Perron’s capacity of random sets
Anthony Gauvan

To cite this version:
Anthony Gauvan. Perron’s capacity of random sets. 2023. hal-04103354

HAL Id: hal-04103354
https://hal.science/hal-04103354
Preprint submitted on 19 Dec 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Perron’s capacity of random sets

Anthony Gauvan

June 8, 2023

Abstract

We answer in a probabilistic setting two questions raised by Stokolos in private communication. Precisely, given a sequence of random variables \( \{X_k : k \geq 1\} \) uniformly distributed in \((0,1)\) and independent, we consider the following random sets of directions

\[
\Omega_{\text{rand,lin}} := \left\{ \frac{\pi X_k}{k} : k \geq 1 \right\}
\]

and

\[
\Omega_{\text{rand,lac}} := \left\{ \frac{\pi X_k}{2^k} : k \geq 1 \right\}.
\]

We prove that almost surely the directional maximal operators associated to those sets of directions are not bounded on \( L^p(\mathbb{R}^2) \) for any \( 1 < p < \infty \).

We denote by \( \mathcal{R} \) the collection of all rectangles in the plane; if \( R \) belongs to \( \mathcal{R} \), we denote by \( \omega_R \in (0, \pi) \) the angle that its longest side makes with the \( Oy \)-axis. Without loss of generality, we will always suppose that we have actually \( 0 \leq \omega_R \leq \frac{\pi}{2} \).

1 Introduction

Given any set of directions \( \Omega \subset S^1 \), one can define the directional family of rectangle \( \mathcal{R}_\Omega \) as

\[
\mathcal{R}_\Omega := \{ R \in \mathcal{R} : \omega_R \in \Omega \}
\]

and then consider the directional maximal operator \( M_\Omega \) defined for \( f : \mathbb{R}^2 \to \mathbb{R} \) locally integrable and \( x \in \mathbb{R}^2 \) as

\[
M_\Omega f(x) := \sup_{x \in R \in \mathcal{R}_\Omega} \frac{1}{|R|} \int_R |f|.
\]

The boundedness property of the operator \( M_\Omega \) is deeply related to the geometric structure of the set of directions \( \Omega \). For example, in the case where \( \Omega = S^1 \), the following obstruction of the Euclidean plane (which is also true in higher dimension) allows us to completely describe the boundedness property of the operator \( M_{S^1} \).
Theorem 1 (Kakeya blow with $\mathcal{R}$). Given any large constant $A \gg 1$, there exists a finite family of rectangles $\{R_i : i \in I\} \subset \mathcal{R}$ such that we have

$$\left| \bigcup_{i \in I} TR_i \right| \geq A \left| \bigcup_{i \in I} R_i \right|.$$ 

Here, we have denoted by $TR$ the rectangle $R$ translated along its longest side by its own length.

The reader can find a proof of this Theorem in [11]: it follows that given any large constant $A \gg 1$, there exists a bounded set $E$ satisfying the following estimate

$$\left| \left\{ M_{S_1} 1_E \geq \frac{1}{2} \right\} \right| \geq A |E|.$$ 

It suffices to set $E = \bigcup_{i \in I} R_i$ and to observe that we have the following inclusion

$$\bigcup_{i \in I} TR_i \subset \left\{ M_{S_1} 1_E \geq \frac{1}{2} \right\}.$$ 

The previous estimate easily implies the following.

Theorem 2. The operator $M_{S_1}$ is not bounded on $L^p$ for any $p < \infty$.

Far from being exotic, Theorem 1 has deep implications in harmonic analysis: for example, it is a central part of Fefferman’s work in [11] where he disproves the famous Ball multiplier conjecture. A natural question is the following: given a set of directions $\Omega$, is it possible to make a Kakeya blow only with the directional family $\mathcal{R}_\Omega$? This question have been investigated by different analyst among which [2], [6], [8], [9] or [7] to cite a few. In [3], Bateman answered this question as he classified the $L^p(\mathbb{R}^2)$ behavior of those operators according to the geometry of the set $\Omega$. Precisely, he proved that the notion of finitely lacunary for a set of directions were the correct one to consider.

