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# MACROSCOPIC DIFFUSIVE FLUCTUATIONS FOR GENERALIZED HARD RODS DYNAMICS

PABLO A. FERRARI AND STEFANO OLLA

ABSTRACT. We study the fluctuations in equilibrium for a dynamics of rods with random length. This includes the classical hard rod elastic collisions, when rod lengths are constant and equal to a positive value. We prove that in the diffusive space-time scaling, an initial fluctuation of density of particles of velocity  $v$ , after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance  $\mathcal{D}(v)$ .

## 1. INTRODUCTION

The mechanical system of one-dimensional hard rods is the simplest non trivial completely integrable dynamics where the macroscopic behavior can be described by generalized hydrodynamics. The density of particles of each given velocity is conserved and in the Euler scaling the macroscopic evolution of such densities have been studied by the pioneristic work of Percus [13] and Boldrighini, Dobrushin and Soukhov [3]. Fluctuations around this Euler limit have been studied by Boldrighini and Wick in [6]. Recently these results have been generalized to a completely integrable dynamics of rods of random length (even negative length) where lengths and velocities are exchanged at collision [9]. The elastic collisions are recovered in the particular case that all rods have the same positive length. Similar dynamics were considered in [7], while in [2] velocities are exchanged but not the lengths (i.e. the classical elastic collision).

In this article we investigate the evolution of the densities fluctuations in the diffusive space-time scale for the generalized dynamics considered in [9]. We consider the system in a *stationary homogeneous* initial condition since typical inhomogeneous non-stationary state will converge to a space-homogeneous stationary state in a large Euler time scale. Consequently in the diffusive time scale we find the system essentially in a stationary state.

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The result we obtain is that the initial fluctuation of the density of particles of velocity  $v$ , after recentering on its Euler evolution, evolve randomly shifted by a Brownian motion of variance  $\mathcal{D}(v)$ . This diffusion coefficient  $\mathcal{D}(v)$  has an explicit expression depending on  $v$  and on the particular stationary measure (cf. (3.2)). In the case of rods of constant length  $\mathcal{D}(v)$  is the same as computed by Spohn in [17], as well as it appears in the first order diffusive correction to the Euler Hydrodynamic limit [4] [5].

This result corresponds to the following partial SDE for the evolution of the macroscopic fluctuation  $\Xi_t(y, v, r)$  of the density of particles of velocity  $v$  and length  $r$  at position  $y$ :

$$\partial_t \Xi_t(y, v, r) = \frac{1}{2} \mathcal{D}(v) \partial_y^2 \Xi_t(y, v, r) + \sqrt{\mathcal{D}(v)} \partial_y \Xi_t(y, v, r) \dot{W}_t(v), \quad (1.1)$$

where  $\{\dot{W}_t(v)\}$  is a centered gaussian field with covariance

$$\mathbb{E}(\dot{W}_t(v) \dot{W}_s(w)) = \delta(t-s) \frac{\Gamma(v, w)}{\sqrt{\mathcal{D}(v) \mathcal{D}(w)}}$$

with  $\Gamma(v, w)$  is given by (3.4). Since  $\Xi_t$  is a distribution in  $(y, v, r)$ , (1.1) should be understood in the weak sense.

Notice that in (1.1) the noise term is only white in time and completely correlated in space. This is in contrast with typical diffusive evolution of fluctuations in chaotic systems, where it is expected an additive space-time white noise driving the fluctuations and the equation would be of the type [15]

$$\partial_t \bar{\Xi}_t(y, v, r) = \frac{1}{2} \mathcal{D}(v) \partial_y^2 \bar{\Xi}_t(y, v, r) + \sqrt{\mathcal{D}(v)} \partial_y \dot{W}_{y,t}(v), \quad (1.2)$$

with  $\mathbb{E}(\dot{W}_{y,t}(v) \dot{W}_{y',s}(w)) = \delta(t-s) \delta(y-y') \delta(v-w)$ . Notice that (1.1) and (1.2) have the same space-time covariance in equilibrium, i.e. the space-time covariance does not give information about the martingale term of the evolution equation.

In fact we believe, as Herbert Spohn suggested us, that (1.1) is a typical (universal) macroscopic behaviour for the diffusive fluctuations of completely integrable many-body systems [16]. In order to understand why (1.1) arise, we follow the behaviour of two tagged quasi-particles with the same velocity. We call here quasi-particles (or impulsions) the particles with the dynamics defined by the exchange of positions at the moment of collision. The standard technique to study such dynamics is to go to the *reduced description* where quasi-particles are mapped to points and evolve without interaction. Then the evolution of the tagged quasi-particle is obtained by the trivial evolution of the corresponding point, shifted by the collisions with quasi-particles of different velocity. Since the points corresponding to the other quasi-particles are distributed by a Poisson field, these

collisions happen at random times and the collisional shifts are independent. Consequently at the Euler scale we have a law of large numbers (Lemma ??) that produce a deterministic evolution of the tagged quasi-particle with an effective velocity given by (2.15) (such ergodicity was proven first in [1]). Recentering the position of the tagged quasi-particle on the Euler deterministic behavior, we have then a functional central limit theorem so that the position converges in law to a Brownian motion of variance  $\mathcal{D}(v)$ . Now consider two tagged quasi-particles with the same velocity and initially located at macroscopic distance: this means that there are typically  $\varepsilon^{-1}$  particles in between, where  $\varepsilon$  is the scaling parameter going to 0. In the diffusive scaling each tagged quasi-particle at time  $t$  has a number of collisions proportional to  $\varepsilon^{-2}t$ , but most of them with the same quasi-particles, except for an order  $\varepsilon^{-1}$  of collisions at the beginning and at the end of the time interval  $[0, t]$ . Consequently the two tagged quasi-particles completely correlate in the limit  $\varepsilon \rightarrow 0$ , i.e. they converge to the same Brownian motion. This causes the rigid motion of the corresponding density fluctuations. Notice that in chaotic system it is expected that the two quasi-particles at initial macroscopic distance converge to two independent Brownian motion, generating the space-time white noise of (1.2).

As far as we know, equation (1.1) for the diffusive fluctuations for hard rods is new in the literature. We recently discovered the article of Presutti and Wick [14] that concerns the diffusive behavior of travelling wave initial conditions in hard rods systems, where in the remark after Theorem 1 they comment about possible diffusive behaviour of fluctuations and they write: “spatially separated fluctuations in the density of rods with the same velocity move with the same Brownian component”. Strangely [14] is never quoted in the following literature about Navier-Stokes corrections for the hard rods hydrodynamics (cf. [5]). In the introduction of [6] it was announced a second article about the *Navier Stokes corrections* for the evolution of the fluctuations, but the authors confirmed us that this has never been written.

In [8] it is investigated the hard rods dynamics with domain wall initial conditions, a particular case of travelling wave. In this article also the Navier-Stokes corrections are studied and the covariance  $\Gamma(v, v')$  is computed from Green-Kubo formula.

As we believe that our approach is more elementary than the one used in the previous literature, we have written this article so that it is completely independent from previous results on hard rods. Basically the only tools we use are the law of large numbers and the central limit theorem for a Poisson field. In Section 2 we prove the macroscopic evolution of the fluctuations in the Euler scaling (recovering the result of [6]). In Section 3 we prove the evolution of the

fluctuations in the diffusive scaling. Finally in [Section 6](#) we prove two lemmas about limits for Poisson field that we need in the proofs.

## 2. EQUILIBRIUM FLUCTUATIONS IN THE EULER SCALING

Let  $X^\varepsilon$  be the Poisson process on  $\mathbb{R}^3$  with intensity  $\varepsilon^{-1}\rho dx d\mu(v, r)$ , where  $\mu$  is a positive finite measure on  $\mathbb{R}^2$  with finite second moments. We should think  $x$  as the *macroscopic* position of the point  $x$ , as the typical distance between points is  $\varepsilon$ . The *macroscopic length of the rod*  $(x, v, r)$  is  $\varepsilon r$ .

