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# Perron's capacity of random sets

Anthony Gauvan

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## Abstract

We answer in a probabilistic setting two questions raised by Stokolos in private communication. Precisely, given a sequence of random variables  $\{X_k : k \geq 1\}$  uniformly distributed in  $(0, 1)$  and independent, we consider the following random sets of directions

$$\Omega_{\text{rand,lin}} := \left\{ \frac{\pi X_k}{k} : k \geq 1 \right\}$$

and

$$\Omega_{\text{rand,lac}} := \left\{ \frac{\pi X_k}{2^k} : k \geq 1 \right\}.$$

We prove that almost surely the directional maximal operators associated to those sets of directions are not bounded on  $L^p(\mathbb{R}^2)$  for any  $1 < p < \infty$ .

We denote by  $\mathcal{R}$  the collection of all rectangles in the plane ; if  $R$  belongs to  $\mathcal{R}$ , we denote by  $\omega_R \in (0, \pi)$  the angle that its longest side makes with the  $Oy$ -axis. Without loss of generality, we will always suppose that we have actually  $0 \leq \omega_R \leq \frac{\pi}{2}$ .

## 1 Introduction

Given any set of directions  $\Omega \subset \mathbb{S}^1$ , one can define the directional family of rectangle  $\mathcal{R}_\Omega$  as

$$\mathcal{R}_\Omega := \{R \in \mathcal{R} : \omega_R \in \Omega\}$$

and then consider the *directional maximal operator*  $M_\Omega$  defined for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  locally integrable and  $x \in \mathbb{R}^2$  as

$$M_\Omega f(x) := \sup_{x \in R \in \mathcal{R}_\Omega} \frac{1}{|R|} \int_R |f|.$$

The boundedness property of the operator  $M_\Omega$  is deeply related to the geometric structure of the set of directions  $\Omega$ . For example, in the case where  $\Omega = \mathbb{S}^1$ , the following obstruction of the Euclidean plane (which is also true in higher dimension) allows us to completely describe the boundedness property of the operator  $M_{\mathbb{S}^1}$ .

**Theorem 1** (Kakeya blow with  $\mathcal{R}$ ). *Given any large constant  $A \gg 1$ , there exists a finite family of rectangles  $\{R_i : i \in I\} \subset \mathcal{R}$  such that we have*

$$\left| \bigcup_{i \in I} TR_i \right| \geq A \left| \bigcup_{i \in I} R_i \right|.$$

Here, we have denoted by  $TR$  the rectangle  $R$  translated along its longest side by its own length

The reader can find a proof of this Theorem in [11]: it follows that given any large constant  $A \gg 1$ , there exists a bounded set  $E$  satisfying the following estimate

$$\left| \left\{ M_{\mathbb{S}^1} \mathbb{1}_E \geq \frac{1}{2} \right\} \right| \geq A |E|.$$

It suffices to set  $E = \cup_{i \in I} R_i$  and to observe that we have the following inclusion

$$\bigcup_{i \in I} TR_i \subset \left\{ M_{\mathbb{S}^1} \mathbb{1}_E \geq \frac{1}{2} \right\}.$$

The previous estimate easily implies the following.

**Theorem 2.** *The operator  $M_{\mathbb{S}^1}$  is not bounded on  $L^p$  for any  $p < \infty$ .*

Far from being exotic, Theorem 1 has deep implications in harmonic analysis: for example, it is a central part of Fefferman's work in [11] where he disproves the famous *Ball multiplier* conjecture. A natural question is the following: given a set of directions  $\Omega$ , is it possible to make a Kakeya blow only with the directional family  $\mathcal{R}_\Omega$ ? This question have been investigated by different analyst among which [2], [6], [8], [9] or [7] to cite a few. In [3], Bateman answered this question as he classified the  $L^p(\mathbb{R}^2)$  behavior of those operators according to the *geometry* of the set  $\Omega$ . Precisely, he proved that the notion of *finitely lacunary* for a set of directions were the correct one to consider.

