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## LAGRANGIANS FOR ELECTROMECHANICAL SYSTEMS

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**Abstract.** Electromechanical systems are very common. The importance of constructing the dynamical equations of motors coupled with mechanical systems suggests a new strategy. In the literature, often, the derivation of the dynamical equations is wrong. One thinks that the standard derivations of the dynamical equations of purely mechanical systems can be mimicked to electromechanical systems. Unfortunately, it cannot. The main reason is that in electromechanical systems one deals with the presence of electromagnetic fields, continuous entities. These fields store electrical and mechanical energies. In purely mechanical systems the conservative mechanical energy is stored as elastic or gravitational energy, and the nonconservative terms enter the equation as nonconservative forces. This cannot be done in electromagnetic systems. This paper shows the right way to derive the dynamical equations applying the results for two systems. Both systems are formed by a motor, a coupling mechanism, and a mechanical subsystem. In one case the coupling is the scotch yoke mechanism and the mechanical part is a cart. In the other the coupling mechanism is a slider crank mechanism and the mechanical system is a pendulum. To explain clearly the ideas, the dynamical equations are derived in a different way, putting in evidence the common errors. The examples were taken from the recent literature, but the same mistake is found in several places.

## 1 INTRODUCTION

Modeling plays a necessary role in the design of dynamical systems. It helps to describe the behavior of a system and to acquire a better understanding of its functionality. This paper is about modeling electromechanical systems using Lagrangians, although it is not a new subject (Wells, 1967) it needs to be discussed. Electromechanical systems are characterized by a mutual influence between mechanical and electrical parts (Dantas et al., 2014, 2016). Once dealing with systems composed of two different natures, a new method to describe dynamical equations has to be used. In this paper electromechanical systems are modeled by an extended Lagrangian. Although mechanical and electrical systems when analysed separately usually show different methods of modeling, it is possible to analyse electrical circuits with a similar approach to the classical Lagrangian. Before introducing electromechanical systems, the objective is to discern two classes of energy functions involved: energy and coenergy. These concepts are akin to the several energy functions used in Thermodynamics according to the independent variables chosen to characterise the systems all energy functions connected by Lagrange transformations. The introduction of this complementary energy will help to identify terms of mutual influence later in the discussion.

One particular aspect of systems with mutual interactions between mechanical and electrical parts is a feature called coupling. In this case, an electromechanical coupling. A very important tool when it comes to formulate an extended Lagrangian. Basically, this paper analyses two electromechanical systems. Both have a DC Motor coupled to a mechanical part. They have the same electromechanical coupling. Their difference lies in the choice of the mechanical coupling mechanism. The mechanical coupling of the first system is made by a mechanism called *scotch-yoke*. The second uses a mechanism called *slider-crank*. These systems have an interesting feature: they work as a sort of master-slave condition. The motor rotates and the mechanical part reacts. The focus is to formulate an extended Lagrangian for both systems. This paper also points out a mistake commonly made in deriving the dynamical equations of electromechanical systems. This procedure gives a wrong interpretation of coupled systems. It formulates a Lagrangian only for the mechanical part, disregarding the electrical subsystem. The authors who commit this first mistake are then led to commit a second unjustified simplification, which has even more serious consequences. This second mistake is to say that a torque provided by the motor to the mechanical system is a function of the generalized coordinates. But, of course, the torque is not a function of only the coordinates, it also depends on the initial conditions. Therefore, it must not be added as a generalized force to the description of the mechanical system.

## 2 ENERGY CONSIDERATIONS

### 2.1 Energy Balance Relationships

In this paper, for a better understanding, electromechanical systems are considered as lumped-parameter systems. They are described by a finite number of mechanical and electrical variables. In these systems, interaction occurs through electrostatic and electromagnetic fields. These fields are called coupling fields. They are common to electrical and mechanical parts. Both subsystems will always interact and the interaction causes energy transfer.

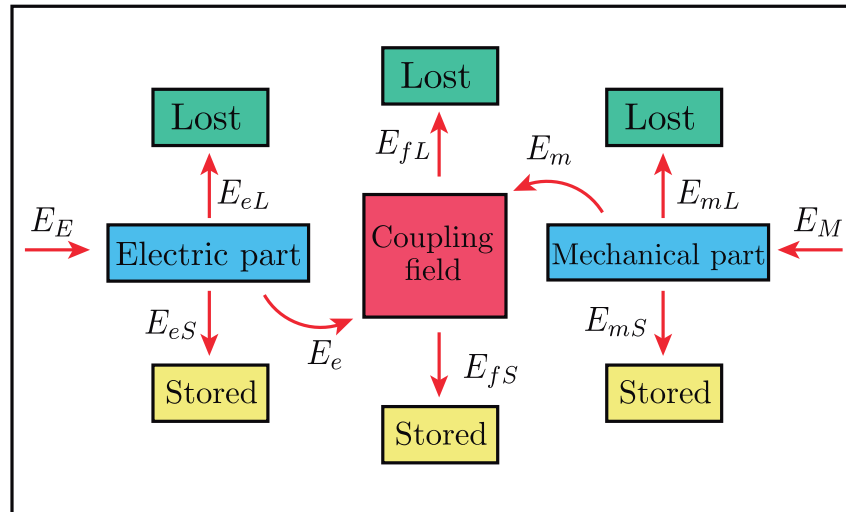


Figure 1: Energy transfer in an electromechanical system.

A simplified form of an electromechanical system can be represented by figure 1. It shows how energy transfers from one point to another. Now it is apparent how electrical and mechanical parts interact in a system. Electromechanical systems are coupled systems by nature. They interact through a coupling field.  $E_m$  is the energy transferred to the coupling field by the mechanical subsystem,  $E_{mL}$  the loss also associated to the mechanical part and  $E_{mS}$  is the energy stored in the mechanical elements. It is possible to describe the total energy supplied by the mechanical part as  $E_M$ . The same can be done for the electrical part.  $E_e$  is the energy transferred to the coupling field by the electrical part,  $E_{eL}$  is the loss also associated to the electric subsystem and  $E_{eS}$  is the energy stored in electric or magnetic fields which are not coupled to the mechanical subsystem. It is possible to describe the energy supplied by the electrical part as  $E_E$ . Analysing the directions of the arrows in figure 1, the following equations are easily obtained.

