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# Macroscopic diffusive fluctuations for generalized hard rods dynamics

Pablo A. Ferrari<sup>1</sup> and Stefano Olla<sup>2</sup>

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**Abstract:** We study the fluctuations in equilibrium for a dynamics of rods with random length. This includes the classical hard rod elastic collisions, when rod lengths are constant and equal to a positive value. We prove that in the diffusive space-time scaling, an initial fluctuation of density of particles of velocity  $v$ , after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance  $\mathcal{D}(v)$ .

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## 1. Introduction

The mechanical system of one-dimensional hard rods is the simplest non trivial completely integrable dynamics where the macroscopic behavior can be described by generalized hydrodynamics. A rod configuration  $Y$  is a set of points  $(y, v, r)$ , viewed as a segment/rod  $[y, y + r]$  travelling at bare speed  $v$ , satisfying  $[y, y + r] \cap [\tilde{y}, \tilde{y} + \tilde{r}]$  is empty for all  $(y, v, r), (\tilde{y}, \tilde{v}, \tilde{r})$  in  $Y$ . Rods travel ballistically with its bare velocity until collision with other rod, when they interchange speed and length. The density of particles of each given velocity is conserved and in the Euler scaling the macroscopic evolution of such densities have been studied by the pioneristic work of Percus [17] and Boldrighini, Dobrushin and Suhov [5]. Fluctuations around this Euler limit have been studied by Boldrighini and Wick in [7]. Recently these results have been generalized to a completely integrable dynamics of rods of random length (even negative length) where lengths and velocities are exchanged at collision [12]. The elastic collisions are recovered in the particular case that all rods have the same positive length. Similar dynamics were considered in [8], while in [2] velocities are exchanged but not the lengths (i.e. the classical elastic collision).

In this article we investigate the evolution of the densities fluctuations in the diffusive space-time scale for the generalized dynamics considered in [12]. We consider the system in a *stationary homogeneous* initial condition. We will

discuss initial inhomogeneous non-stationary state in [Section 4](#).

The result we prove in the present article is that the initial fluctuations of the density of particles of velocity  $v$ , after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance  $\mathcal{D}(v)$ . This diffusion coefficient  $\mathcal{D}(v)$  has an explicit expression depending on  $v$  and on the particular stationary measure (cf. [\(3.4\)](#)). In the case of rods of constant length  $\mathcal{D}(v)$  is the same as computed by Spohn in [\[19\]](#), as well as it appears in the first order diffusive correction to the Euler Hydrodynamic limit [\[4\]](#) [\[6\]](#).

This result corresponds to the following stochastic partial differential equation for the evolution of the macroscopic fluctuation  $\Xi_t(y, v, r)$  of the density of particles of velocity  $v$  and length  $r$  at position  $y$ :

$$\partial_t \Xi_t(y, v, r) = \frac{1}{2} \mathcal{D}(v) \partial_y^2 \Xi_t(y, v, r) + \sqrt{\mathcal{D}(v)} \partial_y \Xi_t(y, v, r) \dot{W}_t(v), \quad (1.1)$$

where  $(\dot{W}_t(v) : t, v \in \mathbb{R})$  is a centered Gaussian field with covariance

$$\mathbb{E}(\dot{W}_t(v) \dot{W}_s(w)) = \delta(t - s) \frac{\Gamma(v, w)}{\sqrt{\mathcal{D}(v) \mathcal{D}(w)}}$$

with  $\Gamma(v, w)$  given in [\(3.7\)](#). Since  $\Xi_t$  is a distribution in  $(y, v, r)$ , [\(1.1\)](#) should be understood in the weak sense.

Notice that in [\(1.1\)](#) the noise term is only white in time and completely correlated in space (i.e.  $\dot{W}_t(v)$  does not depend on  $y$ ). This is in contrast with the typical diffusive evolution of fluctuations in chaotic systems, where it is expected an additive space-time white noise driving the fluctuations and the equation would be of the type [\[20\]](#)

$$\partial_t \bar{\Xi}_t(y, v, r) = \frac{1}{2} \mathcal{D}(v) \partial_y^2 \bar{\Xi}_t(y, v, r) + \sqrt{\mathcal{D}(v)} \partial_y \dot{W}_{y,t}(v), \quad (1.2)$$

with  $\mathbb{E}(\dot{W}_{y,t}(v) \dot{W}_{y',s}(w)) = \delta(t - s) \delta(y - y') \delta(v - w)$ . Notice that the equilibrium solutions of [\(1.1\)](#) and [\(1.2\)](#) have the same space-time covariance, i.e. the space-time covariance does not give informations about the martingale term of the evolution equation. About the uncorrelation in  $r$ , this persists in the macroscopic fluctuations as consequence of the decorrelation at the microscopic scale.

In order to understand why [\(1.1\)](#) arise, we follow the behaviour of two tagged quasi-particles with the same velocity. We call here quasi-particles (or impulsions) the particles with the dynamics defined by the exchange of positions at the moment of collision. The standard technique to study such dynamics is to go to the *reduced description* where quasi-particles are mapped to points and evolve without interaction. Then the evolution of the tagged quasi-particle is obtained by the trivial evolution of the corresponding point, shifted by the collisions with quasi-particles of different velocity. Since the points corresponding to the other quasi-particles are distributed by a Poisson field, these collisions happen at random times and the collisional shifts are independent. Consequently at the Euler

scale we have a law of large numbers (see Section 2.2) that produce a deterministic evolution of the tagged quasi-particle with an effective velocity given by (2.20) (such ergodicity was proven first in [1]). Recentering the position of the tagged quasi-particle on the Euler deterministic behavior, we have then a functional central limit theorem so that the position converges in law to a Brownian motion of variance  $\mathcal{D}(v)$ . Now consider two tagged quasi-particles with the same velocity and initially located at macroscopic distance: this means that there are typically  $\varepsilon^{-1}$  particles in between, where  $\varepsilon$  is the scaling parameter going to 0. In the diffusive scaling each tagged quasi-particle at time  $t$  has a number of collisions proportional to  $\varepsilon^{-2}t$ , but most of them with the same quasi-particles, except for an order  $\varepsilon^{-1}$  of collisions at the beginning and at the end of the time interval  $[0, t]$ . Consequently the two tagged quasi-particles completely correlate in the limit  $\varepsilon \rightarrow 0$ , i.e. they converge to the same Brownian motion. This causes the rigid motion of the corresponding density fluctuations. Notice that in chaotic system it is expected that the two quasi-particles at initial macroscopic distance converge to two independent Brownian motion, generating the space-time white noise present in (1.2).

As far as we know, equation (1.1) for the diffusive fluctuations for hard rods is new in the literature. We recently discovered the article by Presutti and Wick [18] that concerns the diffusive behavior of travelling wave initial conditions in hard rods systems, where in the remark after Theorem 1 they comment about possible diffusive behaviour of fluctuations and they write: “spatially separated fluctuations in the density of rods with the same velocity move with the same Brownian component”. Strangely [18] is never quoted in the following literature about Navier-Stokes corrections for the hard rods hydrodynamics (cf. [6]). In the introduction of [7] it was announced a second article about the *Navier Stokes corrections* for the evolution of the fluctuations, but the authors confirmed us that this has never been written.