Theorem 3 (Bateman). We have the following alternative:

- if $\Omega$ is finitely lacunary then $M_\Omega$ is bounded on $L^p$ for any $p > 1$.
- if $\Omega$ is not finitely lacunary then it is possible to make a Kakeya blow with the family $\mathcal{R}_\Omega$. In particular, the operator $M_\Omega$ is not bounded on $L^p$ for any $p < \infty$.

Let us define the notion of finite lacunarity following a nice presentation made by Kroc and Pramanik [10]: we start by defining the notion of lacunary sequence and then the notion of lacunary set of finite order. We say that a sequence of real numbers $L = \{\ell_k : k \geq 1\}$ is a lacunary sequence converging to $\ell \in \mathbb{R}$ when there exists $0 < \lambda < 1$ such that

$$|\ell - \ell_{k+1}| \leq \lambda|\ell - \ell_k|$$

for any $k$. For example the sequences $\{\frac{1}{2^k} : k \geq 1\}$ and $\{\frac{1}{k^2} : k \geq 2\}$ are lacunary. We define now by induction the notion of lacunary set of finite order.
**Definition 4** (Lacunary set of finite order). A lacunary set of order 0 in \( \mathbb{R} \) is a set which is either empty or a singleton. Recursively, for \( N \in \mathbb{N}^* \), we say that a set \( \Omega \) included in \( \mathbb{R} \) is a lacunary set of order at most \( N + 1 \) — and write \( \Omega \in \Lambda(N + 1) \) — whether there exists a lacunary sequence \( L \) with the following properties: for any \( a, b \in L \) such that \( a < b \) and \( (a, b) \cap L = \emptyset \), the set \( \Omega \cap (a, b) \) is a lacunary set of order at most \( N \) i.e. \( \Omega \cap (a, b) \in \Lambda(N) \).

For example the set
\[
\Omega := \left\{ \frac{\pi}{2^k} + \frac{\pi}{4^l} : k, l \in \mathbb{N}, l \leq k \right\}
\]
is a lacunary set of order 2. In this case, observe that the set \( \Omega \) cannot be re-written as a monotone sequence, since it has several points of accumulation.

We can finally give a definition of a finitely lacunary set.

**Definition 5** (Finitely lacunary set). A set \( \Omega \) in \([0, \pi)\) is said to be finitely lacunary if there exists a finite number of set \( \Omega_1, \ldots, \Omega_M \) which are lacunary of finite order such that
\[
\Omega \subset \bigcup_{k \leq M} \Omega_k.
\]

## 2 Can we apply Bateman’s Theorem?

A classic example of set which is known to be not finitely lacunary is the set
\[
\Omega_{\text{lin}} = \left\{ \frac{\pi}{n} : n \in \mathbb{N}^* \right\}.
\]

Indeed, the classic construction of Perron trees shows that it is possible to make a Kakeya blow with the family \( \mathcal{R}_{\Omega_{\text{lin}}} \); hence an application of Bateman’s Theorem implies that \( \Omega_{\text{lin}} \) is not finitely lacunary. The second classic example of set which is known to be finitely lacunary is the set
\[
\Omega_{\text{lac}} = \left\{ \frac{\pi}{2^n} : n \in \mathbb{N}^* \right\}.
\]

One can see that the set \( \Omega_{\text{lac}} \) is finitely lacunary by definition and it was in [9] that Nagel, Stein and Waigner proved that the maximal operator \( M_{\Omega_{\text{lac}}} \) is bounded on \( L^p(\mathbb{R}^2) \) for any \( 1 < p < \infty \); their proof relies on Fourier analysis.

Also, let us say that the comprehension of the sets \( \Omega_{\text{lin}} \) and \( \Omega_{\text{lac}} \) is important because they are the most simple (and smallest) cases of infinite sets which yield maximal operator having different boundedness properties.

