



HAL
open science

A Discrete Integral Sliding Manifold for a Nonlinear System with Time Delay: An Event-Triggered Scheme

Yassine Benyazid, Mohamed Fnadi, Ahmed Said Nouri

► To cite this version:

Yassine Benyazid, Mohamed Fnadi, Ahmed Said Nouri. A Discrete Integral Sliding Manifold for a Nonlinear System with Time Delay: An Event-Triggered Scheme. *Mathematics*, 2023, 11 (10), pp.2326. 10.3390/math11102326 . hal-04102138

HAL Id: hal-04102138

<https://hal.science/hal-04102138v1>

Submitted on 22 May 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Article

A Discrete Integral Sliding Manifold for a Nonlinear System with Time Delay: An Event-Triggered Scheme

Yassine Benyazid ^{1,*} , Mohamed Fnadi ¹  and Ahmed Said Nouri ² 

¹ Laboratoire d'Informatique Signal et Image de la Côte d'Opale (LISIC), University Littoral Côte d'Opale, UR 4491, F-62100 Calais, France

² Research Laboratory of Numerical Control of Industrial Processes, National Engineering School of Gabes, University of Gabes, Gabes 6072, Tunisia

* Correspondence: yassine.benyazid@univ-littoral.fr

Abstract: This paper presents a new approach to integral sliding mode control for discrete nonlinear systems with time delay. The approach is based on an event-triggered scheme and is applied to Takagi–Sugeno fuzzy models. In the first step, a new integral sliding function is constructed, which avoids the limited assumptions of most existing fuzzy sliding mode control schemes. The design parameter matrices defining the sliding surface are obtained by solving linear matrix inequalities. In the second step, an event trigger-based integral sliding mode control protocol is developed to ensure the state trajectories of the Takagi–Sugeno fuzzy systems with time delays. Finally, the proposed strategies are evaluated through a simulation example to demonstrate their effectiveness.

Keywords: event-triggered control; integral sliding manifold; fuzzy system; time-varying delay; linear matrix inequalities

MSC: 93B12; 93D20



Citation: Benyazid, Y.; Fnadi, M.; Nouri, A.S. A Discrete Integral Sliding Manifold for a Nonlinear System with Time Delay: An Event-Triggered Scheme. *Mathematics* **2023**, *11*, 2326. <https://doi.org/10.3390/math11102326>

Academic Editors: Chang-Hua Lien, Hamid Reza Karimi and Sundarapandian Vaidyanathan

Received: 12 April 2023

Revised: 9 May 2023

Accepted: 13 May 2023

Published: 16 May 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

In recent years, T-S fuzzy models have gained widespread attention for their effectiveness in system control [1–3]. Takagi–Sugeno (T-S) models typically consist of linear models connected together which are enclosed by nonlinear membership functions. These membership functions rely on the premise variables. These models offer a flexible framework for approximating complex systems with the ability to incorporate linguistic variables, making them an appealing choice for control applications. In this context, the multi-model approach is used to describe the nonlinear behavior of physical systems as accurately as possible. In the majority of cases, the stability and stabilization issues of T-S fuzzy models have been analyzed through the direct Lyapunov method, which has been thoroughly investigated for the past 20 years [4,5]. Although the original T-S model formulation does not account for significant delays in the input, output, or system state, several recent research endeavors have focused on studying nonlinear systems with time delay.

Time delay is often considered the primary cause of system instability and poor performance, making its analysis one of the most challenging issues in the control of dynamic systems. To study these systems, a range of approaches have been developed, such as synthesizing control laws [6,7], stability analysis [8–10], and designing observers [11–14]. Traditional control techniques for time delay systems, such as PID control and state and output feedback control, have garnered the attention of numerous researchers [15–17]. The latter can be ineffective when the system model does not accurately describe the real system behavior due to the existence of time delay, varying parameters, and external disturbances. Then, it is advisable to resort to robust control techniques. One of the techniques, which is easy to implement and for obtaining robust control laws, is sliding mode control (SMC).

SMC is a control technique that was introduced in the early 1980s, and it involves a two-step process. The first step involves constructing a sliding surface, while the second step entails designing a feedback control law to ensure that the system trajectory converges to the sliding surface within a specified time. Many researchers have focused on solving the SMC problem in T-S fuzzy systems with time delay, which has led to several proposed approaches [18–20]. However, some of these approaches have limitations, such as the assumption that the input matrices of all linear local models must be equal [18,19,21]. This limited assumption is not suitable for various real systems, including the inverted pendulum [22]. On the other hand, an ISMC scheme was proposed in [20] which identified two necessary assumptions for developing sliding mode controllers for T-S fuzzy systems. The first assumption is that the input matrices of all models in the T-S fuzzy system must be identical, and the second assumption is that the product of a parameter matrix in the sliding variable and the diffusion matrices of all linear models must be zero. In order to remove this restrictive assumption, a novel integral sliding surface is proposed in this work.