We believe that (1.1) is a typical (universal) macroscopic behaviour for the diffusive fluctuations of completely integrable many-body systems [21]. Since this macroscopic behaviour is due to the scattering shift that the quasi particles have when *crossing* particles with a different velocity, we believe that this equations are universal for any extended integrable system. In fact any extended integrable system is equivalent (in a macroscopic limit) to a collisional system (of *quasi-particle* or *impulsions* or *solitons*) and it is characterized macroscopically by its scattering shift [21]. What makes the hard rods system treatable is the fact that the scattering shift does not depend on the velocities of the quasi-particles colliding.

Hard rods dynamics with domain wall initial conditions, a particular case of travelling wave, is investigated in [11]; the paper includes the Navier-Stokes corrections and the computation of the covariance  $\Gamma(v, v')$  from Green-Kubo formula.

Diffusive corrections to the hydrodynamic Euler scaling in general completely integrable systems have been investigated in the physics literature, see the recent review [10] with the references therein, as well as [14] and [9]. These articles

contain general formulas for diffusion coefficients that are in agreement with our formula for the generalized hard rods. A generalization of our equation (1.1) to other integrable systems will give answer about the macroscopic evolution on the diffusive scale of multipoint correlation functions, mentioned in [10] as an open problem.

As we believe that our approach is more elementary than the one used in the previous literature, we have written this article to be completely independent of results on hard rods prior to the paper [12], which is our starting point. Essentially, the only tools we use are the law of large numbers and the central limit theorem for a Poisson field. In Section 2 we prove the macroscopic evolution of the fluctuations in the Euler scaling (recovering the result of [7]). In Section 3 we prove the evolution of the fluctuations in the diffusive scaling. Finally in Section 5 we prove a lemma about limits for Poisson fields that we need in the proofs.

## 2. Equilibrium Fluctuations in the Euler scaling

Let  $X^\varepsilon$  be the Poisson process on  $\mathbb{R}^3$  with intensity  $\varepsilon^{-1}\rho dx d\mu(v,r)$ , where  $\mu$  is a probability on  $\mathbb{R}^2$  with finite second moments. We should think about  $x$  as the *macroscopic* position of the point  $x$ , since the typical distance between points is  $\varepsilon$ . The *macroscopic length of the rod*  $(x,v,r)$  is  $\varepsilon r$ .

We define

$$\begin{aligned}\sigma &:= \rho \iint r d\mu(v,r), && \text{length density,} \\ \pi &:= \rho \iint r v d\mu(v,r), && \text{momentum density.}\end{aligned}$$

Denote the empirical length distribution by

$$N^\varepsilon \varphi := \sum_{(x,v,r) \in X^\varepsilon} \varepsilon r \varphi(x,v,r), \quad N^\varepsilon(A) := N^\varepsilon \mathbb{1}_A.$$

Here and in the following  $\varphi(x,v,r)$  is a generic continuous function on  $\mathbb{R}^3$ , differentiable and compactly supported in  $x$ . The expectation of  $N^\varepsilon$  is

$$\mathbb{E} N^\varepsilon \varphi = \langle\langle \varphi \rangle\rangle,$$

where the length biased measure  $\langle\langle \cdot \rangle\rangle$  is defined by

$$\langle\langle \varphi \rangle\rangle := \rho \iiint r \varphi(x,v,r) dx d\mu(v,r).$$

We have the law of large numbers

$$N^\varepsilon \varphi \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \mathbb{E} N^\varepsilon \varphi = \langle\langle \varphi \rangle\rangle. \quad (2.1)$$

If  $\varphi(x, v, r) = 1_{[0,1]}(x)$ , we have  $\sum_{(x,v,r) \in X^\varepsilon} \varepsilon r \varphi(x) \rightarrow \sigma$ .

The central limit theorem states that

$$\xi^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2}(N^\varepsilon \varphi - \langle\langle \varphi \rangle\rangle) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\varphi) \quad (2.2)$$

where  $\xi^X$  is the centered Gaussian white noise with covariance

$$\mathbb{E}(\xi^X(\varphi)\xi^X(\psi)) = \langle\langle \varphi\psi \rangle\rangle_2 - \langle\langle \varphi \rangle\rangle \langle\langle \psi \rangle\rangle, \quad (2.3)$$

where

$$\langle\langle \varphi \rangle\rangle_2 := \rho \iiint r^2 \varphi(x, v, r) dx d\mu(v, r).$$

Notice that

$$\mathbb{E}((\xi^{X,\varepsilon} \varphi)^2) = \langle\langle \varphi^2 \rangle\rangle_2 - \langle\langle \varphi \rangle\rangle^2,$$

so that for any  $\varphi \in L^2(\langle\langle \cdot \rangle\rangle_2)$  we have the bound

$$\mathbb{E}((\xi^{X,\varepsilon} \varphi)^2) \leq \langle\langle \varphi^2 \rangle\rangle_2.$$

For any  $a, b \in \mathbb{R}$ , define the *mass* (length) measure by

$$m_a^b(X^\varepsilon) = \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r (1_{[x \in [a,b]]} - 1_{[x \in [b,a]]}).$$

Consequently,

$$m_a^b(X^\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \mathbb{E} m_a^b(X^\varepsilon) = (b - a)\sigma. \quad (2.4)$$

To each configuration  $X^\varepsilon$  and a point  $b \in \mathbb{R}$ , there are a dilated point and configuration

$$\begin{aligned} D^\varepsilon(b) &:= b + m_0^b(X^\varepsilon) \\ Y^\varepsilon = D^\varepsilon(X^\varepsilon) &:= \{(D^\varepsilon(x), v, r) : (x, v, r) \in X^\varepsilon\}. \end{aligned}$$

**Remark 2.1.** The distribution of  $X^\varepsilon$  is space shift invariant, but the distribution of the rod configuration  $Y^\varepsilon$  is not because  $Y^\varepsilon$  has no rod containing the origin. Our results can be extended to random rod configurations with space shift invariant distribution, by using Palm transforms [22] and Harris theorem [15]; see for instance [13].

The macroscopic dilation of the point  $b$  is given by

$$\mathbb{E} D^\varepsilon(b) = b(1 + \sigma).$$

Denote the length empirical measure induced by  $Y^\varepsilon$  by

$$K^\varepsilon \varphi := \varepsilon \sum_{(y,v,r) \in Y^\varepsilon} r \varphi(y, v, r).$$

We have the law of large numbers:

$$\begin{aligned}
K^\varepsilon \varphi &= \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x + m_0^x(X^\varepsilon), v, r) \\
&\xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \rho \iiint r \varphi(x + x\sigma, v, r) dx d\mu(v, r) \\
&= \frac{\rho}{1 + \sigma} \iiint r \varphi(y, v, r) dy d\mu(v, r) \\
&= \frac{1}{1 + \sigma} \langle\langle \varphi \rangle\rangle.
\end{aligned} \tag{2.5}$$

### 2.1. Static CLT for the dilated configuration

We define the fluctuation field

$$\xi^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2} (K^\varepsilon \varphi - \mathbb{E}K^\varepsilon \varphi).$$

We have

$$K^\varepsilon \varphi - \mathbb{E}K^\varepsilon \varphi = (K^\varepsilon \varphi - A^\varepsilon \varphi) + (A^\varepsilon \varphi - \mathbb{E}A^\varepsilon \varphi) - (\mathbb{E}K^\varepsilon \varphi - \mathbb{E}A^\varepsilon \varphi). \tag{2.6}$$

where

$$\begin{aligned}
A^\varepsilon \varphi &:= \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x(1 + \sigma), v, r), \\
\mathbb{E}A^\varepsilon \varphi &= \frac{\rho}{1 + \sigma} \iiint r \varphi(y, v, r) dy d\mu(v, r).
\end{aligned}$$