However, even if Bateman’s Theorem is extremely satisfying, it appears to be difficult to decide if a given set \( \Omega \) is finitely lacunary or not. The most striking example was raised by Stokolos : at the present time, it is not known if the set
\[
\Omega_{\text{sin},\text{lac}} := \left\{ \frac{\pi \sin(n)}{2^n} : n \in \mathbb{N}^* \right\}
\]
is finitely lacunary or not. The main problem of this set of directions is that we have a very poor control on the deterministic sequence \( \{ \sin(n) : n \geq 1 \} \) and that initially, the set \( \Omega_{\text{lac}} \) is finitely lacunary: hence, the perturbations are quite difficult to handle. In [1], with D’Aniello and Moonens, we were able to show that the following set

\[
\Omega_{\text{sin,lin}} := \left\{ \frac{\pi \sin(n)}{n} : n \in \mathbb{N}^* \right\}
\]

is not finitely lacunary (this was also a set considered by Stokolos). More precisely, we studied the maximal operator \( M_{\Omega_{\text{sin,lin}}} \) associated and, improving on concrete techniques, we proved that this operator is not bounded on \( L^p(\mathbb{R}^2) \) for any \( 1 < p < \infty \); the heart of the method relied on the introduction of the Perron’s capacity of a set of directions. We need some notations to recall our results: given an infinite set of directions \( \Omega \subset S^1 \) whose only point of accumulation is 0 and we denote, for notational convenience, by \( \Omega^{-1} \) the set \( \left\{ \frac{\pi}{u} : u \in \Omega \right\} \), that is

\[
\Omega^{-1} := \frac{\pi}{\Omega}
\]

and we order \( \Omega^{-1} \) as a strictly increasing sequence \( \{ u_k : k \in \mathbb{N}^* \} \). With those notations, we define the Perron’s factor of \( \Omega \) as

\[
P(\Omega) := \lim \inf_{N \to \infty} \inf_{U \subset \Omega^{-1}} G(\Omega) \in [2, \infty]
\]

where as before

\[
G(\Omega) = \sup_{k,l \geq 1} \left( \frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right)
\]

if \( U = \{ u_1 < \cdots < u_{2^N} \} \). In [1], we proved the following (in contrast with Hare and Ronning Theorem, we do not assume that the set \( \Omega \) is ordered):
Theorem 7 (D’Aniello, G. and Moonens). For any set of directions $\Omega$, if we have
\[
P(\Omega) < \infty
\]
then it is possible to make a Kakeya blow with the family $R_{\Omega}$. In particular, for any $p < \infty$, one has $\|M_{\Omega}\|_p = \infty$.

Loosely speaking, if $P(\Omega) < \infty$ then the set $\Omega$ contains arbitrary large set which are (more or less) uniformly distributed and this geometric pattern prevents the set $\Omega$ to be finitely lacunary. The advantage of Theorem 7 is that it allows us to make effective computation. However, as mentioned earlier, the following case is still unsettled.

Question 1. Is the following set of direction
\[
\Omega_{\sin, lac} := \left\{ \frac{\pi \sin(n)}{2^n} : n \in \mathbb{N}^* \right\}
\]
finely lacunary or not?

3 Results

Our result concerns random sets of directions which are meant to give a generic comprehension of the two classic examples $\Omega_{\text{lin}}$ and $\Omega_{\text{lac}}$ when they are randomly perturbated. Precisely we consider the following random sets of slopes
\[
\Omega_{\text{rand}, \text{lin}} := \left\{ \frac{\pi X_k}{k} : k \geq 1 \right\}
\]
and
\[
\Omega_{\text{rand}, \text{lac}} := \left\{ \frac{\pi X_k}{2^k} : k \geq 1 \right\}
\]
where $\{X_k : k \geq 1\}$ are random variables uniformly distributed in $(0, 1)$ and independent. To begin with, we prove the following Theorem.

Theorem 8. The Perron’s capacities of $\Omega_{\text{rand}, \text{lin}}$ is finite almost surely i.e. we have almoste surely
\[
P(\Omega_{\text{rand}, \text{lin}}) < \infty.
\]

In some sense, Theorem 8 means that if a set $\Omega$ presents structured patterns - like large uniformly distributed sequence - then a small perturbation of $\Omega$ will still exhibit those patterns. The second result reads as follow.

