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Spinoriality: From Projections in a Flat Set of Paths to Elements in Curved Space(-Times)

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► **To cite this version:**

Edoardo Niccolai. Spinoriality: From Projections in a Flat Set of Paths to Elements in Curved Space(-Times). 2023. hal-04102963v1

HAL Id: hal-04102963

<https://hal.science/hal-04102963v1>

Preprint submitted on 22 May 2023 (v1), last revised 28 Dec 2023 (v2)

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**SPINORIALITY:
FROM PROJECTIONS IN A FLAT SET OF PATHS
TO ELEMENTS IN CURVED SPACE(-TIMES)**

EDOARDO NICCOLAI

ABSTRACT. This paper proposes a description of the notion of spinor by restricting attention on its definition in curved space(-time). The starting point is a study on a flat ambient (Euclidean) space, followed by the concept of (isotropic) vector. After a quick specimen of Cliffordian spin group, the question is addressed via tensor-spinor equivalence, using the Pauli–Levi-Civita toolbox, as well as the Riccian algebraic spectrum plus the Riemann curvature tensor. And it ends with the identification of the gravitational spinor.

KEYWORDS: Cliffordian spin group, curved spinorial spaces, (curved) spinor manifold, spin structure, spinor covariant derivative, spinor curvature, spinor-gravity.

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Thierry's Question

Question à toi: Est-il possible de définir un spineur dans un espace-temps courbe ou bien c'est seulement par projection dans l'espace plat qui est localement tangent?

— T. LEHNER, via email to me

1. Memorandum

Let us do a li'l refresher.

1.1. The First, Crude, Definition

É. Cartan talks about himself [3, intro]:

Spinors were first used under that name, by physicists,^a in the field of Quantum Mechanics [but] [i]n their most general mathematical form, spinors were discovered in 1913 by the author of this work in his investigations on the linear representations of simple groups [2, pp. 53-96]; they provide a linear representation of the group of rotations in a space with any number n of dimensions, each spinor having 2^ν components where $n = 2\nu + 1$ or 2ν . Spinors in four-dimensional space occur in Dirac's famous equations for the electron, the four wave functions being nothing other than the components of a spinor.

2. Flat Spinor: Sort of Points of a Line Bundle, & Stick-Spinors

If we decide to work in a (pseudo-)Euclidean space, *flatness* imposes a projective map (representation). For instance, in Euclidean 3-space, the spinors are points of a line bundle, viz. of a line varying from point to point (a curve) in a continuous space, with a 1-dimensional vector space for any point. But beware: if we take the Möbius strip,

$$\mathring{\mathbb{O}} \cong \mathbb{S}^1 \times_{\mathbb{Z}/2} \mathbb{R}, \quad (1)$$

the set of these lines, or rather, the infinite set of points that constitute them, under the imaginative lens of the visual picture, are not very different from the bamboo-sticks (挑竹籤), which characterize the game of spillikins (Fig. 1).

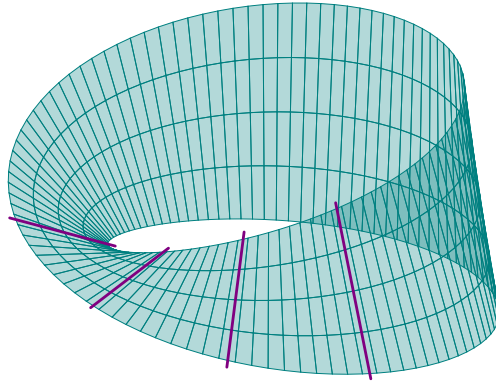


FIGURE 1. Spinorial simplification in 4 sticks depicted as vector elements resting on a Möbius bundle distinguished by spin groups $Spin_{2,1}^+(\mathbb{R}) \cong SL_2(\mathbb{R})$

The spin group, in this case, is the special linear group

$$SL_2(\mathbb{R}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha\delta - \beta\gamma = 1, \quad (2)$$

composed of all matrices $[M]_{\mathbb{R}}^{2 \times 2} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with unit determinant. For convenience, we can think of these sticks as lines of dimension $D = 1$, which rest on a piece of deformable 2-dimensional fabric, i.e. on a vector space, and of infinitesimal thickness.

The stick-spinors have infinitesimal rotations, according to the Lie algebra, which is associated with a linear representation of the Clifford algebra. The Clifford space here is, of course, flat, e.g.

^a The first is P. Ehrenfest, according to S.-I. Tomonaga's testimony [16, p. 129].

$\mathcal{Cl}_s = \mathbb{E}^2$ a/o \mathbb{E}^3 . There is no shortage of spinor representations in Minkowski pseudo-Euclidean 4-space,^a and in such a case, one speaks of spinors in/of 4-dimensional space, of *spinors of space-time*, or of *spinors of Minkowski space*.

Alternatively, the spinor space, in the Cliffordian geometric algebra, can be computed in a 0-dimensional space-time. However, this simplification has a *peccatum originale* behind it, that of starting with an aporetic primitiveness, with the subsequent obligation to ascend to higher dimensions.

2.1. Spinor as an Isotropic Vector, But ...

The foregoing suggest that a spinor is an element comparable to a vector, to an isotropic vector, to be precise. Cartan [3, p. 42, I replaced the prime notation with the (over)dot notation] writes:

Consider a rotation (or reversal) defined by the equations

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + \beta x_2 + \gamma x_3, \\ \dot{x}_2 &= \dot{\alpha} x_1 + \dot{\beta} x_2 + \dot{\gamma} x_3, \\ \dot{x}_3 &= \ddot{\alpha} x_1 + \ddot{\beta} x_2 + \ddot{\gamma} x_3,\end{aligned}$$

where $\alpha\beta\gamma, \dot{\alpha}\dot{\beta}\dot{\gamma}, \ddot{\alpha}\ddot{\beta}\ddot{\gamma}$ are the nine direction cosines of three orthogonal directions. Consider the spinor (ξ_0, ξ_1) associated with an isotropic vector (x_1, x_2, x_3) and one of the spinors $(\dot{\xi}_0, \dot{\xi}_1)$ associated with the transformed vector; then

$$\dot{\xi}_0^2 = \frac{1}{2}[(\alpha - i\dot{\alpha})x_1 + (\beta - i\dot{\beta})x_2 + (\gamma - i\dot{\gamma})x_3] = \frac{1}{2}(\alpha - i\dot{\alpha} + i\beta + \dot{\beta})\xi_0^2 - (\gamma - i\dot{\gamma})\xi_0\xi_1 + \frac{1}{2}(-\alpha + i\dot{\alpha} + i\beta + \dot{\beta})\xi_1^2.$$