The last term in (2.6) gives

$$\begin{aligned}
&\varepsilon^{-1/2} [\mathbb{E}(K^\varepsilon \varphi) - \mathbb{E}(A^\varepsilon \varphi)] \\
&= \mathbb{E} \left( \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r [\varphi(x + m_0^x(X^\varepsilon), v, r) - \varphi(x(1 + \sigma), v, r)] \right)
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&= \mathbb{E} \left( \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\partial_y \varphi)(x(1 + \sigma), v, r) (m_0^x(X^\varepsilon) - x\sigma) \right) + R^\varepsilon \\
&= \iiint r (\partial_y \varphi)(x(1 + \sigma), v, r) \varepsilon^{-1/2} [\mathbb{E}(m_0^x(X^\varepsilon) - x\sigma)] dx d\mu(v, r) + R^\varepsilon
\end{aligned} \tag{2.8}$$

$$= R^\varepsilon, \tag{2.9}$$

where  $R^\varepsilon$  denotes a generic term small with  $\varepsilon$ . Identity (2.8) follows from Slyvniak-Mecke formula (Theorem 3.2 in [16]). Since by (2.4)  $\mathbb{E}(m_0^x(X^\varepsilon) - x\sigma) = 0$ , the identity (2.9) follows.

Applying (2.2) to the second term in (2.6) we have

$$\varepsilon^{-1/2} [A^\varepsilon \varphi - \mathbb{E}(A^\varepsilon \varphi)]$$

$$\begin{aligned}
&= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \check{\varphi}(x, v, r) - \rho \iiint r \check{\varphi}(x, v, r) dx d\mu(v, r) \right] \\
&\xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\check{\varphi}), \tag{2.10}
\end{aligned}$$

where  $\check{\varphi}(x, v, r) := \varphi((1 + \sigma)x, v, r)$ . Notice that

$$\begin{aligned}
&\mathbb{E}(\xi^X(\check{\varphi}) \xi^X(\check{\psi})) \\
&= \rho \iiint r^2 \check{\varphi}(x, v, r) \check{\psi}(x, v, r) dx d\mu(v, r) - \langle\langle \check{\varphi} \rangle\rangle \langle\langle \check{\psi} \rangle\rangle \\
&= \frac{\rho}{1 + \sigma} \iiint r^2 \varphi(y, v, r) \psi(y, v, r) dy d\mu(v, r) - \frac{1}{(1 + \sigma)^2} \langle\langle \varphi \rangle\rangle \langle\langle \psi \rangle\rangle \\
&= \frac{1}{1 + \sigma} \langle\langle \varphi \psi \rangle\rangle_2 - \frac{1}{(1 + \sigma)^2} \langle\langle \varphi \rangle\rangle \langle\langle \psi \rangle\rangle.
\end{aligned}$$

Finally expand the first term of the RHS of (2.6):

$$\begin{aligned}
\varepsilon^{-1/2}(K^\varepsilon \varphi - A^\varepsilon \varphi) &= \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r(\varphi(x + m_0^x(X^\varepsilon), v, r) - \varphi(x(1 + \sigma), v, r)) \\
&= \frac{1}{1 + \sigma} \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r(\partial_x \check{\varphi})(x, v, r)(m_0^x(X^\varepsilon) - x\sigma) + R^\varepsilon.
\end{aligned}$$

By the functional central limit theorem (2.2) we have for any  $z \in \mathbb{R}$

$$\begin{aligned}
B^\varepsilon(z) &:= \varepsilon^{-1/2}(m_0^z(X^\varepsilon) - z\sigma) = \xi^{X,\varepsilon}(1_{[0,z]}1_{z>0} - 1_{[z,0]}1_{z<0}) \\
&\xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(1_{[0,z]}1_{z>0} - 1_{[z,0]}1_{z<0}) =: B(z),
\end{aligned}$$

where  $(B(z) : z \in \mathbb{R})$  is a bilateral Brownian motion. Using Lemma 5.1 we have

$$\begin{aligned}
&\varepsilon^{-1/2}(K^\varepsilon \varphi - A^\varepsilon \varphi) \\
&\xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \frac{\rho}{1 + \sigma} \iiint r(\partial_x \check{\varphi})(x, v, r) B(x) dx d\mu(v, r) \tag{2.11} \\
&= \frac{1}{1 + \sigma} \int dx B(x) \partial_x \left( \rho \iiint r \check{\varphi}(x, v, r) d\mu(v, r) \right) \\
&= \frac{1}{1 + \sigma} \int dx \xi^X(1_{[0,x]}1_{x>0} - 1_{[x,0]}1_{x<0}) \partial_x \left( \rho \iiint r \check{\varphi}(x, v, r) d\mu(v, r) \right) \\
&= \frac{1}{1 + \sigma} \xi^X \left( \int dx (1_{[0,x]}1_{x>0} - 1_{[x,0]}1_{x<0}) \partial_x \left( \rho \iiint r \check{\varphi}(x, v, r) d\mu(v, r) \right) \right) \\
&= -\frac{1}{1 + \sigma} \xi^X \left( \rho \iiint r' \check{\varphi}(\cdot, v', r') d\mu(v', r') \right). \tag{2.12}
\end{aligned}$$

We conclude that

$$\varepsilon^{1/2}(K^\varepsilon \varphi - A^\varepsilon \varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} -\xi^X(P\check{\varphi}), \tag{2.13}$$



where

$$P\check{\varphi}(x) = \frac{\rho}{1+\sigma} \iint r\check{\varphi}(x, v', r') d\mu(v', r') = \frac{\rho}{1+\sigma} \iint r\varphi((1+\sigma)x, v', r') d\mu(v', r'). \quad (2.14)$$

Putting together (2.13), (2.7)-(2.9) and (2.10) we have shown that

$$\xi^{Y, \varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^Y(\varphi) = \xi^X(\check{\varphi} - P\check{\varphi}) = \xi^X(C\check{\varphi}), \quad (2.15)$$

where  $C = I - P$ . This identifies  $\xi^Y$  as the centered Gaussian field with covariance

$$\begin{aligned} \mathbb{E}(\xi^Y(\varphi)\xi^Y(\psi)) &= \mathbb{E}(\xi^X(C\check{\varphi})\xi^X(C\check{\psi})) \\ &= \rho \iiint r^2 C\varphi(x(1+\sigma), v, r) C\psi(x(1+\sigma), v, r) dx d\mu(v, r), \\ &= \frac{\rho}{1+\sigma} \iiint r^2 C\varphi(y, v, r) C\psi(y, v, r) dy d\mu(v, r) \\ &= \frac{\rho}{1+\sigma} \langle\langle C\varphi C\psi \rangle\rangle_2. \end{aligned}$$

That means for the Fourier transforms

$$\begin{aligned} \widehat{\varphi}(k, v, r) &= \int e^{i2\pi ky} \varphi(y, v, r) dy \\ \mathbb{E}(\xi^Y(\varphi)\xi^Y(\psi)) &= \frac{\rho}{1+\sigma} \iiint r^2 C\widehat{\varphi}(k, v, r) * C\widehat{\psi}(k, v, r) dk d\mu(v, r). \end{aligned}$$

i.e. a covariance operator

$$C = \frac{\rho}{1+\sigma} r^2 C^2 = \frac{\rho}{1+\sigma} r^2 \left( I + \left( \frac{\sigma^2}{(1+\sigma)^2} - 2\frac{\sigma}{1+\sigma} \right) P \right).$$