Theorem 9. The Perron’s capacities of $\Omega_{\text{rand}, \text{lac}}$ is finite almost surely i.e. we have almoste surely
\[
P(\Omega_{\text{rand}, \text{lac}}) < \infty.
\]

The proof of Theorems 8 and 9 relies on the possibility to compute effectively the Perron’s capacity of the random sets $\Omega_{\text{rand}, \text{lin}}$ and $\Omega_{\text{rand}, \text{lac}}$. 
4 Proof of Theorem 8

We wish to prove that the Perron’s capacity of $\Omega_{\text{rand,lin}}$ is finite almost surely. We are simply going to prove that the set $\Omega_{\text{rand,lin}}^{-1}$ contains small perturbation of arbitrarily long homogeneous sets. We say that a set $H$ of the form

$$H := H_{a,N} = \{ka : 1 \leq k \leq 2^N\}$$

for some integer $a \in \mathbb{N}^*$ is a homogeneous set. The following claim is easy.

**Claim 1.** For any $a, N \in \mathbb{N}$, one has $G(H_{a,N}) = 2$.

We wish to perturb a little an homogeneous set $H$ into a set $H'$ such that the Perron’s factor of $H'$ is still controlled. Precisely, fix any $a, N \in \mathbb{N}^*$ and let $\epsilon$ be an arbitrary function $\epsilon : H_{a,N} \rightarrow (0, \infty)$.

Define then the set $H_{a,N}(\epsilon)$ as

$$H' := H_{a,N}(\epsilon) := \{(1 + \epsilon(l))l : l \in H_{a,N}\}.$$

If the perturbation $\epsilon$ is small enough compared to the integer $N$, one can control uniformly $G(H')$.

**Proposition 1.** With the previous notations, if we have

$$2^N \|\epsilon\|_{\infty} \leq \frac{1}{2}$$

then we have $G(H_{a,N}(\epsilon)) < 6$.

We are now ready to prove Theorem 8. We fix a large integer $N \in \mathbb{N}$ and consider the following set of indices

$$E_N := \{k \in \mathbb{N} : |X_k - 1| \leq 2^{-N}\}.$$

In other words, an integer $k$ belongs to $E_N$ when $X_k$ is close to 1 with precision $2^{-N}$. We claim that this random set $E_N$ contains almost surely large (with at least $2^N$ points) homogeneous sequences.

**Claim 2.** For any $N \geq 1$, the set $E_N$ contains an homogeneous set of cardinal $2^N$ almost surely.

**Proof.** Observe that for any $a \in \mathbb{N}^*$, the following probability $P(H_{a,N} \subset E_N)$ is independent of $a$: indeed since the random variables $\{X_k : k \geq 1\}$ are independent and uniformly distributed, we have

$$P(H_{a,N} \subset E_N) = \prod_{k \in H_{a,N}} P(|X_k - 1| \leq 2^{-N}) = \frac{1}{2^{N^2}}.$$
Figure 1: We choose the sequence \( \{a_i : i \geq 1\} \) such that \( H_{a_i,N} \cap H_{a_j,N} = \emptyset \) for any \( i \neq j \).

Hence we fix a sequence \( \{a_i\}_{i \geq 1} \) satisfying for any \( i \neq j \)

\[
H_{a_i,N} \cap H_{a_j,N} = \emptyset.
\]

For example, setting \( a_i = 2^{2N(i+1)} \) works since we have \( a_i 2^N < a_{i+1} \) for any \( i \geq 1 \). In particular this means that the events

\[
\{ (H_{a_i,N} \subset E_N) : i \geq 1 \}
\]

are independent and since we have

\[
\sum_{i \geq 1} \mathbb{P}(H_{a_i,N} \subset E_N) = \infty
\]

an application of Borel-Cantelli lemma yields the conclusion.