We define

$$\begin{aligned}\sigma &:= \rho \iint r d\mu(v, r), & \text{length density} \\ \pi &:= \rho \iint r v d\mu(v, r), & \text{momentum density.}\end{aligned}$$

Denote the empirical length distribution by

$$N^\varepsilon \varphi := \sum_{(x,v,r) \in X^\varepsilon} \varepsilon r \varphi(x, v, r), \quad N^\varepsilon(A) := N^\varepsilon \mathbb{1}_A.$$

The expectation of  $N^\varepsilon$  is

$$\mathbb{E} N^\varepsilon \varphi = \langle\langle \varphi \rangle\rangle,$$

where the length biased measure  $\langle\langle \cdot \rangle\rangle$  is defined by

$$\langle\langle \varphi \rangle\rangle := \rho \iiint r \varphi(x, v, r) dx d\mu(v, r).$$

We have the law of large numbers

$$N^\varepsilon \varphi \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \mathbb{E} N^\varepsilon \varphi = \langle\langle \varphi \rangle\rangle,$$

If  $\varphi(x, v, r) = 1_{[0,1]}(x)$ , we have  $\sum_{(x,v,r) \in X^\varepsilon} \varepsilon r \varphi(x) \rightarrow \sigma$ .

The central limit theorem states that

$$\xi^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2} (N^\varepsilon - \langle\langle \varphi \rangle\rangle) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\varphi) \tag{2.1}$$

where  $\xi^X$  is the centered gaussian white noise with covariance

$$\mathbb{E}(\xi^X(\varphi) \xi^X(\psi)) = \langle\langle \varphi \psi \rangle\rangle_2 - \langle\langle \varphi \rangle\rangle \langle\langle \psi \rangle\rangle,$$

where

$$\langle\langle \varphi \rangle\rangle_2 := \rho \iiint r^2 \varphi(x, v, r) dx d\mu(v, r).$$

Notice that

$$\mathbb{E}(\xi^{X,\varepsilon} \varphi)^2 = \langle\langle \varphi^2 \rangle\rangle_2 - \langle\langle \varphi \rangle\rangle^2,$$

so that for any  $\varphi \in L^2(\langle\langle \cdot \rangle\rangle_2)$  we have the bound

$$\mathbb{E}(\xi^{X,\varepsilon} \varphi)^2 \leq \langle\langle \varphi^2 \rangle\rangle_2.$$

Define the *mass* (length) measure by

$$m_a^b(X^\varepsilon) = \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r (1_{[x \in [a,b]]} - 1_{[x \in [b,a]]}).$$

Consequently,

$$m_a^b(X^\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \mathbb{E} m_a^b(X^\varepsilon) = (b-a)\sigma. \quad (2.2)$$

To each configuration  $X^\varepsilon$  and a point  $a \in \mathbb{R}$ , there are a dilated point and configuration

$$\begin{aligned} D_a^\varepsilon(b) &:= b - a + m_a^b(X^\varepsilon) \\ Y^\varepsilon = D_0^\varepsilon(X^\varepsilon) &:= \{(D_0^\varepsilon(b), v, r) : (x, v, r) \in X^\varepsilon\}. \end{aligned}$$

**Remark 2.1.** The distribution of  $X^\varepsilon$  is space shift invariant, but the distribution of the rod configuration  $Y^\varepsilon$  is not because  $Y^\varepsilon$  has no rod containing the origin. Our results can be extended to random rod configurations with space shift invariant distribution, by using Palm transforms [18] and Harris theorem [11]; see for instance [10].

If  $r \geq 0$  for all  $(x, v, r) \in X^\varepsilon$ , and the origin does not belong to a rod of  $Y^\varepsilon$ , then we can define the inverse  $D_a^{-1}(Y^\varepsilon) = X^\varepsilon$ . The macroscopic dilation of the point  $b$  with respect to  $a$  is given by

$$\mathbb{E} D_a^\varepsilon(b) = (b-a)(1+\sigma).$$

Denote the length empirical measure induced by  $Y^\varepsilon$  by

$$K^\varepsilon \varphi := \varepsilon \sum_{(y,v,r) \in Y^\varepsilon} r \varphi(y, v, r).$$

We have the law of large numbers:

$$\begin{aligned} K^\varepsilon \varphi &= \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x + m_a^x(X^\varepsilon), v, r) \\ &\xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \rho \iiint r \varphi(x + (x-a)\sigma, v, r) dx d\mu(v, r) \\ &= \frac{\rho}{1+\sigma} \iiint r \varphi(x, v, r) dx d\mu(v, r) \\ &= \frac{1}{1+\sigma} \langle\langle \varphi \rangle\rangle. \end{aligned} \quad (2.3)$$

**2.1. Static CLT for the dilated configuration.** We define the fluctuation field

$$\xi^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2} (K^\varepsilon \varphi - \mathbb{E} K^\varepsilon \varphi).$$

We have

$$K^\varepsilon \varphi - \mathbb{E} K^\varepsilon \varphi = (K^\varepsilon \varphi - A^\varepsilon \varphi) + (A^\varepsilon \varphi - \mathbb{E} A^\varepsilon \varphi) - (\mathbb{E} K^\varepsilon \varphi - \mathbb{E} A^\varepsilon \varphi). \quad (2.4)$$

where

$$A^\varepsilon \varphi := \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x(1+\sigma), v, r),$$

$$\mathbb{E}A^\varepsilon \varphi = \frac{\rho}{1+\sigma} \iiint r \varphi(x, v, r) dx d\mu(v, r).$$

The last term in (2.4) gives

$$\varepsilon^{-1/2}(\mathbb{E}K^\varepsilon \varphi - \mathbb{E}A^\varepsilon \varphi)$$

$$= \mathbb{E}\left(\varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r [\varphi(x + m_0^x(X^\varepsilon), v, r) - \varphi(x(1+\sigma), v, r)]\right) \quad (2.5)$$

$$= \mathbb{E}\left(\varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\partial_y \varphi)(x(1+\sigma), v, r) (m_0^x(X^\varepsilon) - x\sigma)\right) + R^\varepsilon$$

$$= \iiint r (\partial_y \varphi)(x(1+\sigma), v, r) \varepsilon^{-1/2} [\mathbb{E}(m_0^x(X^\varepsilon) - x\sigma)] dx d\mu(v, r) + R^\varepsilon \quad (2.6)$$

$$= R^\varepsilon, \quad (2.7)$$

where  $R^\varepsilon$  denotes a generic term small with  $\varepsilon$ . Identity (2.6) follows from Slyvniak-Mecke formula (Theorem 3.2 in [12]) and identity (2.7) follows from (2.2).

The second term in (2.4) gives

$$\varepsilon^{1/2}(A^\varepsilon - \mathbb{E}A^\varepsilon)$$

$$= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \check{\varphi}(x, v, r) - \rho \iiint r \check{\varphi}(x, v, r) dx d\mu(v, r) \right]$$

$$\xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\check{\varphi}), \quad (2.8)$$

where  $\check{\varphi}(x, v, r) := \varphi((1+\sigma)x, v, r)$ . Notice that

$$\mathbb{E}(\xi^X(\check{\varphi}) \xi^X(\check{\psi}))$$

$$= \rho \iiint r^2 \check{\varphi}(x, v, r) \check{\psi}(x, v, r) dx d\mu(v, r) - \langle\langle \check{\varphi} \rangle\rangle \langle\langle \check{\psi} \rangle\rangle$$

$$= \frac{\rho}{1+\sigma} \iiint r^2 \varphi(y, v, r) \psi(y, v, r) dy d\mu(v, r) - \frac{1}{(1+\sigma)^2} \langle\langle \varphi \rangle\rangle \langle\langle \psi \rangle\rangle$$

$$= \frac{1}{1+\sigma} \langle\langle \varphi \psi \rangle\rangle_2 - \frac{1}{(1+\sigma)^2} \langle\langle \varphi \rangle\rangle \langle\langle \psi \rangle\rangle.$$

Finally expand the first term of the RHS of (2.4):

$$\varepsilon^{-1/2}(K^\varepsilon \varphi - A^\varepsilon \varphi) = \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\varphi(x + m_a^x(X^\varepsilon), v, r) - \varphi(x(1+\sigma), v, r))$$

$$= \frac{1}{1+\sigma} \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\partial_x \check{\varphi})(x, v, r) (m_0^x(X^\varepsilon) - x\sigma) + R^\varepsilon.$$

By the functional central limit theorem (2.1) we have

$$\varepsilon^{-1/2}(m_0^x(X^\varepsilon) - x\sigma) := B^\varepsilon(x) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} B(x) := \begin{cases} \xi^X(1_{[0,x]}), & x > 0 \\ -\xi^X(1_{[x,0]}), & x < 0 \end{cases}$$