**Theorem 3** (Bateman). *We have the following alternative:*

- *if  $\Omega$  is finitely lacunary then  $M_\Omega$  is bounded on  $L^p$  for any  $p > 1$ .*
- *if  $\Omega$  is not finitely lacunary then it is possible to make a Kakeya blow with the family  $\mathcal{R}_\Omega$ . In particular, the operator  $M_\Omega$  is not bounded on  $L^p$  for any  $p < \infty$ .*

Let us define the notion of *finite lacunarity* following a nice presentation made by Kroc and Pramanik [10]: we start by defining the notion of *lacunary sequence* and then the notion of *lacunary set of finite order*. We say that a sequence of real numbers  $L = \{\ell_k : k \geq 1\}$  is a lacunary sequence converging to  $\ell \in \mathbb{R}$  when there exists  $0 < \lambda < 1$  such that

$$|\ell - \ell_{k+1}| \leq \lambda |\ell - \ell_k|$$

for any  $k$ . For example the sequences  $\{\frac{1}{2^k} : k \geq 2\}$  and  $\{\frac{1}{k!} : k \geq 4\}$  are lacunary. We define now by induction the notion of lacunary set of finite order.

**Definition 4** (Lacunary set of finite order). *A lacunary set of order 0 in  $\mathbb{R}$  is a set which is either empty or a singleton. Recursively, for  $N \in \mathbb{N}^*$ , we say that a set  $\Omega$  included in  $\mathbb{R}$  is a lacunary set of order at most  $N + 1$  — and write  $\Omega \in \Lambda(N + 1)$  — wether there exists a lacunary sequence  $L$  with the following properties : for any  $a, b \in L$  such that  $a < b$  and  $(a, b) \cap L = \emptyset$ , the set  $\Omega \cap (a, b)$  is a lacunary set of order at most  $N$  i.e.  $\Omega \cap (a, b) \in \Lambda(N)$ .*

For example the set

$$\Omega := \left\{ \frac{\pi}{2^k} + \frac{\pi}{4^l} : k, l \in \mathbb{N}, l \leq k \right\}$$

is a lacunary set of order 2. In this case, observe that the set  $\Omega$  cannot be re-written as a monotone sequence, since it has several points of accumulation. We can finally give a definition of a finitely lacunary set.

**Definition 5** (Finitely lacunary set). *A set  $\Omega$  in  $[0, \pi)$  is said to be finitely lacunary if there exists a finite number of set  $\Omega_1, \dots, \Omega_M$  which are lacunary of finite order such that*

$$\Omega \subset \bigcup_{k \leq M} \Omega_k.$$

## 2 Can we apply Bateman's Theorem ?

A classic example of set which is known to be not finitely lacunary is the set

$$\Omega_{\text{lin}} = \left\{ \frac{\pi}{n} : n \in \mathbb{N}^* \right\}.$$

Indeed, the classic construction of *Perron trees* shows that it is possible to make a Kakeya blow with the family  $\mathcal{R}_{\Omega_{\text{lin}}}$ : hence an application of Bateman's Theorem implies that  $\Omega_{\text{lin}}$  is not finitely lacunary. The second classic example of set which is known to be finitely lacunary is the set

$$\Omega_{\text{lac}} = \left\{ \frac{\pi}{2^n} : n \in \mathbb{N}^* \right\}.$$

One can see that the set  $\Omega_{\text{lac}}$  is finitely lacunary by definition and it was in [9] that Nagel, Stein and Waigner proved that the maximal operator  $M_{\Omega_{\text{lac}}}$  is bounded on  $L^p(\mathbb{R}^2)$  for any  $1 < p < \infty$ ; their proof relies on Fourier analysis. *Also, let us say that the comprehension of the sets  $\Omega_{\text{lin}}$  and  $\Omega_{\text{lac}}$  is important because they are the most simple (and smallest) cases of infinite sets which yield maximal operator having different boundedness properties.*

However, even if Bateman's Theorem is extremely satisfying, it appears to be difficult to decide if a given set  $\Omega$  is finitely lacunary or not. The most striking example was raised by Stokolos : at the present time, it is not known if the set

$$\Omega_{\text{sin,lac}} := \left\{ \frac{\pi \sin(n)}{2^n} : n \in \mathbb{N}^* \right\}$$

is finitely lacunary or not. The main problem of this set of directions is that we have a very poor control on the deterministic sequence  $\{\sin(n) : n \geq 1\}$  and that initially, the set  $\Omega_{\text{lac}}$  is finitely lacunary: hence, the perturbations are quite difficult to handle. In [1], with D’Aniello and Moonens, we were able to show that the following set

$$\Omega_{\text{sin,lin}} := \left\{ \frac{\pi \sin(n)}{n} : n \in \mathbb{N}^* \right\}$$

