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► **To cite this version:**

Megala Anandan, Benjamin Boutin, Nicolas Crouseilles. High order asymptotic preserving scheme for linear kinetic equations with diffusive scaling. 2023. hal-04103000

**HAL Id: hal-04103000**

**<https://hal.science/hal-04103000v1>**

Preprint submitted on 24 May 2023

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# HIGH ORDER ASYMPTOTIC PRESERVING SCHEME FOR LINEAR KINETIC EQUATIONS WITH DIFFUSIVE SCALING

M. ANANDAN\*, B. BOUTIN†, AND N. CROUSEILLES‡

**Abstract.** In this work, high order asymptotic preserving schemes are constructed and analysed for kinetic equations under a diffusive scaling. The framework enables to consider different cases: the diffusion equation, the advection-diffusion equation and the presence of inflow boundary conditions. Starting from the micro-macro reformulation of the original kinetic equation, high order time integrators are introduced. This class of numerical schemes enjoys the Asymptotic Preserving (AP) property for arbitrary initial data and degenerates when  $\epsilon$  goes to zero into a high order scheme which is implicit for the diffusion term, which makes it free from the usual diffusion stability condition. The space discretization is also discussed and high order methods are also proposed based on classical finite differences schemes. The Asymptotic Preserving property is analysed and numerical results are presented to illustrate the properties of the proposed schemes in different regimes.

**Key words.** collisional kinetic equation, diffusive scaling, high order Runge-Kutta schemes, asymptotic preserving property.

**MSC codes.** 82C40, 85A25, 65M06, 65L04, 65L06.

**1. Introduction.** In this work, we are concerned with the numerical approximation of linear kinetic transport equations in a diffusive scaling. Such models are widely used in applications such as rarefied gas dynamics, neutron transport, and radiative transfer. Due to the presence of a small parameter  $\epsilon$  (which is the normalized mean free path of the particles), standard schemes suffer from a severe restriction on the numerical parameters, making the simulations very costly. In the last decades, the so-called Asymptotic-Preserving (AP) schemes have been proposed to make possible the numerical passage between the micro and macro scale [14, 15]. Indeed, these AP schemes are uniformly stable and degenerate when  $\epsilon \rightarrow 0$  to a scheme which is consistent with the asymptotic diffusion model. This makes them very attractive to deal with multi-scale phenomena as an alternative to domain decomposition approaches.

The goal of this work is to design high order in time AP schemes for collisional kinetic equations in the diffusive scaling. Several works can be found in the literature on this topic [17, 14, 15, 18, 19, 20, 20, 26, 10, 21, 22, 25, 23, 28]. Our work is based on a micro-macro decomposition as introduced in [25] where the unknown  $f$  of the stiff kinetic equation is split into an equilibrium part  $\rho$  plus a remainder  $g$ . A micro-macro model (equivalent to the original kinetic one) satisfied by  $\rho$  and  $g$  can be derived. This micro-macro strategy turns out to be the starting point of several numerical approximations in phase space (using particles method, Discontinuous Galerkin method or low rank approximation [6, 5, 13, 11, 12]). In addition, a suitable first order semi-implicit time discretization of the micro-macro model is used as in [25] for which however the asymptotic diffusion equation is solved explicitly. This drawback is overcome following [23, 6, 5] in which the AP scheme degenerates into an implicit treatment of the diffusion equation. This improvement enables to get a numerical scheme which is asymptotically free from the usual parabolic condition.

The derivation of high order in time AP schemes for stiff kinetic problem has

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\*Indian Institute of Science, C.V. Raman Road, 560012, Bangalore, India (megalaa@iisc.ac.in).

†Univ Rennes, CNRS, IRMAR UMR 6625, 35000 Rennes, France (benjamin.boutin@univ-rennes.fr).

‡Univ Rennes, CNRS, IRMAR UMR 6625 & centre Inria de l'Université de Rennes (MINGuS) & ENS Rennes, France (nicolas.crouseilles@inria.fr)

43 been performed by several authors [7, 8, 9, 1, 13] using the so-called high order IMEX  
 44 methods [4, 2, 27]. In this work, a family of high order IMEX schemes is proposed  
 45 for linear collisional kinetic equations in the diffusive scaling, which degenerates when  
 46  $\epsilon \rightarrow 0$  to a high order IMEX scheme for the diffusion equation. From the first order  
 47 semi-implicit AP numerical scheme [6], the family of high order schemes proposed in  
 48 this work is obtained using globally stiffly accurate high order IMEX Runge-Kutta  
 49 methods, namely type A and type CK [8, 13].

50 In addition to the standard diffusion scaling, we also consider two other examples  
 51 that enter in our framework. First, we consider a modification of the collision operator  
 52 that enables to derive a transport-diffusion asymptotic model [17, 13]. Second, we  
 53 discuss half moments micro-macro decomposition which naturally incorporates the  
 54 incoming boundary conditions [24].

55 Lastly, we address the space discretization in order to get a fully high order solver  
 56 of the kinetic equation. High order space approximation based on finite difference  
 57 methods is considered. Staggered or non-staggered strategies are proposed to achieve  
 58 high order accuracy in space.

59 The paper is organized as follows. First in Section 2, the kinetic and asymptotic  
 60 diffusion models are introduced. Then in Section 3, high order time integrators (using  
 61 globally stiffly accurate IMEX Runge-Kutta temporal discretization) are proposed,  
 62 and their AP property in the diffusive limit is addressed in Section 4. Section 5 is  
 63 devoted to the space approximation. In Section 6, we discuss some extensions to other  
 64 collision operators and to half moments. In Section 7, numerical results are presented,  
 65 illustrating high order accuracy and the main properties of the schemes.

## 66 **2. Kinetic equation, diffusion limit and micro-macro decomposition.**

67 In this section, we introduce the kinetic model in the diffusive scaling, and recall the  
 68 asymptotic limit. Then, the micro-macro decomposition is performed to derive the  
 69 micro-macro model which serves as a basis for the numerical developments.

70 **2.1. Linear kinetic equation with diffusive scaling.** Let  $\Omega \subset \mathbb{R}^d$  be the  
 71 position space and  $V \subseteq \mathbb{R}^d$  be the velocity space with measure  $d\mu(v)$ . We consider  
 72 the linear kinetic equation with diffusive scaling,

$$73 \quad (2.1) \quad \partial_t f + \frac{1}{\epsilon} v \cdot \nabla_x f = \frac{1}{\epsilon^2} Lf, \quad (t, x, v) \in \mathbb{R}^+ \times \Omega \times V$$

74 where  $f(t, x, v) \in \mathbb{R}$  is the distribution function (depending on time  $t \in \mathbb{R}^+$ , space  
 75  $x \in \Omega \subset \mathbb{R}^d$  and velocity  $v \in V \subset \mathbb{R}^d$ ) and  $\epsilon > 0$  measures the dimensionless mean free  
 76 path of particles or the inverse of relaxation time. We consider the initial condition,

$$77 \quad (2.2) \quad f(0, x, v) = f^{\text{init}}(x, v), \quad (x, v) \in \Omega \times V$$

78 and boundary conditions are imposed in space. In this work, we will consider periodic  
 79 boundary conditions or inflow boundary conditions. The linear collision operator  $L$   
 80 in (2.1) acts only on the velocity dependence of  $f$ , and it relaxes the particles to an  
 81 equilibrium  $M(v)$  which is positive and even. We denote for all velocity dependent  
 82 distribution functions  $h$ ,

$$83 \quad (2.3) \quad \langle h \rangle_V = \frac{\int_V h(v) d\mu}{\int_V M(v) d\mu}.$$

84 In particular, we obtain  $\langle M \rangle_V = 1$  and  $\langle vM \rangle_V = 0$ . Further, the operator  $L$  is non-  
 85 positive and self-adjoint in  $L^2(V, M^{-1}d\mu)$ , with the following null space and range:

$$86 \quad (2.4) \quad \mathcal{N}(L) = \{f : f \in \text{Span}(M)\}, \quad \mathcal{R}(L) = (\mathcal{N}(L))^\perp = \{f : \langle f \rangle_V = 0\}.$$

87 Therefore,  $L$  is invertible on  $\mathcal{R}(L)$  and we denote its pseudo-inverse by  $L^{-1}$ . We also  
 88 assume that  $L$  is invariant under orthogonal transformations of  $\mathbb{R}^d$ .

89 **2.2. Diffusion limit.** In the limit  $\epsilon \rightarrow 0$ , it is seen from (2.1) that  $f \rightarrow f_0$   
 90 where  $f_0$  belongs to  $\mathcal{N}(L)$ . Thus,  $f_0 = \rho(t, x)M$  where  $f_0$  solves  $Lf_0 = 0$  and where  
 91 the limiting density  $\rho$  is the solution of the asymptotic diffusion equation. To derive  
 92 the diffusion equation, a Chapman-Enskog expansion has to be performed to get  
 93  $f = f_0 + \epsilon L^{-1}(vM) \cdot \nabla_x \rho + \mathcal{O}(\epsilon^2)$ . Integrating with respect to the velocity variable  
 94 enables to get the diffusion limit

$$95 \quad (2.5) \quad \partial_t \rho - \nabla_x \cdot (\kappa \nabla_x \rho) = 0 \text{ with } \kappa = -\langle v \otimes L^{-1}(vM) \rangle_V > 0.$$

96 **2.3. Micro-macro decomposition.** In this part, we derive a micro-macro  
 97 model which is equivalent to (2.1), and this is the model that will be discretized  
 98 in the next sections. First, we consider the standard micro-macro decomposition of  
 99 the unknown  $f$  [25, 23],

$$100 \quad (2.6) \quad f = \rho M + g, \quad \text{with } \rho(t, x) = \langle f \rangle_V \text{ and } \langle g \rangle_V = 0.$$

101 We introduce the orthogonal projector  $\Pi$  in  $L^2(V, M^{-1}d\mu)$  onto  $\mathcal{N}(L)$ :  $\Pi h = \langle h \rangle_V M$ ,  
 102 which will be useful to derive the micro-macro model. Substituting (2.6) into (2.1)  
 103 and applying successively  $\Pi$  and  $(I - \Pi)$  enables to get the micro-macro model satisfied  
 104 by  $(\rho, g)$

$$105 \quad (2.7) \quad \partial_t \rho + \frac{1}{\epsilon} \nabla_x \cdot \langle vg \rangle_V = 0,$$

$$106 \quad (2.8) \quad \partial_t g + \frac{1}{\epsilon} (I - \Pi)(v \cdot \nabla_x g) + \frac{1}{\epsilon} vM \cdot \nabla_x \rho = \frac{1}{\epsilon^2} Lg.$$

108 Initial conditions for macro and micro equations become

$$109 \quad (2.9) \quad \rho(0, x) = \rho^{\text{init}}(x) = \langle f^{\text{init}}(x, \cdot) \rangle_V,$$

$$110 \quad (2.10) \quad g(0, x, v) = g^{\text{init}}(x, v) = f^{\text{init}}(x, v) - \rho^{\text{init}}(x)M(v),$$

whereas the boundary conditions for  $\rho$  and  $g$  become periodic if  $f$  is periodic. From  
 the micro part (2.8), a Chapman-Enskog expansion of  $g$  can be performed to get

$$g = -\epsilon (\epsilon^2 \partial_t - L)^{-1} \left( (I - \Pi)(v \cdot \nabla_x g) + vM \cdot \nabla_x \rho \right) = \epsilon L^{-1}(vM) \cdot \nabla_x \rho + \mathcal{O}(\epsilon^2),$$

112 under some suitable smoothness assumptions. Inserting this expression in (2.7) leads  
 113 to (2.5) in the limit  $\epsilon \rightarrow 0$ .

114 **3. Time integrators.** In this part, we present the family of high order time  
 115 integrators for the micro-macro model (2.7)-(2.8). We will keep the phase space  
 116 variables continuous to ease the reading. We first recall the first order temporal  
 117 scheme which leads to the implicit treatment of the asymptotic diffusion model before  
 118 introducing the high order version.

119 **3.1. First order accurate time integrator.** Given  $\rho^n, g^n$  that approximate  
 120  $\rho, g$  at time  $t = n\Delta t$ , we obtain the solution  $\rho^{n+1}, g^{n+1}$  from the following time  
 121 integration of (2.7) and (2.8) respectively. We use the following first order implicit-  
 122 explicit (IMEX) strategy to attain the asymptotic preserving property

$$123 \quad (3.1) \quad \rho^{n+1} = \rho^n - \frac{\Delta t}{\epsilon} \nabla_x \cdot \langle vg^{n+1} \rangle_V,$$

$$124 \quad (3.2) \quad g^{n+1} = g^n - \frac{\Delta t}{\epsilon} (I - \Pi)(v \cdot \nabla_x g^n) - \frac{\Delta t}{\epsilon} vM \cdot \nabla_x \rho^{n+1} + \frac{\Delta t}{\epsilon^2} Lg^{n+1}.$$

126 Let us observe that this scheme is different from the IMEX strategies employed in  
 127 [25, 13], due to our implicit treatment of density gradient in micro equation (3.2) and  
 128 fully implicit treatment of the macro equation. This strategy enables us to get an  
 129 implicit scheme for diffusion equation in the limit  $\epsilon \rightarrow 0$ .

130 Although the macro equation is treated in a fully implicit manner,  $\rho^{n+1}$  and  $g^{n+1}$  can  
 131 be updated using (3.1) and (3.2) in an explicit manner. From (3.2), we get

$$132 \quad (3.3) \quad g^{n+1} = (\epsilon^2 I - \Delta t L)^{-1} (\epsilon^2 g^n - \epsilon \Delta t (I - \Pi) (v \cdot \nabla_x g^n) - \epsilon \Delta t v M \cdot \nabla_x \rho^{n+1}).$$

133 Inserting this in (3.1), we obtain the following implicit scheme for the macro unknown

$$134 \quad \rho^{n+1} = \rho^n - \Delta t \nabla_x \cdot \langle v (\epsilon^2 I - \Delta t L)^{-1} (\epsilon g^n - \Delta t (I - \Pi) (v \cdot \nabla_x g^n) - \Delta t v M \cdot \nabla_x \rho^{n+1}) \rangle_V,$$

135 or, expressing  $\rho^{n+1}$  as quantities at iteration  $n$

136

$$137 \quad \rho^{n+1} = (I - \Delta t^2 \nabla_x \cdot (\mathcal{D}_{\epsilon, \Delta t} \nabla_x))^{-1} \left( \rho^n \right. \\
 138 \quad \left. - \Delta t \nabla_x \cdot \left\langle v (\epsilon^2 I - \Delta t L)^{-1} (\epsilon g^n - \Delta t (I - \Pi) (v \cdot \nabla_x g^n)) \right\rangle_V \right)$$

139

140 with  $\mathcal{D}_{\epsilon, \Delta t} = \langle v \otimes (\epsilon^2 I - \Delta t L)^{-1} (v M) \rangle_V$ . Thanks to this time integrator,  $\rho^{n+1}$  can  
 141 be updated by inverting a diffusion type operator. Following this,  $g^{n+1}$  can be found  
 142 explicitly from the knowledge of  $\rho^{n+1}$ . This first order scheme introduced in [23, 6] is  
 143 the basis of the high order scheme presented below.