Since the discriminant of the quadratic form on the right-hand side is

$$(\gamma - i\dot{\gamma})^2 - (\alpha - i\dot{\alpha} + i\beta + \dot{\beta})(-\alpha + i\dot{\alpha} + i\beta + \dot{\beta}) = (\alpha - i\dot{\alpha})^2 + (\beta - i\dot{\beta})^2 + (\gamma - i\dot{\gamma})^2 = 0,$$

the right-hand side must be a perfect square. Thus the quantity $\dot{\xi}_0$ is linear in ξ_0, ξ_1 , and the same is obviously true for $\dot{\xi}_1$.

This brings us to two immediate considerations.

(1) A spinor is an isotropic vector-like and a Euclidean tensor (which is the title of the Section from which the above passage is taken), cf. Secc. 3.1 and 3.3.

(2) Do spinors build a (topological) vector space, then? If the spinorial representation is vectorial, as happens in quantum mechanics and quantum field theory, the answer is yes. If, on the other hand, one wants an exquisitely algebraic (C. Chevalley) or algebro-geometric construction (Lie group),^b the vectorial structure is lost, and the answer is no;^c except that a Lie group is but a smooth differentiable manifold, and a differentiable manifolds is locally identifiable with, and relatable to, a vector space—endowed with global topological properties in a differentiable way (see the theorems of Cartan–Hadamard, de Rham’s, Bonnet–Myers, and Synge–Weinstein).

2.2. Synoptic Spin-Spectrum; a Specimen of Cliffordian Spin Group

All topics in the previous Sections cling to the representation of the Lie algebra and the spin groups. Here are some of the most frequent.

(1) Spin group on the \mathbb{R} -field (from 1 to 10):

$$\begin{cases} Spin_1(\mathbb{R}) \cong O_1(\mathbb{R}), & (3a) \\ Spin_2(\mathbb{R}) \cong U_1(\mathbb{R}), & (3b) \\ Spin_3(\mathbb{R}) \cong SU_2(\mathbb{R}) \cong Sp_1, & (3c) \\ Spin_4(\mathbb{R}) \cong SU_2(\mathbb{R}) \times SU_2(\mathbb{R}) \cong Sp_1 \times Sp_1, & (3d) \\ Spin_5(\mathbb{R}) \cong Sp_2(\mathbb{H} \cong \mathbb{R}^4), & (3e) \\ Spin_6(\mathbb{R}) \cong SU_4(\mathbb{R}) \subset U_4. & (3f) \end{cases}^d$$

^a The plain characterization of Minkowski space is \mathfrak{M}^n viz. $\mathbb{M}^n = \mathbb{R}^{1, n-1}$ or $\mathbb{R}^{n-1, 1}$ of $n \geq 2$ dimension.

^b See C. Chevalley [4] [5] [6] [7].

^c G. Coddens [8, p. 4, e.m.] is of this opinion: «[S]pinors in $SU(2)$ do not build a vector space but a *curved manifold*. This is almost never clearly spelled out. A consequence of this is that physicists believe that the linearity of the Dirac equation (and the Schrödinger equation) implies the superposition principle in QM, which is wrong because the spinors are not building a vector space. In this respect Cartan stated that physicists are using spinors like vectors. This confusion plays a major rôle in one of the meanest paradoxes of QM, viz. the double-slit experiment».

^d And then come: $Spin_7(\mathbb{R}) \subset O_8$, $Spin_8(\mathbb{R}) \subset O_8 \times O_8$, $Spin_9(\mathbb{R}) \subset O_{16}$, $Spin_{10}(\mathbb{R}) \subset U_{16}$. Here the unitary group (U_n), the special unitary group (SU_n), the symplectic group (Sp_n)—in the \mathbb{H} -algebra of quaternions—appear.

(2) Spin group on the \mathbb{C} -field (from 2 to 6):

$$\begin{cases} Spin_2(\mathbb{C}) \cong \mathbb{C}^\times, & (4a) \\ Spin_3(\mathbb{C}) \cong SL_2(\mathbb{C}), & (4b) \\ Spin_4(\mathbb{C}) \cong SL_2(\mathbb{C}) \times SL_2(\mathbb{C}), & (4c) \\ Spin_5(\mathbb{C}) \cong Sp_4(\mathbb{C}), & (4d) \\ Spin_6(\mathbb{C}) \cong SL_4(\mathbb{C}).^a & (4e) \end{cases}$$

I report the complex spin representation, denoted by the letter \mathcal{D} in its generality, under the Dirac algebra $\text{dir}_{(4,1)}$,

$$\mathcal{D}_{Spin_{\mathbb{C}}}^\pm \cong \left(\mathcal{D}_{Spin_{\mathbb{C}}}^+ \oplus \mathcal{D}_{Spin_{\mathbb{C}}}^- \right) : Spin_4^{\text{dir}}(\mathbb{R}) \rightarrow SU_2(\mathbb{U}_{\mathbb{C}}^\pm), \quad (5)$$

in accordance with the pre-eminence it has in the physico-mathematical area; in it the spin group is a subset of a plexus of Clifford algebra,^b

$$Spin_4^{\text{dir}}(\mathbb{R}) \subset \underbrace{\left(Cl_{4,1,\mathbb{C}\otimes}(\mathbb{R}) \stackrel{\text{viz}}{\cong} Cl_{4\times}(\mathbb{C}) \right)}_{\text{dir}_{(4,1)} \stackrel{\text{eqv}}{\cong} \mathbb{R}_{4,1}} \cong Cl_{1,1} \otimes Cl_{3,0} \cong {}^2\mathbb{R} \otimes {}^2\mathbb{C} \cong {}^4\mathbb{C}, \quad (6)$$