For the energy supplied to both parts, one has

$$E_M = E_m + E_{mL} + E_{mS}, \quad (1)$$

$$E_E = E_e + E_{eL} + E_{eS}. \quad (2)$$

Now for the total energy transferred to the coupling field we must separate the energy stored from the energy dissipated. For  $E_{fS}$  being the total energy stored and  $E_{fL}$  the total energy dissipated in the coupling field, it is possible to define the total energy transferred to the coupling field as:

$$E_F = E_e + E_m = E_{fS} + E_{fL}. \quad (3)$$

To demonstrate the equality of the equation 3 we define  $E_{fS}$  and  $E_{fL}$ . Both are the difference between the total energy supplied and the sum of the energy stored with the energy lost, for each subsystem :

$$E_{fS} = E_e - (E_{eL} + E_{eS}), \quad (4)$$

$$E_{fL} = E_m - (E_{mL} + E_{mS}), \quad (5)$$

Substituting 4 and 5 in 3 we have:

$$E_F = (E_e - E_{eL} - E_{eS}) + (E_m - E_{mL} - E_{mS}), \quad (6)$$

Substituting equations 1 and 2 in 6 gives,

$$E_F = E_e + E_m. \quad (7)$$

The total energy transferred to the coupling field is independent of the energy dissipated in the mechanical or electrical part. It is also independent of the energy stored outside the coupling terminals. Thus, if the losses of the coupling field are not taken into account, the field between the coupling terminals is said to be conservative. It can be compared to a box that contains only an electric or a magnetic field. The value and the energy stored inside is affected by electrical and mechanical variables. The energy put into the system by the mechanical and electrical pairs is stored in the field and can be recovered completely. That is the reason an assumption of a conservative coupling field is valid. Consider two examples of electromechanical systems. One has a magnetic field storage between coupling terminals and the other an electric field storage. Both fields interact with the mechanical subsystem by moving a plate. The electric subsystem is composed by a voltage  $v$ , a current  $\dot{q}$  representing the time derivative of the electric charge  $q$ , an inductance  $l$ , a resistance  $r$  and a coupling voltage denoted by  $e_f$ . The mechanical subsystem has a plate of mass  $m$ , a spring of stiffness  $k$ , a force  $f$  externally applied, a damper with damping coefficient  $b$  and a force of electric or magnetic origin  $f_e$ . In systems  $a$  and  $b$ ,  $x$  is the displacement and  $x_0$  is the equilibrium position of the plate.

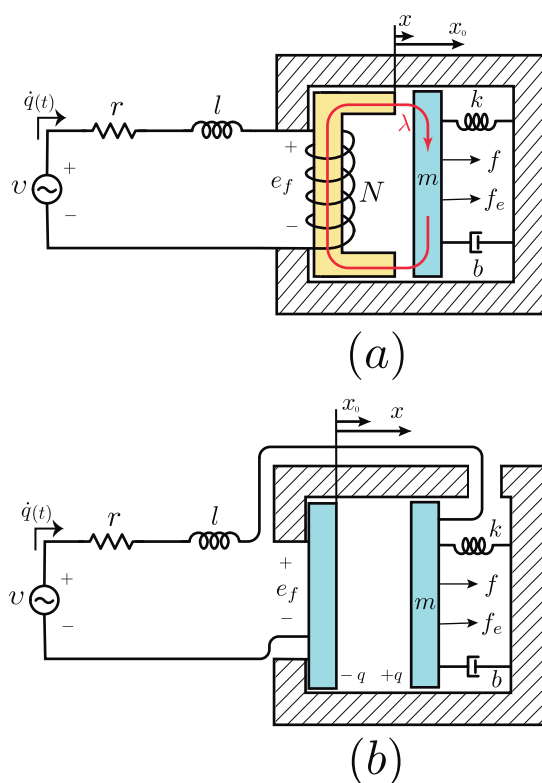


Figure 2: (a) Electromechanical system with a magnetic coupling field (b) Electromechanical system with an electrical coupling field

These two electromechanical systems are very simple. To obtain the equations of the dynamics in this case is trivial. Newton's and Kirchoff's law applied to systems (a) and (b) gives

$$m\ddot{x} + b\dot{x} + k(x - x_0) - f_e = f, \tag{8}$$

$$l\ddot{q} + r\dot{q} + e_f = v, \tag{9}$$

After multiplying equation 9 by the current  $\dot{q}$  and equation 8 by the velocity  $\dot{x}$  we get two powers. Integrating both equations in time, we get two equalities of energy equations for

$$\int f\dot{x}dt = \int m\ddot{x}\dot{x} dt + \int b\dot{x}^2 dt + \int k(x - x_0)\dot{x} dt - \int f_e\dot{x} dt, \tag{10}$$

$$\int v\dot{q} dt = \int l\ddot{q}\dot{q} dt + \int r\dot{q}^2 dt + \int e_f\dot{q} dt. \tag{11}$$

Equations 10 and 11 also represent the energy supplied by the mechanical and the electrical part. Comparing equation 1 to equation 10 leads to

$$E_M = \int f \dot{x} dt, \quad (12)$$

$$E_{mS} = \int m \ddot{x} \dot{x} dt + \int k(x - x_0) \dot{x} dt, \quad (13)$$

$$E_{mL} = \int b \dot{x}^2 dt, \quad (14)$$

$$E_m = - \int f_e \dot{x} dt. \quad (15)$$

Now comparing equation 2 to equation 11 we get

$$E_E = \int v \dot{q} dt, \quad (16)$$

$$E_{eS} = \int l \ddot{q} \dot{q} dt, \quad (17)$$

$$E_{eL} = \int r \dot{q}^2 dt, \quad (18)$$

$$E_e = \int e_f \dot{q} dt. \quad (19)$$

Substituting equations 15 and 16 in 7 gives

$$E_F = \int e_f \dot{q} dt - \int f_e \dot{x} dt, \quad (20)$$

or in differential form:

$$\frac{d}{dt}(E_F) = e_f \dot{q} - f_e \dot{x}. \quad (21)$$

The total energy transferred to the coupling field  $E_F$  for systems (a) and (b) is denoted by equation 21. The examples presented do not include all possibilities. Note that this equation is valid only for one mechanical and one electrical terminal. Aiming to expand the equation to other circumstances, it should not only increase the number of terminals but also include a torque. This mechanical torque, as the mechanical force, can have a magnetic or an electric origin. The rate of change in the angular displacement  $\dot{\alpha}$  must multiply the torque  $\tau_e$  in order to form a power and be included in equation 21. For  $N$  electrical and  $M$  mechanical terminals:

$$\frac{d}{dt}(E_F) = \sum_{i=1}^N (e_f \dot{q})_i - \left( \sum_{i=1}^M (f_e \dot{x})_i + \sum_{i=1}^M (\tau_e \dot{\alpha})_i \right), \quad (22)$$

$$\frac{d}{dt}(E_F) = \sum_{i=1}^N (e_f \dot{q})_i - \left( \sum_{i=1}^M (f_e)_i \frac{dx_i}{dt} + \sum_{i=1}^M (\tau_e)_i \frac{d\alpha_i}{dt} \right). \quad (23)$$