144 **3.2. High order accurate time integrators.** Following previous works [8, 13,  
 145 3], we will consider globally stiffly accurate (GSA) IMEX Runge-Kutta (RK) schemes  
 146 to construct high order time integrators for the micro-macro model (2.7) and (2.8).  
 147 An IMEX RK scheme is represented using the double Butcher tableau [4, 2]

$$148 \quad (3.4) \quad \begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^T \end{array} \quad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

149 where  $\tilde{A} = (\tilde{a}_{ij})$  and  $A = (a_{ij})$  are  $s \times s$  matrices which correspond to the explicit and  
 150 implicit parts of the scheme ( $A$  and  $\tilde{A}$  respectively are lower triangular and strictly  
 151 lower triangular matrices). The coefficients  $\tilde{c}$  and  $c$  are given by  $\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}$ ,  
 152  $c_i = \sum_{j=1}^i a_{ij}$ , and the vectors  $\tilde{b} = (\tilde{b}_j)$  and  $b = (b_j)$  give quadrature weights that  
 153 combine the stages. For GSA IMEX RK scheme, we have

$$154 \quad (3.5) \quad c_s = \tilde{c}_s = 1 \text{ and } a_{sj} = b_j, \tilde{a}_{sj} = \tilde{b}_j, \quad \forall j \in \{1, 2, \dots, s\}.$$

155 An IMEX RK method is type A if the matrix  $A$  is invertible, and it is type CK if the  
 156 first row of matrix  $A$  has zero entries and the square sub-matrix formed by excluding  
 157 the first column and row of  $A$  is invertible. In the special case where the first column  
 158 of  $A$  has zero entries, the scheme is said to be of type CK-ARS. The reader is referred  
 159 to [8] for more details. In this work, we employ both type A and CK-ARS schemes.  
 160 The first order GSA IMEX RK scheme employed in (3.1) and (3.2) follows the type  
 161 CK-ARS double Butcher tableau (known as ARS(1, 1, 1)),

$$162 \quad (3.6) \quad \begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline & 1 & 0 \end{array} \quad \begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline & 0 & 1 \end{array}$$

163 We now use the general IMEX RK scheme from (3.4) with GSA property (3.5) for  
 164 obtaining high order accurate time integration of macro and micro (2.7) and (2.8)  
 165 respectively. We introduce the following notations in the presentation of our scheme.

$$166 \quad (3.7) \quad \mathcal{T}h^{(k)} = (I - \Pi) \left( v \cdot \nabla_x h^{(k)} \right),$$

$$167 \quad (3.8) \quad \mathcal{D}_{\epsilon, \Delta t}^{(j)} = \left\langle v \otimes (\epsilon^2 I - a_{jj} \Delta t L)^{-1} (vM) \right\rangle_V,$$

$$168 \quad (3.9) \quad \mathcal{I}_{\epsilon, \Delta t}^{(j)} = (\epsilon^2 I - a_{jj} \Delta t L)^{-1}.$$

170 We will construct high order IMEX RK schemes following the first order guidelines  
 171 (fully implicit treatment of macro equation, implicit treatment of density gradient and  
 172 relaxation terms and explicit treatment of transport term in micro equation). Given  
 173  $\rho^n, g^n$  that approximate  $\rho, g$  at time  $t = n\Delta t$ , we obtain the internal RK stage values  
 174  $\rho^{(j)}$  and  $g^{(j)}$ ,  $j = 1, \dots, s$  as

$$175 \quad (3.10) \quad \rho^{(j)} = \rho^n - \sum_{k=1}^j a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \left\langle v g^{(k)} \right\rangle_V,$$

$$176 \quad (3.11) \quad g^{(j)} = g^n - \sum_{k=1}^{j-1} \tilde{a}_{jk} \frac{\Delta t}{\epsilon} \mathcal{T}g^{(k)} - \sum_{k=1}^j a_{jk} \frac{\Delta t}{\epsilon} vM \cdot \nabla_x \rho^{(k)} + \sum_{k=1}^j a_{jk} \frac{\Delta t}{\epsilon^2} Lg^{(k)},$$

178 where, as usual, the summation  $\sum_{k=1}^{j-1}$  in the explicit term is zero for  $j = 1$ .  
 179 Although the expressions above are implicit, the stage values  $\rho^{(1)}, g^{(1)}$  can be found  
 180 in an explicit manner by using the known quantities  $\rho^n, g^n$ , and the stage values  $\rho^{(j)}$ ,  
 181  $g^{(j)}$ ,  $\forall j \in \{2, 3, \dots, s\}$  can be found explicitly from  $\rho^n, g^n$  and the previous stage  
 182 values  $\rho^{(l)}, g^{(l)}$ ,  $\forall l \in \{1, 2, \dots, j-1\}$ . Indeed, proceeding similarly as for the first  
 183 order scheme, we get the following expression of  $g^{(j)}$ ,  $j = 1, \dots, s$  from (3.11),

$$184 \quad (3.12) \quad g^{(j)} = \mathcal{I}_{\epsilon, \Delta t}^{(j)} \left( \epsilon^2 g^n - \epsilon \sum_{k=1}^{j-1} \tilde{a}_{jk} \Delta t \mathcal{T}g^{(k)} - \epsilon \sum_{k=1}^j a_{jk} \Delta t vM \cdot \nabla_x \rho^{(k)} + \sum_{k=1}^{j-1} a_{jk} \Delta t Lg^{(k)} \right).$$

185 Further, we write (3.10) by splitting the summation on  $k$  as

$$186 \quad \rho^{(j)} = \rho^n - \sum_{k=1}^{j-1} a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \left\langle v g^{(k)} \right\rangle_V - a_{jj} \frac{\Delta t}{\epsilon} \nabla_x \cdot \left\langle v g^{(j)} \right\rangle_V,$$

187 and inserting (3.12) in the last term leads to the update of  $\rho^{(j)}$  for  $j = 1, \dots, s$

$$188 \quad (3.13) \quad \rho^{(j)} = \left( I - a_{jj}^2 \Delta t^2 \nabla_x \cdot \left( \mathcal{D}_{\epsilon, \Delta t}^{(j)} \nabla_x \right) \right)^{-1} \left( \rho^n - \sum_{k=1}^{j-1} a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \left\langle v g^{(k)} \right\rangle_V \right. \\ 189 \quad \left. - a_{jj} \Delta t \nabla_x \cdot \left\langle v \mathcal{I}_{\epsilon, \Delta t}^{(j)} \left( \epsilon g^n - \sum_{k=1}^{j-1} \tilde{a}_{jk} \Delta t \mathcal{T}g^{(k)} \right. \right. \right. \\ 190 \quad \left. \left. \left. - \sum_{k=1}^{j-1} a_{jk} \Delta t vM \cdot \nabla_x \rho^{(k)} + \frac{1}{\epsilon} \sum_{k=1}^{j-1} a_{jk} \Delta t Lg^{(k)} \right) \right\rangle_V \right),$$

191 where the definition of  $\mathcal{T}, \mathcal{D}_{\epsilon, \Delta t}^{(j)}$  and  $\mathcal{I}_{\epsilon, \Delta t}^{(j)}$  are given by (3.7)–(3.9). After this refor-  
 192 mulation,  $\rho^{(j)}$  can be computed from (3.13) by inverting a linear elliptic type problem

193 and following this,  $g^{(j)}$  can be found from (3.12). The GSA property in (3.5) guaran-  
 194 tees that the solution at  $t^{n+1} = (n+1)\Delta t$  is same as the last RK stage values, that  
 195 is,  $\rho^{n+1} = \rho^{(s)}$  and  $g^{n+1} = g^{(s)}$ .

196 **4. Asymptotic preserving property.** In this section, we show that the time  
 197 integrated scheme (3.13)-(3.12) becomes a consistent scheme for the diffusion equation  
 198 (2.5) in the limit  $\epsilon \rightarrow 0$ . We will discuss the asymptotic preserving property for both  
 199 CK-ARS and type A time integrators as performed in [8] for the fluid limit. First, we  
 200 recall the definition of well-prepared initial data in our context.

201 **DEFINITION 4.1** (Well-prepared initial data). *The initial data  $\rho(0, x)$  and  $g(0, x, v)$*   
 202 *in (2.9) and (2.10) are said to be well-prepared if  $g(0, x, v) = O(\epsilon)$ .*

203 **LEMMA 4.2.** *Assume that  $\epsilon$  is sufficiently small. Let  $\tilde{a}_{jk}$  and  $a_{jk}$  be the coefficients*  
 204 *of the RK method (3.4) applied to the scheme (3.10)-(3.11). Then, the following holds:*

- 205 1. *CK-ARS case: If  $g^n = O(\epsilon)$ , then  $g^{(1)} = g^n = O(\epsilon)$  and*  
 206  $g^{(j)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(j)} + O(\epsilon^2), \forall j \in \{2, \dots, s\}.$
- 207 2. *Type A case:  $g^{(j)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(j)} + O(\epsilon^2), \forall j \in \{1, \dots, s\}.$*

208 *Proof.* Let  $j \in \{1, \dots, s\}$  such that  $a_{jj} \neq 0$ . Observe that the operator  $\mathcal{I}_{\epsilon, \Delta t}^{(j)}$   
 209 defined in (3.9) admits, for small  $\epsilon$ , the following expansion:

$$210 \quad (4.1) \quad \mathcal{I}_{\epsilon, \Delta t}^{(j)} = -(a_{jj} \Delta t L)^{-1} + O(\epsilon^2).$$

211 Consider now an A-type time integrator, so with  $a_{jj} \neq 0$  for any  $j \in \{1, \dots, s\}$ ,  
 212 and assume  $g^n = O(1)$ . From (3.12) and the previous expansion, we obtain

$$213 \quad g^{(1)} = -(a_{11} \Delta t L)^{-1} \left[ -\epsilon a_{11} \Delta t v M \cdot \nabla_x \rho^{(1)} \right] + O(\epsilon^2) = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(1)} + O(\epsilon^2).$$

215 Now, the proof is performed by induction on  $j \in \{2, \dots, s\}$  assuming that for any  
 216  $k \in \{1, \dots, j-1\}$ ,  $g^{(k)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(k)} + O(\epsilon^2)$ . In particular  $g^{(k)} = O(\epsilon)$  and  
 217 the formula (3.12) has therefore the following expansion:

$$218 \quad g^{(j)} = -(a_{jj} \Delta t L)^{-1} \left[ O(\epsilon^2) - \epsilon \sum_{k=1}^j a_{jk} \Delta t v M \cdot \nabla_x \rho^{(k)} + \sum_{k=1}^{j-1} a_{jk} \Delta t L g^{(k)} \right] + O(\epsilon^2).$$

219 Inserting the induction hypothesis in the last sum, most of the terms in the two sums  
 220 eliminate so that finally  $g^{(j)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(j)} + O(\epsilon^2)$ .

221 The case of a CK-ARS time integrator is slightly different. First  $a_{11} = 0$  so that  
 222  $g^{(1)} = g^n = O(\epsilon)$  by the particular well-prepared assumption. Now  $a_{22} \neq 0$  and (3.12)  
 223 has the following expansion for  $j = 2$ :

$$224 \quad g^{(2)} = -(a_{22} \Delta t L)^{-1} \left[ O(\epsilon^2) - \epsilon a_{22} \Delta t v M \cdot \nabla_x \rho^{(2)} \right] + O(\epsilon^2) = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(2)} + O(\epsilon^2).$$

226 Again, the proof is by induction on  $j \in \{3, \dots, s\}$  assuming for any  $k \in \{2, \dots, j-1\}$ ,  
 227  $g^{(k)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(k)} + O(\epsilon^2)$ . The same computation as above is available since  
 228  $g^{(1)} = O(\epsilon)$ . One has (note that  $a_{j1} = 0$  for any  $j$  so that the sums start at  $k = 2$ ):

$$229 \quad g^{(j)} = -(a_{jj} \Delta t L)^{-1} \left[ O(\epsilon^2) - \epsilon \sum_{k=2}^j a_{jk} \Delta t v M \cdot \nabla_x \rho^{(k)} + \sum_{k=2}^{j-1} a_{jk} \Delta t L g^{(k)} \right] + O(\epsilon^2)$$

$$230 \quad = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(j)} + O(\epsilon^2). \quad \square$$

232 Due to the GSA property of both time integrators, we have  $g^{n+1} = g^{(s)} = \epsilon L^{-1}(vM) \cdot$   
 233  $\nabla_x \rho^{(s)} + O(\epsilon^2) = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{n+1} + O(\epsilon^2)$  for sufficiently small  $\epsilon$ . Thus, the  
 234 following are evident from Lemma 4.2:

235 1. For type CK-ARS, if the initial data is well-prepared (that is,  $g^0 = O(\epsilon)$ ),  
 236 then  $g^n = O(\epsilon)$ ,  $\forall n > 0$ .

237 2. For type A, if the initial data is such that  $g^0 = O(1)$ , then  $g^n = O(\epsilon)$ ,  $\forall n > 0$ .

238 As observed in [8], the initial data does not need to be well-prepared for type A time  
 239 integrator, unlike type CK-ARS, to ensure AP property.