We can now conclude the proof: we define a perturbation \( \epsilon \) for any \( n \geq 1 \) as

\[
1 + \epsilon(n) = X_n^{-1}.
\]

We fix a large integer \( N \gg 1 \) and we know that almost surely there exists an integer \( a \in \mathbb{N}^* \) such that \( H_{a,N} \subset E_N \). Observe now that by definition one has the following inclusion

\[
H_{a,N}(\epsilon) \subset \Omega^{-1}_{\text{rand,lin}}.
\]

However since \( H_{a,N} \subset E_N \) and that it is not difficult to see that we have

\[
\|\epsilon|_{H_{a,N}}\|_{\infty} \lesssim 2^{-N}.
\]

Indeed, for \( k \in H_{a,N} \subset E_N \), we have

\[
|1 + \epsilon(k)| = \left| \frac{1}{1 + (X_k - 1)} \right| \leq 1 + 2 |X_k - 1| \leq 1 + 2^{-N}.
\]

It follows that we have \( \mathbb{P}(\Omega^{-1}_{\text{rand,lin}}) < 6 \) almost surely applying Proposition 1.
5 Proof of Theorem 9

We wish to prove that the Perron’s capacity of $\Omega^\text{rand, lac}$ is finite almost surely. Observe that if $U$ is a set in $\mathbb{R}$ who is well distributed then one can control its Perron’s capacity.

Claim 3. If we have $\delta > 0$ and a set

$$U = \{u_1 < \cdots < u_{2^N}\}$$

such that for any $1 \leq i \leq 2^N - 1$ we have

$$\delta \leq u_{i+1} - u_i \leq 3\delta$$

then one has $G(U) \lesssim 1$.

Proof. For any $i, j$ such that $i + j \leq 2^N - 1$, one has $u_{i+j} - u_i \simeq j\delta$. Hence $i, j$ such that $i + 2j \leq 2^N - 1$, we have

$$\frac{u_{i+2j} - u_{i+j}}{u_{i+j} - u_i} + \frac{u_{i+j} - u_i}{u_{i+2j} - u_{i+j}} \simeq \frac{2j\delta}{j\delta} + \frac{j\delta}{2j\delta} \simeq 1.$$  

Hence we obtain $G(U) \lesssim 1$ as claimed. \qed

We are going to prove the following proposition.

Proposition 2. For any $N \geq 1$, there exists almost surely a scale $\delta > 0$ and a set $U \subset \Omega^\text{rand, lac}$ such that

$$U := \{u_1 < \cdots < u_{2N-1}\}$$

and for any $i \leq 2^N - 1$ one has

$$\delta < u_{i+1} - u_i < 3\delta.$$  

Theorem 9 is a consequence of Claim 3 and Proposition 2: for any $N$, we can exhibit almost surely a set $U \subset \Omega^\text{rand, lac}$ of cardinal $2^N$ such that $G(U) \lesssim 1$ and so we obtain

$$P(\Omega^\text{rand, lac}) < \infty$$

as expected. The rest of the section is devoted to the proof of Proposition 2.

Proof of Proposition 2

We consider the following dyadic intervals for $d \in \mathbb{N}$

$$I_d := [2^d, 2^{d+1}]$$

We wish to obtain information on the distribution of the points of the set $\Omega^\text{rand, lac}$ that may be in the interval $I_d$. We fix a large integer $N \gg 1$ and we divide each dyadic interval $I_d$ into $2^N$ intervals of same length $i.e.$ for any $1 \leq l \leq 2^N$ we set

$$I_{d,l} = \left[2^d \left(1 + \frac{l-1}{2^N}\right), 2^d \left(1 + \frac{l}{2^N}\right)\right].$$
Claim 4. For any \( d \geq 2^N + 1 \) and any \( 1 \leq l \leq 2^N \), the probability
\[
P\left( \frac{2d-l}{X_{d-l}} \in I_{d,l} \right) := p_{N,l}
\]
is independent of \( d \).

