show that there is a lax symmetric monoidal natural transformation

$$\phi: D' \text{TC}^{\text{level}} \rightarrow \text{TC} D,$$

later, we complete the proof by showing that  $\phi$  is an equivalence. By adjunction, it is sufficient to obtain a lax symmetric monoidal transformation

$$\text{TC}^{\text{level}} \rightarrow R' \text{TC} D.$$

Since precomposition followed by postcomposition agrees with postcomposition followed by precomposition, we have  $R' \text{TC} \simeq \text{TC}^{\text{level}} R$ . Therefore, it is sufficient to obtain a lax symmetric monoidal transformation

$$\text{TC}^{\text{level}} \rightarrow \text{TC}^{\text{level}} RD$$

and this is given by the unit of the adjunction  $D \dashv R$ . This provides  $\phi$  above. Indeed,  $\phi$  is given by the canonical map

$$\phi_X: \bigvee_{i \in \mathbb{Z}/m} \text{TC}(X_i) \rightarrow \text{TC}(\bigvee_{i \in \mathbb{Z}/m} X_i).$$

Since  $m \neq 0$ , the coproducts above are finite and due to [NS18, Corollary II.1.7], colimits of cyclotomic spectra agree with those of the underlying spectra. Furthermore, both  $\text{CycSp}$  and  $\text{Sp}$  are stable and therefore these coproducts are the corresponding products. As TC commutes with finite products, we obtain that  $\phi$  is an equivalence as desired.  $\square$

What remains is to construct the left hand square in Diagram 5.2 and show that it commutes.

For the rest, we let  $\text{Gr}_m(\mathcal{C})$  denote  $\text{Fun}(\mathbb{Z}/m, \mathcal{C})$  for a given presentably symmetric monoidal  $\infty$ -category  $\mathcal{C}$ . Abusing notation, let

$$(-)^{tC_p}: \text{Gr}_m(\text{Sp}^{BS^1}) \rightarrow \text{Gr}_m(\text{Sp}^{BS^1})$$

also denote the lax symmetric monoidal functor given by levelwise application of  $(-)^{tC_p}$ . Slightly diverting from the notation of [NS18], we let  $\text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p})$  denote the  $\infty$ -category defined as the lax equalizer of the identity endofunctor and the endofunctor  $(-)^{tC_p}$  on  $\text{Gr}_m(\text{Sp}^{BS^1})$  in the sense of [NS18, Definition II.1.4]. The  $\infty$ -category  $\text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p})$  is defined via the following pullback square.

$$\begin{array}{ccc} \text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p}) & \longrightarrow & \text{Gr}_m(\text{Sp}^{BS^1})^{\Delta^1} \\ \downarrow & & \downarrow \text{ev}_0 \times \text{ev}_1 \\ \text{Gr}_m(\text{Sp}^{BS^1}) & \xrightarrow{(id, (-)^{tC_p})} & \text{Gr}_m(\text{Sp}^{BS^1}) \times \text{Gr}_m(\text{Sp}^{BS^1}) \end{array}$$

In particular, the objects of this pullback  $\infty$ -category are given by an object of  $E \in \text{Gr}_m(\text{Sp}^{BS^1})$  and a morphism  $E \rightarrow E^{tC_p}$ .

In [AMMN22, Appendix A], the authors construct THH as a functor on graded ring spectra and show that it fits into the following commuting diagram of symmetric monoidal functors [AMMN22, Proposition A.5 and Corollary A.15].

$$(A.2) \quad \begin{array}{ccc} \text{Alg}_{E_1}(\text{Gr}_m(\text{Sp})) & \xrightarrow{\text{THH}} & \text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p}) \\ \downarrow & & \downarrow D'' \\ \text{Alg}_{E_1}(\text{Sp}) & \xrightarrow{\text{THH}} & \text{CycSp} \end{array}$$

As usual, the vertical arrows are induced by left Kan extension through  $\mathbb{Z}/m \rightarrow 0$ . Here, we omit the functor  $L_p$  (given by left Kan extension through  $\cdot p: \mathbb{Z}/m \rightarrow \mathbb{Z}/m$ ) since  $L_p$  is the identity functor whenever  $p = 1$  in  $\mathbb{Z}/m$ .

**Construction A.3.** Let  $\text{Alg}_{E_1}(\text{Gr}_m(\text{Sp})) \rightarrow \text{Gr}_m(\text{CycSp})$  be the composite of the upper horizontal arrow in Diagram (A.2) with the equivalence provided in the proposition below. This provides the left hand upper horizontal arrow in Diagram (5.2) and the commuting diagram in the following proposition, together with Diagram (A.2) ensures that the left hand square in Diagram (5.2) commutes.

What remains is to prove the following proposition.