240 **THEOREM 4.3.** *Consider the scheme (3.10)-(3.11) approximating the macro-micro*  
 241 *model (2.7)-(2.8), with the RK method (3.4) of type A or of type CK-ARS (with well-*  
 242 *prepared initial data  $g^0 = O(\epsilon)$ ). Then in the limit  $\epsilon \rightarrow 0$ , the scheme (3.10)-(3.11)*  
 243 *degenerates to the following scheme for the diffusion equation*

$$244 \quad (4.2) \quad \rho^{(j)} = \rho^n + \sum_{k=1}^j a_{jk} \Delta t \nabla_x \cdot \left( \kappa \nabla_x \rho^{(k)} \right), \quad \forall j = 1, \dots, s, \quad \kappa = - \left\langle v \otimes L^{-1}(vM) \right\rangle_V.$$

245 *Proof.* Corresponding to each case (CK-ARS or type A), we have the following:

246 **Type CK-ARS** Assumptions in criterion 1 of Lemma 4.2 are satisfied, and its im-  
 247 plications can be utilised. Hence, inserting  $g^{(\ell)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(\ell)} +$   
 248  $O(\epsilon^2)$ ,  $\forall \ell \in \{2, 3, \dots, s\}$  into (3.10), we get (recall that  $a_{j1} = 0$ )

$$249 \quad \begin{aligned} \rho^{(j)} &= \rho^n - \frac{\Delta t}{\epsilon} \sum_{k=2}^j a_{jk} \nabla_x \cdot \left\langle v \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(k)} \right\rangle_V + O(\epsilon), \\ &= \rho^n - \Delta t \sum_{k=2}^j a_{jk} \nabla_x \cdot \left( \left\langle v \otimes L^{-1}(vM) \right\rangle_V \nabla_x \rho^{(k)} \right) + O(\epsilon). \end{aligned}$$

252 **Type A** Assumptions in criterion 2 of Lemma 4.2 are satisfied, and its implications  
 253 can be utilised. Hence, inserting  $g^{(\ell)} = \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(\ell)} + O(\epsilon^2)$ ,  $\forall \ell \in$   
 254  $\{1, 2, \dots, s\}$  into (3.10), we get the required result by following the same sim-  
 255 plification as before. The only difference is that here  $\sum_{k=1}^j$  instead of  $\sum_{k=2}^j$ .  $\square$

256 **Remark 4.4.** For type CK-ARS, if the initial data is not well-prepared, computing  
 257  $g^{(2)}$  from (3.11) involves  $\epsilon \frac{\tilde{a}_{21}}{a_{22}} L^{-1}(I - \Pi)(v \cdot \nabla_x g^{(1)})$  which is not of  $O(\epsilon^2)$ . Thus,

$$258 \quad g^{(2)} = \epsilon \frac{\tilde{a}_{21}}{a_{22}} L^{-1}(I - \Pi)(v \cdot \nabla_x g^{(1)}) + \epsilon L^{-1}(vM) \cdot \nabla_x \rho^{(2)} + O(\epsilon^2),$$

259 and inserting in the macro equation (3.10) for  $j = 2$  leads to (since  $a_{21} = 0$ )

260

$$261 \quad \begin{aligned} \rho^{(2)} &= \rho^n - \frac{\tilde{a}_{21}}{a_{22}} \Delta t \left\langle v \otimes L^{-1} \left( (I - \Pi) v \nabla_x^2 g^{(1)} \right) \right\rangle_V \\ &\quad - a_{22} \Delta t \nabla_x \cdot \left( \left\langle v \otimes L^{-1}(vM) \right\rangle_V \nabla_x \rho^{(2)} \right) + O(\epsilon), \end{aligned}$$

262  
263

264 which is not consistent with the diffusion equation. Thus, for CK-ARS, asymptotic  
 265 consistency cannot be attained if the initial data is not well-prepared.

266 **5. Space and velocity discretization.** In this section, we present the spatial  
 267 (for both non-staggered and staggered grids) and velocity discretization strategies  
 268 that we employ in our numerical scheme.



269 **5.1. Discrete velocity method.** For the velocity discretization, we will follow  
 270 the discrete velocity method [16]. Thus, the velocity domain is truncated as  $v \in$   
 271  $[-v_{\max}, v_{\max}]$ , and a uniform mesh is used  $v_k = -v_{\max} + k\Delta v$ ,  $k = 1, \dots, N_v$  ( $N_v \in \mathbb{N}^*$ )  
 272 and  $\Delta v = 2v_{\max}/N_v$ . Further,  $f(t, x, v)$  and  $M(v)$  are represented as:

$$273 \quad f_k(t, x) := f(t, x, v_k), \quad M_k := M(v_k) \text{ for } k = 1, \dots, N_v.$$

274 Then, according to the definitions (2.3) and (2.6), we have for  $j = 1, \dots, N_v$

$$275 \quad \rho(t, x) \approx \frac{\sum_{k=0}^{N_v-1} f_k \Delta v}{\sum_{k=0}^{N_v-1} M_k \Delta v} \quad \text{and} \quad (\Pi f(t, x, v))_j \approx \frac{\sum_{k=0}^{N_v-1} f_k \Delta v}{\sum_{k=0}^{N_v-1} M_k \Delta v} M_j.$$

277 For the presentation, we will skip the velocity part to focus on space discretization.

278 **5.2. Space discretization using staggered grid.** First, we will consider stag-  
 279 gered grid to approximate  $g^{(j)}$  and  $\rho^{(j)}$  in space following [25]: the two meshes of the  
 280 space interval  $[0, 1]$  are  $x_i = i\Delta x$  and  $x_{i+1/2} = (i+1/2)\Delta x$  for  $i = 0, \dots, N_x$  ( $N_x \in \mathbb{N}^*$ ),  
 281 with  $\Delta x = L/N_x$ . Periodic boundary conditions will be considered in this section.

282 The expressions for  $g^{(j)}$  and  $\rho^{(j)}$  in (3.12)-(3.13) are spatially discretised by con-  
 283 sidering staggered grid:  $\rho^{(j)}$  is stored at  $x_i$  ( $\rho_i^{(j)} \approx \rho^{(j)}(x_i)$ ), and  $g^{(j)}$  is stored at  
 284  $x_{i+1/2}$  ( $g_{i+1/2}^{(j)}(v) \approx g^{(j)}(x_{i+1/2}, v)$ ). The term  $v \cdot \nabla_x g^{(k)}$  in (3.12) and (3.13) is discre-  
 285 tised in an upwind fashion as  $v \cdot \nabla_x \approx v^+ \cdot \mathbf{G}_{\text{upw}}^- + v^- \cdot \mathbf{G}_{\text{upw}}^+$  where  $v^\pm = (v \pm |v|)/2$ ,  
 286  $\mathbf{G}_{\text{upw}}^\pm$  denote the  $N_x \times N_x$  matrices that approximate  $\nabla_x$ . For instance, the first order  
 287 version is

$$288 \quad (5.1) \quad \mathbf{G}_{\text{upw}}^- = \frac{1}{\Delta x} \text{circ}([-1, \underline{1}]), \quad \mathbf{G}_{\text{upw}}^+ = \frac{1}{\Delta x} \text{circ}([\underline{-1}, 1]),$$

289 where the notation circ is defined in Appendix A. With these notations, we get

$$290 \quad \left( v \partial_x g^{(j)} \right)_{x_{i+1/2}} \approx v^+ \frac{g_{i+1/2}^{(j)} - g_{i-1/2}^{(j)}}{\Delta x} + v^- \frac{g_{i+3/2}^{(j)} - g_{i+1/2}^{(j)}}{\Delta x} = \left( (v^+ \mathbf{G}_{\text{upw}}^- + v^- \mathbf{G}_{\text{upw}}^+) g^{(j)} \right)_i,$$

291 where in the last term, the  $i$  index has to be understood as the  $i$ -th component  
 292 of the vector. Instead of first order upwind discretization, one can also use high  
 293 order upwind discretizations so that the matrices  $\mathbf{G}_{\text{upw}}^\pm$  will be different. Further, the  
 294 term  $vM \cdot \nabla_x \rho^{(k)}$  in (3.12)-(3.13) and the terms of the form  $\nabla_x \cdot \langle (\cdot) \rangle_V$  in (3.13) are  
 295 discretised using second order central differences as in [25]. In particular, the term  
 296  $vM \cdot \nabla_x \rho^{(k)}$  is approximated by

$$297 \quad (5.2) \quad \left( vM \partial_x \rho^{(k)} \right)_{x_{i+1/2}} \approx vM \frac{\rho_{i+1}^{(k)} - \rho_i^{(k)}}{\Delta x} = \left( vM \mathbf{G}_{\text{cen}_g} \rho^{(k)} \right)_i, \quad \mathbf{G}_{\text{cen}_g} = \frac{1}{\Delta x} \text{circ}([\underline{-1}, 1]).$$

298 Finally, the gradient terms  $\nabla_x \cdot \langle (\cdot) \rangle_V$  in (3.13) are approximated as follows

$$299 \quad (5.3) \quad \left( \partial_x \langle \cdot \rangle_V \right)_{x_i} = \frac{\langle \cdot \rangle_V|_{x_{i+1/2}} - \langle \cdot \rangle_V|_{x_{i-1/2}}}{\Delta x} = \left( \mathbf{G}_{\text{cen}_\rho} \langle \cdot \rangle_V \right)_i, \quad \mathbf{G}_{\text{cen}_\rho} = \frac{1}{\Delta x} \text{circ}([-1, \underline{1}]).$$

300 Again, high order centered finite differences methods can be used so that it will give  
 301 different expressions for  $\mathbf{G}_{\text{cen}_\rho}$  and  $\mathbf{G}_{\text{cen}_g}$ . Let us remark that the term  $\nabla_x \cdot \nabla_x = \nabla_x^2$   
 302 in (3.13) is approximated by  $\mathbf{G}_{\text{cen}_\rho} \mathbf{G}_{\text{cen}_g}$ , ie  $\mathbf{G}_{\text{cen}_\rho} \mathbf{G}_{\text{cen}_g} = \frac{1}{\Delta x^2} \text{circ}([1, \underline{-2}, 1])$ , which  
 303 gives the standard second order approximation of the Laplacian.

304 To ease the reading, we present the fully discrete scheme for first order ARS(1, 1, 1)  
 305 but the generalization to high order can be done using the elements of Section 3

$$\begin{aligned}
 306 \quad g^{n+1} &= (\epsilon^2 I - \Delta t L)^{-1} (\epsilon^2 g^n - \epsilon \Delta t (I - \Pi) (v^+ \mathbf{G}_{\text{upw}}^- + v^- \mathbf{G}_{\text{upw}}^+) g^n - \epsilon \Delta t v M \mathbf{G}_{\text{cen}_g} \rho^{n+1}) \\
 307 \quad \rho^{n+1} &= \left( I - \Delta t^2 \mathbf{G}_{\text{cen}_\rho} \left( \left\langle v \otimes (\epsilon^2 I - \Delta t L)^{-1} (v M) \right\rangle_V \mathbf{G}_{\text{cen}_g} \right) \right)^{-1} \times \\
 308 \quad &\left( \rho^n - \Delta t \mathbf{G}_{\text{cen}_\rho} \left\langle v (\epsilon^2 I - \Delta t L)^{-1} (\epsilon g^n - \Delta t (I - \Pi) ((v^+ \mathbf{G}_{\text{upw}}^- + v^- \mathbf{G}_{\text{upw}}^+) g^n)) \right\rangle_V \right).
 \end{aligned}$$

309 **5.3. Space discretization using non-staggered grid.** We also address the  
 310 case of non-staggered grids which may be more appropriate when high dimensions are  
 311 considered in space since only one spatial mesh is used:  $x_i = i\Delta x$ , for  $i = 0, 1, \dots, N_x$ ,  
 312 with  $\Delta x = L/N_x$ . Let  $g^{(j)}$  and  $\rho^{(j)}$  in (3.12)-(3.13)  $\forall j \in \{1, 2, \dots, s\}$  be approximated in  
 313 space by  $g_i^{(j)}(v) \approx g^{(j)}(x_i, v)$  and  $\rho_i^{(j)} \approx \rho^{(j)}(x_i)$ . The term  $v \cdot \nabla_x g^{(k)}$  in (3.12)-(3.13) is  
 314 discretised in an upwind fashion as  $v \cdot \nabla_x = v^+ \mathbf{G}_{\text{upw}}^- + v^- \mathbf{G}_{\text{upw}}^+$ , where  $v^\pm = (v \pm |v|)/2$ .  
 315 Here,  $\mathbf{G}_{\text{upw}}^\pm$  denote the matrices that represent an upwind approximation of  $\nabla_x$ . For  
 316 instance, the definition (5.1) can be used, but also its third order version

$$317 \quad (5.4) \quad \mathbf{G}_{\text{upw}}^- = \frac{1}{6\Delta x} \text{circ}([1, -6, \underline{3}, 2]), \quad \mathbf{G}_{\text{upw}}^+ = \frac{1}{6\Delta x} \text{circ}([-2, -\underline{3}, 6, -1]),$$

318 where  $\text{circ}$  represents the matrix notation described in Appendix A can be used. The  
 319 term  $v M \cdot \nabla_x \rho^{(k)}$  in (3.12)-(3.13) and the terms of the form  $\nabla_x \cdot \langle (\cdot) \rangle_V$  in (3.13) are  
 320 discretised in central fashion, since these terms act as source in (3.12) and diffusion  
 321 in (3.13). Here,  $\nabla_x$  is approximated by central differences as in (5.3) or (5.2) but in  
 322 the non-staggered case, the same matrix can be used for both terms. As an example,  
 323 the fourth order central difference produces:

$$324 \quad (5.5) \quad \mathbf{G}_{\text{cen}} = \frac{1}{12\Delta x} \text{circ}([1, -8, \underline{0}, 8, -1]).$$

325 The term  $\nabla_x \cdot \nabla_x = \nabla_x^2$  in (3.13) is discretised as the matrices product  $\mathbf{G}_{\text{cen}}^2 =$   
 326  $\mathbf{G}_{\text{cen}} \mathbf{G}_{\text{cen}}$ . Like in the staggered grid case, we present the fully discrete scheme for  
 327 first order ARS(1, 1, 1) time discretization to ease the reading:

$$\begin{aligned}
 328 \quad g^{n+1} &= (\epsilon^2 I - \Delta t L)^{-1} (\epsilon^2 g^n - \epsilon \Delta t (I - \Pi) (v^+ \mathbf{G}_{\text{upw}}^- + v^- \mathbf{G}_{\text{upw}}^+) g^n - \epsilon \Delta t v M \mathbf{G}_{\text{cen}} \rho^{n+1}) \\
 329 \quad \rho^{n+1} &= \left( I - \Delta t^2 \mathbf{G}_{\text{cen}} \left( \left\langle v \otimes (\epsilon^2 I - \Delta t L)^{-1} (v M) \right\rangle_V \mathbf{G}_{\text{cen}} \right) \right)^{-1} \times \\
 330 \quad &\left( \rho^n - \Delta t \mathbf{G}_{\text{cen}} \left\langle v (\epsilon^2 I - \Delta t L)^{-1} (\epsilon g^n - \Delta t (I - \Pi) ((v^+ \mathbf{G}_{\text{upw}}^- + v^- \mathbf{G}_{\text{upw}}^+) g^n)) \right\rangle_V \right)
 \end{aligned}$$

331 *Remark 5.1.* We know that the term  $\sum_{k=1}^j a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \langle v g^{(k)} \rangle_V$  in (3.10) is split  
 332 into first  $j - 1$  and last  $j$  contributions, and  $g^{(j)}$  is substituted for the last  $j$  contri-  
 333 bution, as in (3.13). The gradient in  $\sum_{k=1}^{j-1} a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \langle v g^{(k)} \rangle_V$  of (3.13) is discretised  
 334 using  $\mathbf{G}_{\text{cen}_\rho}$ . Further, the substitution of  $g^{(j)}$  for the last  $j$  hints the combination of  
 335  $\nabla_x \cdot \nabla_x$  as  $\nabla_x^2$  for the terms of  $g^{(j)}$  involving  $\nabla_x g$  and  $\nabla_x \rho$ . However, if we choose  
 336 a spatial discretization for  $\nabla_x^2$  as  $\mathbf{G}_{\text{diff}}$ , then these terms will experience  $\mathbf{G}_{\text{cen}_\rho} \mathbf{G}_{\text{cen}_g}$   
 337 for the first  $j - 1$  contributions and  $\mathbf{G}_{\text{diff}}$  for the last  $j$  contribution of the  $\rho^{(j)}$  update  
 338 equation. This disrupts the ODE structure present in RK time discretization, and  
 339 hence reduction to first order time accuracy was observed numerically. Therefore,  
 340 in order to retain high order time accuracy, it is important to carry out the space  
 341 discretization carefully. Hence, we do not introduce a different discretization for  $\nabla_x^2$ ,  
 342 and we retain  $\mathbf{G}_{\text{cen}_\rho} \mathbf{G}_{\text{cen}_g}$  even for the last  $j$  contribution of  $\rho^{(j)}$  equation.