from which the following isomorphism and automorphism, respectively, arise,

$$Cl_{4,1,\mathbb{C}\otimes}(\mathbb{R}) \xrightarrow{\varphi_{Cl}} \mathbb{C} \otimes Cl_{1,3}(\mathbb{R}), \quad (7)$$

$$Spin_4^{\text{dir}}(\mathbb{R}) \xrightarrow{\mathcal{D}_{Spin_{\mathbb{C}}}^\pm} \text{aut}_{\mathbb{C}}(\mathbb{U}_{\mathbb{C}}^\pm); \quad (8)$$

and the symbol

$$\mathbb{U}_{\mathbb{C}}^\pm \cong (\mathbb{U}_{\mathbb{C}}^+ \cong \mathbb{C}^2 \oplus \mathbb{U}_{\mathbb{C}}^- \cong \mathbb{C}^2) \cong \mathbb{C}^4 = \mathbb{R}^8 \quad (9)$$

is the complex vector space w.r.t. n -spinors, so that $\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix} \cong \begin{pmatrix} v^1 \\ v^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v^3 \\ v^4 \end{pmatrix}$.

NB. Pay attention to the following two facts.

The last term of (6) is the algebra of $\mathbb{C}^{4 \times 4}$ -matrices, having this chain of congruences:

$${}^4\mathbb{C} \stackrel{\text{viz}}{\cong} \mathbb{C}(4) \cong \mathbb{C} \otimes {}^4\mathbb{R} \cong \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{H}. \quad (10)$$

What lies behind (7) is an endomorphism,

$$Cl_{4,1,\mathbb{C}\otimes}(\mathbb{R}) \stackrel{\text{eqv}}{\cong} \text{end}_{\mathbb{C}}(\mathbb{U}_{\mathbb{C}}^\pm), \quad (11a)$$

$$Cl_{4,1}(\mathbb{R}) \rightarrow \text{end}_{\mathbb{C}}(\mathbb{U}_{\mathbb{C}}^\pm). \quad (11b)$$

(3) Spin of type $\begin{pmatrix} 0 \\ k,p \end{pmatrix}$:

$$\left(Spin_{k,p}^0 \right)_{\substack{1 \leq k \leq q \\ k+q \leq 6}} \begin{cases} Spin_{1,1}^0 \cong \mathbb{R}^\times, & (12a) \\ Spin_{1,2}^0 \cong SL_2(\mathbb{R}), & (12b) \\ Spin_{1,3}^0 \cong SL_2(\mathbb{C}), & (12c) \\ Spin_{1,4}^0 \cong Sp_{1,1}(\mathbb{H} \cong \mathbb{R}^4), & (12d) \\ Spin_{1,5}^0 \cong SL_2(\mathbb{H} \cong \mathbb{R}^4), & (12e) \\ Spin_{2,2}^0 \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R}), & (12f) \\ Spin_{2,3}^0 \cong Sp_4(\mathbb{R}), & (12g) \\ Spin_{2,4}^0 \cong SU_{2,2},^c & (12h) \\ Spin_{3,3}^0 \cong SL_4(\mathbb{R}). & (12i) \end{cases}$$

^a With the appearance of the special linear group (SL_n).

^b A synopsis on the spinoriality of Cliffordian (geometric) algebra is in [12, sec. 3.6]

3. How to Build Spinors in Curved Space-Time?

The classical way to build spinors in curved space-time is, as a first step, to introduce—from quantum mechanics—2-component spinors (column vectors with two complex components), or Pauli-like spinors, videlicet 2-component wave functions like this

$$\psi = \begin{pmatrix} \psi^\alpha \\ \psi^\beta \end{pmatrix}, \quad \psi_+^\alpha = \begin{pmatrix} \psi^\alpha \\ 0 \end{pmatrix}, \quad \psi_-^\beta = \begin{pmatrix} 0 \\ \psi^\beta \end{pmatrix}, \quad \psi \in \mathbb{C}^2 \cong \mathfrak{S}, \quad (13)$$

at each point of a complex tangent 2-space. Secondly, associate each tensor of this space with a 2-component spinor.

NB. It is not possible to make a reverse combination: not every spinor is translatable to a tensor. Why? That is because there is no isomorphism, but only homomorphism (\mathfrak{u}), between the group $SO_{1,3}^+(\mathbb{R}) = \mathcal{J}_+^\uparrow$ and its double covering group $SL_2(\mathbb{C})$, according to an irreducible and non-unitary representation of the latter group. Specifically,

$SO_{1,3}^+(\mathbb{R})$ is the indefinite special orthogonal group of linear transformations of the Minkowski space-time $\mathfrak{M}^4 = \mathbb{R}_{1,3}^4$, and \mathcal{J}_+^\uparrow is the restricted Lorentz group,

$SL_2(\mathbb{C})$ is the special linear group of

$$[M]_{\mathbb{C}}^{2 \times 2} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \quad (14)$$

it is a 3-dimensional complex Lie group and a real 6-dimensional Lie group; lastly, $SL_2(\mathbb{C})$ is simply connected.

With a diagrammatic condensation:

$$\begin{array}{ccc} \{SL_2(\mathbb{C}) \cong Spin_{1,3}^+(\mathbb{R}) \hookrightarrow \mathbb{R}_{1,3}^4 \times SL_2(\mathbb{C}) \rightarrow \mathbb{R}_{1,3}^4\} & \xrightarrow{\varsigma} & SO_{1,3}^+(\mathbb{R}) = \mathcal{J}_+^\uparrow \\ \downarrow \mathfrak{u} \circ \varsigma & \swarrow \mathfrak{u} & \\ GL(\mathfrak{M} \stackrel{\text{viz}}{=} \mathbb{R}^{1,n-1}) & & \end{array}$$

with the representation of \mathcal{J}_+^\uparrow on \mathfrak{M} (real vector space à la Minkowski) as a homomorphism \mathfrak{u} of $SO_{1,3}^+(\mathbb{R})$ into a general linear group $GL(\mathfrak{M})$. The spinor map

$$\varsigma: \left(SL_2(\mathbb{C}) \cong Spin_{1,3}^+(\mathbb{R}) \right) \longrightarrow \left\{ SO_{1,3}^+(\mathbb{R}) = \mathcal{J}_+^\uparrow \hookrightarrow \mathbb{R}_{1,3}^4 \times \left(SO_{1,3}^+(\mathbb{R}) = \mathcal{J}_+^\uparrow \right) \rightarrow \mathbb{R}_{1,3}^4 \right\} \quad (15)$$

is equivalent to a 2-fold covering, viz. a double cover, of $SO_{1,3}^+(\mathbb{R}) = \mathcal{J}_+^\uparrow$. Namely, $SL_2(\mathbb{C})$ is the universal covering group of the Lorentz group \mathcal{J}_+^\uparrow .