Energy generated can be stored, but power can not. Conserving power is a way to conserve the rate at which the energy is transmitted. Equation 23 shows that the rate of the energy transferred to the coupling field is conserved, see Woodson and Melcher (1968). In the systems presented, the displacement  $x$  represents the influence of the mechanical part in the coupling field. It could also be represented by an angular displacement  $\alpha$ . These coordinates  $x$  and  $\alpha$  are not related through a constitutive equation. It is true that, when there is a constraint, one can be written as a function of the other, but their relation is not always imposed. It is not necessary. Therefore, it is possible to have one of them ( $x$  or  $\alpha$ ), or both, to represent the state of the mechanical part. Note that for the electrical part, the choice is not so trivial. The electric charge  $q$  and the flux linkage  $\lambda$  are often chosen as coordinates to represent an electrical system. Once the electric coordinates are related through constitutive equations and as the derivative of the flux linkage ( $d\lambda/dt$ ) equals the voltage between the terminals of the coupling field ( $e_f$ ). The constitutive equations for both possibilities are

$$q = c \frac{d\lambda}{dt}, \quad (24)$$

$$\lambda = l \frac{dq}{dt}. \quad (25)$$

Only one coordinate is necessary. The choice of the electrical variable must be unique for any number of coupling terminals. Usually for systems with magnetic energy storage as (a), the choice of  $\lambda$  seems to be motivated by how an increasing of the flux linkage leads to an increasing of the energy transmitted to the system. The same approach is made for systems like (b) with an electric energy storage, the choice of  $q$  is motivated by how incremental changes in the coupling field influences in the energy transmission. For now, this choice is suitable to understand the principles of the energy transfer. It is chosen  $\lambda$  for the system with magnetic energy storage and  $q$  for the system with electric energy storage. Later in this paper, the choice of only one coordinate has to be made to formulate an extended Lagrangian. It is not convenient to work with two coordinates for a system while having a constraint, although working with two coordinates seems to be possible, see interesting formulation of Haas et al. (2000). For now, it is possible to write the equations for a magnetic and an electric energy storage represented by each of its chosen coordinates. Being  $\mathcal{T}_m$  and  $\mathcal{V}_e$  the energy stored in a magnetic and in an electric field, respectively.

For the energy transferred to a magnetic coupling field one has

$$\frac{d}{dt}(\mathcal{T}_m) = \sum_{i=1}^N (\dot{q})_i \frac{d\lambda_i}{dt} - \left( \sum_{i=1}^M (f_e)_i \frac{dx_i}{dt} + \sum_{i=1}^M (\tau_e)_i \frac{d\alpha_i}{dt} \right). \quad (26)$$

and for the energy transferred to an electric coupling field

$$\frac{d}{dt}(\mathcal{V}_e) = \sum_{i=1}^N (e_f)_i \frac{dq_i}{dt} - \left( \sum_{i=1}^M (f_e)_i \frac{dx_i}{dt} + \sum_{i=1}^M (\tau_e)_i \frac{d\alpha_i}{dt} \right). \quad (27)$$



The left side of the equations 26 and 27 shows a term that represents the time rate of increase in energy stored. In the right side, the difference between two other terms. The first is the power input at the electrical terminals and the second is the power input at the mechanical terminals. Second term has a negative sign because the force  $f_e$  and the torque  $\tau_e$  are defined as acting on the mechanical node.

## 2.2 Energy and Coenergy Relations

In the study of electromechanical systems, the definition of coenergy must be presented. Energy and coenergy complements one another (Jeltsema and Scherpen, 2009). Take an example in the mechanical domain: a particle with mass  $m$  moving in the positive  $x$  direction. This model represents a non-relativistic situation. We are able to write the kinetic energy and the kinetic coenergy of the particle with a constitutive law. This constitutive law will relate the momentum  $p$  with the velocity  $v$ . By substituting the constitutive law in the energy equation, the coenergy is obtained:

$$\text{Energy: } \frac{p^2}{2m}, \quad (28)$$

$$\text{Constitutive law: } p = mv, \quad (29)$$

$$\text{Coenergy: } \frac{mv^2}{2}. \quad (30)$$

When plotting the constitutive law  $p = mv$ , with  $p$  in the y-axis and  $v$  in the x-axis, the area underneath the line represents the energy and the area above the line the coenergy. Notice that the kinetic coenergy is often called kinetic energy in most literatures. The term coenergy is omitted. The reason that happens is that for linear systems the energy equals the coenergy, making the introduction of a new concept unnecessary. For relativistic situations, the constitutive law changes. Now, a nonlinear relation is analysed. Energy and coenergy now differ. The linear equation for the momentum  $p$  used to give the problem similar areas and consequently similar energies. The relativistic nonlinear equation gives rise to different areas and different quantities. Finally, we define energy functions for electrical systems as

Energy:

$$\mathcal{T}_m = \mathcal{T}_m(\lambda), \quad (31)$$

$$\mathcal{W}_e = \mathcal{W}_e(q). \quad (32)$$

Coenergy:

$$\mathcal{W}_m = \mathcal{W}_m(\dot{q}), \quad (33)$$

$$\mathcal{V}_e = \mathcal{V}_e(v). \quad (34)$$

The magnetic energy  $\mathcal{T}_m$  is described as a function of the flux linkage  $\lambda$ .  $\mathcal{W}_m$  is the magnetic coenergy described as a function of the current  $\dot{q}$ . The electric coenergy  $\mathcal{V}_e$  varies with the voltage  $v$ . The electric energy  $\mathcal{W}_e$  varies with the charge  $q$ . Although their difference in the choice of independent variables, energy functions  $\mathcal{T}_m$  and  $\mathcal{W}_m$ , as well as  $\mathcal{V}_e$  and  $\mathcal{W}_e$  have the same units. Note that the magnetic energy and coenergy are functions that can be related through a constitutive law and it depends on the characteristics of the system, just like the example of the moving particle in the mechanical domain. The same holds true to electric energy and coenergy functions. Again, it is important to emphasize that these functions, for magnetic and electric parts separately, may not always be the same, the equality is a characteristic of linear systems only. A benefit of the coenergy is the advantage to calculate mechanical torques and forces of magnetic or electrical origin. It is also interesting to analyze the relation between energies with a graphical approach.