**Proposition A.4.** *Let  $m > 0$  such that  $p = 1$  in  $\mathbb{Z}/m$ . There is an equivalence of symmetric monoidal  $\infty$ -categories:*

$$(A.5) \quad \text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p}) \xrightarrow{\cong} \text{Gr}_m(\text{CycSp})$$

such that the following diagram commutes.

$$\begin{array}{ccc} \text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p}) & \xrightarrow{\cong} & \text{Gr}_m(\text{CycSp}) \\ \downarrow D'' & \swarrow D & \\ \text{CycSp} & & \end{array}$$

*Proof.* We construct an equivalence in the opposite direction. Due to [NS18, Construction IV.2.1], giving a symmetric monoidal functor

$$T: \text{Gr}_m(\text{CycSp}) \rightarrow \text{Leq}(\text{Gr}_m(\text{Sp}^{BS^1}), (-)^{tC_p})$$

is equivalent to giving a symmetric monoidal functor

$$F: \mathrm{Gr}_m(\mathrm{CycSp}) \rightarrow \mathrm{Gr}_m(\mathrm{Sp}^{BS^1})$$

together with a lax symmetric monoidal transformation  $F \rightarrow (-)^{tC_p} \circ F$ .

Applying [NS18, Construction IV.2.1] to the identity functor of  $\mathrm{CycSp}$ , one obtains a symmetric monoidal functor:

$$H: \mathrm{CycSp} \rightarrow \mathrm{Sp}^{BS^1},$$

together with a lax symmetric monoidal transformation  $H \rightarrow (-)^{tC_p} \circ H$ . By [Nik16, Corollary 3.7], this provides the desired symmetric monoidal functor  $F$  above. Furthermore, the lax symmetric monoidal transformation  $H \rightarrow (-)^{tC_p} \circ H$  applied to the following lemma provides the desired lax symmetric monoidal transformation  $F \rightarrow (-)^{tC_p} \circ F$ . This natural transformation and  $F$  provides the symmetric monoidal functor  $T$  above. Since  $\mathrm{Fun}(\mathbb{Z}/m, -)$  commutes with limits, it commutes with the pullback square defining  $\mathrm{CycSp}$  as a lax equalizer; therefore,  $T$  is an equivalence as desired. The functor claimed in (A.5) is now given by  $T^{-1}$ .

For the second statement in the proposition, it is sufficient to show that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Gr}_m(\mathrm{CycSp}) & \xrightarrow{T} & \mathrm{Leq}(\mathrm{Gr}_m(\mathrm{Sp}^{BS^1}), (-)^{tC_p}) \\ \downarrow D & & \swarrow D'' \\ \mathrm{CycSp} & & \end{array}$$

Let  $R$  and  $R''$  denote the right adjoints of  $D$  and  $D''$  respectively. These are given by the corresponding restriction functors along  $\mathbb{Z}/m \rightarrow 0$ . Since  $T$  is an equivalence,  $T^{-1}R''$  is a right adjoint to  $D''T$  and therefore, it is sufficient to show that  $R \simeq T^{-1}R''$ , i.e. the right adjoints of  $D$  and  $D''T$  agree. For this, it is sufficient to show that  $TR \simeq R''$ . This follows by the fact that the functor  $F$  and the lax transformation  $F \rightarrow (-)^{tC_p} \circ F$  are defined levelwise.  $\square$

**Lemma A.6.** *Let  $\eta: T \rightarrow S$  be a lax symmetric monoidal transformation of lax symmetric monoidal functors between presentably symmetric monoidal  $\infty$ -categories  $\mathcal{C}$  and  $\mathcal{D}$ . Then applying  $\eta$  levelwise induces a lax symmetric monoidal transformation between the induced lax symmetric monoidal functors from  $\mathrm{Gr}_m(\mathcal{C})$  to  $\mathrm{Gr}_m(\mathcal{D})$ .*

*Proof.* We follow closely [Nik16, Section 3]. Since the  $\infty$ -category of lax symmetric monoidal functors is a full subcategory of the  $\infty$ -category of functors over  $\mathrm{NFin}_*$ , it is sufficient to show that  $\eta$  provides a map  $\Delta^1 \rightarrow \mathrm{Map}_{\mathrm{NFin}_*}(\mathrm{Gr}_m(\mathcal{C})^\otimes, \mathrm{Gr}_m(\mathcal{D})^\otimes)$  of simplicial sets where the vertices of  $\Delta^1$  correspond to the lax symmetric monoidal functors induced by  $T$  and  $S$ . Using the universal property defining  $\mathrm{hom}_{\mathrm{NFin}_*}(-, -)$  in [Nik16, Section 3], one obtains the second map below:

$$\Delta^1 \rightarrow \mathrm{Map}_{\mathrm{NFin}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \rightarrow \mathrm{Map}_{\mathrm{NFin}_*}(\mathrm{hom}_{\mathrm{NFin}_*}(\mathbb{Z}/m^\otimes, \mathcal{C}^\otimes), \mathrm{hom}_{\mathrm{NFin}_*}(\mathbb{Z}/m^\otimes, \mathcal{D}^\otimes)),$$

where the first map represents  $\eta$ . Using the definition of  $\mathrm{Gr}_m(-)$  as a full simplicial subset of  $\mathrm{hom}_{\mathrm{NFin}_*}(\mathbb{Z}/m^\otimes, -)$  and [Nik16, Corollary 3.7], we deduce that the 1-simplex in the composite above lies in  $\mathrm{Map}_{\mathrm{NFin}_*}(\mathrm{Gr}_m(\mathcal{C})^\otimes, \mathrm{Gr}_m(\mathcal{D})^\otimes)$  with vertices corresponding to the lax symmetric monoidal functors induced by  $T$  and  $S$  as desired.  $\square$

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