The rapid evolution of information technology has rendered conventional control systems inadequate in meeting actual production needs, particularly in the industrial field. To address this problem, network control systems have become essential, as they offer benefits such as easy resource sharing, simple installation and maintenance, and quick fault diagnosis. However, hardware and network technology limitations may lead to issues such as data packet loss, time delays, and communication limitations, which greatly affect system behavior and make it challenging to analyze control systems comprehensively. As a solution to these issues, and to make the best use of network resources, the event-triggered (ET) technique has garnered considerable attention in the literature. The ET framework is carried out only when the trigger condition is fulfilled. In fact, this technique is characterized by a reasonable and effective use of resources, and the extended use time of the controller is reduced. Several works have been developed the T-S fuzzy system based the ET mechanism, such as the finite-time H_∞ control problem for switched fuzzy systems [23], observer-based ET control for T-S fuzzy system [24], and fault-tolerant, control-based aperiodic adaptive event triggered for nonlinear systems [25]. Furthermore, several researchers have been interested in the event-trigger-based sliding mode control (ET-SMC) approach, where the latter decreases the chattering caused by the discontinuous term. In [26], the author investigated the ET mechanism based on novel dynamic SMC for continuous-time nonlinear systems. The design of event trigger-based ISMC for a T-S fuzzy model was studied in [27]. In this paper, we employ the event detector method introduced in [28], which aims to reduce network burdens and conserve communication resources, compared with the schemes proposed in [29,30]. In addition, this method has the potential to save transmission energy in wireless networks, leading to a longer battery lifespan for nodes.

This study was motivated by the aforementioned work, and it aims to propose a discrete event trigger-based integral sliding mode control (ET-ISMC) approach for a nonlinear system represented by coupled multi-models. The main contributions of this article can be summarized as follows:

1. We propose a novel, discrete integral sliding manifold for nonlinear systems to avoid the restrictive assumptions in most existing T-S fuzzy SMC methods.
2. The event-triggered scheme is combined with the ISMC design approach. In this scheme, data in the system are transmitted only when the trigger condition is met, which prevents redundant information, saves valuable network resources, and reduces the chattering phenomenon.

The rest of this paper is organized as follows. The description of the system model is given in Section 2. Section 3 focuses on the construction of novel integral sliding manifolds. In Section 4, the design of event-triggered communication is presented. The control law is synthesized in Section 5. A simulation example is used to illustrate the effectiveness of the proposed approaches in Section 6. Finally, the paper is concluded in Section 7.

Notations: For a matrix, A^{-1} and A^T denote the inverse and transpose of matrix A , respectively, and $\text{sym}\{A\}$ stands for the expression $A + A^T$. In addition, $\text{diag}\{\dots\}$ represents a block-diagonal matrix, and the symbol $*$ stands for the symmetric term in a symmetric block matrix. S^n denotes the set of $n \times n$ real symmetric positive definite matrices. \mathbb{R}^n stands for the n -dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices. Any symmetric matrix means $P = P^T$, and $P > 0$ (≥ 0) represents a positive (positive semi) definite matrix; that is, $x^T P x > 0$ (≥ 0) for all $x \in \mathbb{R}^n$, and $\|\cdot\|$ is the Euclidean norm of a vector and its induced norm of a matrix.

2. Problem Formulation

The given system can be described as a T-S fuzzy model with r plant rules, where the system is discrete and includes a time-varying delay:

Plant Rule i : IF $\alpha(t)^1$ is μ_1^i , and \dots , and $\alpha(t)^p$ is μ_p^i , THEN

$$\begin{cases} x(t+1) = A_i x(t) + A_{\tau_i} x(t - \tau(t)) + B_i u(t) \\ \quad + D_i d(t) \\ y(t) = C_i x(t) + C_{\tau_i} x(t - \tau(t)) \\ x(t) = \psi(t), \quad \forall t = -\tau_M, -\tau_M + 1, \dots, -\tau_m \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$, and $d(t) \in \mathbb{R}^p$ represent the state vector, control input, measured output, and exogenous disturbance, respectively. The initial state is represented by the variable $\psi(t)$, and the time-varying delay is denoted by $\tau(t)$, which must satisfy the condition that $\tau_m \leq \tau(t) \leq \tau_M$. The matrices $A_i \in \mathbb{R}^{n \times n}$, $A_{\tau_i} \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{q \times n}$, $C_{\tau_i} \in \mathbb{R}^{q \times n}$, and $D_i \in \mathbb{R}^{p \times n}$ have appropriate dimensions for the i th local model such that $i = 1, 2, \dots, r$, where r represents the number of plant rules. Additionally, $\alpha(t) = [\alpha^1(t), \dots, \alpha^p(t)]$ represents the measurable premise variables, and μ_j^i represents the number of fuzzy sets, with j ranging from 1 to p . The fuzzy basis functions can be given by