**Remark 2.2.** *This is in agreement with formula (7.61) in [20].*

**Example 2.3.** *In the case  $d\mu(v, r) = \frac{1}{2}(\delta_{v_0}(dv) + \delta_{-v_0}(dv))\delta_a(dr)$  we have  $\sigma = \rho a$  and  $\pi = 0$ . Then,*

$$\begin{aligned} P\varphi(x) &= \frac{\rho a}{2(1+\rho a)} (\varphi(x, v_0) + \varphi(x, -v_0)) \\ C\varphi(x, \pm v_0) &= \frac{2+\rho a}{2(1+\rho a)} \varphi(x, \pm v_0) - \frac{\rho a}{2(1+\rho a)} \varphi(x, \mp v_0). \end{aligned}$$

## 2.2. Equilibrium fluctuations in the Euler scaling

Recall that the interparticle distance is of order  $\varepsilon$ , i.e. the coordinates  $(x, v, r)$  are already rescaled in the Euler scale. Let  $X_t^\varepsilon$  denote the free gas configuration at time  $t$ :

$$X_t^\varepsilon := \{(x + vt, v, r) : (x, v, t) \in X^\varepsilon\}.$$

For any  $(x, v) \in \mathbb{R}^2$  and  $t \geq 0$  define the flow

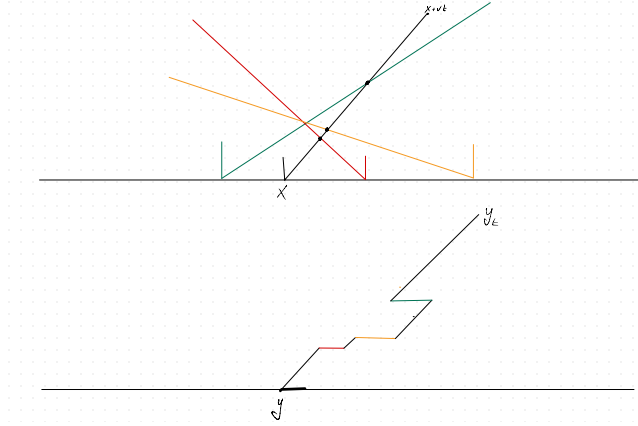
$$j^\varepsilon(x, v, t) := \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \tilde{r} \left( \mathbf{1}_{[\tilde{v} < v]} \mathbf{1}_{[x < \tilde{x} < x + (v - \tilde{v})t]} - \mathbf{1}_{[\tilde{v} > v]} \mathbf{1}_{[x + (v - \tilde{v})t < \tilde{x} < x]} \right). \quad (2.16)$$

For any fixed  $(x, v)$ , the expectation of  $j^\varepsilon(x, v, t)$  is given by

$$\begin{aligned} j(x, v, t) &:= \mathbb{E} j^\varepsilon(x, v, t) \\ &= \iiint \rho \, dx \, d\mu(\tilde{v}, \tilde{r}) \, \tilde{r} \left( \mathbf{1}_{[\tilde{v} < v]} \mathbf{1}_{[x < \tilde{x} < x + (v - \tilde{v})t]} - \mathbf{1}_{[\tilde{v} > v]} \mathbf{1}_{[x + (v - \tilde{v})t < \tilde{x} < x]} \right) \\ &= \rho \int \tilde{r} \int_v^{+\infty} (v - \tilde{v}) \, t \, d\mu(\tilde{v}, \tilde{r}) + \rho \int \tilde{r} \int_{-\infty}^v (v - \tilde{v}) \, t \, d\mu(\tilde{v}, \tilde{r}) \\ &= tv\sigma - t\pi. \end{aligned} \quad (2.17)$$

We have the following limit as a consequence of the law of large numbers

$$j^\varepsilon(x, v, t) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} j(x, v, t) = t(v\sigma - \pi). \quad (2.18)$$



For any  $(x, v) \in \mathbb{R}^2$  and  $t \geq 0$  we define

$$y_{v,t}^\varepsilon(x) := x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x, v, t) \quad (2.19)$$

and its average is

$$y_{v,t}(x) := \mathbb{E} y_{v,t}^\varepsilon(x) = x(1 + \sigma) + vt + j(x, v, t) = x(1 + \sigma) + tv(1 + \sigma) - t\pi.$$

We say that  $y_{v,t}^\varepsilon(x)$  is the position at time  $t$  of the *quasi-particle* with velocity  $v$  starting from  $x$  (see Figure 2.2). Notice that this definition is valid for all  $x, v$ , it does not require that  $(x, v)$  are points in  $X^\varepsilon$ .

By (2.18)

$$y_{v,t}^\varepsilon - y \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} v^{\text{eff}}(v)t,$$

where the effective velocity is given by

$$v^{\text{eff}}(v) := v(1 + \sigma) - \pi. \quad (2.20)$$

We have that, by (2.1), the free gas empirical length measure at time  $t$  satisfies

$$N_t^\varepsilon \varphi := \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x + vt, v, r) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \rho \iiint r \varphi(x + vt, v, r) dx d\mu(v, r) = \langle\langle \varphi \rangle\rangle.$$

The  $X$ -fluctuation field at time  $t$  satisfies

$$\xi_t^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2} (N_t^\varepsilon \varphi - \langle\langle \varphi \rangle\rangle) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\varphi_t), \quad (2.21)$$

where

$$\varphi_t(x, v, r) := \varphi(x + tv, v, r).$$

The hard rod configuration and empirical measure at time  $t$  are given by

$$\begin{aligned} Y_t^\varepsilon &:= \{(y_{v,t}^\varepsilon(x), v, r) : (x, v, r) \in X^\varepsilon\}, \\ K_t^\varepsilon \varphi &:= \varepsilon \sum_{(y,v,r) \in Y_t^\varepsilon} r \varphi(y, v, r) = \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(y_{v,t}^\varepsilon(x), v, r). \end{aligned}$$

Using (2.5) and similar argument used at the beginning of section 2.1, we have the LLN for  $K_t^\varepsilon$ :

$$\begin{aligned} K_t^\varepsilon \varphi &\xrightarrow[\varepsilon \rightarrow 0]{} \rho \iiint r \varphi(y_{v,t}^\varepsilon(x), v, r) dx d\mu(v, r) \\ &= \rho \iint r \left( \int \varphi(x(1 + \sigma) + v^{\text{eff}}(v)t, v, r) dx \right) d\mu(v, r) \\ &= \frac{\rho}{1 + \sigma} \iiint r \varphi(y, v, r) dy d\mu(v, r) = \frac{\rho}{1 + \sigma} \langle\langle \varphi \rangle\rangle. \end{aligned}$$

We define the  $Y$ -fluctuation field at time  $t$  by

$$\xi_t^{Y,\varepsilon}(\varphi) := \varepsilon^{-1/2} (K_t^\varepsilon \varphi - \mathbb{E} K_t^\varepsilon \varphi).$$

In order to shorten notations in the following we set  $\pi = 0$ , the extension to the case  $\pi \neq 0$  is straightforward. We have

$$K_t^\varepsilon \varphi - \mathbb{E} K_t^\varepsilon \varphi = (K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) + (A_t^\varepsilon \varphi - \mathbb{E} A_t^\varepsilon \varphi) - (\mathbb{E} K_t^\varepsilon \varphi - \mathbb{E} A_t^\varepsilon \varphi). \quad (2.22)$$

where

$$A_t^\varepsilon \varphi := \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi((x + vt)(1 + \sigma), v, r).$$