Since  $\{B(x), x \in \mathbb{R}\}$  is a bilateral Brownian motion, using Lemma 6.1 we have

$$\begin{aligned} & \varepsilon^{-1/2}(K^\varepsilon\varphi - A^\varepsilon\varphi) \\ & \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \frac{\rho}{1+\sigma} \iiint r(\partial_x \check{\varphi})(x, v, r) B(x) dx d\mu(v, r) \\ & = \frac{1}{1+\sigma} \int dx B(x) \partial_x \left( \rho \iint r \check{\varphi}(x, v, r) d\mu(v, r) \right) \\ & = \frac{1}{1+\sigma} \int dx \xi^X(1_{[0,x]}1_{x>0} - 1_{[x,0]}1_{x<0}) \partial_x \left( \rho \iint r \check{\varphi}(x, v, r) d\mu(v, r) \right) \\ & = \frac{1}{1+\sigma} \xi^X \left( \int dx (1_{[0,x]}1_{x>0} - 1_{[x,0]}1_{x<0}) \partial_x \left( \rho \iint r \check{\varphi}(x, v, r) d\mu(v, r) \right) \right) \\ & = -\frac{1}{1+\sigma} \xi^X \left( \rho \iint r \check{\varphi}(\cdot, v', r') d\mu(v', r') \right). \end{aligned}$$

We conclude that

$$\varepsilon^{1/2}(K^\varepsilon\varphi - A^\varepsilon\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} -\xi^X(P\check{\varphi}), \quad (2.9)$$

where

$$P\varphi(x) = \frac{\rho}{1+\sigma} \iint r\varphi(x, v', r') d\mu(v', r'). \quad (2.10)$$

Putting together (2.9), (2.5)-(2.7) and (2.8) we have shown that

$$\xi^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^Y(\varphi) = \xi^X(\check{\varphi} - P\check{\varphi}) = \xi^X(C\check{\varphi}),$$

where  $C = I - P$ . This identifies  $\xi^Y$  as the centered gaussian field with covariance

$$\begin{aligned} \mathbb{E}(\xi^Y(\varphi)\xi^Y(\psi)) & = \mathbb{E}(\xi^X(C\check{\varphi})\xi^X(C\check{\psi})) \\ & = \rho \iiint r^2 C\varphi(x(1+\sigma), v, r) C\psi(x(1+\sigma), v, r) dx d\mu(v, r), \\ & = \frac{\rho}{1+\sigma} \iiint r^2 C\varphi(y, v, r) C\psi(y, v, r) dy d\mu(v, r) \\ & = \frac{\rho}{1+\sigma} \langle\langle C\varphi C\psi \rangle\rangle_2. \end{aligned}$$

That means for the Fourier transforms

$$\begin{aligned} \widehat{\varphi}(k, v, r) & = \int e^{i2\pi ky} \varphi(y, v, r) dy \\ \mathbb{E}(\xi^Y(\varphi)\xi^Y(\psi)) & = \frac{\rho}{1+\sigma} \iiint r^2 C\widehat{\varphi}(k, v, r)^* C\widehat{\psi}(k, v, r) dk d\mu(v, r). \end{aligned}$$

i.e. a covariance operator

$$\mathcal{C} = \frac{\rho}{1+\sigma} r^2 C^2 = \frac{\rho}{1+\sigma} r^2 \left( I + \left( \frac{\sigma^2}{(1+\sigma)^2} - 2 \frac{\sigma}{1+\sigma} \right) P \right).$$

**Remark 2.2.** *This is in agreement with formula (7.61) in Spohn's book [15].*

*Example:* in the case  $d\mu(v, r) = \frac{1}{2}(\delta_{v_0}(dv) + \delta_{-v_0}(dv))\delta_a(dr)$  we have  $\sigma = \rho a$  and  $\pi = 0$ . Then  $P\varphi(x) = \frac{1}{2}\varphi(x, v_0) + \frac{1}{2}\varphi(x, -v_0)$  and

$$C\varphi(x, v_0) = \varphi(x, v_0) - \frac{\rho a}{1+\rho a} \frac{1}{2}\varphi(x, v_0) = \frac{2+\rho a}{2(1+\rho a)}\varphi(x, v_0),$$

and

$$\mathcal{C} = \frac{\rho a^2}{1+\rho a} \frac{(2+\rho a)^2}{4(1+\rho a)^2}.$$

**2.2. Equilibrium fluctuations in the Euler scaling.** The tag  $v$  represent the macroscopic velocity of the rod in the Euler scaling, so it is of order 1 and does not need to be rescaled.

Let  $X_t^\varepsilon$  denote the free gas configuration at time  $t$ :

$$X_t^\varepsilon := \{(x + vt, v, r) : (x, v, t) \in X^\varepsilon\}.$$

Define the flow

$$j^\varepsilon(x, v, t) := \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \tilde{r} \left( 1_{[\tilde{v} < v]} 1_{[x < \tilde{x} < x + (v - \tilde{v})t]} - 1_{[\tilde{v} > v]} 1_{[x + (v - \tilde{v})t < \tilde{x} < x]} \right) \quad (2.11)$$

$$j(x, v, t) := \mathbb{E} j^\varepsilon(x, v, t) \quad (2.12)$$

$$\begin{aligned} &= \iiint \rho dx d\mu(\tilde{v}, \tilde{r}) \tilde{r} \left( 1_{[\tilde{v} < v]} 1_{[x < \tilde{x} < x + (v - \tilde{v})t]} - 1_{[\tilde{v} > v]} 1_{[x + (v - \tilde{v})t < \tilde{x} < x]} \right) \\ &= \rho \int \tilde{r} \int_v^{+\infty} (v - \tilde{v})t d\mu(\tilde{v}, \tilde{r}) + \rho \int \tilde{r} \int_{-\infty}^v (v - \tilde{v})t d\mu(\tilde{v}, \tilde{r}) \\ &= tv\sigma - t\pi. \end{aligned}$$

Here  $j^\varepsilon(x, v, t)$  is the ideal gas length flow along the segment  $(x + vs)_{s \in [0, t]}$ , and  $j(x, v, t)$  is its expectation.

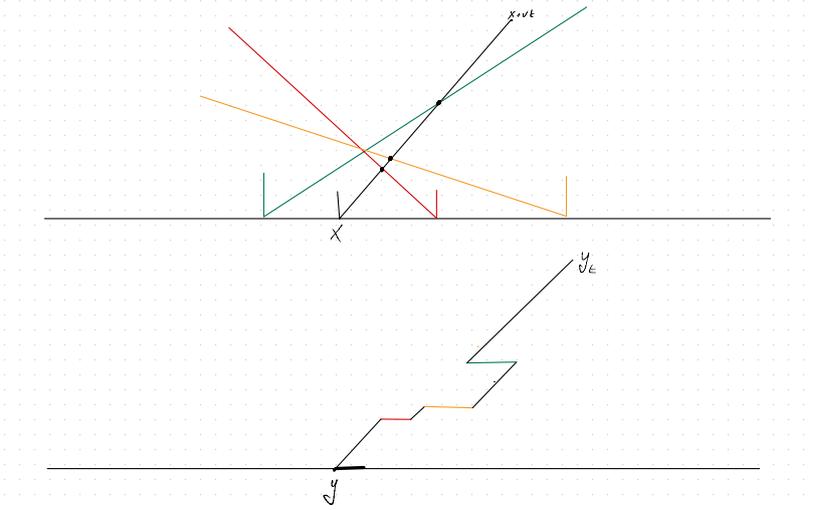
The position of the quasi particle  $y_{v,t}^\varepsilon(x)$  associated to  $(x, v, r)$  is given by

$$y_{v,t}^\varepsilon(x) := D_0^\varepsilon(x) + vt + j^\varepsilon(x, v, t) \quad (2.13)$$

$$y_{v,t}(x) := \mathbb{E} y_{v,t}^\varepsilon(x) = (1 + \sigma)x + vt + j(x, v, t) = x(1 + \sigma) + vt + j(x, v, t).$$

We have the LLN

$$y_{v,t}^\varepsilon - y \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} v^{\text{eff}}(v)t, \quad (2.14)$$



where the effective velocity is given by

$$v^{\text{eff}}(v) := \frac{1}{t} \mathbb{E}(y_t^\varepsilon - vt) = v(1 + \sigma) - \pi. \quad (2.15)$$

To see (2.14), observe that by (2.13), if  $y \in Y^\varepsilon$ , there is an  $x \in X^\varepsilon$  such that  $y = D_0^\varepsilon(x)$  and  $y_{v,t}^\varepsilon - y = j^\varepsilon(x, v, t)$ , implying that (2.14) is equivalent to

$$\frac{1}{t} j^\varepsilon(x, v, t) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \frac{1}{t} j(x, v, t) = v(1 + \sigma) - \pi.$$