is not finitely lacunary (this was also a set considered by Stokolos). More precisely, we studied the maximal operator  $M_{\Omega_{\text{sin,lin}}}$  associated and, improving on concrete techniques, we proved that this operator is not bounded on  $L^p(\mathbb{R}^2)$  for any  $1 < p < \infty$ : the heart of the method relied on the introduction of the *Perron’s capacity* of a set of directions. We need some notations to recall our results: given an infinite set of directions  $\Omega \subset \mathbb{S}^1$  whose only point of accumulation is 0 and we denote, for notational convenience, by  $\Omega^{-1}$  the set  $\{\frac{\pi}{u} : u \in \Omega\}$ , that is

$$\Omega^{-1} := \frac{\pi}{\Omega}$$

and we order  $\Omega^{-1}$  as a strictly increasing sequence  $\{u_k : k \in \mathbb{N}^*\}$ . With those notations, we define the *Perron’s factor* of  $\Omega$  as

$$G(\Omega) := \sup_{\substack{k \geq 1 \\ l \leq k}} \left( \frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right).$$

In [7], Hare and Ronning proved the following Theorem.

**Theorem 6** (Hare and Ronning). *If we have  $G(\Omega) < \infty$  then it is possible to make a Keakeya blow with the family  $\mathcal{R}_\Omega$ .*

It turns out that it is difficult to compute the Perron factor of the set

$$\Omega_{\text{sin,lin}} = \left\{ \frac{\pi \sin(n)}{n} : n \geq 1 \right\}$$

since the oscillation of the cosine prevent us to obtain a good description of the increasing sequence  $\{u_k : k \in \mathbb{N}^*\}$  associated to  $\Omega_{\text{sin,lin}}$ . Based on a careful read of the proof of Theorem 6, for an arbitrary set of directions  $\Omega$  included in  $\mathbb{S}^1$ , we define its *Perron’s capacity* as

$$P(\Omega) := \liminf_{N \rightarrow \infty} \inf_{\substack{U \subset \Omega^{-1} \\ \#U=2^N}} G(\Omega) \in [2, \infty]$$

where as before

$$G(\Omega) = \sup_{\substack{k, l \geq 1 \\ k+2l \leq 2^N}} \left( \frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right)$$

if  $U = \{u_1 < \dots < u_{2^N}\}$ . In [1], we proved the following (in contrast with Hare and Ronning Theorem, we do not assume that the set  $\Omega$  is ordered):

**Theorem 7** (D’Aniello, G. and Moonens). *For any set of directions  $\Omega$ , if we have*

$$P(\Omega) < \infty$$

*then it is possible to make a Kekeya blow with the family  $\mathcal{R}_\Omega$ . In particular, for any  $p < \infty$ , one has  $\|M_\Omega\|_p = \infty$ .*

Loosely speaking, if  $P(\Omega) < \infty$  then the set  $\Omega$  contains arbitrary large set which are (more or less) uniformly distributed and this geometric pattern prevents the set  $\Omega$  to be finitely lacunary. The advantage of Theorem 7 is that it allows us to make *effective* computation. However, as mentioned earlier, the following case is still unsettled.

**Question 1.** *Is the following set of direction*

$$\Omega_{\text{sin,lac}} := \left\{ \frac{\pi \sin(n)}{2^n} : n \in \mathbb{N}^* \right\}$$

*finitely lacunary or not ?*

### 3 Results

Our result concerns random sets of directions which are meant to give a *generic* comprehension of the two classic examples  $\Omega_{\text{lin}}$  and  $\Omega_{\text{lac}}$  when they are randomly perturbed. Precisely we consider the following random sets of slopes

$$\Omega_{\text{rand,lin}} := \left\{ \frac{\pi X_k}{k} : k \geq 1 \right\}$$

and

$$\Omega_{\text{rand,lac}} := \left\{ \frac{\pi X_k}{2^k} : k \geq 1 \right\}$$

where  $\{X_k : k \geq 1\}$  are random variables uniformly distributed in  $(0, 1)$  and independent. To begin with, we prove the following Theorem.

**Theorem 8.** *The Perron’s capacities of  $\Omega_{\text{rand,lin}}$  is finite almost surely i.e. we have almoste surely*

$$P(\Omega_{\text{rand,lin}}) < \infty.$$

In some sense, Theorem 8 means that if a set  $\Omega$  presents structured patterns - like large uniformly distributed sequence - then a small perturbation of  $\Omega$  will still exhibit those patterns. The second result reads as follow.