The last term in (2.22) gives

$$\begin{aligned}
& \varepsilon^{-1/2}(\mathbb{E}K_t^\varepsilon \varphi - \mathbb{E}A_t^\varepsilon \varphi) \\
&= \mathbb{E}\left(\varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r[\varphi(x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x, v, t), v, r) \right. \\
&\quad \left. - \varphi((x + vt)(1 + \sigma), v, r)]\right) \tag{2.23} \\
&= \mathbb{E}\left(\varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r(\partial_y \varphi)((x + vt)(1 + \sigma), v, r) \right. \\
&\quad \left. \times (m_0^x(X_t^\varepsilon) + j^\varepsilon(x, v, t) - (x + vt)\sigma)\right) + R_t^\varepsilon \\
&= \iiint r(\partial_y \varphi)((x + vt)(1 + \sigma), v, r) \varepsilon^{-1/2} \\
&\quad \times \mathbb{E}[m_0^x(X^\varepsilon) + j^\varepsilon(x, v, t) - (x + vt)\sigma] dx d\mu(v, r) + R_t^\varepsilon \\
&= R_t^\varepsilon, \tag{2.24}
\end{aligned}$$

where  $R_t^\varepsilon$  is of smaller order in  $\varepsilon$ . The last identity follows from the Slyvniak-Mecke formula and from (2.4) and (2.17). By (2.2) the second term in (2.22) gives

$$\varepsilon^{1/2}(A_t^\varepsilon - \mathbb{E}A_t^\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi_t^X(\check{\varphi}_t),$$

where

$$\check{\varphi}_t(x, v, r) := \varphi((x + vt)(1 + \sigma), v, r).$$

Finally, the first term in (2.22) gives

$$\begin{aligned}
& \varepsilon^{1/2}(K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) \\
&= \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r[\varphi(x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x, v, t), v, r) - \varphi((x + vt)(1 + \sigma), v, r)] \\
&= \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r(\partial_x \varphi)((x + vt)(1 + \sigma), v, r)(m_0^x(X^\varepsilon) + j^\varepsilon(x, v, t) - \sigma(x + vt)) + R_t^\varepsilon \\
&= \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r(\partial_x \varphi)((x + vt)(1 + \sigma), v, r) \varepsilon^{-1/2}(m_0^x(X^\varepsilon) + j^\varepsilon(x, v, t) - \sigma(x + vt)) + R_t^\varepsilon. \tag{2.25}
\end{aligned}$$

where  $R_t^\varepsilon$  is smaller order. It can be easily shown that

$$\varepsilon^{-1/2} j^\varepsilon(0, 0, t) := \varepsilon^{1/2} \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \tilde{r}(1_{[\tilde{v} < 0]} 1_{[0 < \tilde{x} < -\tilde{v}t]} - 1_{[\tilde{v} > 0]} 1_{[-\tilde{v}t < \tilde{x} < 0]}) \tag{2.26}$$

converges in law to a Wiener process  $\tilde{B}(t, 0)$  with a finite variance. By applying

Lemma 5.1 we have

$$\begin{aligned} & \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r(\partial_x \varphi)((x+vt)(1+\sigma), v, r) \varepsilon^{-1/2} j^\varepsilon(0, 0, t) \\ & \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \tilde{B}(t, 0) \iint d\mu(v, r) r \int (\partial_x \varphi)((x+vt)(1+\sigma), v, r) dx = 0. \end{aligned}$$

Consequently we can write (2.25) as

$$\begin{aligned} \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r(\partial_x \varphi)((x+vt)(1+\sigma), v, r) \varepsilon^{-1/2} (m_0^x(X^\varepsilon) + j^\varepsilon(x, v, t) - j^\varepsilon(0, 0, t) - \sigma(x+vt)) \\ + R_t^\varepsilon. \end{aligned} \quad (2.27)$$

Remark that

$$m_0^x(X^\varepsilon) + j^\varepsilon(x, v, t) = j^\varepsilon(0, 0, t) + m_0^{x+vt}(X_t^\varepsilon),$$

and the first term in (2.27) can be written as

$$\varepsilon \sum_{(x,v,r) \in X^\varepsilon} r(\partial_x \varphi)((x+vt)(1+\sigma), v, r) \varepsilon^{-1/2} (m_0^{x+vt}(X_t^\varepsilon) - \sigma(x+vt)). \quad (2.28)$$

By (2.21) we have

$$\varepsilon^{-1/2} (m_0^x(X_t^\varepsilon) - \sigma(x)) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi_t^X (1_{[0,x]} 1_{[x>0]} - 1_{[x,0]} 1_{[x<0]}) =: \bar{B}(t, x),$$

where, for any  $t \geq 0$ ,  $\bar{B}(t, x)$  us a double sided Wiener process in  $x \in \mathbb{R}$ . Then, by Lemma 5.1, (2.28) converges in law to

$$\begin{aligned} & \rho \iiint r(\partial_x \varphi)((x+vt)(1+\sigma), v, r) \bar{B}(t, x+vt) dx d\mu(v, r) \\ & = \rho \iiint r(\partial_x \varphi)((x)(1+\sigma), v, r) \bar{B}(t, x) dx d\mu(v, r) \\ & = \frac{\rho}{1+\sigma} \iiint \xi_t^X (1_{[0,x]} 1_{[x>0]} - 1_{[x,0]} 1_{[x<0]}) r \partial_x \check{\varphi}(x, v, r) dx d\mu(v, r) \\ & = -\frac{1}{1+\sigma} \xi_t^X \left( \rho \iint r \check{\varphi}(\cdot, \tilde{v}, \tilde{r}) d\mu(\tilde{v}, \tilde{r}) \right) \\ & = -\frac{1}{1+\sigma} \xi^X \left( \rho \iint r \check{\varphi}_t(\cdot, \tilde{v}, \tilde{r}) d\mu(\tilde{v}, \tilde{r}) \right), \end{aligned}$$

where  $\check{\varphi}_t(x, \tilde{v}, \tilde{r}) = \check{\varphi}_t(x + \tilde{v}t, \tilde{v}, \tilde{r})$ .

Recalling  $P\varphi$  defined in (2.14), we conclude that

$$\varepsilon^{1/2} (K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} -\xi^X (P\check{\varphi}_t). \quad (2.29)$$

Putting together (2.29), (2.23)-(2.24) we have shown that

$$\xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi_t^Y(\varphi) := \xi^X(\check{\varphi}_t - P\check{\varphi}_t) = \xi^X(C\check{\varphi}_t),$$

where  $C := I - P$ . Recalling (2.15), we have proven that

$$\xi_t^Y(\varphi) = \xi_0^Y(\varphi_t),$$

i.e.

$$\partial_t \xi_t^Y(\varphi) = \xi_0^Y(v^{\text{eff}} \partial_x \varphi_t) = \xi_t^Y(v^{\text{eff}} \partial_x \varphi).$$

In other words, in a weak sense  $\xi_t^Y$  satisfies the equation

$$\partial_t \xi_t^Y + v^{\text{eff}} \partial_x \xi_t^Y = 0,$$

which is the expected equation.