343 *Remark 5.2.* The matrices introduced for spatial discretization do not change the  
 344 Chapman-Enskog expansion so that the AP property is still true in the fully discrete  
 345 form. Thus, we have  $g^{(k)} = \epsilon L^{-1}(vM)\mathbf{G}_{\text{cen}_g}\rho^{(k)} + O(\epsilon^2)$  for  $k \in \{1, \dots, s\}$  by using  
 346 type A. For CK-ARS with well-prepared data, we have  $g^{(k)} = \epsilon L^{-1}(vM)\mathbf{G}_{\text{cen}_g}\rho^{(k)} +$   
 347  $O(\epsilon^2)$  for  $k \in \{2, \dots, s\}$ . Inserting this in macro equation, we get the corresponding  
 348 RK scheme for the diffusion

$$349 \quad \rho^{(j)} = \rho^n - \Delta t \sum_{k=1}^j a_{jk} \mathbf{G}_{\text{cen}_\rho} \left( \langle v \otimes L^{-1}(vM) \rangle_V \mathbf{G}_{\text{cen}_g} \rho^{(k)} \right) + O(\epsilon).$$

351 **6. Extensions to other collision operator and inflow boundary prob-**  
 352 **lems.** In this section, we show that our high order AP schemes can be extended to  
 353 other problems involving advection-diffusion asymptotics and inflow boundaries.

354 **6.1. Advection-diffusion asymptotics.** In this part, an advection-diffusion  
 355 collision operator is considered (see [17, 13]),

$$356 \quad (6.1) \quad \mathcal{L}f := Lf + \epsilon vM \cdot A \langle f \rangle_V, \quad A \in \mathbb{R}^d, \quad |\epsilon A| < 1,$$

357 where  $L$  denotes a collision satisfying the properties listed in Section 2. A famous  
 358 simple example is  $Lf = \langle f \rangle_V M - f$ .

359 Using the notations introduced in Section 2, we can derive the micro-macro model  
 360 satisfied by  $\rho = \langle f \rangle_V$  and  $g = f - \rho M$  by applying  $\Pi$  and  $I - \Pi$  to (2.1) with collision  
 361  $\mathcal{L}$  to get the macro and micro equations in this context

$$362 \quad (6.2) \quad \partial_t \rho + \frac{1}{\epsilon} \nabla_x \cdot \langle v g \rangle_V = 0,$$

$$363 \quad (6.3) \quad \partial_t g + \frac{1}{\epsilon} (I - \Pi)(v \cdot \nabla_x g) + \frac{1}{\epsilon} vM \cdot \nabla_x \rho = \frac{1}{\epsilon^2} Lg + \frac{1}{\epsilon} vM \cdot A\rho.$$

364 A Chapman-Enskog expansion can be performed to get  $g = \epsilon L^{-1}(vM) \cdot \nabla_x \rho -$   
 365  $\epsilon L^{-1}(vM) \cdot A\rho + \mathcal{O}(\epsilon^2)$ . Inserting this in the macro equation (6.2) enables to ob-  
 366 tain an advection-diffusion equation in the limit  $\epsilon \rightarrow 0$ :

$$367 \quad (6.4) \quad \partial_t \rho + \nabla_x \cdot \left( \langle v \otimes L^{-1}(vM) \rangle_V \nabla_x \rho \right) - \nabla_x \cdot \left( \langle v \otimes L^{-1}(vM) \rangle_V A\rho \right) = 0.$$

368 The goal is to design a uniformly stable high order time integrators for (6.2)-(6.3)  
 369 so that they degenerate into a high order time integrator for (6.4) as  $\epsilon \rightarrow 0$ . The  
 370 extension of the schemes introduced in Section 3 will lead to an IMEX discretization  
 371 of the asymptotic model (6.4), where the advection term is treated explicitly while  
 372 the diffusion term is implicit.

373 **6.1.1. High order time integrator.** In this subsection, we present the dis-  
 374 cretization of macro and micro equations (6.2)-(6.3). As in Section 3, in the micro  
 375 equation, we treat  $\frac{1}{\epsilon^2} Lg$  implicitly to ensure uniform stability and the additional term  
 376  $\frac{1}{\epsilon} vM \cdot A\rho$  explicitly since it will be stabilized by the implicit treatment of the stiffest  
 377 term. Regarding the macro equation and the remaining terms in micro equation, we  
 378 follow the lines from previous Section 3. We thus obtain the following high order  
 379 IMEX RK scheme to approximate (6.2)-(6.3)

$$380 \quad (6.5) \quad \rho^{(j)} = \rho^n - \sum_{k=1}^j a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \langle v g^{(k)} \rangle_V,$$

$$381 \quad (6.6) \quad g^{(j)} = g^n - \frac{\Delta t}{\epsilon} \left[ \sum_{k=1}^{j-1} \tilde{a}_{jk} \mathcal{T}g^{(k)} + \sum_{k=1}^j a_{jk} vM \cdot \nabla_x \rho^{(k)} - \sum_{k=1}^j \frac{a_{jk}}{\epsilon} Lg^{(k)} - \sum_{k=1}^{j-1} \tilde{a}_{jk} vM \cdot A\rho^{(k)} \right],$$

382

383 where the coefficients  $a_{jk}, \tilde{a}_{jk}$  are given by the Butcher tableaux. As in Section 3,  
 384 some calculations are required to make the algorithm explicit. First, we have  
 385

$$386 \quad (6.7) \quad g^{(j)} = \mathcal{I}_{\epsilon, \Delta t}^{(j)} \left( \epsilon^2 g^n - \epsilon \Delta t \left[ \sum_{k=1}^{j-1} \tilde{a}_{jk} \mathcal{T} g^{(k)} + \sum_{k=1}^j a_{jk} vM \cdot \nabla_x \rho^{(k)} \right. \right. \\ \left. \left. - \frac{1}{\epsilon} \sum_{k=1}^{j-1} a_{jk} L g^{(k)} - \sum_{k=1}^{j-1} \tilde{a}_{jk} vM \cdot A \rho^{(k)} \right] \right),$$

387  
388

389 with  $\mathcal{T} g^{(k)} = (I - \Pi) (v \cdot \nabla_x g^{(k)})$  and  $\mathcal{I}_{\epsilon, \Delta t}^{(j)} = (\epsilon^2 I - a_{jj} \Delta t L)^{-1}$ . Then,  $\rho^{(j)}$  is obtained  
 390 by inserting  $g^{(j)}$  given by (6.7) in the macro equation (6.5) to get

$$391 \quad (6.8) \quad \rho^{(j)} = \left( I - a_{jj}^2 \Delta t^2 \nabla_x \cdot \left( \mathcal{D}_{\epsilon, \Delta t}^{(j)} \nabla_x \right) \right)^{-1} \left( \rho^n - \sum_{k=1}^{j-1} a_{jk} \frac{\Delta t}{\epsilon} \nabla_x \cdot \left\langle v g^{(k)} \right\rangle_V \right. \\ 392 \quad \left. - a_{jj} \Delta t \nabla_x \cdot \left\langle v \mathcal{I}_{\epsilon, \Delta t}^{(j)} \left( \epsilon g^n - \sum_{k=1}^{j-1} \tilde{a}_{jk} \Delta t \mathcal{T} g^{(k)} - \sum_{k=1}^{j-1} a_{jk} \Delta t vM \cdot \nabla_x \rho^{(k)} \right. \right. \right. \\ 393 \quad \left. \left. \left. + \frac{1}{\epsilon} \sum_{k=1}^{j-1} a_{jk} \Delta t L g^{(k)} + \sum_{k=1}^{j-1} \tilde{a}_{jk} \Delta t vM \cdot A \rho^{(k)} \right) \right\rangle_V \right),$$

394 where  $\mathcal{D}_{\epsilon, \Delta t}^{(j)} = \langle v \otimes (\epsilon^2 I - a_{jj} \Delta t L)^{-1} (vM) \rangle_V$ . Thus,  $\rho^{(j)}$  can be updated by using  
 395 (6.8) and  $g^{(j)}$  can be found explicitly by using (6.7).

396 **6.1.2. Asymptotic preserving property.** This part is dedicated to the as-  
 397 ymptotic preserving property of the scheme (6.8)-(6.7). We first show the AP prop-  
 398 erty of type A time integrator, and we later remark how this property is true for the  
 399 CK-ARS time integrator with well-prepared initial data. First we have

400 LEMMA 6.1. *If  $g^n = O(1)$  and  $g^{(k)} = O(\epsilon), \forall k \in \{1, 2, \dots, j-1\}$ , then  $g^{(j)} =$   
 401  $O(\epsilon), \forall j \in \{2, 3, \dots, s\}$  for small  $\epsilon$ . In particular, we have  $\forall j \in \{2, 3, \dots, s\}$   
 (6.9)*

$$402 \quad g^{(j)} = \epsilon \sum_{k=1}^j \frac{a_{jk}}{a_{jj}} L^{-1} (vM) \cdot \nabla_x \rho^{(k)} - \sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} g^{(k)} - \epsilon \sum_{k=1}^{j-1} \frac{\tilde{a}_{jk}}{a_{jj}} L^{-1} (vM) \cdot A \rho^{(k)} + O(\epsilon^2).$$

403 *Proof.* Plugging in (6.7) the expansion (4.1) of  $\mathcal{I}_{\epsilon, \Delta t}^{(j)}$  given by (3.9), along with the  
 404 assumptions stated in the Lemma, we obtain (6.9) from which we deduce  $g^{(j)} = \mathcal{O}(\epsilon)$   
 405 for all  $j \in \{2, 3, \dots, s\}$ .  $\square$

406 *Remark 6.2.* For type A time integrator, if  $g^n = \mathcal{O}(1)$ , we have from (6.7):

$$407 \quad g^{(1)} = \epsilon \frac{a_{11}}{a_{11}} vM \cdot \nabla_x \rho^{(1)} + O(\epsilon^2) = O(\epsilon).$$

408 This satisfies the induction hypothesis in Lemma 6.1. Further, (6.9) holds by omitting  
 409  $\sum_{k=1}^{j-1}$  terms for  $j = 1$ . Thus, (6.9) is true for  $j \in \{1, 2, \dots, s\}$ .

410 Lemma 6.1 enables to get an expansion of  $g^{(j)}$  that can be inserted in (6.8) to identify  
 411 the time discretization of the asymptotic limit. However, this leads to quite involved  
 412 calculations which requires to introduce some notations.

413 DEFINITION 6.3. For  $j \in \{1, 2, \dots, s\}$  and  $k_1, m \in \{1, 2, \dots, j\}$  we define

414 (6.10) 
$$\Pi_{j,k_1}^m = \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} (\mathcal{S}^{k_0} \mathcal{S}^{k_1} \mathcal{S}^{k_2} \dots \mathcal{S}^{k_{m-1}}) (\mathcal{R}^{k_m}) \right\rangle_V,$$

415 with

416 
$$\mathcal{S}^{k_0} = 1, \quad \mathcal{S}^{k_l} = \sum_{k_{l+1}=1}^{k_l-1} \frac{a_{k_l k_{l+1}}}{a_{k_{l+1} k_{l+1}}} \text{ for } l \in \{1, 2, \dots, m-1\}, \quad m \geq 2,$$

417 
$$\mathcal{R}^{k_m} = \sum_{k_{m+1}=1}^{k_m} a_{k_m k_{m+1}} L^{-1}(vM) \cdot \nabla_x \rho^{(k_{m+1})} - \sum_{k_{m+1}=1}^{k_m-1} \tilde{a}_{k_m k_{m+1}} L^{-1}(vM) \cdot A \rho^{(k_{m+1})}.$$

418

419 As usual, we will use the convention  $\sum_{j=1}^q \equiv 0$  if  $q \in \mathbb{Z} \setminus \mathbb{N}$ .

420 The term  $\Pi_{j,k_1}^m$  will be useful in the following study and deserves some remarks:  
 421 the index  $m$  denotes the depth of the embedded sums,  $j$  corresponds to the current  
 422 stage and  $k_1$  corresponds to the indexing over previous stages. We continue with the  
 423 following lemma which gives an induction relation on  $\Pi_{j,k_1}^m$ .

424 LEMMA 6.4. For  $j \geq 2$ , we have

425 
$$\Pi_{j,j}^m = \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^{m-1} \text{ for } m \in \{2, 3, \dots, j\}, \text{ and } \Pi_{j,k_1}^j = 0 \text{ for } k_1 \in \{1, 2, \dots, j-1\}.$$

426 *Proof.* For the first relation, considering  $k_1 = j$  (with  $j \geq 2$ ) in (6.10) leads to

427 
$$\Pi_{j,j}^m = \left\langle v (\mathcal{S}^{k_0} \mathcal{S}^j \mathcal{S}^{k_2} \dots \mathcal{S}^{k_{m-1}}) (\mathcal{R}^{k_m}) \right\rangle_V,$$

428 since  $a_{jj} \neq 0$ . Further, since  $\mathcal{S}^{k_1=j} = \sum_{k_2=1}^{j-1} \frac{a_{jk_2}}{a_{k_2 k_2}}$ , we get

429 
$$\Pi_{j,j}^m = \left\langle v \sum_{k_2=1}^{j-1} \frac{a_{jk_2}}{a_{k_2 k_2}} (\mathcal{S}^{k_0} \mathcal{S}^{k_2} \dots \mathcal{S}^{k_{m-1}}) (\mathcal{R}^{k_m}) \right\rangle_V$$

430 By employing the change of variables as  $k_\ell \rightarrow k_{\ell-1}$  for  $\ell \in \{2, 3, \dots, m\}$  in the right  
 431 hand side of above expression, we get

432 
$$\Pi_{j,j}^m = \left\langle v \sum_{k_1=1}^{j-1} \frac{a_{jk_1}}{a_{k_1 k_1}} (\mathcal{S}^{k_0} \mathcal{S}^{k_1} \dots \mathcal{S}^{k_{m-2}}) (\mathcal{R}^{k_{m-1}}) \right\rangle_V$$

433 
$$= \sum_{k_1=1}^{j-1} \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} (\mathcal{S}^{k_0} \mathcal{S}^{k_1} \dots \mathcal{S}^{k_{m-2}}) (\mathcal{R}^{k_{m-1}}) \right\rangle_V = \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^{m-1},$$

434 which proves the first identity.