Marginalia 3.1. Suggestions from some of my writings.

(1) About the spinorial representation of the orthogonal group on a 3-space, and the Pauli-like spinors, see [12, secc. 2.8.2 and 2.8.3].

(i) Do not forget that a rotation of 360° in a $SO_3(\mathbb{R})$ -space is not homotopic to the identity, c'est-à-dire to a null rotation, but a rotation of 720° is homotopic to the identity/to a null rotation, see Fig. 1 in relation to the Möbius strip.

Let $M \stackrel{\text{viz}}{=} [M]^{2 \times 2} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ be a (2×2) -matrix, $\alpha, \beta \in \mathbb{C}$, referring to the special unitary 2-group, and write

$$M = \begin{pmatrix} e^{-\frac{1}{2}i\theta^3} & 0 \\ 0 & e^{\frac{1}{2}i\theta^3} \end{pmatrix} \in SU_2(\mathbb{C}) \longrightarrow R_3^\theta = \begin{pmatrix} \cos \theta^3 & -\sin \theta^3 & 0 \\ \sin \theta^3 & \cos \theta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO_3(\mathbb{R}), \quad (16)$$

where R^θ is the rotation (matrix) through an angle θ . The subsequent values, $\hat{x}_1 = \cos \theta^3 x_1 - \sin \theta^3 x_2$, $\hat{x}_2 = \sin \theta^3 x_1 + \cos \theta^3 x_2$, $\hat{x}^3 = x_3$, are outlined. A coordinate system comes back to its original state after a rotation about, say, the x_3 -axis thru an angle $\theta^3 = 2\pi = 360^\circ$. A spinor,

^c $SU_{2,2}$ is the pseudo-unitary (spin) group.

however, returns to its original state (identity, or neutral, symmetry) after two full rotations, to wit, after a rotation of $4\pi = 720^\circ$ about the x_3 -axis.

(ii) The covering map

$$\varsigma: SU_2(\mathbb{C}) \rightarrow SO_3(\mathbb{R}) \cong Spin_3(\mathbb{R}) \quad (17)$$

sets out a smooth surjective homomorphism between $SU_2(\mathbb{C})$ and $SO_3(\mathbb{R})$. As we know, $SU_2(\mathbb{C})$ is compact and simply connected, and $SO_3(\mathbb{R})$ is compact and connected but not simply connected; the meaning of the map arises from this discrepancy, which has a mathematical bottom and a deep-seated physical motives. The set $SU_2(\mathbb{C})$ is the simply connected universal covering group of $SO_3(\mathbb{R})$.

(2) About the spinor (structure) bundle, see [12, sec. 3.5] and [13, sec. 3.2].

3.1. The Tensor-Spinor Association: Pauli–Levi-Civita Toolbox

It starts with four $\mathbb{C}^{2 \times 2}$ Hermitian matrices, say $X \stackrel{\text{viz}}{=} [X]^{2 \times 2}$, ou seja four complex (2×2) -matrices, each of which is allied to a 4-vector $x^\mu = (x^0, x^1, x^2, x^3)$,

$$X = \{x^\mu = (x^0, x^1, x^2, x^3) \cdot \sigma_\mu = (\sigma_0, \vec{\sigma})\} = x^0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix}, \quad x \in \mathbb{R}, \quad (18)$$

where $\vec{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ is the spin 4-vector whose distinctiveness consists of the Pauli (spin) matrices,

$$\sigma_0 \stackrel{\text{viz}}{=} \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (19)$$

The 2-component complex column vectors on which such (2×2) -matrices act—by matrix multiplication—are precisely the Paulian spinors (13).

With a more synthetic formalism, one fixes

$$\sigma_{j\dot{k}}^\mu = \bar{\sigma}_{k\dot{j}}^\mu = \bar{\sigma}_{\dot{k}j}^\mu, \quad (20)$$

letting $\mu = 0, 1, 2, 3$ be the space-time tensor indices, and $j = 0, 1, \dot{j} = \dot{0}, \dot{1}$, $k = 0, 1, \dot{k} = \dot{0}, \dot{1}$, be the spinor indices, whilst the short overline (via `\bar` command) is for the complex conjugate. A chance to bring out the spinorial attribute, is to employ the aforementioned Pauli matrices (19), so as to satisfy two relations of equality (equivalence, to be exact) together with the metric tensor (field) $g_{\mu\nu}$, in this way,

$$\left\{ \begin{array}{l} g_{\mu\nu} \sigma_{j\dot{k}}^\mu \sigma_{l\dot{m}}^\nu = \varepsilon_{jl} \varepsilon_{\dot{k}\dot{m}}, \end{array} \right. \quad (21a)$$

$$\left\{ \begin{array}{l} \sigma_{j\dot{k}}^\mu \sigma_{\nu\dot{l}}^\mu = g^{\nu\dot{l}}, \end{array} \right. \quad (21b)$$

$$\left\{ \begin{array}{l} \sigma_{j\dot{k}}^\mu \sigma_{\nu\dot{l}}^\mu = \delta_{\nu\dot{l}}^\mu, \end{array} \right. \quad (21c)$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the *skew symmetric Levi-Civita (metric) spinor* à la M. Carmeli [1, p. 120], obtained from the Levi-Civita symbol [11, pp. 180-182], which is a pseudo-tensor or tensor density, and δ is the Kronecker delta, here as a spinor. The Levi-Civita spinor respects this identity,

$$\varepsilon_{jk} \varepsilon_{lm} + \varepsilon_{jl} \varepsilon_{mk} + \varepsilon_{jm} \varepsilon_{kl} = 0. \quad (22)$$