### 3. Equilibrium Fluctuations in the diffusive scaling

#### 3.1. Quasi-particles in the diffusing scaling

Recall the definition of  $j^\varepsilon(x, v, t)$  given by (2.26) for any  $(x, v) \in \mathbb{R}^2$ . and the corresponding convergence (2.18). We look now at this flow in a larger, diffusive, time scale, and we have that

$$j^\varepsilon(x, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} B_t(v). \quad (3.1)$$

where  $(B_t(v))_{v \in \mathbb{R}} =: \mathbb{B}_t$  is a family of Brownian motions in  $t$  with covariance given by (3.7) and initial value  $B_0(v) = 0$  for all  $v$ . Notice in (3.1) that  $B_t(v)$  does not depend on  $x$ . In fact we will see below that, given two points  $x$  and  $\tilde{x}$ , the variance of the difference  $j_{v,t}^\varepsilon(x, v, \varepsilon^{-1}t) - j_{v,t}^\varepsilon(\tilde{x}, v, \varepsilon^{-1}t)$  is of order  $2\varepsilon|x - \tilde{x}|$ , as there are around  $\varepsilon^{-1}|x - \tilde{x}|$  particles between  $x$  and  $\tilde{x}$ , and between  $x + vt$  and  $\tilde{x} + vt$ , and each of those particles contributes  $\varepsilon^2$  to the variance, see (3.5). This explains why the limit in (3.1) does not depend on  $x$ .

Given a point  $(x, v) \in \mathbb{R}^2$  recall that  $y_{v,t}^\varepsilon(x)$  is defined by (2.19). As a consequence of (3.1) we have

$$y_{v,\varepsilon^{-1}t}^\varepsilon(x) - v^{\text{eff}}(v)\varepsilon^{-1}t \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} y + B_t(v), \quad (3.2)$$

where  $y = x(1 + \sigma)$ . Observe that

$$y_{v,\varepsilon^{-1}t}^\varepsilon(x) - v^{\text{eff}}(v)\varepsilon^{-1}t = x + m_0^x(X^\varepsilon) - (1 + \sigma)x + j^\varepsilon(x, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t.$$

Since  $m_0^x \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \sigma x$ , the limit in (3.2) is equivalent to (3.1)

The surface  $(t, vt) \mapsto B_t(v)$  is a two dimensional Gaussian process described by a Lévy–Chentsov field [12]. We compute explicitly the covariances.

By (2.19) Observe that

$$\varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \tilde{r} 1_{[\tilde{v} < v]} 1_{[x < \tilde{x} < x + (v - \tilde{v})\varepsilon^{-1}t]} = \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^{\varepsilon^2}} \tilde{r} 1_{[\tilde{v} < v]} 1_{[\varepsilon x < \tilde{x} < \varepsilon x + (v - \tilde{v})t]},$$

where  $X^{\varepsilon^2} := \{(\varepsilon\tilde{x}, \tilde{v}, \tilde{r}) : (\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon\}$ , is obtained from  $X^\varepsilon$  by rescaling all positions by a factor  $\varepsilon$ , so that  $X^{\varepsilon^2}$  is a Poisson process of intensity measure  $\varepsilon^{-2}\rho d\mu(v, r)$ . We have that

$$\mathbb{E}\left(\varepsilon^2 \sum_{(x', v', r') \in X^{\varepsilon^2}} r' 1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v - v')t]}\right) = t\rho \int r \int_{-\infty}^v (v - v') d\mu(v', r).$$

Define the fluctuation field

$$\tilde{\xi}^\varepsilon(\varphi) = \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^{\varepsilon^2}} \varphi(x, v, r) - \varepsilon^{-1} \langle\langle \varphi \rangle\rangle = \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \varphi(\varepsilon x, v, r) - \varepsilon^{-1} \langle\langle \varphi \rangle\rangle.$$

As for  $\xi^\varepsilon$ , cf. (2.2),  $\tilde{\xi}^\varepsilon$  converge to a centered gaussian field  $\tilde{\xi}$  with the same covariance as  $\xi$ , cf. (2.3). But  $\tilde{\xi}$  and  $\xi$  are independent. In fact consider two test function  $\varphi, \psi$  such that  $\langle\langle \varphi \rangle\rangle = 0$  and  $\langle\langle \psi \rangle\rangle = 0$ :

$$\mathbb{E}\left(\tilde{\xi}^\varepsilon(\varphi)\xi^\varepsilon(\psi)\right) = \varepsilon^{1/2} \iint d\mu(v, r) r^2 \int \varphi(\varepsilon x, v, r) \psi(x, v, r) dx \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.3)$$

Consider the function

$$\varphi_{\varepsilon x, v, t}(x', v') = 1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v - v')t]} - 1_{[v' > v]} 1_{[\varepsilon x + (v - v')t < x' < \varepsilon x]},$$

where  $(x', v', r') \in X^{\varepsilon^2}$ . Then

$$j_{X^\varepsilon}(x, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t = \tilde{\xi}^{X, \varepsilon}(\varphi_{\varepsilon x, v, t})$$

For  $x = 0$  we have

$$j_{X^\varepsilon}(0, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \tilde{\xi}^X(\varphi_{0, v, t}),$$

which has variance

$$\begin{aligned} & \rho \iiint \tilde{r}^2 (1_{[\tilde{v} < v]} 1_{[0 < \tilde{x} < (v - \tilde{v})t]} + 1_{[\tilde{v} > v]} 1_{[(v - \tilde{v})t < \tilde{x} < 0]}) d\tilde{x} d\mu(\tilde{v}, \tilde{r}) \quad (3.4) \\ & = \rho \int \tilde{r}^2 \int_v^{+\infty} (v - \tilde{v})t d\mu(\tilde{v}, \tilde{r}) - \rho \int \tilde{r}^2 \int_{-\infty}^v (v - \tilde{v})t d\mu(\tilde{v}, \tilde{r}) \\ & = t\rho \iint \tilde{r}^2 |v - \tilde{v}| d\mu(\tilde{v}, \tilde{r}) := t\mathcal{D}(v). \end{aligned}$$

Computing the correlation for different initial position, assuming  $x < \bar{x}$ :

$$\begin{aligned}
& \mathbb{E}(\xi^{X,\varepsilon}(\varphi_{\varepsilon x,v,t})\xi^{X,\varepsilon}(\varphi_{\varepsilon \bar{x},v,t})) \tag{3.5} \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} - 1_{[v' > v]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]}) \\
&\quad \times (1_{[v' < v]} 1_{[\varepsilon \bar{x} < x' < \varepsilon \bar{x} + (v-v')t]} - 1_{[v' > v]} 1_{[\varepsilon \bar{x} + (v-v')t < x' < \varepsilon \bar{x}]}) dx' d\mu(v', r) \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} 1_{[\varepsilon \bar{x} < x' < \varepsilon \bar{x} + (v-v')t]} \\
&\quad + 1_{[v' > v]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]} 1_{[\varepsilon \bar{x} + (v-v')t < x' < \varepsilon \bar{x}]}) dx' d\mu(v', r) \\
&= \rho \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} [(v-v')t - \varepsilon(\bar{x}-x)] d\mu(v', r) \\
&\quad + \rho \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} (\varepsilon(x-\bar{x}) - (v-v')t) d\mu(v', r) \\
&= \rho [vt - \varepsilon(\bar{x}-x)] \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} d\mu(v', r) - \rho t \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} v' d\mu(v', r) \\
&\quad + \rho [\varepsilon(x-\bar{x}) - vt] \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} d\mu(v', r) + \rho t \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} v' d\mu(v', r) \\
&= \rho [vt - \varepsilon(\bar{x}-x)] \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} d\mu(v', r) - \rho t \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} v' d\mu(v', r) \\
&\quad - \rho [vt + \varepsilon(\bar{x}-x)] \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} d\mu(v', r) + \rho t \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} v' d\mu(v', r) \\
&\quad \xrightarrow{\varepsilon \rightarrow 0} t\rho \iint (r')^2 |v - v'| d\mu(v', r') = t\mathcal{D}(v).
\end{aligned}$$