435 For the second relation, considering  $m = j$  in (6.10) leads to

436 
$$\Pi_{j,k_1}^j = \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} (\mathcal{S}^{k_0} \mathcal{S}^{k_1} \mathcal{S}^{k_2} \dots \mathcal{S}^{k_{j-1}}) (\mathcal{R}^{k_j}) \right\rangle_V$$

437 We first prove the relation for  $j = 2$ . It is clear from Definition 6.3 that the summation  
 438 in  $\mathcal{S}^{k_1}$  goes from  $k_2 = 1$  to  $k_2 = k_1 - 1$ . For  $k_1 = 1$ , the summation goes to

439  $k_2 = k_1 - 1 = 0$ . Thus, since  $\mathcal{S}^{k_1}$  involves  $\sum_1^0$  for  $k_1 = 1$ , it is zero according to the  
 440 convention. Hence  $\Pi_{j,k_1}^j = 0$  for  $k_1 = 1$ .

441 We now prove the relation for  $j > 2$ . From Definition 6.3, it can be seen that the  
 442 summations in  $\mathcal{S}^{k_1}$  and  $\mathcal{S}^{k_2}$  go from  $k_2 = 1$  to  $k_2 = k_1 - 1$  and  $k_3 = 1$  to  $k_3 = k_2 - 1$   
 443 respectively. Thus, the summation in  $\mathcal{S}^{k_2}$  can go to atmost  $k_3 = k_2 - 1 = (k_1 - 1) - 1 =$   
 444  $k_1 - 2$ . Proceeding in this manner, we see that the summation in  $\mathcal{S}^{k_{j-1}}$  can go to  
 445 atmost  $k_j = k_1 - (j - 1)$ .

446 For  $k_1 \in \{1, 2, \dots, j-1\}$ ,  $k_j = k_1 - (j - 1) \in \mathbb{Z} \setminus \mathbb{N}$  so that  $\mathcal{S}^{k_{j-1}} = 0$  and hence  $\Pi_{j,k_1}^j = 0$   
 447 for  $k_1 \in \{1, 2, \dots, j-1\}$  which ends the proof.  $\square$

448 Now, we can use the previous Lemma to identify the asymptotic numerical scheme.

449 LEMMA 6.5. *When  $\epsilon \rightarrow 0$ , the numerical scheme (6.5)-(6.6) degenerates into*

$$450 \quad (6.11) \quad \rho^{(j)} = \rho^n + \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left( \sum_{\ell=1}^j (-1)^\ell \Pi_{j,k_1}^\ell \right) \quad \text{for } j \in \{1, 2, \dots, s\},$$

451 where  $\Pi_{j,k_1}^\ell$  is given by Definition 6.3.

452 *Proof.* We start with the macro equation in (6.5)

$$453 \quad \rho^{(j)} = \rho^n - \sum_{k_1=1}^j a_{jk_1} \frac{\Delta t}{\epsilon} \nabla_x \cdot \langle v g^{(k_1)} \rangle_V,$$

454 in which we insert  $g^{(k_1)}$  given by (6.9) to get

$$\begin{aligned} 455 \quad \rho^{(j)} &= \rho^n - \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \sum_{k_2=1}^{k_1} a_{k_1 k_2} L^{-1}(vM) \cdot \nabla_x \rho^{(k_2)} - \sum_{k_2=1}^{k_1-1} \tilde{a}_{k_1 k_2} L^{-1}(vM) \cdot A \rho^{(k_2)} \right) \right\rangle_V \\ 456 \quad &+ \frac{\Delta t}{\epsilon} \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \sum_{k_2=1}^{k_1-1} a_{k_1 k_2} g^{(k_2)} \right) \right\rangle_V + O(\epsilon) \\ 457 \quad &= \rho^n - \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} (\mathcal{S}^{k_0} \mathcal{R}^{k_1}) \right\rangle_V + \frac{\Delta t}{\epsilon} \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \sum_{k_2=1}^{k_1-1} a_{k_1 k_2} g^{(k_2)} \right) \right\rangle_V + O(\epsilon) \\ 458 \quad &= \rho^n - \Delta t \sum_{k_1=1}^j \nabla_x \cdot \Pi_{j,k_1}^1 + \frac{\Delta t}{\epsilon} \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \sum_{k_2=1}^{k_1-1} a_{k_1 k_2} g^{(k_2)} \right) \right\rangle_V + O(\epsilon). \end{aligned}$$

459 Inserting  $g^{(k_2)}$  from (6.9) in the above equation and simplifying as before, we get,  $\square$

$$460 \quad \rho^{(j)} = \rho^n - \Delta t \sum_{k_1=1}^j \nabla_x \cdot (\Pi_{j,k_1}^1 - \Pi_{j,k_1}^2) - \frac{\Delta t}{\epsilon} \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \sum_{k_2=1}^{k_1-1} \frac{a_{k_1 k_2}}{a_{k_2 k_2}} \sum_{k_3=1}^{k_2-1} a_{k_2 k_3} g^{(k_3)} \right) \right\rangle_V + O(\epsilon). \quad \blacksquare$$

461 This procedure can be continued  $(j - 1)$  times to finally get,

$$\begin{aligned}
462 \quad \rho^{(j)} &= \rho^n + \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left( \sum_{\ell=1}^{j-1} (-1)^\ell \Pi_{j,k_1}^\ell \right) \\
463 \quad &\quad - (-1)^{j-1} \frac{\Delta t}{\epsilon} \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \sum_{k_2=1}^{k_1-1} \frac{a_{k_1 k_2}}{a_{k_2 k_2}} \dots \sum_{k_{j-1}=1}^{k_{j-2}-1} \frac{a_{k_{j-2} k_{j-1}}}{a_{k_{j-1} k_{j-1}}} \sum_{k_j=1}^{k_{j-1}-1} a_{k_{j-1} k_j} g^{(k_j)} \right) \right\rangle_V + O(\epsilon) \\
464 \quad &= \rho^n + \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left( \sum_{\ell=1}^{j-1} (-1)^\ell \Pi_{j,k_1}^\ell \right) \\
465 \quad &\quad - (-1)^{j-1} \frac{\Delta t}{\epsilon} \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} \left( \mathcal{S}^{k_0} \mathcal{S}^{k_1} \dots \mathcal{S}^{k_{j-2}} \sum_{k_j=1}^{k_{j-1}-1} a_{k_{j-1} k_j} g^{(k_j)} \right) \right\rangle_V + O(\epsilon).
\end{aligned}$$

466 We know from Definition 6.3 that the summations in  $\mathcal{S}^{k_1}$  and  $\mathcal{S}^{k_2}$  go from  $k_2 = 1$  to  
467  $k_2 = k_1 - 1$  and  $k_3 = 1$  to  $k_3 = k_2 - 1$  respectively. Thus, the summation in  $\mathcal{S}^{k_2}$  can go  
468 to atmost  $k_3 = k_2 - 1 = (k_1 - 1) - 1 = k_1 - 2$ . Proceeding in this manner, we see that  
469 the summations in  $\mathcal{S}^{k_{j-2}}$  and  $\sum_{k_j=1}^{k_{j-1}-1} a_{k_{j-1} k_j} g^{(k_j)}$  go to atmost  $k_{j-1} = k_1 - (j - 2)$   
470 and  $k_j = k_1 - (j - 1)$  respectively.

471 Since the summation in  $k_1$  goes to atmost  $j$  in the above equation,  $k_j$  in the term  
472  $\sum_{k_j=1}^{k_{j-1}-1} a_{k_{j-1} k_j} g^{(k_j)}$  goes to atmost  $k_j = k_1 - (j - 1) = j - (j - 1) = 1$ , and  $k_{j-1}$   
473 in  $\mathcal{S}^{k_{j-2}}$  goes to atmost  $k_{j-1} = k_1 - (j - 2) = j - (j - 2) = 2$  and so on. Thus,  
474 only  $k_j = 1$  remains in the last summation so that  $\sum_{k_j=1}^{k_{j-1}-1} a_{k_{j-1} k_j} g^{(k_j)} = a_{21} g^{(1)} =$   
475  $\epsilon a_{21} L^{-1}(vM) \cdot \nabla_x \rho^{(1)} + O(\epsilon^2) = \frac{a_{21}}{a_{11}} \epsilon a_{11} L^{-1}(vM) \cdot \nabla_x \rho^{(1)} + O(\epsilon^2) = \epsilon \mathcal{S}^{k_{j-1}} \mathcal{R}^{k_j} + O(\epsilon^2)$ .  
476 Thus, we have

$$\begin{aligned}
477 \quad \rho^{(j)} &= \rho^n + \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left( \sum_{\ell=1}^{j-1} (-1)^\ell \Pi_{j,k_1}^\ell \right) \\
478 \quad &\quad - (-1)^{j-1} \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left\langle v \frac{a_{jk_1}}{a_{k_1 k_1}} (\mathcal{S}^{k_0} \mathcal{S}^{k_1} \dots \mathcal{S}^{k_{j-1}} \mathcal{R}^{k_j}) \right\rangle_V + O(\epsilon) \\
479 \quad &= \rho^n + \Delta t \sum_{k_1=1}^j \left[ \nabla_x \cdot \left( \sum_{\ell=1}^{j-1} (-1)^\ell \Pi_{j,k_1}^\ell \right) + \nabla_x \cdot \left( (-1)^j \Pi_{j,k_1}^j \right) \right] + O(\epsilon).
\end{aligned}$$

480 We can now prove the asymptotic property of the scheme (6.5)-(6.6).

481 **THEOREM 6.6.** *When  $\epsilon \rightarrow 0$ , the scheme (6.5)-(6.6) degenerates into*

482

$$\begin{aligned}
483 \quad (6.12) \quad \rho^{(j)} &= \rho^n - \Delta t \sum_{k=1}^j a_{jk} \nabla_x \cdot \left( \langle v \otimes L^{-1}(vM) \rangle_V \nabla_x \rho^{(k)} \right) \\
484 \quad &\quad + \Delta t \sum_{k=1}^{j-1} \tilde{a}_{jk} \nabla_x \cdot \left( \langle v \otimes L^{-1}(vM) \rangle_V A \rho^{(k)} \right), \text{ for } j \in \{1, 2, \dots, s\}. \\
485
\end{aligned}$$

486 *Proof.* From Lemma 6.5, the asymptotic limit  $\epsilon \rightarrow 0$  of the macro equation in

487 (6.5) is (for  $j \in \{1, 2, \dots, s\}$ )

$$\begin{aligned}
 488 \quad \rho^{(j)} &= \rho^n + \Delta t \sum_{k_1=1}^j \nabla_x \cdot \left( \sum_{\ell=1}^j (-1)^\ell \Pi_{j,k_1}^\ell \right) = \rho^n + \Delta t \nabla_x \cdot \left( \sum_{\ell=1}^j (-1)^\ell \left( \Pi_{j,j}^\ell + \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^\ell \right) \right) \\
 489 \quad &= \rho^n + \Delta t \nabla_x \cdot \left( -\Pi_{j,j}^1 + \sum_{\ell=2}^j (-1)^\ell \Pi_{j,j}^\ell + \sum_{\ell=1}^j (-1)^\ell \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^\ell \right). \quad \blacksquare
 \end{aligned}$$

491 Using the recurrence relation given by Lemma 6.4 and a change of indices lead to

$$\begin{aligned}
 492 \quad \rho^{(j)} &= \rho^n + \Delta t \nabla_x \cdot \left( -\Pi_{j,j}^1 + \sum_{\ell=2}^j (-1)^\ell \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^{\ell-1} + \sum_{\ell=1}^j (-1)^\ell \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^\ell \right) \\
 493 \quad &= \rho^n + \Delta t \nabla_x \cdot \left( -\Pi_{j,j}^1 - \sum_{\ell=1}^{j-1} (-1)^\ell \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^\ell + \sum_{\ell=1}^j (-1)^\ell \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^\ell \right) \\
 494 \quad &= \rho^n + \Delta t \nabla_x \cdot \left( -\Pi_{j,j}^1 + (-1)^j \sum_{k_1=1}^{j-1} \Pi_{j,k_1}^j \right). \\
 495
 \end{aligned}$$

496 From Lemma 6.4, we have  $\sum_{k_1=1}^{j-1} \Pi_{j,k_1}^j = 0$ , so that from Definition 6.3 we get

$$\begin{aligned}
 497 \quad \rho^{(j)} &= \rho^n + \Delta t \nabla_x \cdot (-\Pi_{j,j}^1) = \rho^n - \Delta t \nabla_x \cdot \left( \left\langle v \frac{a_{jj}}{a_{jj}} \mathcal{S}^{k_0} \mathcal{R}^{k_1=j} \right\rangle_V \right) \\
 498 \quad &= \rho^n - \Delta t \nabla_x \cdot \left( \left\langle v \left( \sum_{k_2=1}^{k_1} a_{k_1 k_2} L^{-1}(vM) \cdot \nabla_x \rho^{(k_2)} - \sum_{k_2=1}^{k_1-1} \tilde{a}_{k_1 k_2} L^{-1}(vM) \cdot A \rho^{(k_2)} \right) \right\rangle_V \right)_{k_1=j} \\
 499 \quad &= \rho^n - \Delta t \sum_{k_2=1}^j a_{j k_2} \nabla_x \cdot \left( \langle v \otimes L^{-1}(vM) \rangle_V \nabla_x \rho^{(k_2)} \right) + \Delta t \sum_{k_2=1}^{j-1} \tilde{a}_{j k_2} \nabla_x \cdot \left( \langle v \otimes L^{-1}(vM) \rangle_V A \rho^{(k_2)} \right), \quad \blacksquare \\
 500
 \end{aligned}$$

501 which ends the proof.  $\square$

502 *Remark 6.7.* For CK-ARS schemes with well-prepared initial data, we obtain  
 503  $g^{(1)} = g^n = O(\epsilon)$  and  $\rho^{(1)} = \rho^n$ . The presentation in this section will apply for  
 504 CK-ARS from the second RK stage onwards. For instance, Definition 6.3 applies for  
 505 CK-ARS with the following change in indexes:  $j \in \{2, 3, \dots, s\}$ ,  $k_1, m \in \{2, 3, \dots, j\}$  and  
 506 all the summations involved start from 2 instead of 1 since  $a_{11} = 0$ . The lemmas and  
 507 theorems that follow also undergo the corresponding change in indexes, and the AP  
 508 property for CK-ARS can be observed for  $j \in \{2, 3, \dots, s\}$ .