A caption-illustration of Eq. (21a):

$$\begin{array}{ccc} \xrightarrow{\text{1st lhs expr.}} & \underbrace{g_{\mu\nu}} & \\ & \text{metric tensor field } g & \\ \xrightarrow{\text{2nd lhs expr.}} & \underbrace{\sigma_{j\dot{k}}^\mu \sigma_{l\dot{m}}^\nu} & \\ & \text{Pauli (spin) matrices} & \\ \xrightarrow{\text{rhs expr.}} & \underbrace{\varepsilon_{jl} \varepsilon_{\dot{k}\dot{m}}} & \\ & \text{Levi-Civita (metric) spinor.} & \end{array}$$

^a The matrix $\sigma_0 = \mathbb{I}$ is the Pauli identity matrix.

3.2. Spinorial Behavior

Let ζ and χ be two 2-spinors of generic value. One has:

$$\zeta\text{- and } \chi\text{-spinors} \begin{cases} \zeta^j = \varepsilon^{jk} \zeta_k, \chi^j = \varepsilon^{jk} \chi_k, & (23a) \\ \zeta_j = \zeta^k \varepsilon_{kj}, \chi_j = \chi^k \varepsilon_{kj}, & (23b) \\ \zeta^j \chi_j = -\zeta_k \chi^k \text{ (step by step: } \zeta^j \chi_j = \zeta^j \chi^k \varepsilon_{kj} = -\zeta^j \varepsilon_{jk} \chi^k = -\zeta_k \chi^k \text{)}. & (23c) \end{cases}$$

3.3. The Tensor-Spinor Equivalence

The equivalence will concern two types of tensors, T -tensor and g -tensor.

(1) Let $T_{\xi\varrho}$ be a tensor, without a specific definition; its equivalent spinor will therefore be (based on what we have seen above)

$$T_{jklm} \stackrel{\text{eqv}}{=} \sigma_{jk}^{\xi} \sigma_{lm}^{\varrho} T_{\xi\varrho}. \quad (24)$$

The matrix group acting isometrically on $\mathbb{R}_{1,3}^4$, within Pauli–Levi-Civita’s scheme, is the $SL_2(\mathbb{C}) \cong Spin_{1,3}^+(\mathbb{R})$ group, see the map (15).

From Eqq. (21a) (21b) (21c) (24), we get

$$\sigma_{\mu}^{jk} \sigma_{\nu}^{lm} T_{jklm} \stackrel{\text{eqv}}{=} \sigma_{\mu}^{jk} \sigma_{\nu}^{lm} \sigma_{jk}^{\xi} \sigma_{lm}^{\varrho} T_{\xi\varrho} \stackrel{\text{eqv}}{=} \delta_{\mu}^{\xi} \delta_{\nu}^{\varrho} T_{\xi\varrho} \stackrel{\text{eqv}}{=} T_{\mu\nu}. \quad (25)$$

(2) By Eq. (21a), we shall proceed to the spinorial equivalence with the metric tensor g ,

$$g_{jklm} \stackrel{\text{eqv}}{=} \sigma_{jk}^{\xi} \sigma_{lm}^{\varrho} g_{\xi\varrho} \stackrel{\text{eqv}}{=} \varepsilon_{jl} \varepsilon_{km}, \quad (26a)$$

$$g^{jklm} \stackrel{\text{eqv}}{=} \sigma_{\xi}^{jk} \sigma_{\varrho}^{lm} g^{\xi\varrho} \stackrel{\text{eqv}}{=} \varepsilon^{jl} \varepsilon^{km}. \quad (26b)$$

NB. The space-time metric tensors are, explicitly, such matrices,

$$g_{jklm} = g^{jklm} = \left\{ \begin{array}{cc} 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \end{array} \right\}, \quad (27)$$

and

$$g_{jk}^{lm} \stackrel{\text{eqv}}{=} \sigma_{\xi}^{jk} \sigma_{\varrho}^{lm} \stackrel{\text{eqv}}{=} \delta_{jk}^{lm} \left\{ \begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right\}. \quad (28)$$

To seek for a correlation with the metric tensor of Minkowski space-time $\eta_{\mu\nu}$, take the Pauli matrices (19), and divide them by the square root of 2, i.e., in the position of lowering indices,

$$\sigma_{jk}^{0123} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\sigma^0}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\sigma^1}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\sigma^2}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\sigma^3} \right\}, \quad (29)$$

and, in the position of raising indices,

$$\sigma^{0123jk} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\sigma^0}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}_{\sigma^1}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\sigma^2}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{\sigma^3} \right\}. \quad (30)$$

The tensor operation of raising indices,^a leads to

$$\sigma_{\mu}^{jk} \sigma_{\nu}^{lm} g_{jklm} = \eta_{\mu\nu}. \quad (31)$$

^a From σ_{jk}^{0123} to σ^{0123jk} .

3.4. Spinor (Field) Covariant Derivative

The *spinor (field) covariant derivative* (32a) (32c), and its complex conjugate version (32b) (32d), employing a ζ -spinor (see Section 3.2), can be established, axiomatically, as

$$\left\{ \begin{array}{l} \nabla_\mu \zeta^j = \frac{\partial \zeta^j}{\partial x^\mu} + \Gamma_{k\mu}^j \zeta^k, \\ \nabla_\mu \bar{\zeta}^j = \frac{\partial \bar{\zeta}^j}{\partial x^\mu} + \bar{\Gamma}_{k\mu}^j \bar{\zeta}^k, \\ \nabla_\mu \zeta_j = \frac{\partial \zeta_j}{\partial x^\mu} - \Gamma_{j\mu}^k \zeta_k, \\ \nabla_\mu \bar{\zeta}_j = \frac{\partial \bar{\zeta}_j}{\partial x^\mu} - \bar{\Gamma}_{j\mu}^k \bar{\zeta}_k, \end{array} \right. \quad \begin{array}{l} (32a) \\ (32b) \\ (32c) \\ (32d) \end{array}$$

with the presence of the Christoffel symbols, here for a *spinorially metric connection* acting on spinor (tensor) fields. If we are dealing with a 2-index spinor, and this is what happens in a Minkowski-like frame, the spinorial derivative will be

$$\nabla_\mu \eta^{jk} = \frac{\partial \eta^{jk}}{\partial x^\mu} + \Gamma_{l\mu}^j \eta^{lk} + \bar{\Gamma}_{l\mu}^k \bar{\eta}^{jl}. \quad (33)$$