Proof. One has \( \frac{2d-l}{X_{d-l}} \in I_{d,l} \) if and only if
\[
2^l \left( 1 + \frac{l-1}{2^N} \right) \leq \frac{1}{X_{d-l}} \leq 2^l \left( 1 + \frac{l}{2^N} \right).
\]
Since the random variable \( X_{d-l} \) is uniformly in \((0,1)\), the probability \( P\left( \frac{2d-l}{X_{d-l}} \in I_{d,l} \right) \) is independent of \( d \) as claimed.

We fix an extraction \( \{d_s : s \geq 1\} \) satisfying the following property
\[
d_{s+1} - d_s > 2^N + 1
\]
for any \( s \geq 1 \); this growth condition will assure that the events we will consider are independent and we will be able to apply Borel-Cantelli lemma when needed. Thanks to Claim 4 and independence, one can see that for any \( s \geq 1 \), the following probability is independent of \( s \)
\[
P\left( \bigcap_{l \leq 2^N} \left( \frac{2d_{s-l}}{X_{d_{s-l}}} \in I_{d_{s-l}} \right) \right) = \prod_{l \leq 2^N} p_{N,l} := \eta_N
\]
Now for any \( d \geq 1 \), we consider the following event \( A_{d,N} \) defined as
\[
A_{d,N} := \bigcap_{l \leq 2^N} \left\{ \Omega_{\text{rand},d}^{-1} \cap I_{d,l} \neq \emptyset \right\}.
\]
In other words, the event \( A_{d,N} \) occurs when the random set \( \Omega_{\text{rand},d}^{-1} \) fills each sub-intervals \( \{ I_{d,l} : 1 \leq l \leq 2^N \} \) with at least one point. In particular, observe that we have
\[
\bigcap_{l \leq 2^N} \left( \frac{2d-l}{X_{d-l}} \in I_{d,l} \right) \subset A_{d,N}.
\]
We claim that the union of those events

\[ B_N := \bigcup_{d \geq 1} A_{d,N} \]

occurs almost surely.

Claim 5. One has \( \mathbb{P}(B_N) = 1 \).

Proof. Indeed we have

\[
\sum_s \mathbb{P} \left( \bigcap_{l \leq 2^N} \left( \frac{2d_s - l}{X_{d_s - l}} \in I_{d_s,l} \right) \right) = \sum_s \eta_N = \infty.
\]

Using Borel-Cantelli lemma, one obtains

\[
\mathbb{P} \left( \bigcup_{s \geq 1} A_{N,d_s} \right) = 1.
\]

In particular, one has \( \mathbb{P}(B_N) = 1 \).

\[ \Omega_{r2} \cap I_d \]

Figure 3: The random set \( \Omega_{\text{rand, lac}} := \Omega_{r2} \) contains almost surely uniformly distributed subset of arbitrarily large cardinal.

We can now prove Proposition 2 since we have

\[
\mathbb{P} \left( \bigcap_{N \geq 1} B_N \right) = 1.
\]

Precisely, for any \( N \geq 1 \), the event \( B_N \) occurs almost surely and this means that there exists a (for each \( N \), we just need one) dyadic interval \( I_d \) such that

\[ \Omega_{\text{rand, lac}}^{-1} \cap I_{d,t} \neq \emptyset \]

for any \( 1 \leq l \leq 2^N \). We let \( u_l \) be one point in \( \Omega_{\text{rand, lac}}^{-1} \cap I_{d,t} \) and we claim that the set

\[ U := \{ u_{2^l} : 1 \leq l \leq 2^{N-1} \} \]

satisfy the condition of Proposition 2 with \( \delta \simeq 2^{d-N} \). In particular, we have \( G(U) \lesssim 1 \) for \( U \subset \Omega_{\text{rand, lac}}^{-1} \) with arbitrary large cardinal. This yields almost surely

\[ \mathbb{P}(\Omega_{\text{rand, lac}}) \lesssim 1 \]

which concludes the proof.
References

[1] E. D’Aniello, A. Gauvan, L. Moonens, (Un)boundedness of directional maximal operators through a notion of "Perron capacity" and an application, accepted for publication in Proceedings of the AMS.