We have that the free gas empirical length measure at time  $t$  satisfies

$$N_t^\varepsilon \varphi := \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x + vt, v, r) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \mathbb{E} N_t^\varepsilon \varphi = \langle\langle \varphi \rangle\rangle.$$

The  $X$ -fluctuation field at time  $t$  satisfies

$$\xi_t^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2} (N_t^\varepsilon \varphi - \langle\langle \varphi \rangle\rangle) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\varphi_t),$$

where

$$\varphi_t(x, v, r) := \varphi(x + tv, v, r).$$

The hard rod configuration and empirical measure at time  $t$  are given by

$$Y_t^\varepsilon := \{(y_t^\varepsilon(x), v, r) : (x, v, r) \in X^\varepsilon\},$$

$$K_t^\varepsilon \varphi := \varepsilon \sum_{(y,v,r) \in Y_t^\varepsilon} r \varphi(y, v, r) = \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(y_t^\varepsilon(x), v, r).$$

Using (2.3) we have the LLN for  $K_t^\varepsilon$ :

$$K_t^\varepsilon \varphi \xrightarrow[\varepsilon \rightarrow 0]{} \rho \iiint r \varphi(y_{v,t}(x), v, r) dx d\mu(v, r)$$

$$\begin{aligned}
&= \rho \iint r \left( \int \varphi(x(1+\sigma) + v^{\text{eff}}(v)t, v, r) dx \right) d\mu(v, r) \\
&= \frac{\rho}{1+\sigma} \iiint r \varphi(x, v, r) dx d\mu(v, r) = \frac{\rho}{1+\sigma} \langle\langle \varphi \rangle\rangle.
\end{aligned}$$

We define the  $Y$ -fluctuation field at time  $t$  by

$$\xi_t^{Y, \varepsilon}(\varphi) := \varepsilon^{-1/2} (K_t^\varepsilon \varphi - \mathbb{E} K_t^\varepsilon \varphi).$$

We have

$$K_t^\varepsilon \varphi - \mathbb{E} K_t^\varepsilon \varphi = (K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) + (A_t^\varepsilon \varphi - \mathbb{E} A_t^\varepsilon \varphi) - (\mathbb{E} K_t^\varepsilon \varphi - \mathbb{E} A_t^\varepsilon \varphi). \quad (2.16)$$

where

$$A_t^\varepsilon \varphi := \varepsilon \sum_{(x, v, r) \in X^\varepsilon} r \varphi(x(1+\sigma) + vt + j^\varepsilon(x, v, t), v, r).$$

The last term in (2.16) gives

$$\begin{aligned}
&\varepsilon^{-1/2} (\mathbb{E} K_t^\varepsilon \varphi - \mathbb{E} A_t^\varepsilon \varphi) \\
&= \mathbb{E} \left( \varepsilon^{1/2} \sum_{(x, v, r) \in X^\varepsilon} r \left[ \varphi(x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x, v, t), v, r) \right. \right. \\
&\quad \left. \left. - \varphi(x(1+\sigma) + vt + j^\varepsilon(x, v, t), v, r) \right] \right) \quad (2.17)
\end{aligned}$$

$$= \mathbb{E} \left( \varepsilon^{1/2} \sum_{(x, v, r) \in X^\varepsilon} r (\partial_y \varphi)(x(1+\sigma) + vt + j^\varepsilon(x, v, t), v, r) (m_0^x(X^\varepsilon) - x\sigma) \right) + R_t^\varepsilon$$

$$= \iiint r (\partial_y \varphi)(x(1+\sigma) + vt + j^\varepsilon(x, v, t), v, r) \quad (2.18)$$

$$\times \varepsilon^{-1/2} \mathbb{E} [m_0^x(X^\varepsilon) - x\sigma] dx d\mu(v, r) + R_t^\varepsilon = R_t^\varepsilon, \quad (2.19)$$

where  $R_t^\varepsilon$  is of smaller order; identity (2.18) follows from the Slyvniak-Mecke formula and identity (2.19) follows from (2.2).

By Lemma 6.2 the second term in (2.16) gives

$$\begin{aligned}
&\varepsilon^{1/2} (A_t^\varepsilon - \mathbb{E} A_t^\varepsilon) \\
&= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x, v, r) \in X^\varepsilon} r \varphi(x(1+\sigma) + vt + j^\varepsilon(x, v, t), v, r) \right. \\
&\quad \left. - \rho \iiint r \varphi(x(1+\sigma) + vt + j^\varepsilon(x, v, t), v, r) dx d\mu(v, r) \right] \\
&\xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi_t^X(\check{\varphi}_t), \quad (2.20)
\end{aligned}$$

where

$$\check{\varphi}_t(x, v, r) := \varphi(x(1+\sigma) + v^{\text{eff}}(v)t, v, r).$$

Finally, the first term in (2.16) gives

$$\begin{aligned}
& \varepsilon^{1/2}(K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) \\
&= \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r [\varphi(x + m_0^x(X^\varepsilon) + vt + j^\varepsilon(x, v, t), v, r) \\
&\quad - \varphi(x(1 + \sigma) + vt + j^\varepsilon(x, v, t), v, r)] \\
&= \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\partial_x \varphi)(x(1 + \sigma) + vt + j^\varepsilon(x, v, t), v, r) (m_0^x(X^\varepsilon) - \sigma x) + R_t^\varepsilon \\
&= \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r (\partial_x \varphi)(x(1 + \sigma) + vt + j^\varepsilon(x, v, t), v, r) \varepsilon^{-1/2} (m_0^x(X^\varepsilon) - \sigma x) + R_t^\varepsilon,
\end{aligned}$$

and combining Lemmas 6.1 and 6.2 we obtain that the limit in law of this last process is

$$\begin{aligned}
& \rho \iiint r (\partial_x \varphi)(x(1 + \sigma) + v^{\text{eff}}(v)t, v, r) B(x) dx d\mu(v, r) \\
&= \frac{1}{1 + \sigma} \int dx B(x) \partial_x \left( \rho \iint r \check{\varphi}_t(x, v, r) d\mu(v, r) \right) \\
&= \frac{1}{1 + \sigma} \int dx \xi^X (1_{[0,x]} 1_{x>0} - 1_{[x,0]} 1_{x<0}) \partial_x \left( \rho \iint r \check{\varphi}_t(x, v, r) d\mu(v, r) \right) \\
&= \frac{1}{1 + \sigma} \xi^X \left( \int dx (1_{[0,x]} 1_{x>0} - 1_{[x,0]} 1_{x<0}) \partial_x \left( \rho \iint r \check{\varphi}_t(x, v, r) d\mu(v, r) \right) \right) \\
&= -\frac{1}{1 + \sigma} \xi^X \left( \rho \iint r \check{\varphi}_t(\cdot, \tilde{v}, \tilde{r}) d\mu(\tilde{v}, \tilde{r}) \right).
\end{aligned}$$

where we used  $\partial_y \varphi = \frac{1}{1 + \sigma} \partial_x \check{\varphi}_t$ , and that  $R_t^\varepsilon$  is smaller order.

Recalling  $P\varphi$  defined in (2.10), we conclude that

$$\varepsilon^{1/2}(K_t^\varepsilon \varphi - A_t^\varepsilon \varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} -\xi^X(P\check{\varphi}_t). \quad (2.21)$$

Putting together (2.21), (2.17)-(2.19) and (2.20) we have shown that

$$\xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi_t^Y(\varphi) := \xi^X(\check{\varphi}_t - P\check{\varphi}_t) = \xi^X(C\check{\varphi}_t),$$

where  $C := I - P$ . We have proven that

$$\xi_t^Y(\varphi) = \xi_0^Y(\varphi_t),$$

i.e.

$$\partial_t \xi_t^Y(\varphi) = \xi_0^Y(v^{\text{eff}} \partial_x \varphi_t) = \xi_t^Y(v^{\text{eff}} \partial_x \varphi).$$

In other words, in a weak sense  $\xi_t^Y$  satisfies the equation

$$\partial_t \xi_t^Y + v^{\text{eff}} \partial_x \xi_t^Y = 0,$$

which is the expected equation.