$$m_i(\alpha(t)) = \frac{\prod_{j=1}^p \mu_j^i(\alpha^j(t))}{\sum_{i=1}^r \prod_{j=1}^p \mu_j^i(\alpha^j(t))} \tag{2}$$

where $\mu_j^i(\alpha^j(t))$ represents the grade of membership of $\alpha^j(t)$ in μ_j^i . In addition, for all t , we have $m_i(\alpha(t)) \geq 0$ ($i = 1, 2, \dots, r$), and $\sum_{i=1}^r m_i(\alpha(t)) = 1$.

The dynamic fuzzy model in Equation (1) can be expressed using the regular fuzzy inference methodology described in [31]. This methodology involves adopting a singleton fuzzifier, weighted average defuzzifier, and product fuzzy inference. The resulting global model can be represented as follows:

$$\begin{cases} x(t+1) = \sum_{i=1}^r m_i(\alpha(t)) [A_i x(t) + A_{\tau_i} x(t - \tau(t)) \\ \quad + B_i u(t) + D_i d(t)] \\ y(t) = \sum_{i=1}^r m_i(\alpha(t)) [C_i x(t) + C_{\tau_i} x(t - \tau(t))] \\ x(t) = \psi(t), \quad \forall t = -\tau_M, -\tau_M + 1, \dots, -\tau_m \end{cases} \tag{3}$$

For simplicity, we will replace $m_i(\alpha(t))$ with m_i , and for a series of matrices Y_i ($i = 1, \dots, r$), the single and double sums can be written as $Y_m = \sum_{i=1}^r m_i Y_i$ and $Y_{mm} = \sum_{i=1}^r m_i \sum_{j=1}^r m_j Y_i Y_j$, respectively. Then, Equation (3) can describe them as follows:

$$\begin{cases} x(t+1) = A_m x(t) + A_{\tau_m} x(t - \tau(t)) + B_m u(t) \\ \quad + D_m d(t) \\ y(t) = C_m x(t) + C_{\tau_m} x(t - \tau(t)) \end{cases} \tag{4}$$

Note that in the fuzzy SMC scheme proposed in [18,19,21], the below assumptions are required:

- A1: The input matrices B_i ($i = 1, \dots, r$) must be equal.
- A2: There must exist a matrix M such that $\det(MB_i) \neq 0$.

These assumptions constrain the application of the methods. In Section 6, we demonstrate that the model which describes the balancing of the inverted pendulum on a cart fails to satisfy both of the above assumptions. In this context, the present methods in [18,19,21] are not applicable. One contribution of this work is the development of a suitable sliding mode scheme for T-S fuzzy systems. Notably, the proposed scheme does not rely on the two assumptions mentioned earlier.

To fulfill this aim, the following lemmas are required for the study of the stabilization and stability criteria:

Lemma 1 ([32]). *For a given symmetric positive definite matrix $R \in \mathbb{S}^n$, integers $d_1 \leq d_2$, and vector function $\mu : \{d_1, d_1 + 1, \dots, d_2\} \rightarrow \mathbb{R}^n$ such that the following sum is well defined, then*

$$d \sum_{h=-d_2+1}^{-d_1} \mu^T(h)R\mu(h) \leq \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3\lambda(d)R \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \tag{5}$$

where $\lambda(d) = \left(\frac{d+1}{d-1}\right)$ and

$$\begin{cases} \mu(h) = \mu(h+1) - \mu(h), & d = d_2 - d_1 \\ \zeta_1 = x(-d_1) - x(-d_2) \\ \zeta_2 = x(-d_1) + x(-d_2) - \frac{2}{d+1} \sum_{h=-d_2}^{-d_1} x(h) \end{cases}$$

In some cases with time-varying delays, the factor $\left(\frac{d+1}{d-1}\right)$ might raise some computational burden. Thus, the following alternative lemma is introduced:

Lemma 2 ([32]). *For a given symmetric positive definite $R \in \mathbb{S}^n$, integers $d_1 \leq d_2$, and vector function $\mu : \{d_1, d_1 + 1, \dots, d_2\} \rightarrow \mathbb{R}^n$ such that the following sum is well defined, then*

$$d \sum_{h=-d_2+1}^{-d_1} \mu^T(h)R\mu(h) \leq \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \tag{6}$$

where $\mu(h)$, ζ_1 and ζ_2 are defined in Lemma 1.