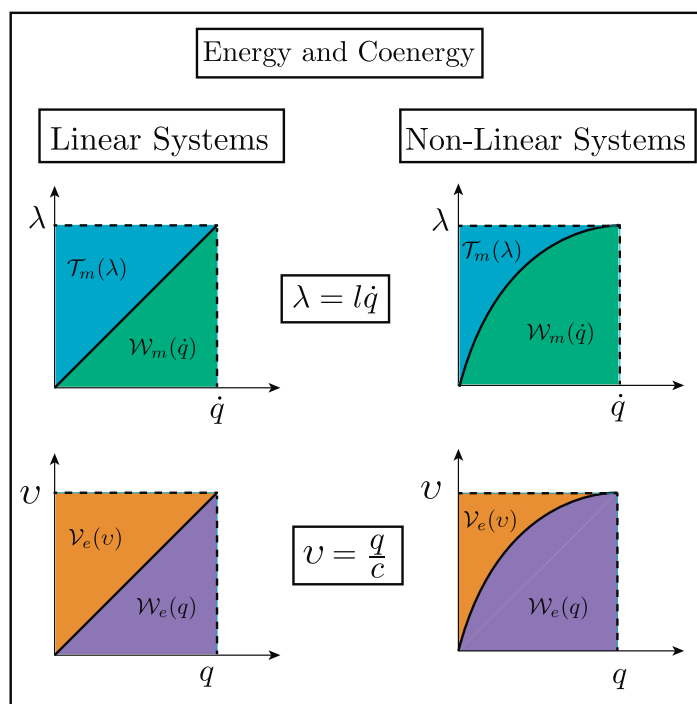


Figure 3: Graphics of energy and coenergy for linear and non-linear systems .

These functions are related through their sum, which equals the sum of the multiplication of two independent variables, which are the chosen possibilities to describe electrical systems:  $q$  and its derivative  $\dot{q}$  and  $\lambda_f$  and its derivative  $e_f$ . To verify this information an analysis of the graphics is sufficient. For  $N$  electrical terminals:

$$\mathcal{T}_m + \mathcal{W}_m = \sum_{i=1}^N (\lambda_f)_i \dot{q}_i, \tag{35}$$

$$\mathcal{V}_e + \mathcal{W}_e = \sum_{i=1}^N (e_f)_i q_i. \tag{36}$$

### 3 MODELING TECHNIQUE FOR ELECTROMECHANICAL SYSTEMS

#### 3.1 Electromechanical Coupling

One of the most important subjects when electromechanical systems comes to mind is the electromechanical coupling (Woodson and Melcher, 1968) and how it should appear in the equations of the system. By a sum of energy functions of the electrical part (they are presented in section 2.2), it is possible to obtain the electromechanical coupling of the system. The energy being discussed in these functions is the energy between coupling terminals. A coupling terminal can store magnetic or electrical energy, or both.

First, it is shown how to obtain the electromechanical coupling of a system which stores magnetic energy, then the same method is applied to a system with electrical energy stored. To simplify calculus, it is supposed that the system has only one coupled terminal:

$$\mathcal{T}_m + \mathcal{W}_m = \dot{q}\lambda, \quad (37)$$

$$\frac{d}{dt}(\mathcal{T}_m + \mathcal{W}_m) = \frac{d}{dt}(\dot{q}\lambda), \quad (38)$$

$$\frac{d}{dt}\mathcal{T}_m + \frac{d}{dt}\mathcal{W}_m = \frac{d}{dt}(\dot{q}\lambda), \quad (39)$$

$$\left(\dot{q}\frac{d\lambda}{dt} - f_e\frac{dx}{dt} - \tau_e\frac{d\alpha}{dt}\right) + \left(\frac{d\mathcal{W}_m d\dot{q}}{d\dot{q} dt} + \frac{d\mathcal{W}_m dx}{dx dt} + \frac{d\mathcal{W}_m d\alpha}{d\alpha dt}\right) = \left(\dot{q}\frac{d\lambda}{dt} + \lambda\frac{d\dot{q}}{dt}\right), \quad (40)$$

$$\left(\frac{d\mathcal{W}_m}{d\dot{q}} - \lambda\right)\frac{d\dot{q}}{dt} + \left(\frac{d\mathcal{W}_m}{dx} - f_e\right)\frac{dx}{dt} + \left(\frac{d\mathcal{W}_m}{d\alpha} - \tau_e\right)\frac{d\alpha}{dt} = 0 \quad (41)$$

Now, the same is done for the electric part:

$$\mathcal{V}_e + \mathcal{W}_e = vq, \quad (42)$$

$$\frac{d}{dt}(\mathcal{V}_e + \mathcal{W}_e) = \frac{d}{dt}(vq), \quad (43)$$

$$\frac{d}{dt}\mathcal{V}_e + \frac{d}{dt}\mathcal{W}_e = \frac{d}{dt}(vq), \quad (44)$$

$$\left(v\frac{dq}{dt} - f_e\frac{dx}{dt} - \tau_e\frac{d\alpha}{dt}\right) + \left(\frac{d\mathcal{W}_e dv}{dv dt} + \frac{d\mathcal{W}_e dx}{dx dt} + \frac{d\mathcal{W}_e d\alpha}{d\alpha dt}\right) = \left(v\frac{dq}{dt} + q\frac{dv}{dt}\right), \quad (45)$$

$$\left(\frac{d\mathcal{W}_e}{dv} - q\right)\frac{dv}{dt} + \left(\frac{d\mathcal{W}_e}{dx} - f_e\right)\frac{dx}{dt} + \left(\frac{d\mathcal{W}_e}{d\alpha} - \tau_e\right)\frac{d\alpha}{dt} = 0. \quad (46)$$

It is possible to satisfy the equations as  $dv$ ,  $dx$ ,  $d\dot{q}$  and  $d\alpha$  can have arbitrary values. By requiring the coefficients to be zero one gets

- $\frac{d\mathcal{W}_e}{dv} = q,$
- $\frac{d\mathcal{W}_e}{dx} = f_e,$
- $\frac{d\mathcal{W}_e}{d\alpha} = \tau_e,$
- $\frac{d\mathcal{W}_m}{d\dot{q}} = \lambda,$
- $\frac{d\mathcal{W}_m}{dx} = f_e,$
- $\frac{d\mathcal{W}_m}{d\alpha} = \tau_e,$

Depending on the characteristics of the system, the coupling can be obtained by the equations above and if the stored energy is known, the electrical and mechanical terminal relations can now be calculated. To differ from the magnetic coenergy and the electric energy,  $\mathcal{W}_m$  and  $\mathcal{W}_e$ , we represent the coupling term as  $\mathcal{W}^*$ .