It follows from (3.5) that two tagged quasi-particles with the same velocity are asymptotically completely correlated and

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left( \left| \tilde{\xi}^{X,\varepsilon}(\varphi_{\varepsilon x,v,t}) - \tilde{\xi}^{X,\varepsilon}(\varphi_{0,v,t}) \right|^2 \right) = 0.$$

Considering the correlation at different velocities  $v < \bar{v}$ :

$$\begin{aligned}
& \mathbb{E}(\xi^{X,\varepsilon}(\varphi_{\varepsilon x,v,t})\xi^{X,\varepsilon}(\varphi_{\varepsilon x,\bar{v},t})) \tag{3.6} \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} - 1_{[v' > v]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]}) \\
&\quad \times (1_{[v' < \bar{v}]} 1_{[\varepsilon x < x' < \varepsilon x + (\bar{v}-v')t]} - 1_{[v' > \bar{v}]} 1_{[\varepsilon x + (\bar{v}-v')t < x' < \varepsilon x]}) dx' d\mu(v', r) \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} 1_{[\varepsilon x < x' < \varepsilon x + (\bar{v}-v')t]} \\
&\quad + 1_{[v' > \bar{v}]} 1_{[\varepsilon x + (\bar{v}-v')t < x' < \varepsilon x]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]} \\
&\quad - 1_{[v < v' < \bar{v}]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]} 1_{[\varepsilon x < x' < \varepsilon x + (\bar{v}-v')t]}) dx' d\mu(v', r)
\end{aligned}$$



$$\begin{aligned}
&= \rho \iiint r^2 \left( \mathbf{1}_{[v' < v]} \mathbf{1}_{[\varepsilon x < x' < \varepsilon x + (v - v')t]} + \mathbf{1}_{[v' > \bar{v}]} \mathbf{1}_{[\varepsilon x + (\bar{v} - v')t < x' < \varepsilon x]} \right) dx' d\mu(v', r) \\
&= t\rho \iint r^2 (\mathbf{1}_{[v' < v]}(v - v') + \mathbf{1}_{[v' > \bar{v}]}(v' - \bar{v})) d\mu(v', r) \\
&=: t\Gamma(v, \bar{v}).
\end{aligned}$$

Noting that

$$\mathbf{1}_{[v' < v]}(v - v') + \mathbf{1}_{[v' > \bar{v}]}(v' - \bar{v}) = \frac{1}{2}(|v - v'| + |v' - \bar{v}| - (\bar{v} - v)),$$

we have that

$$\Gamma(v, \bar{v}) = \frac{1}{2} \left( \mathcal{D}(v) + \mathcal{D}(\bar{v}) - (\bar{v} - v)\rho \int r^2 d\mu(v', r) \right). \quad (3.7)$$

This last expression for the covariance corresponds to the Lévy Chentsov field, as shown in [12].

**Remark 3.1.** From (3.6) we can see immediately that  $\Gamma(v, \bar{v}) \geq 0$ . In the particular case where there are only two velocities admitted, for example  $d\mu(v, r) = \delta_a(dr) \frac{1}{2} (\delta_{-1}(dv) + \delta_1(dv))$ , we have  $\mathcal{D}(\pm 1) = \rho a^2$  and  $\Gamma(1, -1) = 0$ . Typically this decorrelation happens only when two velocities at most are present.

### 3.2. Diffusive evolution of density fluctuations

Define the fluctuation field at diffusive scaling as

$$\Xi_t^{Y, \varepsilon}(\varphi) := \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x, v, r) \in X^\varepsilon} r \varphi [y_{v, \varepsilon^{-1}t}^\varepsilon(x) - v^{\text{eff}}(v) \varepsilon^{-1}t, v, r] - \frac{1}{1 + \sigma} \langle\langle \varphi \rangle\rangle \right].$$

Notice that this is recentered on the Euler characteristics.

Define

$$\varphi_{\mathbb{B}_t}(y, v, r) := \varphi(y + B_t(v), v, r).$$

**Theorem 3.2.**

$$\Xi_t^{Y, \varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \Xi_t^Y(\varphi) = \Xi_0^Y(\varphi_{\mathbb{B}_t}) = \xi^X(C\check{\varphi}_{\mathbb{B}_t})$$

where  $\mathbb{B}_t = (B_t(v))_{v \in \mathbb{R}}$  is a family of Wiener processes with covariance

$$\mathbb{E}(B_t(v), B_t(w)) = t\Gamma(v, w)$$

with  $\Gamma$  defined in (3.6). Furthermore  $\mathbb{B}_t$  is independent from  $\Xi_0^Y$ . More formally

$$\begin{aligned}
\Xi_t^Y(\varphi) &= \iiint r \varphi(y + B_t(v), v, r) d\xi_0^Y(y, v, r) \\
&= \iiint r C\varphi((1 + \sigma)x + B_t(v), v, r) d\xi_0^X(x, v, r)
\end{aligned} \quad (3.8)$$

We prove this theorem after some comments. By (3.8),  $\Xi_t^Y$  solves the stochastic differential equation

$$d\Xi_t^Y(\varphi) = \frac{1}{2}\Xi_t^Y(\mathcal{D}\partial_y^2\varphi)dt - \iiint (\partial_y\varphi)(y, v, r)dB_t(v)d\Xi_t^Y(y, v, r)$$

or in the time integrated form:

$$\begin{aligned}\Xi_t^Y(\varphi) &= \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}\partial_y^2\varphi)ds \\ &\quad - \int_0^t \iiint (\partial_y\varphi)(y, v, r)dB_s(v)d\Xi_s^Y(y, v, r) \\ &= \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}\partial_y^2\varphi)ds - \int_0^t \Xi_s^Y(\partial_y\varphi dB_s),\end{aligned}$$

where the last term is a martingale with quadratic variation

$$\int_0^t \left( \iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y, v, r)d\Xi_s^Y(y, v, r) \right)^2 ds = \int_0^t \Xi_s^Y(\sqrt{\mathcal{D}}(\partial_y\varphi))^2 ds. \quad (3.9)$$

Notice that

$$\begin{aligned}\mathbb{E}(\Xi_t^{Y,\varepsilon}(\varphi)^2) &= \frac{\rho}{1+\sigma} \iiint r^2 \mathbb{E}[(C\varphi)^2(y + B_t(v), v, r)] dy d\mu(v, r) \\ &= \frac{\rho}{1+\sigma} \iiint r^2 \int (C\varphi)^2(y + z, v, r) \frac{e^{-\frac{z^2}{2t\mathcal{D}(v)}}}{\sqrt{2\pi t\mathcal{D}(v)}} dz dy d\mu(v, r) \\ &= \frac{\rho}{1+\sigma} \iiint r^2 (C\varphi)^2(y, v, r) dy d\mu(v, r),\end{aligned}$$

independent of  $t$ , in agreement with the stationarity of the process.