3.5. Spinor Curvature

Let us introduce the spinor value

$$\nabla_\mu \zeta_s = \partial_\mu \zeta_s - \Gamma_{s\mu}^r \zeta_r, \quad (34)$$

and, from Eq. (32c), determine that

$$\nabla_\nu \nabla_\mu \zeta_s = \partial_\nu (\nabla_\mu \zeta_s) - \Gamma_{\nu\mu}^\xi \nabla_\xi \zeta_s - \Gamma_{s\nu}^k \nabla_\mu \zeta_k, \quad (35)$$

and

$$\nabla_\nu \nabla_\mu \zeta_s = \partial_\nu \partial_\mu \zeta_s - \Gamma_{s\mu}^r \partial_\nu \zeta_r - \partial_\nu \Gamma_{s\mu}^r \zeta_r - \Gamma_{\nu\mu}^\xi \partial_\xi \zeta_s + \Gamma_{\nu\mu}^\xi \Gamma_{s\xi}^r \zeta_r - \Gamma_{s\nu}^k \partial_\mu \zeta_k + \Gamma_{s\nu}^k \Gamma_{k\mu}^r \zeta_r. \quad (36)$$

Eq. (36) contains the quantity we are looking for. It gives us

$$C_{\beta}^r{}_{s\mu\nu} \zeta^r = \zeta_s (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu), \quad (37)$$

where

$$C_{\beta}^r{}_{s\mu\nu} = \Gamma_{s\mu,\nu}^r - \Gamma_{s\nu,\mu}^r + \Gamma_{s\mu}^k \Gamma_{k\nu}^r - \Gamma_{s\nu}^k \Gamma_{k\mu}^r. \quad (38)$$

is nothing but what we can call *spinor curvature*. For a spinor ζ^s , Eq. (37) becomes

$$C_{\beta}^s{}_{r\mu\nu} \zeta^r = \zeta^s (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu). \quad (39)$$

NB. By adopting the bemolle–diesis convention w.r.t. Eqq. (37) (39), something like this comes out,

$$C_{\beta} \left(\flat(\zeta_s) \right) \stackrel{\text{viz}}{=} C_{\beta} \zeta_{\flat}, \quad (40)$$

$$C_{\beta} \left(\sharp(\zeta_s) \right) \stackrel{\text{viz}}{=} C_{\beta} \zeta^{\sharp}, \quad (41)$$

which are a combination of those equations.

If the spinor has two indices, say T_r^s , the spinor curvature formula, along with the commutator (with the subtraction of nabla operators), is

$$C_{\beta}^j{}_{r\mu\nu} T_j^s = T_r^s (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) + C_{\beta}^s{}_{j\mu\nu} T_r^j. \quad (42)$$

From Eqq. (37) (39) it is possible to easily arrive at

$$C_{\beta r s \mu \nu} \zeta^r = \zeta_s (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu), \quad (43)$$

$$C_{\beta s r \mu \nu} \zeta^r = \zeta_s (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu), \quad (44)$$

^a Or $C_{\beta}^r{}_{s\mu\nu} + \Gamma_{s\nu,\mu}^r + \Gamma_{s\nu}^k \Gamma_{k\mu}^r = \Gamma_{s\mu,\nu}^r + \Gamma_{s\mu}^k \Gamma_{k\nu}^r$.

which reveals the symmetrical constitution of the spinor curvature:

$$C_{\beta\text{-symmetry (of spinor curvature)}} \begin{cases} C_{\beta_{rs\mu\nu}} = C_{\beta_{sr\mu\nu}} & (45a) \\ C_{\beta_{rs\mu\nu}} = -C_{\beta_{rs\nu\mu}} & (45b) \end{cases}$$

3.6. Algebro-geometric Spinor: Inside Ricci Calculus

Continuing the discussion of the previous Section, we can approach the Ricci calculus. And so be it,

$$C_{\beta^j_{t\mu\nu}} T_{rj\cdots}^{su\cdots} = T_{rt\cdots}^{su\cdots} (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) + C_{\beta^s_{j\mu\nu}} T_{rt\cdots}^{ju\cdots} + C_{\beta^u_{j\mu\nu}} T_{rt\cdots}^{sj\cdots} - C_{\beta^j_{r\mu\nu}} T_{jt\cdots}^{su\cdots}, \quad (46)$$

$$\bar{C}_{\beta^j_{s\mu\nu}} T^r_j = T^r_s (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) + C_{\beta^r_{j\mu\nu}} T^j_s, \quad (47)$$

$$\bar{C}_{\beta^j_{s\mu\nu}} T_{rj} = T_{r\dot{s}} (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) - C_{\beta^j_{r\mu\nu}} T_{j\dot{s}}, \quad (48)$$

$$\bar{C}_{\beta^s_{j\mu\nu}} T^{rj} = T^{r\dot{s}} (\nabla_{\nu} \nabla_{\mu} - \nabla_{\mu} \nabla_{\nu}) - C_{\beta^r_{j\mu\nu}} T^{j\dot{s}}, \quad (49)$$

for a $\binom{su}{rt}$ -spinor $T_{rt\cdots}^{su\cdots}$, as in (46), and for $\binom{r}{s}$ -spinors T^r_s , $T_{r\dot{s}}$, $T^{r\dot{s}}$, as in (47), (48), (49), respectively.