### 3.2 Lagrangians for Electromechanical Systems

By simply grouping the mechanical kinetic coenergy  $T$  and potential energy  $V$  with the magnetic coenergy  $\mathcal{W}_m$  and electrical energy  $\mathcal{W}_e$  in the Lagrange's equations, a formulation for electromechanical system can be developed (Bishop, 2008). For a generalized coordinate  $z$  one gets

$$\frac{d}{dt} \left( \frac{\partial}{\partial \dot{z}_i} (T + \mathcal{W}_m) \right) - \frac{\partial}{\partial z_i} (T + \mathcal{W}_m) + \frac{\partial}{\partial z_i} (V + \mathcal{W}_e) = 0, \quad (47)$$

$$\frac{d}{dt} \left( \frac{\partial}{\partial \dot{z}_i} (T + \mathcal{W}_m) \right) - \frac{\partial}{\partial z_i} (T + \mathcal{W}_m - V - \mathcal{W}_e) = 0. \quad (48)$$

Introducing the Lagrangian  $\mathcal{L}$  for electromechanical systems

$$\mathcal{L} = T - V + \mathcal{W}_m - \mathcal{W}_e, \quad (49)$$

where the Lagrangian for mechanical systems  $L$  and electrical systems  $\mathcal{W}$  are defined by:

$$L = T - V, \quad (50)$$

$$\mathcal{W} = \mathcal{W}_m - \mathcal{W}_e. \quad (51)$$

As a result an extended Lagrangian is obtained:

$$\mathcal{L} = L + \mathcal{W}. \quad (52)$$

By substituting in equation 48 we get:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{z}_i} \right) - \frac{\partial \mathcal{L}}{\partial z_i} = 0. \quad (53)$$

Now, a generalized function  $Q$  must be added to the right side of the equation 53. A clever way to represent this generalized function is with an index  $i$ . For example, a  $Q_1$  could represent the imposed forces and the dissipation of the mechanical part and a  $Q_2$ , the imposed voltages and the dissipation of the electrical part. We define

$$Q_i = F_i - \frac{\partial D_i}{\partial \dot{z}_i}. \quad (54)$$

The dynamical equations for electromechanical systems are obtained from

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{z}_i} \right) - \frac{\partial \mathcal{L}}{\partial z_i} = Q_i, \quad (55)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{z}_i} \right) - \frac{\partial \mathcal{L}}{\partial z_i} = F_i - \frac{\partial D_i}{\partial \dot{z}_i}. \quad (56)$$

### 3.3 Step by Step: Lagrangians for Electromechanical Systems

#### [Step 1: Identifying the problem]

First step consists in identify how many mechanical and electrical variables the system needs to be described. In this paper the electrical part is described by the coordinate  $q$  (for electric charges) and the mechanical part is described by the coordinates  $x$  (for translational displacements) and  $\alpha$ ,  $\phi$ , or  $\theta$  (for angular displacements).

#### [Step 2: Electromechanical Coupling]

Step 2 consists in determining the electromechanical coupling of the system. Electromechanical couplings are not always evident. An identification of which terminals demonstrate interactions between subsystems seems to be a good start. After finding the coupling terminals, some other questions might need answers. It is important to know from what kind of energy the coupling is originated and what equation to use.

Couplings of electric origin can be identified by the equations

$$\frac{d\mathcal{W}_e^*}{dv} = q, \quad (57)$$

$$\frac{d\mathcal{W}_e^*}{dx} = f_e, \quad (58)$$

$$\frac{d\mathcal{W}_e^*}{d\alpha} = \tau_e. \quad (59)$$

Couplings of magnetic origin can be identified by the equations:

$$\frac{d\mathcal{W}_m^*}{d\dot{q}} = \lambda, \quad (60)$$

$$\frac{d\mathcal{W}_m^*}{dx} = f_e, \quad (61)$$

$$\frac{d\mathcal{W}_m^*}{d\alpha} = \tau_e. \quad (62)$$

After the coupling is obtained, it must be inserted in  $\mathcal{W}$ . The sign of the coupling will depend of its origin. If it is originated from a magnetic coupling field, the sign should be positive. If it is originated from an electric coupling field, the sign should be negative. The coupling must be added to its respective energy, namely

$$\mathcal{W} = (\mathcal{W}_m + \mathcal{W}^*) - \mathcal{W}_e \quad (\text{For a magnetic coupling}), \quad (63)$$

$$\mathcal{W} = \mathcal{W}_m - (\mathcal{W}_e + \mathcal{W}^*) \quad (\text{For an electrical coupling}). \quad (64)$$

### [Step 3: Kinetic Coenergy and Potential Energy]

Describe kinetic coenergy and potential energy of the mechanical part.

Translational and rotational kinetic coenergy are represented by  $T$ .

Gravitational and elastic potential energy are represented by  $V$ .

### [Step 4: Magnetic Coenergy and Electric Energy]

Describe magnetic coenergy and electrical energy of the non-coupled electrical part.

Magnetic coenergy is represented by  $\mathcal{W}_m$ .

Electrical potential energy is represented by  $\mathcal{W}_e$ .

### [Step 5: Lagrangian for Electromechanical Systems]

We write the Lagrangian:

$$\mathcal{L} = L + \mathcal{W}, \quad (65)$$

$$\mathcal{L} = T - V + \mathcal{W}_m - \mathcal{W}_e \pm \mathcal{W}^*. \quad (66)$$

### [Step 6: Generalized Forces]

We introduce the function  $Q$ .

Where  $D$  is a dissipative function for resistances and viscous frictions.  $F$  includes imposed forces and voltages. For each degree of freedom  $i$ , the  $Q$  function takes the form:

$$Q_i = F_i - \frac{\partial D_i}{\partial \dot{z}_i}. \quad (67)$$

### [Step 7: Lagrange's Equations]

Being  $z_i$  a generalized coordinate:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{z}_i} \right) - \frac{\partial \mathcal{L}}{\partial z_i} = F_i - \frac{\partial D_i}{\partial \dot{z}_i}. \quad (68)$$

## 4 DYNAMICS OF ELECTROMECHANICAL SYSTEMS

### 4.1 Scotch-Yoke Mechanism

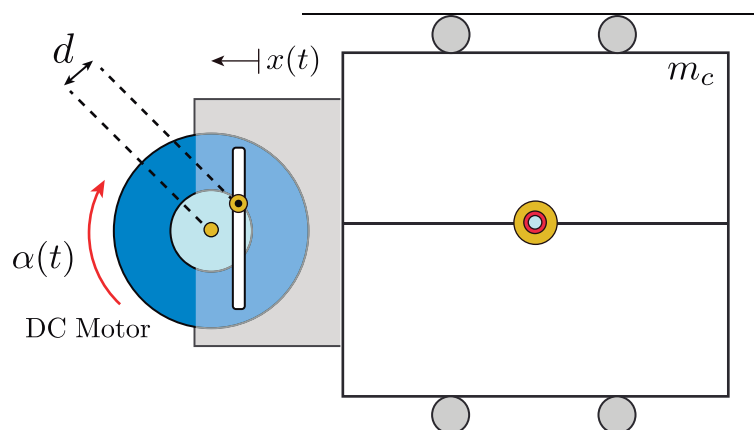


Figure 4: Cart-motor system with a scotch-yoke mechanism.