**Theorem 9.** *The Perron’s capacities of  $\Omega_{\text{rand,lac}}$  is finite almost surely i.e. we have almoste surely*

$$P(\Omega_{\text{rand,lac}}) < \infty.$$

The proof of Theorems 8 and 9 relies on the possibility to compute effectively the Perron’s capacity of the random sets  $\Omega_{\text{rand,lin}}$  and  $\Omega_{\text{rand,lac}}$ .

## 4 Proof of Theorem 8

We wish to prove that the Perron's capacity of  $\Omega_{\text{rand,lin}}$  is finite almost surely. We are simply going to prove that *the set  $\Omega_{\text{rand,lin}}^{-1}$  contains small perturbation of arbitrarily long homogeneous sets*. We say that a set  $H$  of the form

$$H := H_{a,N} = \{ka : 1 \leq k \leq 2^N\}$$

for some integer  $a \in \mathbb{N}^*$  is an *homogeneous set*. The following claim is easy.

**Claim 1.** *For any  $a, N \in \mathbb{N}$ , one has  $G(H_{a,N}) = 2$ .*

We wish to perturb a little an homogeneous set  $H$  into a set  $H'$  such that the Perron's factor of  $H'$  is still controlled. Precisely, fix any  $a, N \in \mathbb{N}^*$  and let  $\epsilon$  be an arbitrary function

$$\epsilon : H_{a,N} \rightarrow (0, \infty).$$

Define then the set  $H_{a,N}(\epsilon)$  as

$$H' := H_{a,N}(\epsilon) := \{(1 + \epsilon(l))l : l \in H_{a,N}\}.$$

If the perturbation  $\epsilon$  is small enough compared to the integer  $N$ , one can control uniformly  $G(H')$ .

**Proposition 1.** *With the previous notations, if we have*

$$2^N \|\epsilon\|_\infty \leq \frac{1}{2}$$

*then we have  $G(H_{a,N}(\epsilon)) < 6$ .*

We are now ready to prove Theorem 8. We fix a large integer  $N \in \mathbb{N}$  and consider the following set of indices

$$E_N := \{k \in \mathbb{N} : |X_k - 1| \leq 2^{-N}\}.$$

In other words, an integer  $k$  belongs to  $E_N$  when  $X_k$  is close to 1 with precision  $2^{-N}$ . We claim that this random set  $E_N$  contains almost surely large (with at least  $2^N$  points) homogeneous sequences.

**Claim 2.** *For any  $N \geq 1$ , the set  $E_N$  contains an homogeneous set of cardinal  $2^N$  almost surely.*

*Proof.* Observe that for any  $a \in \mathbb{N}^*$ , the following probability  $\mathbb{P}(H_{a,N} \subset E_N)$  is independent of  $a$ : indeed since the random variables  $\{X_k : k \geq 1\}$  are independent and uniformly distributed, we have

$$\mathbb{P}(H_{a,N} \subset E_N) = \prod_{k \in H_{a,N}} \mathbb{P}(|X_k - 1| \leq 2^{-N}) = \frac{1}{2^{N2^N}}.$$

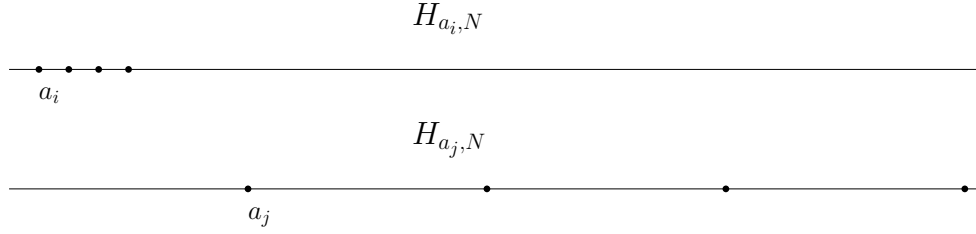


Figure 1: We choose the sequence  $\{a_i : i \geq 1\}$  such that  $H_{a_i, N} \cap H_{a_j, N} = \emptyset$  for any  $i \neq j$ .