### 3. EQUILIBRIUM FLUCTUATIONS IN THE DIFFUSIVE SCALING

**3.1. Quasiparticles in the diffusing scaling.** Given a point  $(x, v, r) \in X^\varepsilon$ , recall that  $y_{v,t}^\varepsilon(x)$  is the position at time  $t$  of the quasiparticle  $(y, v, r)$ , defined by (2.13). We will show that

$$y_{\varepsilon^{-1}t}^\varepsilon(x) - v^{\text{eff}}(v)\varepsilon^{-1}t \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} y + \sqrt{\mathcal{D}(v)}W_t(v), \quad (3.1)$$

where  $W_t(v)$  is a Wiener process in  $t$ . The limit process  $W_t(v)$  does not depend on the initial position  $x$ . The family of processes  $\{W_t(v), v \in \mathbb{R}\}$  are jointly gaussian, and related to a Levy-Chentsov field [9]. We compute explicitly the covariances.

By (2.13) and (2.12), we have

$$y_{v,\varepsilon^{-1}t}^\varepsilon(x) - v^{\text{eff}}(v)\varepsilon^{-1}t = D_0^\varepsilon(x) - (1 + \sigma)x + j^\varepsilon(x, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t.$$

Since  $D_0^\varepsilon(x) \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} (1 + \sigma)x$ , the limit in (3.1) is equivalent to

$$j^\varepsilon(x, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \sqrt{\mathcal{D}(v)}W_t(v);$$

Observe that

$$\varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \tilde{r} \mathbf{1}_{[\tilde{v} < v]} \mathbf{1}_{[x < \tilde{x} < x + (v - \tilde{v})\varepsilon^{-1}t]} = \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^{\varepsilon^2}} \tilde{r} \mathbf{1}_{[\tilde{v} < v]} \mathbf{1}_{[\varepsilon x < \tilde{x} < \varepsilon x + (v - \tilde{v})t]},$$

where  $X^{\varepsilon^2} := \{(\varepsilon x, v, r) : (x, v, r) \in X^\varepsilon\}$ , is obtained from  $X^\varepsilon$  by rescaling all positions by a factor  $\varepsilon$ , so that  $X^{\varepsilon^2}$  is a Poisson process of intensity measure  $\varepsilon^{-2}\rho d\mu(v, r)$ . We have that

$$\mathbb{E}\left(\varepsilon^2 \sum_{(x', v', r') \in X^{\varepsilon^2}} r' \mathbf{1}_{[v' < v]} \mathbf{1}_{[\varepsilon x < x' < \varepsilon x + (v - v')t]}\right) = t\rho \int r \int_{-\infty}^v (v - v') d\mu(v, r).$$

Applying the CLT (2.1) to the function

$$\varphi_{\varepsilon x, v, t}(x', v') = \mathbf{1}_{[v' < v]} \mathbf{1}_{[\varepsilon x < x' < \varepsilon x + (v - v')t]} - \mathbf{1}_{[v' > v]} \mathbf{1}_{[\varepsilon x + (v - v')t < x' < \varepsilon x]},$$

we have that

$$j_{X^\varepsilon}(x, v, \varepsilon^{-1}t) - (v\sigma - \pi)\varepsilon^{-1}t \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\varphi_{0, v, t}),$$

which has variance

$$\begin{aligned} & \rho \iiint \tilde{r}^2 (\mathbf{1}_{[\tilde{v} < v]} \mathbf{1}_{[x < \tilde{x} < x + (v - \tilde{v})t]} + \mathbf{1}_{[\tilde{v} > v]} \mathbf{1}_{[x + (v - \tilde{v})t < \tilde{x} < x]}) d\tilde{x} d\mu(\tilde{v}, \tilde{r}) \\ &= \rho \int \tilde{r}^2 \int_v^{+\infty} (v - \tilde{v})t d\mu(\tilde{v}, \tilde{r}) - \rho \int \tilde{r}^2 \int_{-\infty}^v (v - \tilde{v})t d\mu(\tilde{v}, \tilde{r}) \\ &= t\rho \iint \tilde{r}^2 |v - \tilde{v}| d\mu(\tilde{v}, \tilde{r}) := t\mathcal{D}(v). \end{aligned} \quad (3.2)$$

About the correlation for different initial position, assuming  $x < \bar{x}$ :

$$\begin{aligned}
& \mathbb{E}(\xi^{X,\varepsilon}(\varphi_{\varepsilon x,v,t})\xi^{X,\varepsilon}(\varphi_{\varepsilon \bar{x},v,t})) \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} - 1_{[v' > v]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]}) \\
&\quad \times (1_{[v' < v]} 1_{[\varepsilon \bar{x} < x' < \varepsilon \bar{x} + (v-v')t]} - 1_{[v' > v]} 1_{[\varepsilon \bar{x} + (v-v')t < x' < \varepsilon \bar{x}]}) dx' d\mu(v', r) \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} 1_{[\varepsilon \bar{x} < x' < \varepsilon \bar{x} + (v-v')t]} \\
&\quad + 1_{[v' > v]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]} 1_{[\varepsilon \bar{x} + (v-v')t < x' < \varepsilon \bar{x}]}) dx' d\mu(v', r) \\
&= \rho \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} [(v-v')t - \varepsilon(\bar{x}-x)] d\mu(v', r) \\
&\quad + \rho \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} (\varepsilon(x-\bar{x}) - (v-v')t) d\mu(v', r) \\
&= \rho [vt - \varepsilon(\bar{x}-x)] \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} d\mu(v', r) \\
&\quad - \rho t \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} v' d\mu(v', r) \\
&\quad + \rho [\varepsilon(x-\bar{x}) - vt] \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} d\mu(v', r) \\
&\quad + \rho t \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} v' d\mu(v', r) \\
&= \rho [vt - \varepsilon(\bar{x}-x)] \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} d\mu(v', r) \\
&\quad - \rho t \int r^2 \int_{-\infty}^{v-\varepsilon(\bar{x}-x)/t} v' d\mu(v', r) \\
&\quad - \rho [vt + \varepsilon(\bar{x}-x)] \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} d\mu(v', r) \\
&\quad + \rho t \int r^2 \int_{v+\varepsilon(\bar{x}-x)/t}^{+\infty} v' d\mu(v', r) \\
&\xrightarrow{\varepsilon \rightarrow 0} t\rho \iint (r')^2 |v - v'| d\mu(v', r') = t\mathcal{D}(v).
\end{aligned}$$

Considering the correlation at different velocities  $v < \bar{v}$ :

$$\begin{aligned}
& \mathbb{E}(\xi^{X,\varepsilon}(\varphi_{\varepsilon x,v,t})\xi^{X,\varepsilon}(\varphi_{\varepsilon x,\bar{v},t})) \tag{3.3} \\
&= \rho \iiint r^2 (1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} - 1_{[v' > v]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]}) \\
&\quad \times (1_{[v' < \bar{v}]} 1_{[\varepsilon x < x' < \varepsilon x + (\bar{v}-v')t]} - 1_{[v' > \bar{v}]} 1_{[\varepsilon x + (\bar{v}-v')t < x' < \varepsilon x]}) dx' d\mu(v', r)
\end{aligned}$$

$$\begin{aligned}
&= \rho \iiint r^2 \left( 1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} 1_{[\varepsilon x < x' < \varepsilon x + (\bar{v}-v')t]} \right. \\
&\quad \left. + 1_{[v' > \bar{v}]} 1_{[\varepsilon x + (\bar{v}-v')t < x' < \varepsilon x]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]} \right. \\
&\quad \left. - 1_{[v < v' < \bar{v}]} 1_{[\varepsilon x + (v-v')t < x' < \varepsilon x]} 1_{[\varepsilon x < x' < \varepsilon x + (\bar{v}-v')t]} \right) dx' d\mu(v', r) \\
&= \rho \iiint r^2 \left( 1_{[v' < v]} 1_{[\varepsilon x < x' < \varepsilon x + (v-v')t]} + 1_{[v' > \bar{v}]} 1_{[\varepsilon x + (\bar{v}-v')t < x' < \varepsilon x]} \right) dx' d\mu(v', r) \\
&= t\rho \iint r^2 \left( 1_{[v' < v]} (v - v') + 1_{[v' > \bar{v}]} (v' - \bar{v}) \right) d\mu(v', r) \\
&:= t\Gamma(v, \bar{v}).
\end{aligned}$$

Noting that

$$1_{[v' < v]}(v - v') + 1_{[v' > \bar{v}]}(v' - \bar{v}) = \frac{1}{2}(|v - v'| + |v' - \bar{v}| - (\bar{v} - v)),$$

we have that

$$\Gamma(v, \bar{v}) = \frac{1}{2} \left( \mathcal{D}(v) + \mathcal{D}(\bar{v}) - (\bar{v} - v)\rho \int r^2 d\mu(v', r) \right). \quad (3.4)$$

This last expression for the covariance corresponds to the Levy Chentsov field, as shown in [9].