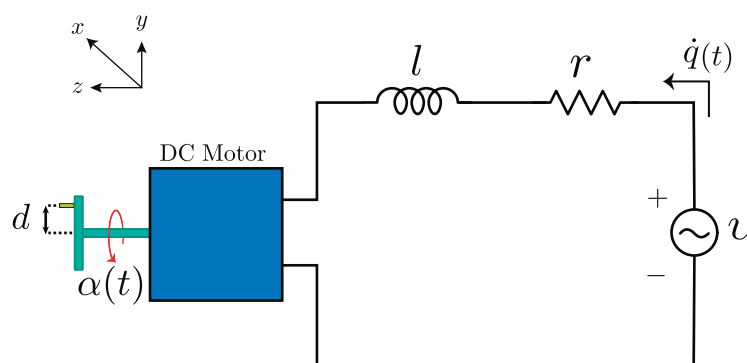


Figure 5: DC motor.

As previously stated, the first electromechanical system analysed in this paper is composed by a cart driven by a DC motor coupled by a mechanism called *scotch-yoke*. This system has been an object of study to analyze the behavior of electromechanical systems, see (Lima and Sampaio, 2016, 2012). Where  $v$  is the voltage source,  $\dot{q}$  is the electric current,  $\alpha$  is the angular displacement of the disk,  $l$  is the electric inductance and  $r$  is the electric resistance. The motor is coupled to the cart through a pin that slides into a slot machined in a plate that is attached to the cart,  $d$  is the distance related to the eccentricity of the pin. In this system, there are two coupling mechanisms. One of them is the electromechanical coupling. It exists when a disk is attached to a motor. The other is a mechanical coupling. It exists when a mechanical part is coupled to another mechanical part. In this case, the *scotch-yoke* mechanism is transforming the rotational movement of the disk in a translational movement of the cart.

### [Step 1: Identifying the problem]

This system has three movements involved. The rotational movement of the disk, the translational movement of the cart and the moving charges in the electrical part. It has one constraint due to the *scotch-yoke* mechanism. It has two degrees of freedom. The electrical part is described by the charge  $q$  and the mechanical part by the angular displacement  $\alpha$ .

### [Step 2: Electromechanical coupling]

For the electromechanical coupling, as it stores magnetic energy in the coupling terminal, it is interesting to analyse the equation

$$\frac{d\mathcal{W}_m^*}{d\alpha} = \tau_e. \quad (69)$$

In most DC motors, the torque  $\tau_e$  is proportional to the armature current  $\dot{q}$  and the strength of the magnetic field denoted by  $k_e$ , which is the motor electromagnetic force constant (Hughes and Drury, 2013).

$$\tau_e = k_e \dot{q}. \quad (70)$$

Substituting equation 70 in 69 gives

$$\frac{d\mathcal{W}_m^*}{d\alpha} = k_e \dot{q}, \quad (71)$$

$$d\mathcal{W}_m^* = k_e \dot{q} d\alpha. \quad (72)$$

Now, the magnetic energy stored in the terminals of the coupling equals the electromechanical coupling of the system. For  $\mathcal{W}_m^* = \mathcal{W}^*$ :

$$\mathcal{W}^* = k_e \dot{q} \alpha. \quad (73)$$

### [Step 3: Kinetic coenergy and potential energy]

$$T = \frac{j_m \dot{\alpha}^2}{2} + \frac{m_c \dot{x}^2}{2}. \quad (74)$$



Where  $j_m$  is the inertia moment of the motor. Besides the electromechanical coupling, there is another one. The coupling between the motor and the cart is made by a mechanism called *scotch yoke*, which transforms the rotational movement of the motor in the translation motion of the cart, leading to the relations

$$x = d\cos(\alpha), \quad (75)$$

$$\dot{x} = -d\sin(\alpha)\dot{\alpha}, \quad (76)$$

Substituting 76 in 74 gives

$$T = \frac{j_m\dot{\alpha}^2}{2} + \frac{m_c[-d\sin(\alpha)\dot{\alpha}]^2}{2}. \quad (77)$$

This system has no potential energy for the mechanical part, i. e.

$$V = 0. \quad (78)$$

#### [Step 4: Magnetic coenergy and electric energy]

For the electrical subsystem we write the magnetic coenergy:

$$\mathcal{W}_m = \frac{l\dot{q}^2}{2} + \mathcal{W}^*, \quad (79)$$

Substituting 73 in 79 leads to

$$\mathcal{W}_m = \frac{l\dot{q}^2}{2} + k_e\dot{q}\alpha, \quad (80)$$

This system has no electrical potential energy.

$$\mathcal{W}_e = 0, \quad (81)$$

#### [Step 5: Lagrangian for electromechanical systems]

$$\mathcal{L} = L + \mathcal{W}, \quad (82)$$

$$\mathcal{L} = T - V + \mathcal{W}_m - \mathcal{W}_e, \quad (83)$$

$$\mathcal{L} = \frac{j_m\dot{\alpha}^2}{2} + \frac{m_c[-d\sin(\alpha)\dot{\alpha}]^2}{2} + \frac{l\dot{q}^2}{2} + k_e\dot{q}\alpha. \quad (84)$$

**[Step 6: Generalized forces]**

For the mechanical coordinate  $\alpha$ :

$$F_1 = 0, \quad (85)$$

$$D_1 = \frac{b_m \dot{\alpha}^2}{2}, \quad (86)$$

$$Q_1 = F_1 - \frac{\partial D_1}{\partial \dot{\alpha}}, \quad (87)$$

$$Q_1 = -b_m \dot{\alpha}. \quad (88)$$

For the electrical coordinate  $q$ :

$$F_2 = v, \quad (89)$$

$$D_2 = \frac{r \dot{q}^2}{2}, \quad (90)$$

$$Q_2 = F_2 - \frac{\partial D_2}{\partial \dot{q}}, \quad (91)$$

$$Q_2 = v - r \dot{q}. \quad (92)$$

**[Step 7: Lagrange's equations]**

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \alpha} = Q_1, \quad (93)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = Q_2. \quad (94)$$