Hence we fix a sequence  $\{a_i\}_{i \geq 1}$  satisfying for any  $i \neq j$

$$H_{a_i, N} \cap H_{a_j, N} = \emptyset.$$

For example, setting  $a_i = 2^{2N(i+1)}$  works since we have  $a_i 2^N < a_{i+1}$  for any  $i \geq 1$ . In particular this means that the events

$$\{(H_{a_i, N} \subset E_N) : i \geq 1\}$$

are independent and since we have

$$\sum_{i \geq 1} \mathbb{P}(H_{a_i, N} \subset E_N) = \infty$$

an application of Borel-Cantelli lemma yields the conclusion.  $\square$

We can now conclude the proof : we define a perturbation  $\epsilon$  for any  $n \geq 1$  as

$$1 + \epsilon(n) = X_n^{-1}.$$

We fix a large integer  $N \gg 1$  and we know that almost surely there exists an integer  $a \in \mathbb{N}^*$  such that  $H_{a, N} \subset E_N$ . Observe now that by definition one has the following inclusion

$$H_{a, N}(\epsilon) \subset \Omega_{\text{rand, lin}}^{-1}.$$

However since  $H_{a, N} \subset E_N$  and that it is not difficult to see that we have

$$\|\epsilon|_{H_{a, N}}\|_{\infty} \lesssim 2^{-N}.$$

Indeed, for  $k \in H_{a, N} \subset E_N$ , we have

$$|1 + \epsilon(k)| = \left| \frac{1}{1 + (X_k - 1)} \right| \lesssim 1 + 2|X_k - 1| \lesssim 1 + 2^{-N}.$$

It follows that we have  $P(\Omega_{\text{rand, lin}}) < 6$  almost surely applying Proposition 1.



## 5 Proof of Theorem 9

We wish to prove that the Perron's capacity of  $\Omega_{\text{rand,lac}}$  is finite almost surely. Observe that if  $U$  is a set in  $\mathbb{R}$  who is well distributed then one can control its Perron's capacity.

**Claim 3.** *If we have  $\delta > 0$  and a set*

$$U = \{u_1 < \dots < u_{2^N}\}$$

*such that for any  $1 \leq i \leq 2^N - 1$  we have*

$$\delta \leq u_{i+1} - u_i \leq 3\delta$$

*then one has  $G(U) \lesssim 1$ .*

*Proof.* For any  $i, j$  such that  $i + j \leq 2^N - 1$ , one has  $u_{i+j} - u_i \simeq j\delta$ . Hence  $i, j$  such that  $i + 2j \leq 2^{N-1}$ , we have

$$\frac{u_{i+2j} - u_{i+j}}{u_{i+j} - u_i} + \frac{u_{i+j} - u_i}{u_{i+2j} - u_{i+j}} \simeq \frac{2j\delta}{j\delta} + \frac{j\delta}{2j\delta} \simeq 1.$$

Hence we obtain  $G(U) \lesssim 1$  as claimed  $\square$

We are going to prove the following proposition.

**Proposition 2.** *For any  $N \geq 1$ , there exists almost surely a scale  $\delta > 0$  and a set  $U \subset \Omega_{\text{rand,lac}}^{-1}$  such that*

$$U := \{u_1 < \dots < u_{2^{N-1}}\}$$

*and for any  $i \leq 2^{N-1}$  one has*

$$\delta < u_{i+1} - u_i < 3\delta.$$

Theorem 9 is a consequence of Claim 3 and Proposition 2: for any  $N$ , we can exhibit almost surely a set  $U \subset \Omega_{\text{rand,lac}}^{-1}$  of cardinal  $2^N$  such that  $G(U) \lesssim 1$  and so we obtain

$$P(\Omega_{\text{rand,lac}}) < \infty$$

as expected. The rest of the section is devoted to the proof of Proposition 2.

### Proof of Proposition 2

We consider the following dyadic intervals for  $d \in \mathbb{N}$

$$I_d := [2^d, 2^{d+1}].$$

We wish to obtain information on the distribution of the points of the set  $\Omega_{\text{rand,lac}}$  that may be in the interval  $I_d$ . We fix a large integer  $N \gg 1$  and we divide each dyadic interval  $I_d$  into  $2^N$  intervals of same length *i.e.* for any  $1 \leq l \leq 2^N$  we set

$$I_{d,l} = \left[ 2^d \left( 1 + \frac{l-1}{2^N} \right), 2^d \left( 1 + \frac{l}{2^N} \right) \right].$$

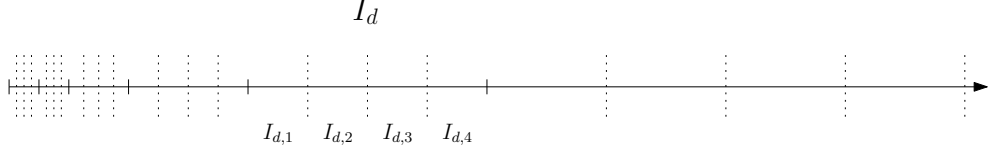


Figure 2: For  $N = 2$ , each intervals  $I_d$  is divided in  $2^N = 4$  equal parts.