509 *Remark 6.8.* Upon incorporating the spatial matrices corresponding to staggered  
 510 grid in place of the continuous gradient operator, we obtain in the limit  $\epsilon \rightarrow 0$ ,

$$\begin{aligned}
 512 \quad (6.13) \quad \rho^{(j)} &= (I + a_{jj} \Delta t \mathbf{G}_{\text{cen}_\rho} (\langle v \otimes L^{-1}(vM) \rangle_V \mathbf{G}_{\text{cen}_g}))^{-1} \times \\
 513 \quad &\left( \rho^n - \sum_{k=1}^{j-1} a_{jk} \Delta t \mathbf{G}_{\text{cen}_\rho} (\langle v \otimes L^{-1}(vM) \rangle_V \mathbf{G}_{\text{cen}_g} \rho^{(k)}) \right. \\
 514 \quad &\quad \left. + \sum_{k=1}^{j-1} \tilde{a}_{jk} \Delta t \mathbf{G}_{\text{cen}_\rho} (\langle v \otimes L^{-1}(vM) \rangle_V \mathbf{G}_{\text{avg}_g} A \rho^{(k)}) \right). \\
 515
 \end{aligned}$$



516 The matrices  $\mathbf{G}_{\text{cen}_\rho}$ ,  $\mathbf{G}_{\text{cen}_g}$  are given in subsection 5.2 and  $\mathbf{G}_{\text{avg}_g} = \frac{1}{2} \text{circ}([1, 1])$ . Thus,  
 517  $A(\rho^{(k)})_{x_{i+1/2}} = \frac{1}{2} A(\rho_{i+1}^{(k)} + \rho_i^{(k)}) = (\mathbf{G}_{\text{avg}_g} A\rho^{(k)})_i$ . This results in a central discretiza-  
 518 tion of the advection term in the macro equation. Thus, we obtain a consistent internal  
 519 RK stage approximation of the advection-diffusion equation in the limit  $\epsilon \rightarrow 0$ .

520 **6.2. Inflow Boundaries.** So far, periodic boundary conditions were considered.  
 521 In this part, we consider inflow boundary conditions for  $f$  solution to (2.1)

$$522 \quad (6.14) \quad f(t, x, v) = f_b(t, x, v), \quad (x, v) \in \partial\Omega \times V \text{ such that } v \cdot n(x) < 0, \quad \forall t,$$

523 where  $f_b$  is a given function and  $n(x)$  denotes the unitary outgoing normal vector to  
 524  $\partial\Omega$ . As mentioned in [25, 24], such boundary conditions cannot be adapted naturally  
 525 to the standard micro-macro unknown  $\rho(t, x)$  and  $g(t, x, v)$  solution to (2.6). To  
 526 overcome this drawback, another micro-macro decomposition is introduced in [24]

$$527 \quad (6.15) \quad f = \bar{\rho}M + \bar{g}, \quad \bar{\rho}(t, x) = \langle f(t, x, \cdot) \rangle_{V_-}, \quad \langle \bar{g}(t, x, \cdot) \rangle_{V_-} = 0, \quad \langle f \rangle_{V_-} = \frac{\int_{V_-} f d\mu}{\int_{V_-} M d\mu},$$

528 where the velocity domain  $V_-$  is defined by

$$529 \quad (6.16) \quad V_-(x) = \{v \in V, \omega(x, v) < 0\}, \quad V_+(x) = V \setminus V_-(x).$$

530 The function  $\omega(x, v)$  extends  $v \cdot n(x)$  in the interior of domain. Some examples of  
 531  $\omega(x, v)$  for different geometries are provided in [24]. It can be seen that the bound-  
 532 ary conditions for  $\bar{\rho}(t, x)$  and  $\bar{g}(t, x, v)$  can be evaluated from the inflow boundary  
 533 condition in (6.14). Indeed, for  $(x, v) \in \partial\Omega \times V$  such that  $v \cdot n(x) < 0, \forall t$ , we define

$$534 \quad (6.17) \quad \bar{\rho}_b(t, x) = \langle f_b(t, x, \cdot) \rangle_{V_-}, \quad \bar{g}_b(t, x, v) = f_b(t, x, v) - \bar{\rho}_b(t, x)M(v).$$

535 The derivation of the micro-macro model needs to be adapted to this decomposi-  
 536 tion. The projector  $\Pi^-$  is defined as  $\Pi^- h = \langle h \rangle_{V_-} M$ . Then, substituting (6.15) into  
 537 (2.1) and applying  $\Pi^-$  and  $I - \Pi^-$  enable to get the macro and micro equations:

$$538 \quad (6.18) \quad \partial_t \bar{\rho} + \frac{1}{\epsilon} \langle vM \rangle_{V_-} \cdot \nabla_x \bar{\rho} + \frac{1}{\epsilon} \nabla_x \cdot \langle v\bar{g} \rangle_{V_-} = \frac{1}{\epsilon^2} \langle L\bar{g} \rangle_{V_-},$$

$$539 \quad (6.19) \quad \partial_t \bar{g} + \frac{1}{\epsilon} (I - \Pi^-) (v \cdot \nabla_x \bar{g}) + \frac{1}{\epsilon} (I - \Pi^-) vM \cdot \nabla_x \bar{\rho} = \frac{1}{\epsilon^2} \tilde{L}\bar{g},$$

541 where  $\tilde{L} = (I - \Pi^-) L$ . Moreover, it can be seen that  $\tilde{L} = (I - \Pi^-) L (I - \Pi^-) =$   
 542  $(I - \Pi^-) L (I - \Pi)$  since  $\Pi^- h, \Pi h \in \mathcal{N}(L), \forall h$ .

543 The macro equation (6.18) turns out to be more complicated than the one obtained  
 544 for standard micro-macro decomposition. It can be made simpler by using  $\rho = \bar{\rho} +$   
 545  $\langle \bar{g} \rangle_V, f = \rho M - \langle \bar{g} \rangle_V M + \bar{g}$ , obtained from the decompositions (2.6) and (6.15).  
 546 Applying  $\Pi$  to (2.1) instead of  $\Pi^-$ , we obtain the simpler macro equation,

$$547 \quad (6.20) \quad \partial_t \rho + \frac{1}{\epsilon} \nabla_x \cdot \langle v\bar{g} \rangle_V = 0,$$

548 and the micro-macro system that we will consider in the sequel is (6.19)-(6.20).

549 **6.2.1. Numerical scheme.** In this part, we present the fully discretized scheme  
 550 to approximate (6.19)-(6.20). The boundary conditions on  $\bar{\rho}_b$  and  $\bar{g}_b$  in (6.17) will be  
 551 utilised along with the relation  $\rho = \bar{\rho} + \langle \bar{g} \rangle_V$  that allows to link  $\rho$  and  $\bar{\rho}$  in the interior  
 552 of the domain. We will use a staggered grid in space following [24] and a high order  
 553 scheme in time, following the strategy developed previously. To ease the reading, only  
 554 the first order version will be presented.

555 First, we present the space approximation based on a staggered grid. Let us consider  
 556 the space interval  $[0, L]$  with two grids:  $x_i = i\Delta x$  and  $x_{i+1/2} = (i + 1/2)\Delta x$ ,  $\Delta x =$   
 557  $L/(N_x - 1)$ . The 'interior' variables such as  $\rho, \bar{\rho}$  are stored at grid points  $x_i$  with  
 558  $i = 1, \dots, N_x - 2$  and  $\bar{g}$  is stored at  $i + 1/2 = 1/2, \dots, N_x - 3/2$ . We also use  
 559 the variable  $\bar{g}_{cl} = \bar{g} \cup \bar{g}_b \in \mathbb{R}^{N_x+1}$ . The whole domain including boundary will be  
 560 considered for the micro unknown  $\bar{g}$  so that the components of  $\bar{g}_{cl}$  correspond to the  
 561 grid indices  $i + 1/2 = -1/2, \dots, N_x - 1/2$ . The matrices corresponding to spatial  
 562 operators are given by

$$563 \quad (6.21) \quad \mathbf{B}_{\text{upw}}^- = \frac{1}{\Delta x} \text{circ}([-1, 1])_{(N_x-1) \times (N_x+1)}, \mathbf{B}_{\text{upw}}^+ = \frac{1}{\Delta x} \text{circ}([0, -1, 1])_{(N_x-1) \times (N_x+1)},$$

$$564 \quad (6.22) \quad \mathbf{B}_{\text{cen}_\rho} = \frac{1}{\Delta x} \text{circ}([-1, 1])_{(N_x-2) \times (N_x-1)}, \mathbf{B}_{\text{avg}} = \frac{1}{2} \text{circ}([1, 1])_{(N_x-2) \times (N_x-1)},$$

$$565 \quad (6.23) \quad \mathbf{B}_{\text{cen}_g} = \frac{1}{\Delta x} \text{circ}_b([-1, 1])_{(N_x-1) \times (N_x-2)}.$$

566 The  $\text{circ}_b$  definition is presented in Appendix A. Further, we also introduce a vector  
 567 containing the boundary values of  $\bar{\rho}$  as  $\bar{\rho}_{bd} = \frac{1}{\Delta x} [-\bar{\rho}_{b_{i=0}}, 0, 0, \dots, 0, \bar{\rho}_{b_{i=N_x-1}}]_{(N_x-1) \times 1}^T$ .  
 568 We now present our scheme by using this matrix notation. For simplicity, we assume  
 569 that  $\bar{\rho}_{bd}$  is time invariant. We also use the following notations:

$$570 \quad \bar{\mathcal{T}}h = (I - \Pi^-) (v^+ \mathbf{B}_{\text{upw}}^- + v^- \mathbf{B}_{\text{upw}}^+) h, \bar{\mathcal{D}}_{\epsilon, \Delta t} = \langle v (\epsilon^2 I - \Delta t \tilde{L})^{-1} \Delta t (I - \Pi^-) (vM) \rangle_V,$$

$$572 \quad \bar{\mathcal{E}}_{\epsilon, \Delta t} = \langle (\epsilon^2 I - \Delta t \tilde{L})^{-1} \Delta t (I - \Pi^-) (vM) \rangle_V, \quad \bar{\mathcal{I}}_{\epsilon, \Delta t} = (\epsilon^2 I - \Delta t \tilde{L})^{-1}, \quad \bar{\mathcal{J}} = (I - \Pi^-) (vM). \blacksquare$$

573 The micro equation (6.19) is discretised in time as in the previous (periodic) case

$$574 \quad (6.24) \quad \bar{g}^{n+1} = \bar{\mathcal{I}}_{\epsilon, \Delta t} (\epsilon^2 \bar{g}^n - \epsilon \Delta t \bar{\mathcal{T}} \bar{g}_{cl}^n - \epsilon \Delta t \bar{\mathcal{J}} \mathbf{B}_{\text{cen}_g} \bar{\rho}^{n+1} - \epsilon \Delta t \bar{\mathcal{J}} \bar{\rho}_{bd}),$$

575 and for the macro equation (6.20), we obtain

$$576 \quad \frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\epsilon} \langle v \mathbf{B}_{\text{cen}_\rho} \bar{g}^{n+1} \rangle_V = 0$$

577 Substituting  $\bar{g}^{n+1}$  in the above equation, we get

$$578 \quad (6.25) \quad \rho^{n+1} = \rho^n - \Delta t \mathbf{B}_{\text{cen}_\rho} \langle v \bar{\mathcal{I}}_{\epsilon, \Delta t} (\epsilon \bar{g}^n - \Delta t \bar{\mathcal{T}} \bar{g}_{cl}^n - \Delta t \bar{\mathcal{J}} \mathbf{B}_{\text{cen}_g} \bar{\rho}^{n+1} - \Delta t \bar{\mathcal{J}} \bar{\rho}_{bd}) \rangle_V.$$

579 In index notation, we use  $\rho_i^{n+1} = \bar{\rho}_i^{n+1} + \frac{1}{2} \langle \bar{g}_{i-1/2}^{n+1} + \bar{g}_{i+1/2}^{n+1} \rangle_V$  (since  $\rho = \bar{\rho} + \langle \bar{g} \rangle_V$ ) to  
 580 match the two grids. In matrix notation, this becomes  $\rho^{n+1} = \bar{\rho}^{n+1} + \mathbf{B}_{\text{avg}} \langle \bar{g}^{n+1} \rangle_V$   
 581 with  $\mathbf{B}_{\text{avg}}$  given by (6.22). Substituting this into the above equation and inserting the  
 582 expression for  $\bar{g}^{n+1}$  into  $\mathbf{B}_{\text{avg}} \langle \bar{g}^{n+1} \rangle_V$  enable to update the interior macro unknown  
 583

$$584 \quad (6.26) \quad \bar{\rho}^{n+1} = (I - \epsilon \mathbf{B}_{\text{avg}} (\bar{\mathcal{E}}_{\epsilon, \Delta t} \mathbf{B}_{\text{cen}_g}) - \Delta t \mathbf{B}_{\text{cen}_\rho} (\bar{\mathcal{D}}_{\epsilon, \Delta t} \mathbf{B}_{\text{cen}_g}))^{-1} \times \\
 585 \quad (\rho^n - \mathbf{B}_{\text{avg}} \langle \bar{\mathcal{I}}_{\epsilon, \Delta t} (\epsilon^2 \bar{g}^n - \epsilon \Delta t \bar{\mathcal{T}} \bar{g}_{cl}^n - \epsilon \Delta t \bar{\mathcal{J}} \bar{\rho}_{bd}) \rangle_V \\
 586 \quad - \Delta t \mathbf{B}_{\text{cen}_\rho} \langle v \bar{\mathcal{I}}_{\epsilon, \Delta t} (\epsilon \bar{g}^n - \Delta t \bar{\mathcal{T}} \bar{g}_{cl}^n - \Delta t \bar{\mathcal{J}} \bar{\rho}_{bd}) \rangle_V).$$

588 The right hand side of above expression involves only known quantities so that  $\bar{\rho}^{n+1}$   
 589 can be updated from (6.26) which can then be used to update  $\bar{g}^{n+1}$  in (6.24). Then,  
 590 we update  $\bar{g}_{cl}^{n+1}$  thanks to the boundary conditions (6.17), and finally  $\rho^{n+1}$  can be  
 591 computed from  $\rho^{n+1} = \bar{\rho}^{n+1} + \mathbf{B}_{\text{avg}} \langle \bar{g}^{n+1} \rangle_V$ . In the limit  $\epsilon \rightarrow 0$ , the above equation  
 592 becomes,

$$593 \quad \bar{\rho}^{n+1} = \left( I + \Delta t \mathbf{B}_{\text{cen}_\rho} \left( \left\langle v \otimes \tilde{L}^{-1} \bar{\mathcal{J}} \right\rangle_V \mathbf{B}_{\text{cen}_g} \right) \right)^{-1} \left( \rho^n - \Delta t \mathbf{B}_{\text{cen}_\rho} \left( \left\langle v \otimes \tilde{L}^{-1} \bar{\mathcal{J}} \right\rangle_V \bar{\rho}_{bd} \right) \right)$$

594 This is a consistent discretization of the diffusion equation in (2.5) since  $\langle v \otimes \tilde{L}^{-1} \bar{\mathcal{J}} \rangle_V =$   
 595  $\langle v \otimes L^{-1}(vM) \rangle_V = -\kappa$ . Further, the high order scheme in time can be constructed  
 596 in a similar manner as before.