For equations 93 and 94, respectively:

$$\ddot{\alpha} [j_m + m_c d^2 \sin^2(\alpha)] + \dot{\alpha} [b_m + m_c d^2 \sin(\alpha) \cos(\alpha) \dot{\alpha}] - k_e \dot{q} = 0, \quad (95)$$

$$l \ddot{q} + r \dot{q} + k_e \dot{\alpha} = v, \quad (96)$$

## 4.2 Crank-Slider Mechanism

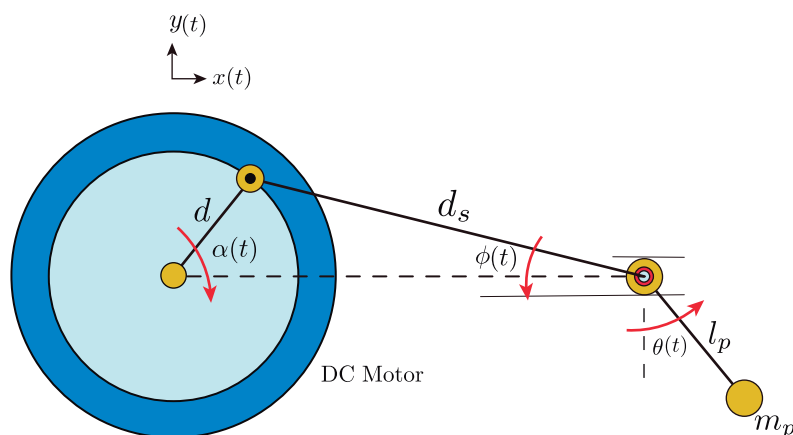


Figure 6: Electromechanical system with a crank-slider mechanism.

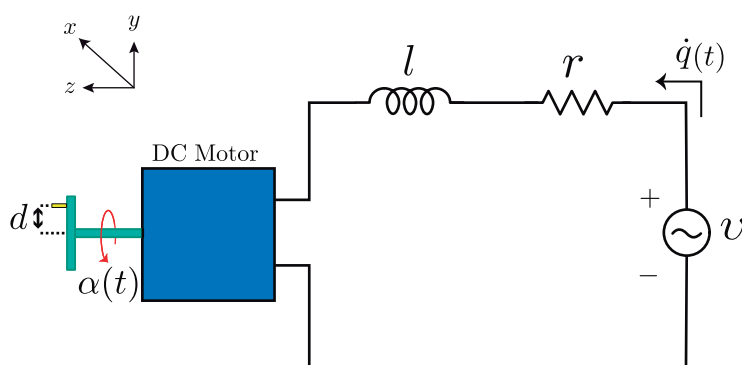


Figure 7: DC motor.

The second system analysed in this paper consists of a crank-slider mechanism linked to a pendulum of mass  $m_p$  and length  $l_p$ . There is not much difference to the previous system when it comes to the electrical subsystem. The coordinates of the electrical subsystem are the same. The particle with no mass in the mechanical part substitutes the cart. This system has also a DC motor. There are viscous frictions in the disk coupled to the motor ( $b_m$ ), in the displacement of the particle with no mass ( $b_c$ ) and in the movement of the pendulum ( $b_p$ ). It differs from the first in the choice of coupling mechanism. The distance related to the eccentricity of the crank is  $d$ . The length of the shaft is  $d_s$ . The angles in the crank and in the shaft are given by  $\alpha(t)$  and  $\phi(t)$ , respectively. The angular displacement of the pendulum is given by  $\theta(t)$ .

### [Step 1: Identifying the problem]

This system has four movements involved. The rotational movement of the disk, the rotational movement of the shaft, the rotational movement of the pendulum and the moving charges in the electrical part. It has one constraint due to the *crank-slider* mechanism. It has three degrees of freedom. The electrical part is described by the charge  $q(t)$  and the mechanical part by the angular displacements  $\alpha(t)$  and  $\theta(t)$ .

**[Step 2: Electromechanical coupling]**

Comparing to the first system analysed, the first coupling do not change:

$$\mathcal{W}^* = k_e \dot{q} \alpha. \quad (97)$$

**[Step 3: Kinetic coenergy and potential energy]**

For the kinetic coenergy and the potential energy of the mechanical part:

$$T = \frac{j_m \dot{\alpha}^2}{2} + \frac{m_p (\dot{x}^2 + \dot{y}^2)}{2}, \quad (98)$$

besides the electromechanical coupling, there is another one made by the *crank-slider* mechanism. First, we begin by writing the position of the pendulum:

$$x = d \cos(\alpha) + d_s \cos(\phi) + l_p \sin(\theta), \quad (99)$$

$$y = -l_p \cos(\theta). \quad (100)$$

The angles  $\alpha$  and  $\phi$  are related by

$$\cos(\phi) = \left( 1 - \frac{d^2}{d_s^2} \sin(\alpha)^2 \right)^{\frac{1}{2}}. \quad (101)$$

Substituting in the equation and deriving in time gives

$$\dot{x} = -d\dot{\alpha} \left( \sin(\alpha) + \frac{d \sin(\alpha) \cos(\alpha)}{d_s \left( 1 - \frac{d^2}{d_s^2} \sin(\alpha)^2 \right)^{\frac{1}{2}}} \right) + l_p \dot{\theta} \cos(\theta), \quad (102)$$

$$\dot{y} = l_p \dot{\theta} \sin(\theta), \quad (103)$$

introducing the function  $\mathcal{K}(\alpha)$ :

$$\mathcal{K}(\alpha) = \sin(\alpha) + \frac{d \sin(\alpha) \cos(\alpha)}{d_s \left( 1 - \frac{d^2}{d_s^2} \sin(\alpha)^2 \right)^{\frac{1}{2}}}. \quad (104)$$

As a result, it is possible to simplify  $\dot{x}$ . For the horizontal and vertical velocities of the pendulum:

$$\dot{x} = -d\dot{\alpha} \mathcal{K} + l_p \dot{\theta} \cos(\theta), \quad (105)$$

$$\dot{y} = l_p \dot{\theta} \sin(\theta). \quad (106)$$

Rewriting the kinetic coenergy:

$$T = \frac{j_m \dot{\alpha}^2}{2} + \frac{m_p}{2} [(l_p \dot{\theta} \cos(\theta) - d \dot{\alpha} \mathcal{K})^2 + (l_p \dot{\theta} \sin(\theta))^2]. \quad (107)$$

For the potential energy of the system we write the gravitational potential energy:

$$V = -m_p g l_p \cos(\theta). \quad (108)$$

#### [Step 4: Magnetic coenergy and electric energy]

For the magnetic coenergy of the system:

$$\mathcal{W}_m = \frac{l \dot{q}^2}{2} + \mathcal{W}^*, \quad (109)$$

$$\mathcal{W}_m = \frac{l \dot{q}^2}{2} + k_e \dot{q} \alpha. \quad (110)$$

For the electric potential energy of the system:

$$\mathcal{W}_e = 0. \quad (111)$$

#### [Step 5: Lagrangian for electromechanical systems]

$$\mathcal{L} = L + \mathcal{W}, \quad (112)$$

$$\mathcal{L} = T - V + \mathcal{W}_m - \mathcal{W}_e, \quad (113)$$

$$\mathcal{L} = \frac{j_m \dot{\alpha}^2}{2} + \frac{m_p}{2} [(l_p \dot{\theta} \cos(\theta) - d \dot{\alpha} \mathcal{K})^2 + (l_p \dot{\theta} \sin(\theta))^2] + m_p g l_p \cos(\theta) + \frac{l \dot{q}^2}{2} + k_e \dot{q} \alpha. \quad (114)$$