3.7. The Riemann Tensor Graft

If Eq. (48) is multiplied by $\sigma_{\xi}^{r\dot{s}}$, it comes out that

$$T_{l\dot{m}} \left(C_{\beta^l_{r\mu\nu}} \sigma_{\xi}^{r\dot{m}} + \bar{C}_{\beta^{\dot{m}}_{s\mu\nu}} \sigma_{\xi}^{l\dot{s}} \right) = T_{r\dot{s}} \sigma_{\xi}^{r\dot{s}} (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}), \text{ where } T_{r\dot{s}} \sigma_{\xi}^{r\dot{s}} = T_{\xi}, \quad (50)$$

$$T_{\xi} (\nabla_{\nu} \nabla_{\mu} - \nabla_{\mu} \nabla_{\nu}) = R^{l\dot{m}}_{\xi\mu\nu} T_{l\dot{m}} = R^{\rho}_{\xi\mu\nu} T_{\rho}, \quad (51)$$

where $R^{l\dot{m}}_{\xi\mu\nu} = \left(C_{\beta^l_{r\mu\nu}} \sigma_{\xi}^{r\dot{m}} + \bar{C}_{\beta^{\dot{m}}_{s\mu\nu}} \sigma_{\xi}^{l\dot{s}} \right)$, and $R^{\rho}_{\xi\mu\nu} = R^{l\dot{m}}_{\xi\mu\nu} \sigma_{l\dot{m}}^{\rho}$, which is the Riemann curvature tensor.

4. (Mathematical) Cues for a Spinor-Gravity

In Sections 3.1 and 3.3, we saw that there is a tensor-spinor equivalence, especially in the metric g -tensor field version, which is a bridge to the spinor afferent to the gravitational force of geometrical order. Another piece to represent this bridge is to set, again, the curvature tensor in an equivalence union with the spinorial part:

$$\begin{aligned} R_{jkl\dot{m}jkl\dot{m}} &= \sigma_{j\dot{k}}^{\mu} \sigma_{l\dot{m}}^{\nu} \sigma_{j\dot{k}}^{\xi} \sigma_{l\dot{m}}^{\rho} R_{\mu\nu\xi\rho} = \frac{1}{2} \left(\varepsilon_{jl} R_{rkr\dot{m}jkl\dot{m}} + R_{jrlr\dot{m}jkl\dot{m}} \varepsilon_{k\dot{m}} \right) \\ &= \frac{1}{4} \varepsilon_{jl} \left(R_{rkr\dot{m}jkl\dot{m}} \varepsilon_{k\dot{m}} + R_{rkr\dot{m}j\dot{o}l\dot{o}} \varepsilon_{k\dot{m}} \right) + \frac{1}{4} \varepsilon_{k\dot{m}} \left(R_{jrlr\dot{m}jkl\dot{m}} \varepsilon_{k\dot{m}} + R_{jrlr\dot{m}j\dot{o}l\dot{o}} \varepsilon_{k\dot{m}} \right). \end{aligned} \quad (52)$$

Attention, please:

- the two series of indices in #000000- and #E30382-colors are distinct; they have nothing in common; indicated colors serve to differentiate them without introducing new letters;
- the superscript ‡-diesis symbol serves to remind us that the index-letter is raised.

From Eq. (52) we arrive at such definable

$$4\text{-indices } \zeta\text{- and } \chi\text{-spinors} \begin{cases} \zeta_{jljl} = -\frac{1}{4} R_{jrlr\dot{m}j\dot{o}l\dot{o}} \stackrel{\text{eqv}}{=} -\frac{1}{2} C_{\beta_{jlj\dot{o}l\dot{o}}}, & (53a) \\ \chi_{jl\dot{k}\dot{m}} = -\frac{1}{4} R_{jrlr\dot{m}j\dot{o}l\dot{o}} \stackrel{\text{eqv}}{=} -\frac{1}{2} C_{\beta_{jl\dot{k}\dot{m}}}. & (53b) \end{cases}$$

A spinor decomposition reveals, as a result, that

$$R_{jkl\dot{m}jkl\dot{m}} = -\zeta_{jljl} \varepsilon_{k\dot{m}} \varepsilon_{k\dot{m}} - \chi_{jl\dot{k}\dot{m}} \varepsilon_{k\dot{m}} \varepsilon_{jl} - \varepsilon_{jl} \bar{\chi}_{k\dot{m}jl} \varepsilon_{k\dot{i}} - \varepsilon_{jl} \varepsilon_{jl} \bar{\zeta}_{k\dot{m}k\dot{m}}. \quad (54)$$

The equivalence between the Riemann tensor and the spinor manifold passes from spinors (53a) and (53b). They have symmetry properties because the symmetry pertains to the curvature tensor

just mentioned:^a

$$\begin{cases} \zeta_{jklm} = \zeta_{kltm} = \zeta_{jklm} = \zeta_{lmjk}, & (55a) \\ \chi_{jkl\dot{m}} = \chi_{kjl\dot{m}} = \chi_{jkl\dot{m}} = \bar{\chi}_{\dot{m}jkl}, & (55b) \end{cases}$$

with the occurrence of the usual complex conjugate in the χ -spinor.

The ζ -spinor is characterized by 11 real independent components, but no more than 6 complex components; the χ -spinor has 3 real et 3 complex components, 9 in total—it is a Hermitian-like (3×3) -matrix. The *spinor-gravity* is exemplified by the ζ_{jklm} -spinor, as it is able to define the Weyl spinor [12, sec. 3.5.2.2] et the Ricci scalar curvature [12, sec. 3.3.2.2].

4.1. Spinor Decomposition: Two Theorems

We return to the previous Section.