**Remark 3.1.** From (3.3) we can see immediately that  $\Gamma(v, \bar{v}) \geq 0$ . In the particular case where there are only two velocities admitted, for example  $d\mu(v, r) = \delta_a(dr) \frac{1}{2} (\delta_{-1}(dv) + \delta_1(dv))$ , we have  $\mathcal{D}(\pm 1) = \rho a^2$  and  $\Gamma(1, -1) = 0$ . Typically this decorrelation happens only when two velocities at most are present.

**3.2. Diffusive evolution of density fluctuations.** Define the fluctuation field at diffusive scaling as

$$\Xi_t^{Y, \varepsilon}(\varphi) := \varepsilon^{-1/2} \left[ \varepsilon \sum_{(y, v, r) \in Y^\varepsilon} r \varphi[y_{v, \varepsilon^{-1}t} - v^{\text{eff}}(v) \varepsilon^{-1}t, v, r] - \frac{1}{1 + \sigma} \langle\langle \varphi \rangle\rangle \right]. \quad (3.5)$$

Notice that this is recentered on the Euler characteristics. Define

$$\varphi_{W_t}(y, v, r) := \varphi(y + \sqrt{\mathcal{D}(v)} W_t(v), v, r).$$

**Theorem 3.2.**

$$\Xi_t^{Y, \varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{law} \Xi_t^Y(\varphi) = \Xi^Y(\varphi_{W_t}) = \xi^X(C \widehat{\varphi}_{W_t})$$

where  $\{W_t(v), v \in \mathbb{R}\}$  is a family of Wiener processes with covariance

$$\mathbb{E}(W_t(v), W_t(w)) = \frac{t\Gamma(v, w)}{\sqrt{\mathcal{D}(v)\mathcal{D}(w)}}$$

with  $\Gamma$  defined in (3.3). More formally

$$\begin{aligned}\Xi_t^Y(\varphi) &= \iiint r\varphi(y + \sqrt{\mathcal{D}(v)}W_t(v), v, r)d\xi_0^Y(y, v, r) \\ &= \iiint rC\varphi((1 + \sigma)(x + \sqrt{\mathcal{D}(v)}W_t(v)), v, r)d\xi_0^X(y, v, r)\end{aligned}\quad (3.6)$$

We prove this theorem after some comments. By (3.6),  $\Xi_t^Y$  solves the stochastic differential equation

$$d\Xi_t^Y(\varphi) = \frac{1}{2}\Xi_t^Y(\mathcal{D}\partial_y^2\varphi)dt - \iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y, v, r)dW_t(v)d\Xi_t^Y(y, v, r)$$

or in the time integrated form:

$$\begin{aligned}\Xi_t^Y(\varphi) &= \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}\partial_y^2\varphi)ds \\ &\quad - \int_0^t \iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y, v, r)dW_s(v)d\Xi_s^Y(y, v, r) \\ &= \Xi_0^Y(\varphi) + \int_0^t \frac{1}{2}\Xi_s^Y(\mathcal{D}\partial_y^2\varphi)ds - \int_0^t \Xi_s^Y(\sqrt{\mathcal{D}}\partial_y\varphi dW_s),\end{aligned}$$

where the last term is a martingale with quadratic variation

$$\int_0^t \left( \iiint \sqrt{\mathcal{D}(v)}(\partial_y\varphi)(y, v, r)d\Xi_s^Y(y, v, r) \right)^2 ds = \int_0^t \Xi_s^Y(\sqrt{\mathcal{D}}(\partial_y\varphi))^2 ds. \quad (3.7)$$

Notice that

$$\begin{aligned}\mathbb{E}(\Xi_t^{Y,\varepsilon}(\varphi)^2) &= \frac{\rho}{1 + \sigma} \iiint r^2 \mathbb{E}[(C\varphi)^2(y + \sqrt{\mathcal{D}(v)}W_t(v), v, r)] dy d\mu(v, r) \\ &= \frac{\rho}{1 + \sigma} \iiint r^2 \int (C\varphi)^2(y(1 + \sqrt{\mathcal{D}(v)}z, v, r) \frac{e^{-z^2/2}}{\sqrt{2\pi t}} dz dy d\mu(v, r) \\ &= \frac{\rho}{1 + \sigma} \iiint r^2 (C\varphi)^2(y, v, r) dy d\mu(v, r),\end{aligned}$$

independent of  $t$ , in agreement with the stationarity of the process.

Similarly the expectation of the quadratic variation (3.7) is given by

$$t \frac{\rho}{1 + \sigma} \iiint \mathcal{D}(v)r^2(\partial_y\varphi)^2(y, v, r) dy d\mu(v, r).$$

*Proof of Theorem 3.2.* We can express (3.5) as

$$\begin{aligned}\Xi_t^{Y,\varepsilon}(\varphi) & \quad (3.8) \\ &= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r\varphi[x + m_0^x(X^\varepsilon) + j^\varepsilon(x, v, \varepsilon^{-1}t) - v^{\text{eff}}(v)\varepsilon^{-1}t, v, r] - \frac{1}{1 + \sigma} \langle\langle \varphi \rangle\rangle \right] \\ &= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r\varphi(x + m_0^x(X^\varepsilon) + z(\varepsilon^{-1}t, x, v; X^\varepsilon), v, r) - \frac{1}{1 + \sigma} \langle\langle \varphi \rangle\rangle \right]\end{aligned}$$

$$\begin{aligned}
&= \varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x(1+\sigma) + z(\varepsilon^{-1}t, x, v; X^\varepsilon), v, r) - \frac{1}{1+\sigma} \langle\langle \varphi \rangle\rangle \right] \\
&\quad + \varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\partial_y \varphi)(x(1+\sigma) + z(\varepsilon^{-1}t, x, v; X^\varepsilon), v, r) (m_0^x(X^\varepsilon) - \sigma x) + R_t^\varepsilon.
\end{aligned}$$

Applying Lemma 6.2 to the first term on the rhs of (3.8) we have

$$\varepsilon^{-1/2} \left[ \varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x(1+\sigma) + z(\varepsilon^{-1}t, 0, v; X^\varepsilon), v, r) - \frac{1}{1+\sigma} \langle\langle \varphi \rangle\rangle \right] \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(\check{\varphi}_{\bar{B}_t}),$$

where we have denoted

$$\check{\varphi}_{W_t}(x, v, r) = \varphi_{W_t}(x(1+\sigma), v, r) = \varphi(x(1+\sigma) + \sqrt{\mathcal{D}(v)} W_t(v), v, r).$$

Combining Lemma 6.2 and the same argument used in (2.9) the second term converges in law:

$$\varepsilon^{1/2} \sum_{(x,v,r) \in X^\varepsilon} r (\partial_y \varphi)(x(1+\sigma) + z(\varepsilon^{-1}t, x, v; X^\varepsilon), v, r) (m_0^x(X^\varepsilon) - \sigma x) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} -\xi^X(P\check{\varphi}_{W_t}).$$

Putting the two terms together we conclude that

$$\Xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \xi^X(C\check{\varphi}_{W_t}) = \xi^Y(\varphi_{W_t}).$$

□

Formally, choosing  $\varphi_{k,w}(x, v, r) = e^{i2\pi x k} \varphi(r) \delta(v-w)$ , we have that

$$\begin{aligned}
\widehat{\varphi}(k, w, t) &:= \Xi_t^{Y,\varepsilon}(\varphi_{k,w}) = \int d\xi^Y(y, r, v) e^{i2\pi k(y + \sqrt{\mathcal{D}(w)} W_t)} \varphi(r) \delta(v-w) \\
&= \xi^Y(e^{i2\pi k(+\sqrt{\mathcal{D}(w)} W_t)} \varphi(\cdot) \delta(\cdot - w))
\end{aligned}$$

satisfies the SDE

$$d\widehat{\varphi}(k, w, t) = -\frac{(2\pi k)^2}{2} \mathcal{D}(w) \widehat{\varphi}(k, w, t) + i2\pi k \sqrt{\mathcal{D}(w)} \widehat{\varphi}(k, w, t) dW_t(w).$$

Notice that  $|\widehat{\varphi}(k, w, t)|^2 = |\widehat{\varphi}(k, w, 0)|^2$  for any  $k$ , a persistence on the macroscopic scale of the complete integrability of the dynamics also at the level of these fluctuations.