597 **7. Numerical results.** In this section, we present the numerical validation of  
 598 our high order asymptotic preserving schemes in different configurations.

599 **7.1. Diffusion asymptotics.** First, we check time and space accuracy for the  
 600 micro-macro scheme in the diffusion limit.

601 **7.1.1. Time order of accuracy.** The spatial domain  $L = [0, 2\pi]$  of the prob-  
 602 lem is discretized using  $N_x = 50$  grid points. The velocity domain is truncated to  
 603  $[-v_{\text{max}}, v_{\text{max}}]$  with  $v_{\text{max}} = 5$  and we take  $\Delta v = 1$ . The initial condition is:

$$604 \quad \rho(0, x) = 1 + \cos(x)$$

$$605 \quad \text{Well-prepared data (WP): } g(0, x, v) = \epsilon^2 (I - \Pi) (v^2 M) \rho(0, x)$$

$$606 \quad \text{Non-well prepared data (N-WP): } g(0, x, v) = (I - \Pi) (v^2 M) \rho(0, x),$$

608 with  $M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ . Periodic boundary conditions are used on both  $\rho$  and  $g$ .  
 609 The spatial terms are discretised by using the atmost-third order accurate matric-  
 610 es on non-staggered grid presented in subsection 5.3. The final time is  $T = 0.5$ ,  
 611 and the following  $\Delta t$  are considered to validate the different high order time in-  
 612 tegrators:  $\Delta t = 0.5, 0.1, 0.05, 0.01, 0.005, 0.001$ . The type A micro-macro schemes  
 613 constructed using the Butcher tableau corresponding to DP-A(1, 2, 1), DP2-A(2, 4, 2)  
 614 and DP1-A(2, 4, 2) are considered. Although DP1-A(2, 4, 2) is second order accurate,  
 615 the implicit part of it when used separately is third order accurate. Further, we also  
 616 consider the type CK-ARS micro-macro schemes constructed using Butcher tableau  
 617 corresponding to ARS(1, 1, 1), ARS(2, 2, 2) and ARS(4, 4, 3). The Butcher tableau of  
 618 different time integrators utilised are presented in Appendix B.

619  
 620 In Figure 1, we plot the time error for the different time integrators in both WP  
 621 and N-WP cases and for different values of  $\epsilon$ . Note that the reference solution for  
 622 each curve is obtained by using the same micro-macro scheme corresponding to that  
 623 curve with  $\Delta t = 10^{-4}$ . For  $\epsilon = 1$ , the required orders of accuracy are recovered for  
 624 type A schemes with both N-WP and WP initial data, as observed in Figures 1a  
 625 and 1b. For  $\epsilon = 10^{-4}$ , due to the asymptotic degeneracy of our scheme into a fully-  
 626 implicit scheme for diffusion equation, only the implicit part of the Butcher tableau  
 627 plays a role. Hence DP1-A(2, 4, 2) becomes third order accurate in time, while DP-  
 628 A(1, 2, 1) and DP2-A(2, 4, 2) are first and second order accurate respectively. This is  
 629 shown in Figures 1c and 1d. On the other hand, CK-ARS schemes with both N-WP  
 630 and WP initial data for  $\epsilon = 1$  recover the required orders of accuracy as shown in  
 631 Figures 1e and 1f. However, for  $\epsilon = 10^{-4}$ , orders of accuracy are observed only when  
 632 WP initial data are used (Figure 1h). As shown in the analyses presented in previous

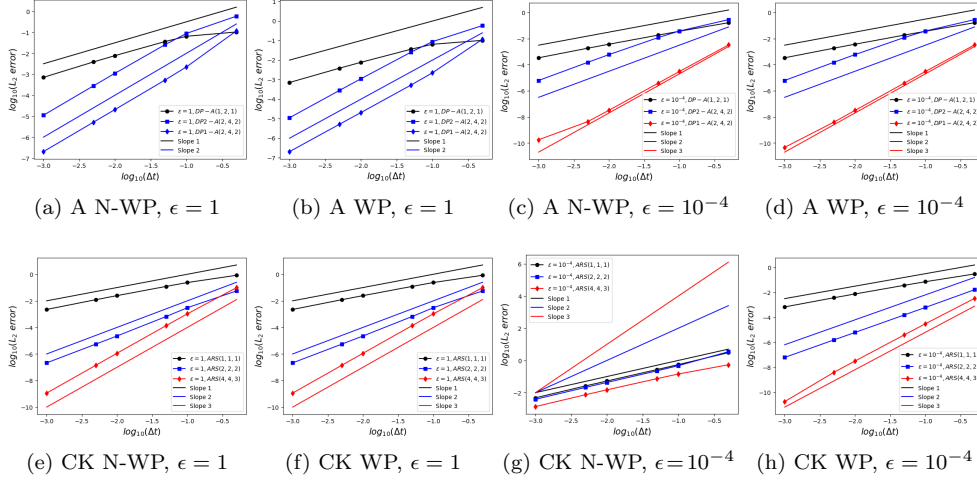


FIG. 1. Accuracy in time for different type A and CK-ARS time integrators (both WP and N-WP initial data). The reference solution is obtained from the micro-macro with  $\Delta t = 10^{-4}$ .

633 sections, usage of N-WP initial data for CK-ARS time integrators does not allow the  
 634 asymptotic accuracy (Figure 1g), as discussed in [8].

635 Since we proved the asymptotic preserving property, the diffusion solution is used as  
 636 reference solution in the asymptotic regime ( $\epsilon = 10^{-4}$ ) with  $\Delta t = 10^{-4}$  (in Figure 2)  
 637 to check the orders of accuracy of high order integrators. The results are similar to  
 638 the ones obtained for  $\epsilon = 10^{-4}$  in Figure 1, except that here we observe a plateau  
 639 for third order scheme and small  $\Delta t$ . This is due to the  $\mathcal{O}(\epsilon^2)$  difference between the  
 640 schemes based on micro-macro and diffusion models. This error dominates  $\mathcal{O}(\Delta t^3)$   
 641 error, and hence it is observed.

642 **7.1.2. Space order of accuracy.** The problem set-up is the same as described  
 643 in the previous subsection, except for the following changes. Here, we consider the  
 644 final time to be  $T = 0.01$  and  $\Delta t = 0.001$  so that the error in time is small enough to  
 645 study the spatial accuracy. To do so, we consider the following number of points in  
 646 space:  $N_x = 20, 24, 30, 40$  and  $60$ . The reference solution is obtained with  $N_x = 120$ .  
 647 Since the spatial accuracy plots obtained from different time integrators are quite  
 648 similar, we present only the plots obtained by using DP1-A(2, 4, 2) and ARS(4, 4, 3)  
 649 for different values of  $\epsilon$  ( $\epsilon = 10^{-4}, 0.2, 1$ ) in Figures 3a and 3b. For the spatial  
 650 discretization, we only show the results obtained by the third order spatial matrices  
 651 on non-staggered grid presented in subsection 5.3 so that the scheme is expected to  
 652 be third order accurate in space. In Figures 3a and 3b, the expected order is observed  
 653 for the two time integrators and for the three considered values of  $\epsilon$ .

654 **7.1.3. Qualitative results.** In this part, we compare the density obtained by  
 655 the micro-macro equation (MM), the linear kinetic equation with BGK collision oper-  
 656 ator (BGK) and the asymptotic diffusion equation, for different values of  $\epsilon$ . The  
 657 MM scheme described in previous sections is utilised, the BGK is discretized using an  
 658 IMEX (implicit treatment of collision term and explicit treatment of transport term)  
 659 scheme whereas for the diffusion model, an implicit scheme is used. For all three  
 660 models, the Butcher tableau corresponding to DP1-A(2, 4, 2) time integrator is used.  
 661 For the spatial discretization, we use third order scheme on non-staggered grid.

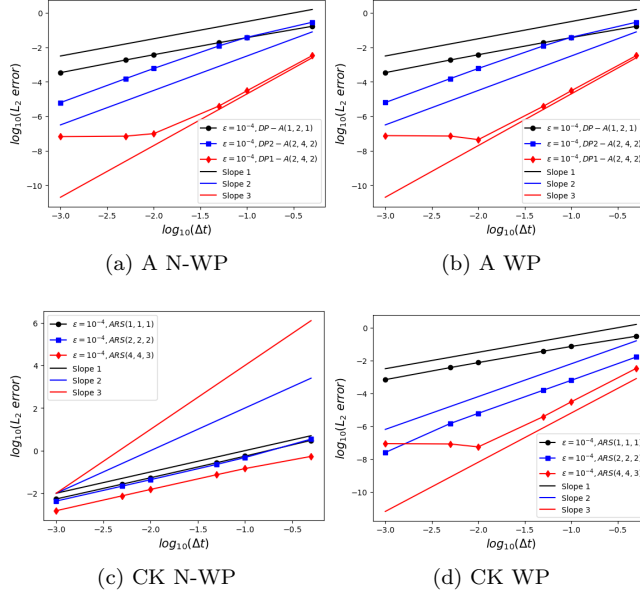


FIG. 2. Accuracy in time for different type A and CK-ARS time integrators (both WP and N-WP initial data). The reference solution is obtained from the diffusion equation with  $\Delta t = 10^{-4}$ .

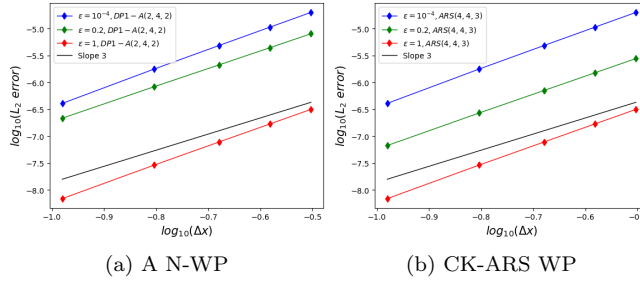


FIG. 3. Accuracy in space for the third order spatial scheme coupled with DP1-A(2,4,2) (left) and ARS(4,4,3) (right) for the time approximation.

662 The problem domain  $L = [0, 2\pi]$  is discretised using  $N_x = 20$  grid points for all the  
 663 three models. The final time is  $T = 0.5$ , and  $\Delta t = 0.005$ . We use the same N-WP  
 664 initial and boundary conditions described in the previous subsection. Further, we also  
 665 consider the same velocity discretization as before for both MM and BGK models.  
 666 In Figure 4a for rarefied regime ( $\epsilon = 1$ ), the MM and BGK models compare very  
 667 well, while the diffusion model is different as expected. In the intermediate regime  
 668 ( $\epsilon = 0.2$ ), the BGK and MM models match very well while the diffusion model is  
 669 slightly different. For  $\epsilon = 10^{-4}$ , we only compare MM and the diffusion in Figure 4c  
 670 and illustrate the AP property of the time integrators used for MM.

671 **7.2. Advection-diffusion asymptotics.** In this subsection, we present the  
 672 time accuracy of our high order micro-macro scheme for the advection-diffusion case.  
 673 As in the diffusion case, the spatial domain  $L = [0, 2\pi]$  is discretised using  $N_x = 20$   
 674 grid points whereas the velocity domain is  $[-v_{\max}, v_{\max}]$  with  $v_{\max} = 5$  and  $\Delta v = 1$ .

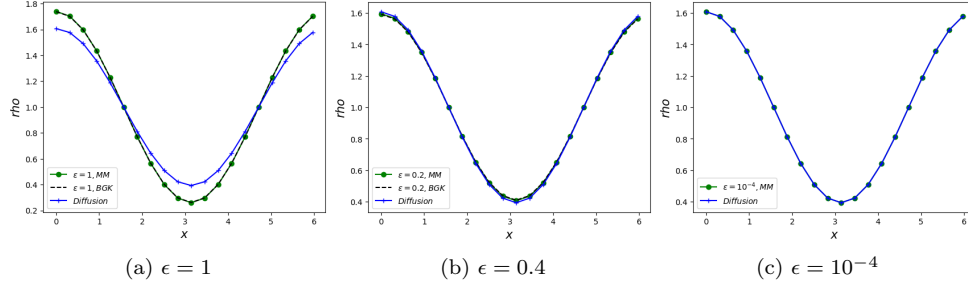
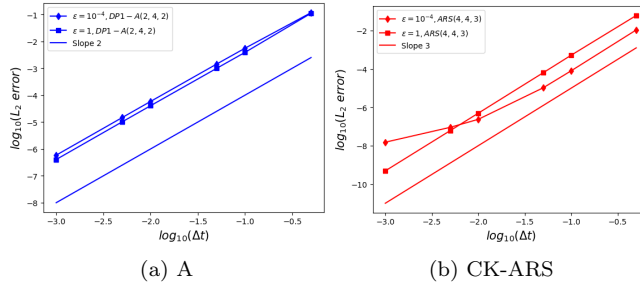


FIG. 4. Qualitative results for diffusion asymptotics


 FIG. 5. Accuracy in time. Left: DP1-A(2,4,2) (N-WP initial data). Right: ARS(4,4,3) (WP initial data). The reference solution is obtained from the micro-macro scheme with  $\Delta t = 10^{-4}$ .

675 The initial condition for the problem is:

676 (7.1) 
$$\rho(0, x) = \sin(x)$$

677 (7.2) Well-prepared data (WP):  $g(0, x, v) = \epsilon^2(I - \Pi)(v^2 M)\rho(0, x)$

678 (7.3) Non-well prepared data (N-WP):  $g(0, x, v) = (I - \Pi)(v^2 M)\rho(0, x),$

680 with  $M(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$ . Periodic boundary conditions are used on both  $\rho$  and  $g$ .  
 681 The spatial terms are discretised by using the atmost-first order accurate matrices  
 682 on staggered grid presented in subsection 5.2. The final time is  $T = 0.5$ , and the  
 683 following time steps are considered:  $\Delta t = 0.5, 0.1, 0.05, 0.01, 0.005, 0.001$ . We observe  
 684 the time order of accuracy for both  $\epsilon = 1$  and  $\epsilon = 10^{-4}$ . We choose the highest order  
 685 time integrator in both type A and CK-ARS schemes for studying the time accuracy.  
 686 Hence, we consider DP1-A(2, 4, 2) and ARS(4, 4, 3) with N-WP and WP data respec-  
 687 tively.