**Claim 4.** For any  $d \geq 2^N + 1$  and any  $1 \leq l \leq 2^N$ , the probability

$$\mathbb{P}\left(\frac{2^{d-l}}{X_{d-l}} \in I_{d,l}\right) := p_{N,l}$$

is independent of  $d$ .

*Proof.* One has  $\frac{2^{d-l}}{X_{d-l}} \in I_{d,l}$  if and only if

$$2^l \left(1 + \frac{l-1}{2^N}\right) \leq \frac{1}{X_{d-l}} \leq 2^l \left(1 + \frac{l}{2^N}\right).$$

Since the random variable  $X_{d-l}$  is uniformly in  $(0, 1)$ , the probability  $\mathbb{P}\left(\frac{2^{d-l}}{X_{d-l}} \in I_{d,l}\right)$  is independent of  $d$  as claimed.  $\square$

We fix an extraction  $\{d_s : s \geq 1\}$  satisfying the following property

$$d_{s+1} - d_s > 2^N + 1$$

for any  $s \geq 1$ ; this growth condition will assure that the events we will consider are independent and we will be able to apply Borel-Cantelli lemma when needed. Thanks to Claim 4 and independence, one can see that for any  $s \geq 1$ , the following probability is independent of  $s$

$$\mathbb{P}\left(\bigcap_{l \leq 2^N} \left(\frac{2^{d_s-l}}{X_{d_s-l}} \in I_{d_s,l}\right)\right) = \prod_{l \leq 2^N} p_{N,l} := \eta_N$$

Now for any  $d \geq 1$ , we consider the following event  $A_{d,N}$  defined as

$$A_{d,N} := \bigcap_{l \leq 2^N} \left\{ \Omega_{\text{rand,lac}}^{-1} \cap I_{d,l} \neq \emptyset \right\}.$$

In other words, the event  $A_{d,N}$  occurs when the random set  $\Omega_{\text{rand,lac}}^{-1}$  fills each sub-intervals  $\{I_{d,l} : 1 \leq l \leq 2^N\}$  with at least one point. In particular, observe that we have

$$\bigcap_{l \leq 2^N} \left(\frac{2^{d-l}}{X_{d-l}} \in I_{d,l}\right) \subset A_{d,N}.$$

We claim that the union of those events

$$B_N := \bigcup_{d \geq 1} A_{d,N}$$

occurs almost surely.

**Claim 5.** *One has  $\mathbb{P}(B_N) = 1$ .*

*Proof.* Indeed we have

$$\sum_s \mathbb{P} \left( \bigcap_{l \leq 2^N} \left( \frac{2^{d_s-l}}{X_{d_s-l}} \in I_{d_s,l} \right) \right) = \sum_s \eta_N = \infty.$$

Using Borel-Cantelli lemma, one obtains

$$\mathbb{P} \left( \bigcup_{s \geq 1} A_{N,d_s} \right) = 1.$$

In particular, one has  $\mathbb{P}(B_N) = 1$ . □



Figure 3: The random set  $\Omega_{\text{rand,lac}} := \Omega_{r_2}$  contains almost surely uniformly distributed subset of arbitrarily large cardinal.

We can now prove Proposition 2 since we have

$$\mathbb{P} \left( \bigcap_{N \geq 1} B_N \right) = 1.$$

Precisely, for any  $N \geq 1$ , the event  $B_N$  occurs almost surely and this means that there exists a (for each  $N$ , we just need one) dyadic interval  $I_d$  such that

$$\Omega_{\text{rand,lac}}^{-1} \cap I_{d,l} \neq \emptyset$$

for any  $1 \leq l \leq 2^N$ . We let  $u_l$  be one point in  $\Omega_{\text{rand,lac}}^{-1} \cap I_{d,l}$  and we claim that the set

$$U := \{u_{2^l} : 1 \leq l \leq 2^{N-1}\}$$

satisfy the condition of Proposition 2 with  $\delta \simeq 2^{d-N}$ . In particular, we have  $G(U) \lesssim 1$  for  $U \subset \Omega_{\text{rand,lac}}^{-1}$  with arbitrary large cardinal. This yields almost surely

$$P(\Omega_{\text{rand,lac}}) \lesssim 1$$

which concludes the proof.

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