#### [Step 6: Generalized forces]

For the mechanical coordinate  $\alpha$ :

$$F_1 = 0, \quad (115)$$

$$D_1 = \frac{b_c (-d \dot{\alpha} \mathcal{K})^2}{2} + \frac{b_m \dot{\alpha}^2}{2}, \quad (116)$$

$$Q_1 = F_1 - \frac{\partial D_1}{\partial \dot{\alpha}}, \quad (117)$$

$$Q_1 = -b_c d^2 \mathcal{K}^2 \dot{\alpha} - b_m \dot{\alpha}. \quad (118)$$

For the mechanical coordinate  $\theta$ :

$$F_2 = 0, \quad (119)$$

$$D_2 = \frac{b_p((l_p\dot{\theta}\sin(\theta))^2 + (l_p\dot{\theta}\sin(\theta))^2)}{2}, \quad (120)$$

$$Q_2 = F_2 - \frac{\partial D_2}{\partial \dot{\theta}}, \quad (121)$$

$$Q_2 = -b_p l_p^2 \dot{\theta}. \quad (122)$$

For the electrical coordinate  $q$ :

$$F_3 = v, \quad (123)$$

$$D_3 = \frac{r\dot{q}^2}{2}, \quad (124)$$

$$Q_3 = F_3 - \frac{\partial D_3}{\partial \dot{q}}, \quad (125)$$

$$Q_3 = v - r\dot{q}. \quad (126)$$

**[Step 7: Lagrange's equations]**

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \alpha} = Q_1, \quad (127)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_2, \quad (128)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = Q_3. \quad (129)$$

For equations 127, 128 and 129, respectively:

$$\ddot{\alpha}[j_m + m_p d^2 \mathcal{K}^2] + \dot{\alpha}[b_m + b_c d^2 \mathcal{K}^2 + m_p d^2 \mathcal{K} \dot{\mathcal{K}}] + \ddot{\theta}[-m_p l_p d \cos(\theta) \mathcal{K}] + \dot{\theta}[m_p l_p d \sin(\theta) \dot{\mathcal{K}}] - k_e q = 0, \quad (130)$$

$$\ddot{\theta}[m_p l_p^2] + \dot{\theta}[b_p l_p^2] + \ddot{\alpha}[-m_p l_p d \cos(\theta) \dot{\mathcal{K}}] + \dot{\alpha}[-m_p l_p d \cos(\theta) \mathcal{K}] + m_p l_p g \sin(\theta) = 0, \quad (131)$$

$$l\ddot{q} + r\dot{q} + k_e \dot{\alpha} = v. \quad (132)$$

## 5 A COMMON MISTAKE IN THE LAGRANGIAN FORMULATION FOR ELECTROMECHANICAL SYSTEMS

Many references claim to write a proper Lagrangian for electromechanical systems, please see (Belato et al., 2001; Avanço et al., 2017; Cveticanin et al., 2018; Avanço, 2015; Belato, 2002) by adding a torque, said to be imposed, to the generalized forces in the Lagrangian's equations. This paper shows this is not correct and explains the mistake. The torque added wrongly in the equations is arbitrary, it depends only on the coordinates of the system and it must not be added as a generalized force. The standard derivations of the dynamical equation of purely mechanical systems do not apply in this situation. Also, the derivation of a Lagrangian that only provides mechanical terms of an electromechanical system is already an error. There must be only one Lagrangian to describe the whole system. As an example of a wrong method, take this solution proposed by the recent literature (Avanço et al., 2018) for the very last electromechanical system analysed in this paper:

**[Error 1: Formulates a Lagrangian only for the mechanical part]**

$$T = \frac{j_m \dot{\alpha}^2}{2} + \frac{m_p}{2} [(l_p \dot{\theta} \cos(\theta) - d \dot{\alpha} \mathcal{K})^2 + (l_p \dot{\theta} \sin(\theta))^2], \quad (133)$$

$$V = -m_p g l_p \cos(\theta). \quad (134)$$

Presents only two Lagrange's equations ignoring the electrical part:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q_1, \quad (135)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \alpha} = Q_2. \quad (136)$$

**[Error 2: Adds an arbitrary torque to the generalized forces]**

$$Q_1 = -b_c d^2 \mathcal{K}^2 \dot{\alpha} + M_{motor}(\dot{q}, \dot{\alpha}). \quad (137)$$

Where  $M_{motor}$  is a function:

$$M_{motor}(\dot{q}, \dot{\alpha}) = \tau_e(\dot{q}) - b_m \dot{\alpha}. \quad (138)$$

In (Avanço et al., 2018) the torque  $M_{motor}$  is given by equations (20) and (21) (the equation numbers in this paragraph correspond to (Avanço et al., 2018)), so as a solution of a coupled system because (20) cannot be solved alone. With the hypothesis made, one gets (22) and (23) uncoupling the system. Now  $M_{motor}$  is only function of the coordinates and the initial condition associated to (20) is lost. We stress the main point, when the hypothesis that the torque is only function of the generalized coordinates is made the initial condition is lost, the electrical system is uncoupled, and now depends of the mechanical system. This mistake is a common one and, unfortunately, it pervades the literature. The mistake has a bonus, if one uncouples the system it is easy to do a lot of *computations* since now one has only an ode to solve. The reference (Lima and Sampaio, 2018) shows the mistake clearly.

## 6 CONCLUSIONS

The objective of this paper was to formulate a Lagrangian for electromechanical systems. It had also the purpose to correct common mistakes in formulations. It is not possible to deal with electromechanical systems without knowing the electrical part. This incorrect method consists in mimicking the standard derivations of purely mechanical systems to electromechanical systems. There are many references that say they are able to find the equations of electromechanical systems through a Lagrangian. Frequently, they solve problems only for the mechanical part or make mistakes in the formulation that leads to the correct solution. When systems composed by different parts are analysed, the term that makes the connection between the two must be discussed. The electromechanical coupling is an essential tool to formulate a Lagrangian. To introduce the electromechanical coupling, it was necessary to present the difference between energy and coenergy and how energy is transferred from one place to another in an electromechanical model.

## 7 ACKNOWLEDGMENTS

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