**Remark 3.3.** Since we consider also systems where lengths  $r$  can be negative, in the case that  $\sigma = 0$  and  $\pi = 0$  the macroscopic evolution of fluctuations in the Euler scaling is the same as the independent point particles. But in the diffusive scaling the fluctuations have non trivial behaviour.

#### 4. A REMARK ABOUT INHOMOGENEOUS INITIAL DISTRIBUTION.

Let  $\rho(x)$  a nice non-negative bounded function on  $\mathbb{R}$  and  $X^\varepsilon$  the Poisson process on  $\mathbb{R}^3$  with intensity  $\varepsilon^{-1}f_0(x, v, r)dx dv dr$ .

In the Euler scaling, the empirical distribution converges to the solution of

$$\partial_t f_t(x, v, r) + v\partial_x f_t(x, v, r) = 0,$$

with initial condition given by  $f_0$ . For the rods density this corresponds to the equation

$$\begin{aligned} \partial_t g_t(y, v, r) + \partial_y(v^{\text{eff}}(y, v, t)g_t(y, v, r)) &= 0, \\ v^{\text{eff}}(y, v, t) &= v + \frac{\iint r(v-w)g_t(y, w, r)dwdr}{1 - \iint rg_t(y, w, r)dwdr}. \end{aligned}$$

as proven in [9].

For *generic* initial conditions, we can guess that the initial density  $f$  is absolutely continuous in the  $x$  and  $v$  coordinates, then it satisfies that the limit as  $t \rightarrow \infty$  of  $f_t(x, v, r)$  is constant in  $x$ , that is,  $f_t(x, v, r) \rightarrow \rho\bar{f}(v, r)$ , for some  $\bar{f}$ . This suggests that in a diffusive time scale the system essentially behaves like if it is a stationary state determined by a Poisson point field  $\rho\bar{f}(v, r)dx dv dr$ , and the analysis for the macroscopic fluctuations of [Section 3](#) applies.

#### 5. FLUCTUATIONS FOR GENERAL COLLISION RULES

We can consider different completely integrable dynamics constructed from the free motion of points. For example, instead of (2.11) we can define

$$j^\varepsilon(x, v, r, t) := \varepsilon \sum_{(\tilde{x}, \tilde{v}, \tilde{r}) \in X^\varepsilon} \phi(\tilde{r}, r, \tilde{v}, v) (1_{[\tilde{v} < v]} 1_{[x < \tilde{x} < x + (v - \tilde{v})t]} - 1_{[\tilde{v} > v]} 1_{[x + (v - \tilde{v})t < \tilde{x} < x]}),$$

where  $\phi$  is the collision rule. We have studied the case  $\phi(\tilde{r}, r, \tilde{v}, v) = \tilde{r}$ . When the collision rule is different, we do not have anymore the interpretation of  $j^\varepsilon(x, v, r, t)$  as scattering shift of hard rods. However, we can still compute expectations and limits as  $\varepsilon \rightarrow 0$  of the length flows:

$$\begin{aligned} j(x, v, r, t) &:= \mathbb{E}j^\varepsilon(x, v, r, t) \\ &= \iiint \rho dx d\mu(\tilde{v}, \tilde{r}) \phi(\tilde{r}, r, \tilde{v}, v) (1_{[\tilde{v} < v]} 1_{[x < \tilde{x} < x + (v - \tilde{v})t]} - 1_{[\tilde{v} > v]} 1_{[x + (v - \tilde{v})t < \tilde{x} < x]}) \\ &= \rho \iint_v^{+\infty} \phi(\tilde{r}, r, \tilde{v}, v) (v - \tilde{v}) t d\mu(\tilde{v}, \tilde{r}) + \rho \iint_{-\infty}^v \phi(\tilde{r}, \tilde{v}, v) (v - \tilde{v}) t d\mu(\tilde{v}, \tilde{r}) \\ &= t\rho \iint \phi(\tilde{r}, r, \tilde{v}, v) (v - \tilde{v}) d\mu(\tilde{v}, \tilde{r}), \end{aligned}$$

that gives an *effective velocity*

$$v^{\text{eff}}(v) = v + \rho \iint \phi(\tilde{r}, r, \tilde{v}, v)(v - \tilde{v}) d\mu(\tilde{v}, \tilde{r}).$$

Corresponding fluctuations in Euler and diffusive time scales can be proven following the same lines as in the previous sections.

## 6. TWO LIMIT THEOREMS FOR POISSON PROCESS

**Lemma 6.1.** *Let  $\varphi(x, v, r)$  a smooth function on  $\mathbb{R}^3$  with compact support in  $x$ . Then*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sum_{(x, v, r) \in X^\varepsilon} r \varphi(x, v, r) B^\varepsilon(x) \stackrel{\text{law}}{=} \rho \iiint r \varphi(x, v, r) B(x) dx d\mu(v, r).$$

*Proof.* Since  $\varphi$  is a smooth function we can approximate both sides by step functions, so that it is enough to prove that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sum_{(x, v, r) \in X^\varepsilon} 1_{[0 \leq x < 1]} B^\varepsilon(x) \stackrel{\text{law}}{=} \rho \int_0^1 B(x) dx.$$

Since in this limit  $(v, r)$  are not involved, to simplify notation we will ignore them. We can write

$$\begin{aligned} & \varepsilon \sum_{(x, v, r) \in X^\varepsilon} 1_{[0 \leq x < 1]} B^\varepsilon(x) - \rho \int_0^1 B(x) dx \\ &= \varepsilon \sum_{(x, v, r) \in X^\varepsilon} 1_{[0 \leq x < 1]} B^\varepsilon(x) - \rho \int_0^1 B^\varepsilon(x) dx + \rho \int_0^1 B^\varepsilon(x) dx, \end{aligned}$$

and since, by the FCLT,  $\rho \int_0^1 B^\varepsilon(x) dx \xrightarrow[\varepsilon \rightarrow 0]{\text{law}} \rho \int_0^1 B(x) dx$ , we only need to prove that the other difference is small in probability.

We first approximate  $\int_0^1 B^\varepsilon(x) dx$  by a Riemann sum:

$$\begin{aligned} & \int_0^1 B^\varepsilon(x) dx - \varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} B^\varepsilon(\varepsilon(j-1)) \\ &= \sum_{j=1}^{[\varepsilon^{-1}]} \int_{\varepsilon(j-1)}^{\varepsilon j} (B^\varepsilon(x) - B^\varepsilon(\varepsilon(j-1))) dx \\ &= \int_0^\varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} (B^\varepsilon(x + \varepsilon(j-1)) - B^\varepsilon(\varepsilon(j-1))) dx. \end{aligned}$$

Notice that for any fixed  $x \in [0, \varepsilon]$  the sum is over independent random variables. Then the variance of the last line is bounded by

$$\varepsilon \int_0^\varepsilon \mathbb{E} \left( \left[ \sum_{j=1}^{[\varepsilon^{-1}]} (B^\varepsilon(x + \varepsilon(j-1)) - B^\varepsilon(\varepsilon(j-1))) \right]^2 \right) dx \leq \int_0^\varepsilon x dx = \frac{1}{2} \varepsilon^2.$$

We are left to estimate the difference

$$\begin{aligned} & \varepsilon \sum_{x \in X^\varepsilon} 1_{[0 \leq x < 1]} B^\varepsilon(x) - \varepsilon \rho \sum_{j=1}^{[\varepsilon^{-1}]} B^\varepsilon(\varepsilon(j-1)) \\ &= \varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} \left( \sum_{x \in X^\varepsilon} 1_{[\varepsilon(j-1) \leq x < \varepsilon j]} B^\varepsilon(x) - \rho B^\varepsilon(\varepsilon(j-1)) \right) \\ &= \varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} \sum_{x \in X^\varepsilon} 1_{[\varepsilon(j-1) \leq x < \varepsilon j]} [B^\varepsilon(x) - B^\varepsilon(\varepsilon(j-1))] \\ &+ \varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} \left( \sum_{x \in X^\varepsilon} 1_{[\varepsilon(j-1) \leq x < \varepsilon j]} - \rho \right) B^\varepsilon(\varepsilon(j-1)). \end{aligned} \tag{6.1}$$