688 Asymptotically, our micro-macro scheme degenerates to a consistent scheme for the  
 689 advection-diffusion equation with advection and diffusion terms being treated explic-  
 690 itly and implicitly respectively. Hence, unlike the case of diffusion asymptotics for  
 691 which an extra order is observed asymptotically, DP1-A(2, 4, 2) remains second order  
 692 accurate for  $\epsilon = 10^{-4}$  since both explicit and implicit matrices of the Butcher tableau  
 693 are involved here (Figure 5a). For  $\epsilon = 1$ , the required second order accuracy is ob-  
 694 served. Further, the required third order accuracy of ARS(4, 4, 3) is observed for both  
 695  $\epsilon = 10^{-4}, 1$  in Figure 5b, since well-prepared initial data is considered.

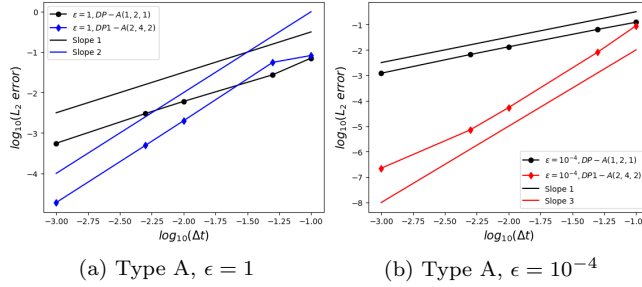


FIG. 6. Accuracy in time with type A schemes for  $\epsilon = 1$  (left) and  $\epsilon = 10^{-4}$  (right). The reference solution is obtained from the micro-macro for inflow boundaries scheme with  $\Delta t = 10^{-4}$ .

696 **7.3. Inflow boundary condition.** In this subsection, the high order numerical  
 697 scheme for micro-macro model that allows inflow boundary conditions is validated  
 698 numerically. We first present the time accuracy results for high order schemes. Then,  
 699 some qualitative plots are shown for two tests with zero inflow at the right boundary,  
 700 and equilibrium and non-equilibrium inflows respectively at the left boundary.

701 **7.3.1. Time order of accuracy.** If the domain of the problem is a half-plane,  
 702  $\omega(x, v) = [-v, 0, 0, \dots]$  can be chosen  $\forall x$  as described in [24]. Here, for numerical  
 703 purposes, we consider a domain of  $L = [0, 2]$  and assume that the right boundary  
 704 does not influence the dynamics.

705 The spatial domain is discretised using  $N_x = 20$  grid points and the velocity domain  
 706 is  $[-v_{\max}, v_{\max}]$  with  $v_{\max} = 5$  with  $\Delta v = 1$ . The initial conditions at all interior  
 707 points and right boundary conditions for the variables  $\rho, \bar{\rho}$  and  $\bar{g}$  are considered to be  
 708 0. The left boundary conditions (for  $v_k > 0$ ) are:

$$709 \quad (7.4) \quad f(t, x_i = 0, v_k) = M(v_k), \quad \bar{\rho}(t, x_i = 0) = 1, \quad \bar{g}(t, x_{i+1/2} = -\Delta x/2, v_k) = 0,$$

710 with  $M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$ . The final time is  $T = 0.1$ , and the following time steps are  
 711 considered to check the accuracy in time:  $\Delta t = 0.1, 0.05, 0.01, 0.005, 0.001$ . Like in the  
 712 previous problems, we observe the time order of accuracy for both  $\epsilon = 1$  and  $\epsilon = 10^{-4}$ .  
 713 The time integrators considered are DP-A(1, 2, 1) and DP1-A(2, 4, 2). The reference  
 714 solution for each curve in Figure 6 is obtained by using the same micro-macro scheme  
 715 corresponding to that curve with  $\Delta t = 10^{-4}$ . For type A time integrators with  $\epsilon = 1$   
 716 in Figure 6a, first and second order accuracies of DP-A(1, 2, 1) and DP1-A(2, 4, 2)  
 717 are observed. In Figure 6b for  $\epsilon = 10^{-4}$ , first and third order accuracies of DP-  
 718 A(1, 2, 1) and DP1-A(2, 4, 2) respectively are observed. As for the (periodic) diffusion  
 719 case, DP1-A(2, 4, 2) turns out to be third order accurate since only the implicit part  
 720 of Butcher tableau is involved asymptotically. For ARS(2, 2, 2) and ARS(4, 4, 3) time  
 721 integrators (not shown here), order reduction to first order for  $\epsilon = 1$  (due to the initial  
 722 condition). However, for  $\epsilon = 10^{-4}$ , the required second and third orders respectively  
 723 are observed.

724 **7.3.2. Qualitative results for equilibrium inflow.** In this part, we consider  
 725 the same problem as before and present a comparison of density plots obtained by  
 726 using schemes based on micro-macro (MM), full-kinetic (BGK) and diffusion models,  
 727 for different regimes of  $\epsilon$ . The boundary conditions for diffusion model  $\rho(t, x = 0) = 1$   
 728 and  $\rho(t, x = 2) = 0$ . The final time is  $T = 0.1$ ,  $N_x = 40$  and  $\Delta t = 0.001$ . Further,

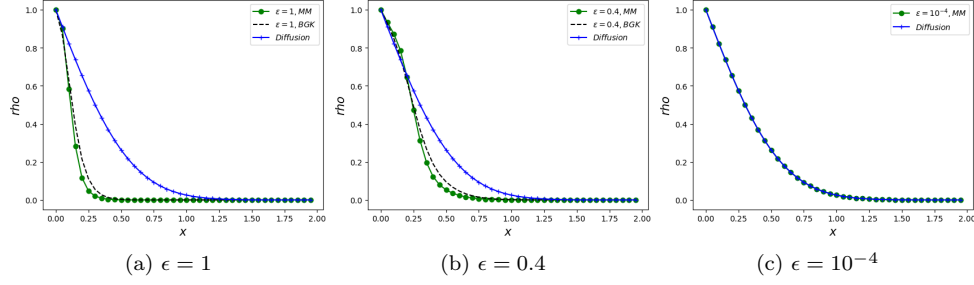


FIG. 7. Qualitative results for equilibrium inflow at the left boundary.

729 we consider the same velocity discretization as before for both MM and BGK models.  
 730 The results for MM are obtained by DP1-A(2, 4, 2) time integrator.  
 731 In Figure 7a for rarefied regime ( $\epsilon = 1$ ), the MM and BGK results are in good  
 732 agreement. In the intermediate regime ( $\epsilon = 0.4$ ) in Figure 7b, the MM and BGK  
 733 results are still close, and still different from the diffusion one. For  $\epsilon = 10^{-4}$ , only  
 734 MM and the diffusion are plotted and are found to be in very good agreement, thereby  
 735 illustrating the AP property of the numerical scheme for MM.

736 **7.3.3. Qualitative results for non-equilibrium inflow.** In this part, we con-  
 737 sider the same problem as before, but the left boundary condition is chosen as (for  
 738  $v_k > 0$ )

$$739 \quad (7.5) \quad f(t, x_i=0, v_k) = v_k M_k, \quad \bar{\rho}(t, x_i=0) = \langle f(t, x_i=0, v_k) \rangle_{V_-}$$

$$740 \quad (7.6) \quad \bar{g}(t, x_{i+1/2} = -\frac{\Delta x}{2}, v_k) = 2(f(t, x_i=0, v_k) - \bar{\rho}(t, x_i=0) M_k) - \bar{g}(t, x_{i+1/2} = \frac{\Delta x}{2}, v_k).$$

742 The number of grid points, velocity discretization, final time and time step are the  
 743 same as in the previous (equilibrium inflow) case. Here, we present a comparison of  
 744 plots obtained by using schemes based on MM, BGK and diffusion models, for different  
 745 regimes of  $\epsilon$ . The scheme described in subsection 6.2.1 is used for the micro-macro  
 746 model and a standard BGK approximation where only inflow boundary condition is  
 747 needed serves as a reference. For diffusion, the diffusion term is treated implicitly and  
 748 the left boundary condition for diffusion model is obtained from [18] which translates  
 749 in our context

$$750 \quad \rho(t, x_i=0) = \frac{\sum_{v_k > 0} v_k f(t, x_i=0, v_k) \Delta v}{\sum_{v_k > 0} v_k M_k \Delta v}$$

$$751 \quad + \frac{1}{\kappa \sum_{v_k} M_k \Delta v} \sum_{v_k > 0} v_k^2 \left( f(t, x_i=0, v_k) - M_k \frac{\sum_{v_k > 0} v_k f(t, x_i=0, v_k) \Delta v}{\sum_{v_k > 0} v_k M_k \Delta v} \right) \Delta v.$$

754 In Figure 8a for rarefied regime ( $\epsilon = 1$ ), the MM and BGK models compare very  
 755 well, while the diffusion model is driven by the macro boundary condition. In the  
 756 intermediate regime ( $\epsilon = 0.4$ ) in Figure 8b, in the MM and BGK results (which are  
 757 in a good agreement), a boundary layer starts to be created whereas it is not the  
 758 case for the diffusion model. For  $\epsilon = 10^{-4}$ , one can see that MM model develops a  
 759 boundary layer at the left boundary before aligning with the diffusion model in the



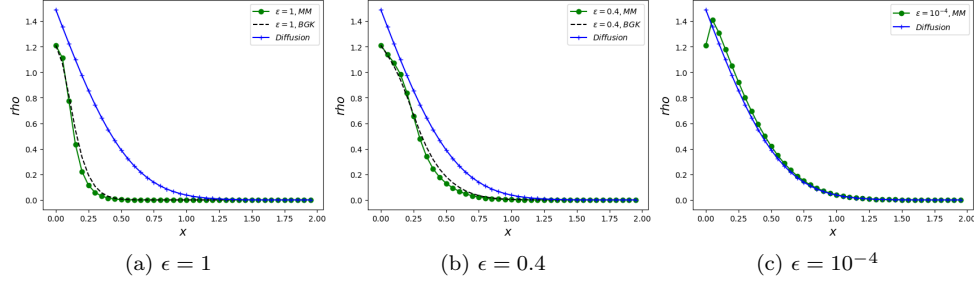


FIG. 8. Qualitative results for non-equilibrium inflow at the left boundary.

760 interior of the domain. This is consistent with the results observed in the literature  
 761 [18, 24, 25, 6].

762 **Appendix A. Appendix: Matrix notation.** The circ function is given by:

$$763 \quad (A.1) \quad \text{circ}([a_1, a_2, \dots, a_m, \dots, a_M]) = \begin{bmatrix} a_m & a_{m+1} & \dots & a_M & 0 & \dots & 0 & a_1 & \dots & a_{m-1} \\ a_{m-1} & a_m & a_{m+1} & \dots & a_M & 0 & \dots & 0 & a_1 & \dots \\ a_{m+2} & \dots & a_M & 0 & \dots & 0 & a_1 & \dots & a_m & a_{m+1} \\ a_{m+1} & \dots & a_M & 0 & \dots & 0 & a_1 & \dots & a_{m-1} & a_m \end{bmatrix}$$

765 The  $\text{circ}_b([-1, \underline{1}]_{(N_x-1) \times (N_x-2)})$  function is given by:

$$766 \quad (A.2) \quad \text{circ}_b([-1, \underline{1}]_{(N_x-1) \times (N_x-2)}) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots \\ \dots & \dots & -1 & 1 \\ \dots & \dots & \dots & -1 \end{bmatrix}_{(N_x-1) \times (N_x-2)}$$

768 **Appendix B. Appendix: Butcher tableau.** The following is the 2-stage  
 769 second order accurate Butcher tableau ARS(2, 2, 2):

$$770 \quad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & \gamma & 0 & 0 \\ 1 & \delta & 1-\delta & 0 \\ \hline & \delta & 1-\delta & 0 \end{array} \quad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 1 & 0 & 1-\gamma & \gamma \\ \hline & 0 & 1-\gamma & \gamma \end{array}$$

771 Here,  $\gamma = 1 - \frac{1}{\sqrt{2}}$  and  $\delta = 1 - \frac{1}{2\gamma}$ .

772 The following is the 4-stage third order accurate Butcher tableau ARS(4, 4, 3):

$$773 \quad \begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 2/3 & 11/18 & 1/18 & 0 & 0 \\ 1/2 & 5/6 & -5/6 & 1/2 & 0 \\ 1 & 1/4 & 7/4 & 3/4 & -7/4 \\ \hline & 1/4 & 7/4 & 3/4 & -7/4 \end{array} \quad \begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 2/3 & 0 & 1/6 & 1/2 & 0 \\ 1/2 & 0 & -1/2 & 1/2 & 0 \\ 1 & 0 & 3/2 & -3/2 & 1/2 \\ \hline & 0 & 3/2 & -3/2 & 1/2 \end{array}$$

774 For type A, we use 2-stage first order accurate Butcher tableau DP-A(1, 2, 1) ( $\gamma \geq \frac{1}{2}$ )

775

$$776 \quad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1 & 0 \end{array} \quad \begin{array}{c|cc} \gamma & \gamma & 0 \\ 1 & 1-\gamma & \gamma \\ \hline & 1-\gamma & \gamma \end{array}$$

777 The following is the 4-stage second order accurate Butcher tableau DP2-A(2, 4, 2):

778	0	0	0	0	0	$\gamma$	$\gamma$	0	0	0
	0	0	0	0	0	0	$-\gamma$	$\gamma$	0	0
	1	0	1	0	0	1	0	$1 - \gamma$	$\gamma$	0
	1	0	1/2	1/2	0	1	0	1/2	1/2 - $\gamma$	$\gamma$
		0	1/2	1/2	0		0	1/2	1/2 - $\gamma$	$\gamma$

779 The following is the 4-stage second order accurate Butcher tableau DP1-A(2, 4, 2)  
 780 which achieves third order accuracy on the DIRK part:

781	0	0	0	0	0	1/2	1/2	0	0	0
	1/3	1/3	0	0	0	2/3	1/6	1/2	0	0
	1	1	0	0	0	1/2	$-1/2$	1/2	1/2	0
	1	1/2	0	1/2	0	1	3/2	1 - 3/2	1/2	1/2
		1/2	0	1/2	0		3/2	1 - 3/2	1/2	1/2

782 **Acknowledgement.** Megala Anandan sincerely acknowledges the indispensable  
 783 support provided by Prof. S. V. Raghurama Rao, whose encouragement was instru-  
 784 mental in the successful completion of this research work.

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