Theorem 4.1. *The spinor decomposition of the Riemann curvature tensor (useful for isolating the spinor-gravity) is provided by Eq. (52).*

Proof. The demonstration can be simplified by reducing the number of indices to 2. From Eq. (22) this chain of equalities is reached,

$$2\zeta_{[jk]} = \begin{cases} \zeta_{jk} - \zeta_{kj}, & (56a) \\ \zeta_l^l \varepsilon_{jk}, \text{ in which } \zeta_l^l = \varepsilon^{lm} \zeta_{lm}, & (56b) \end{cases}$$

just multiply what we have seen to be pertinent to the Levi-Civita spinor identity by this ζ^{lm} -spinor. And so, if we pass to 4 indices, for a ζ_{jklm} -spinor, the chain of formulæ becomes

$$2\zeta_{[jk]lm} = \begin{cases} \zeta_{jklm} - \zeta_{kjlm}, & (57a) \\ \zeta_k^k{}_{lm} \varepsilon_{jk}. & (57b) \end{cases}$$

The continuation of the demonstration is implicit in Eq. (52), which is already written in the form of a result. \square

The proof can also be carried out for the properties of the dual curvature tensor, say $R_{\mu\nu\xi\varrho}^*$, in Riemann space.

Theorem 4.2. *Let $R_{\mu\nu\xi\varrho}^*$ be the dual to $R_{\mu\nu\xi\varrho}$. Its spinor counterpart will be*

$$R_{jkl\dot{m}jkl\dot{m}}^* = \left(\zeta_{jl\dot{l}} \varepsilon_{k\dot{m}} \varepsilon_{k\dot{m}} - \chi_{jl\dot{k}\dot{m}} \varepsilon_{k\dot{m}} \varepsilon_{j\dot{l}} + \varepsilon_{j\dot{l}} \bar{\chi}_{k\dot{m}j\dot{l}} \varepsilon_{k\dot{i}} - \varepsilon_{j\dot{l}} \varepsilon_{j\dot{l}} \bar{\zeta}_{k\dot{m}k\dot{m}} \right)_i \quad (58a)$$

$$= \sigma_{jk}^\mu \sigma_{l\dot{m}}^\nu \sigma_{jk}^\xi \sigma_{l\dot{m}}^\varrho R_{\mu\nu\xi\varrho}^* \quad (58b)$$

cf. Eq. (54) relating to Eq. (58a). We want to show that Eqq. (58a) (58b) are the decomposition of such a dual tensor.

Proof. If the dual curvature tensor in Riemann space is matched to

$$R_{\mu\nu\xi\varrho}^* = \frac{1}{2} \left\{ (\sqrt{-g} R_{\xi\varrho}{}^{\mu\nu} \varepsilon_{\mu\nu\varsigma\tau}) = (R_{\xi\varrho\mu\nu} \varepsilon_{\varsigma\tau}^{\mu\nu}) \right\}, \quad (59)$$

then

$$R_{jkl\dot{m}jkl\dot{m}}^* = \frac{1}{2} R_{jkl\dot{m}n\dot{o}p\dot{q}} \varepsilon_{jkl\dot{m}}^{n\dot{o}p\dot{q}}. \quad (60)$$

From Eq. (54) plus a ε -spinor of these features

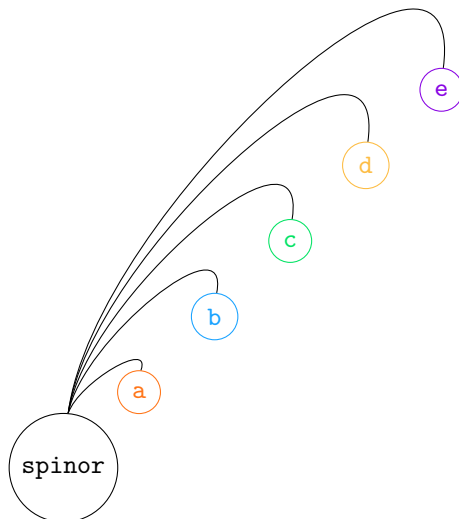
$$\varepsilon_{n\dot{o}n\dot{o}}^{jkl\dot{m}} = \left(\delta_{n\dot{o}}^{j\dot{k}} \delta_{n\dot{o}}^{l\dot{m}} - \delta_{n\dot{o}}^{j\dot{k}} \delta_{n\dot{o}}^{l\dot{m}} \right)_i, \quad (61)$$

it is nimble to reconstruct Eq. (58a). \square

^a $R_{\mu\nu\xi\varrho} = -R_{\nu\mu\xi\varrho} = -R_{\mu\nu\varrho\xi} = R_{\xi\varrho\mu\nu}$, and $R_{\mu\nu\xi\varrho} + R_{\mu\xi\varrho\nu} + R_{\mu\varrho\nu\xi} = 0$.

5. In Corde Quæstionis

(1) The answer to the initial question has been given. For the sake of synthesis, we offer a brief summary:



- (a) spinor as an element of a vector space,
- (b) spinor as a part of Clifford algebra/space,
- (c) spinor representations of Lie algebras, in which curved spaces are conceivable,
- (d) spinor curvature C_β ,
- (e) spinor-gravity ζ_{jklm} (in general relativity), adoptable for flat (absence of massive body) or curved spaces.

(2) Let us not disremember, nonetheless, that mathematics is not a mere answering-*robot* of questions; but it is, above all, a movement de l'esprit to reflect on the issues that fall under its investigation, and create new questions & generate new problems.

(3) This paper skips many steps, because the latter are already present in my other works. (For this reason the bibliography is concise). Readers will excuse me.

One of the pioneering studies on how to conceive the spinor in space-time can be found in R. Geroch [9] [10]. To mention, then, at least two classical books: R. Penrose & W. Rindler [14] [15].

CODA: BUSILLIS

The allure of spinors is probably hidden in their algebro-geometric mystery yet to be fully revealed and understood. M.F. Atiyah gives us a testimony:^a

No one fully understands spinors. Their algebra is formally understood but their geometrical significance is mysterious. In some sense they describe the 'square-root' of geometry and, just as understanding the concept of the square root of -1 took centuries, the same might be true of spinors.

^a E-mail (15 July 2007) from M.F. Atiyah to G. Farmelo; the excerpt is reported in G. Farmelo, *The Strangest Man: The Hidden Life of Paul Dirac, Mystic of the Atom*, Basic Books, New York, 2009, p. 430.

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DEDICATION

To my maternal uncle, Roberto,
who left this flagitious society (22 May 2023)
after years of psycho-physical devastation brought on by Parkinson's disease.
Now, he has boundless clearings of peace ahead of him.