The variance of the first term on the RHS of (6.1) is bounded by

$$\begin{aligned} & \mathbb{E} \left( \left[ \sum_{x \in X^\varepsilon} 1_{[0 \leq x < \varepsilon]} B^\varepsilon(x) \right]^2 \right) = \mathbb{E} \left( \sum_{x, y \in X^\varepsilon} 1_{[0 \leq x < \varepsilon]} 1_{[0 \leq y < \varepsilon]} B^\varepsilon(x) B^\varepsilon(y) \right) \\ & \sim \varepsilon^{-2} \rho^2 \iint 1_{[0 \leq x < \varepsilon]} 1_{[0 \leq y < \varepsilon]} x \wedge y dx dy = 2\varepsilon^{-2} \rho^2 \int_0^\varepsilon dx \int_0^x y dy = \frac{\rho^2}{3} \varepsilon, \end{aligned}$$

while the variance for the last term is bounded by

$$\begin{aligned} & \varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} \mathbb{E} \left( \left[ \sum_{x \in X^\varepsilon} 1_{[\varepsilon(j-1) \leq x < \varepsilon j]} - \rho \right]^2 \right) \mathbb{E} (B^\varepsilon(\varepsilon(j-1)))^2 \\ & \leq \varepsilon \sum_{j=1}^{[\varepsilon^{-1}]} \varepsilon \rho \varepsilon(j-1) \sim O(\varepsilon). \end{aligned} \quad \square$$

**Lemma 6.2.** *Let  $\tilde{X}^\varepsilon$  be a homogeneous Poisson process on  $\mathbb{R}$  with density  $\varepsilon^{-1} \tilde{\rho}$  and  $\{\tilde{B}^\varepsilon(x), x \in \mathbb{R}\}$  be a family of processes independent of  $\tilde{X}^\varepsilon$  and such that they converge in law to the same random variable  $\tilde{B}$ . Then for any smooth compact support function  $\varphi(x)$  on  $\mathbb{R}$ ,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \left( \varepsilon \sum_{x \in \tilde{X}^\varepsilon} \varphi(x + \tilde{B}^\varepsilon(x)) - \tilde{\rho} \int \varphi(x) dx \right) \stackrel{law}{=} \xi^{\tilde{X}}(\varphi_{\tilde{B}}), \tag{6.2}$$

where  $\varphi_{\tilde{B}}(x) = \varphi(x + \tilde{B})$ .

*Proof.* In order to shorten notation let's assume  $\int \varphi(x)dx = 0$ . Denote

$$\mathcal{Z}_\varepsilon := \varepsilon^{1/2} \sum_{x \in \tilde{X}^\varepsilon} \varphi(x + \tilde{B}^\varepsilon(x)).$$

Its characteristic function is

$$\begin{aligned} \mathbb{E}[e^{ik\mathcal{Z}_\varepsilon}] &= \mathbb{E}[\mathbb{E}(e^{ik\mathcal{Z}_\varepsilon}|B^\varepsilon)] = \mathbb{E}\left[\exp\left(\rho\varepsilon^{-1} \int (e^{ik\varepsilon^{1/2}\varphi(x+\tilde{B}^\varepsilon(x))} - 1)dx\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\rho k^2 \int \varphi(x + \tilde{B}^\varepsilon(x))^2 dx\right)\right] + O(\varepsilon^{1/2}) \\ &\xrightarrow{\varepsilon \rightarrow 0} \mathbb{E}\left[\exp\left(-\rho k^2 \int \varphi(x + \tilde{B})^2 dx\right)\right] = \mathbb{E}\left[\exp\left(-\rho k^2 \int \varphi(x)^2 dx\right)\right]. \end{aligned}$$

The joint characteristic function of the couple  $(\xi^{\varepsilon, X}, \mathcal{Z}_\varepsilon)$  is

$$\begin{aligned} \mathbb{E}\left[e^{ik_1\xi^{\varepsilon, X}(\varphi) + ik_2\mathcal{Z}_\varepsilon}\right] &= \mathbb{E}\left[\exp\left(-\rho \int \left[k_1\varphi(x) + k_2\varphi(x + \tilde{B}^\varepsilon(x))\right]^2 dx\right)\right] + O(\varepsilon^{1/2}) \\ &\xrightarrow{\varepsilon \rightarrow 0} \mathbb{E}\left[\exp\left(-\rho \int \left[k_1\varphi(x) + k_2\varphi(x + \tilde{B})\right]^2 dx\right)\right] = \mathbb{E}\left[e^{ik_1\xi^{\tilde{X}}(\varphi) + ik_2\xi^{\tilde{X}}(\varphi_{\tilde{B}})}\right], \end{aligned}$$

i.e. (6.2). □

## REFERENCES

- [1] Michael Aizenman, Sheldon Goldstein, and Joel L. Lebowitz. Ergodic properties of an infinite one dimensional hard rod system. *Communications in Mathematical Physics*, 39(4):289–301, 1975. [3](#)
- [2] Michael Aizenman, Joel Lebowitz, and Joaquin Marro. Time-displaced correlation functions in an infinite one-dimensional mixture of hard rods with different diameters. *Journal of Statistical Physics*, 18(2):179–190, 1978. [1](#)
- [3] C. Boldrighini, R. L. Dobrushin, and Yu. M. Sukhov. One-dimensional hard rod caricature of hydrodynamics. *J. Statist. Phys.*, 31(3):577–616, 1983. [1](#)
- [4] C Boldrighini, RL Dobrushin, and Yu M Suhov. One-dimensional hard-rod caricature of hydrodynamics: Navier-Stokes correction. Technical report, Dublin Institute for Advances Studies, 1990. Preprint. [2](#)
- [5] C. Boldrighini and Yu. M. Suhov. One-dimensional hard-rod caricature of hydrodynamics: “Navier-Stokes correction” for local equilibrium initial states. *Comm. Math. Phys.*, 189(2):577–590, 1997. [2](#), [3](#)
- [6] C. Boldrighini and W. David Wick. Fluctuations in a one-dimensional mechanical system. I. The Euler limit. *J. Statist. Phys.*, 52(3-4):1069–1095, 1988. [1](#), [3](#)
- [7] John Cardy and Benjamin Doyon.  $\overline{TT}$  deformations and the width of fundamental particles. *J. High Energy Phys.*, (4):Paper No. 136, 26, 2022. [1](#)
- [8] Benjamin Doyon and Herbert Spohn. Dynamics of hard rods with initial domain wall state. *J. Stat. Mech. Theory Exp.*, (7):073210, 21, 2017. [3](#)
- [9] Pablo A Ferrari, Chiara Franceschini, Dante GE Grevino, and Herbert Spohn. Hard rod hydrodynamics and the Levy Chentsov field. *arXiv preprint arXiv:2211.11117*, 2022. [1](#), [12](#), [14](#), [17](#)
- [10] Pablo A. Ferrari, Chi Nguyen, Leonardo T. Rolla, and Minmin Wang. Soliton decomposition of the box-ball system. *Forum Math. Sigma*, 9:Paper No. e60, 37, 2021. [5](#)

- [11] T. E. Harris. Random measures and motions of point processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 18:85–115, 1971. [5](#)
- [12] Jesper Møller and Rasmus Plenge Waagepetersen. *Statistical inference and simulation for spatial point processes*, volume 100 of *Monographs on Statistics and Applied Probability*. Chapman & Hall/CRC, Boca Raton, FL, 2004. [6](#)
- [13] JK Percus. Exact solution of kinetics of a model classical fluid. *The Physics of Fluids*, 12(8):1560–1563, 1969. [1](#)
- [14] Errico Presutti and W. David Wick. Macroscopic stochastic fluctuations in a one-dimensional mechanical system. *J. Statist. Phys.*, 51(5-6):871–876, 1988. New directions in statistical mechanics (Santa Barbara, CA, 1987). [3](#)
- [15] H. Spohn. *Large Scale Dynamics of Interacting Particles*. Springer-Verlag, Heidelberg, 1991. [2](#), [8](#)
- [16] H. Spohn. Hydrodynamic scales of integrable many-particle systems. arXiv:2301.08504v1, 2023. [2](#)
- [17] Herbert Spohn. Hydrodynamical theory for equilibrium time correlation functions of hard rods. *Ann. Physics*, 141(2):353–364, 1982. [2](#)
- [18] Hermann Thorisson. *Coupling, stationarity, and regeneration*. Probab. Appl. New York, NY: Springer, 2000. [5](#)

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