Similarly the expectation of the quadratic variation (3.9) is given by

$$t \frac{\rho}{1+\sigma} \iiint \mathcal{D}(v) r^2 (\partial_y\varphi)^2(y, v, r) dy d\mu(v, r).$$

*Proof of Theorem 3.2.* In order to simplify notation we set  $\pi = 0$ , the extension to general  $\pi \neq 0$  is straightforward. Denote

$$B_t^\varepsilon(x, v) = j^\varepsilon(x, v, \varepsilon^{-1}t) - \sigma v \varepsilon^{-1}t, \quad \mathbb{B}_t^\varepsilon = \mathbb{B}_t^\varepsilon(X^\varepsilon) = \{B_t^\varepsilon(x, v), (x, v) \in \mathbb{R}^2\}.$$

By (3.1)  $\mathbb{B}_t^\varepsilon$  converges in law to the field of Brownian motions

$$\mathbb{B}_t^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \mathbb{B}_t = (B_t(v))_{v \in \mathbb{R}}. \quad (3.10)$$

Then we can rewrite

$$\begin{aligned}\Xi_t^{Y,\varepsilon}(\varphi) &= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r\varphi[x(1+\sigma) + B_t^\varepsilon(x, v), v, r] - \frac{1}{1+\sigma} \langle\langle \varphi \rangle\rangle \right] \\ &\quad + \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r(\partial_y\varphi)[x(1+\sigma) + B_t^\varepsilon(x, v), v, r] \varepsilon^{-1/2} (m_0^x(X_0^\varepsilon) - \sigma x) + R_t^\varepsilon,\end{aligned} \quad (3.11)$$

where  $R_t^\varepsilon \rightarrow 0$  in law.

We deal with the first term on the RHS of (3.11).

By (2.2) and (3.10), the couple  $(\xi^{X,\varepsilon}, \mathbb{B}_t^\varepsilon)$  converges jointly in law to  $(\xi^X, \mathbb{B}_t)$ . By (3.3) they are independent. By the mapping theorem ([3], chapter 1, section 2) we have that the first term on the RHS of (3.11) converges in law to  $\xi^X(\check{\varphi}_{\mathbb{B}_t})$  where

$$\check{\varphi}_{\mathbb{B}_t}(x, v, r) = \varphi(x(1 + \sigma) + B_t(v), v, r).$$

About the second term of (3.11), combining the argument above and the one used in (2.11) to (2.12) this converges in law to  $-\xi^X(P\check{\varphi}_{\mathbb{B}_t})$ .  $\square$

Formally, choosing  $\varphi_{k,w}(x, v, r) = e^{i2\pi xk} \varphi(r) \delta(v - w)$ , we have that

$$\begin{aligned} \hat{\varphi}(k, w, t) &:= \Xi_t^Y(\varphi_{k,w}) = \int d\xi^Y(y, r, v) e^{i2\pi k(y + B_t(w))} \varphi(r) \delta(v - w) \\ &= \xi^Y\left(e^{i2\pi k(\cdot + B_t(w))} \varphi(\cdot) \delta(\cdot - w)\right) \end{aligned}$$

satisfies the SDE

$$d\hat{\varphi}(k, w, t) = -\frac{(2\pi k)^2}{2} \mathcal{D}(w) \hat{\varphi}(k, w, t) + i2\pi k \hat{\varphi}(k, w, t) dB_t(w).$$

Notice that  $|\hat{\varphi}(k, w, t)|^2 = |\hat{\varphi}(k, w, 0)|^2$  for any  $k$ , a persistence on the macroscopic scale of the complete integrability of the dynamics also at the level of these fluctuations.

**Remark 3.3.** Since we consider also systems where lengths  $r$  can be negative, in the case that  $\sigma = 0$  and  $\pi = 0$  the macroscopic evolution of fluctuations in the Euler scaling is the same as the independent point particles. But in the diffusive scaling the fluctuations have non trivial behaviour.

#### 4. A remark about inhomogeneous initial distribution

Let  $f_0(x, v, r)$  be a nice non-negative bounded function on  $\mathbb{R}^3$  and  $X^\varepsilon$  the Poisson process on  $\mathbb{R}^3$  with intensity  $\varepsilon^{-1} f_0(x, v, r) dx dv dr$ .

In the Euler scaling, the empirical distribution of the free gas converges to the solution of

$$\partial_t f_t(x, v, r) + v \partial_x f_t(x, v, r) = 0,$$

with initial condition given by  $f_0$ . For the rods density this corresponds to the equation

$$\begin{aligned} \partial_t g_t(y, v, r) + \partial_y(v^{\text{eff}}(y, v, t) g_t(y, v, r)) &= 0, \\ v^{\text{eff}}(y, v, t) &= v + \frac{\iint r(v - w) g_t(y, w, r) dw dr}{1 - \iint r g_t(y, w, r) dw dr}. \end{aligned}$$

as proven in [12].

For *generic* initial conditions, we can guess that, if the initial density  $f$  is absolutely continuous in the  $x$  and  $v$  coordinates, then it satisfies that the limit as  $t \rightarrow \infty$  of  $f_t(x, v, r)$  is constant in  $x$ , that is,  $f_t(x, v, r) \rightarrow \rho \bar{f}(v, r)$ , for some  $\rho \in \mathbb{R}_+$  and  $\bar{f}(v, r)$ . This suggests that in a diffusive time scale the system essentially behaves like if it is a stationary state determined by a Poisson point field  $\rho \bar{f}(v, r) dx dv dr$ , and the analysis for the macroscopic fluctuations of [Section 3](#) applies.

## 5. A limit theorem for Poisson process

**Lemma 5.1.** *Let  $\varphi(x, v, r)$  a smooth function on  $\mathbb{R}^3$  with compact support in  $x$ , and  $\{B^\varepsilon(x), x \in \mathbb{R}\}$  a process converging in law to  $\{B(x), x \in \mathbb{R}\}$ . Then*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sum_{(x, v, r) \in X^\varepsilon} r \varphi(x, v, r) B^\varepsilon(x) \stackrel{\text{law}}{=} \rho \iiint r \varphi(x, v, r) B(x) dx d\mu(v, r).$$

*Proof.* Since  $\varphi$  is a smooth function we can approximate both sides by step functions, so that it is enough to prove that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sum_{(x, v, r) \in X^\varepsilon} 1_{[0 \leq x < 1]} B^\varepsilon(x) \stackrel{\text{law}}{=} \rho \int_0^1 B(x) dx. \quad (5.1)$$

Since in this limit  $(v, r)$  are not involved, to simplify notation we will ignore them. The positive random measure  $M^\varepsilon$  on  $[0, 1]$  defined by

$$M^\varepsilon(\psi) = \varepsilon \sum_{(x, v, r) \in X^\varepsilon} 1_{[0 \leq x < 1]} \psi(x), \quad \psi \in \mathcal{C}(0, 1),$$

converges a.s. to the Lebesgue measure  $dx$  on  $[0, 1]$ . Then, by the generalization of Slutsky's theorem in Theorem 2.7 (v) in [\[23\]](#), the couple  $(M^\varepsilon, B^\varepsilon)$  converges in law to  $(dx, B)$ .

Let  $F(\mu, \psi)$  a continuous function on  $\mathcal{M}_+([0, 1]) \times \mathcal{C}(0, 1)$ . Then  $F(M^\varepsilon, B^\varepsilon) \rightarrow F(dx, B)$  in law. Apply this to

$$F(\mu, \psi) = \int_0^1 \psi(x) d\mu(x),$$

and [\(5.1\)](#) follows. □

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