Lemma 3 ([33]). *Given two matrices $\mathbb{Z}_1 \in \mathbb{S}^{n_1}$ and $\mathbb{Z}_2 \in \mathbb{S}^{n_2}$, if there exists a matrix $\mathbb{N} \in \mathbb{R}^{n_1 \times n_2}$, then the inequality*

$$\begin{bmatrix} \frac{1}{\omega} \mathbb{Z}_1 & 0 \\ 0 & \frac{1}{1-\omega} \mathbb{Z}_2 \end{bmatrix} \geq \begin{bmatrix} \mathbb{Z}_1 & \mathbb{N} \\ * & \mathbb{Z}_2 \end{bmatrix}$$

holds with any scalar $\omega \in (0, 1)$ such that

$$\begin{bmatrix} \mathbb{Z}_1 & \mathbb{N} \\ * & \mathbb{Z}_2 \end{bmatrix} \geq 0$$

where $(*)$ denotes the transpose.

3. Discrete Fuzzy Integral Sliding Surface Design

In the first example, a discrete integral sliding manifold is introduced to avoid the limited assumptions A1 and A2 for the T-S fuzzy system in Equation (3). Therefore, the integral sliding manifold is given as follows:

$$\begin{cases} S(t) = Mx(t) - Mx(0) - \beta(t) \\ \beta(t) = \beta(t - 1) - \delta(t - 1) \end{cases} \tag{7}$$

where $\delta(t)$ is given by

$$\delta(t) = M(A_m + B_m K_m - I)x(t) + M A_{\tau m} x(t - \tau(t)) \tag{8}$$

where $K_m \triangleq \sum_{i=1}^r m_i K_i$ is the real matrix to be conceived and $M \in \mathbb{R}^{m \times n}$ is the parameter matrix that ensures the non-singularity of MB_m .

For this purpose, the approach in [34] can be used. By defining $B = \frac{1}{r} \sum_{i=1}^r B_i$ it follows that

$$B_m = B + HF(\bar{m}(\alpha(t)))G \tag{9}$$

where $\bar{m} = [m_1, m_2, \dots, m_r]$ and

$$H = \frac{1}{2}[B - B_1, B - B_2, \dots, B - B_r], F(\bar{m}) = \text{diag}\{(1 - 2m_1)I, (1 - 2m_2)I, \dots, (1 - 2m_r)I\}, G = \underbrace{[I, I, \dots, I]}_r^T$$

Hence, the following result can be inferred from the method in [34]:

Lemma 4 ([34]). *There exists Q, g_1, g_2, g_3 with $Q > 0$ such that the following LMIs hold:*

$$\begin{aligned} \begin{bmatrix} -I & * \\ g_1 H & -I \end{bmatrix} < 0, \quad \begin{bmatrix} Q & * \\ I & g_2 I \end{bmatrix} > 0, \quad Q < g_3 I \\ \begin{bmatrix} 2g_1 \sqrt{\lambda_{\min}(B^T B)} & * & * \\ rg_2 & rg_1 & * \\ rg_3 & 0 & rg_1 \end{bmatrix} > 0 \end{aligned} \tag{10}$$

Then, there exists a parameter matrix

$$M = (B^T Q^{-1} B)^{-1} B^T Q^{-1}$$

such that MB_m is non-singular.

Remark 1.

- The matrix B can be selected as the convex combination of B_i such that $i = 1, \dots, r$; that is, $B = \sum_{i=1}^r \zeta_i B_i$ with $\zeta_i > 0$ and $\sum_{i=1}^r \zeta_i = 1$. From the property of convex combinations, it follows that if only one of the B_i is non-singular, then there should be a collection of scalars $\zeta_i, i = 1, 2, \dots, r$ such that the non-singularity of B can be ensured. In this situation, set

$$H_c = \frac{1}{2}[B - r\zeta_1 B_1, B - r\zeta_2 B_2, \dots, B - r\zeta_r B_r, 2\zeta I],$$

$$F_c(\bar{m}) = \text{diag}\{(1 - 2m_1)I, (1 - 2m_2)I, \dots,$$

$$(1 - 2m_r)I, \frac{1}{\zeta} \sum_{i=1}^r m_i (1 - r\zeta_i) B_i\} G_c = [G^T \quad I]^T$$

$$\text{where } \zeta = \left\| \sum_{i=1}^r m_i(1 - r\zeta_i)B_i \right\|.$$

It can be demonstrated that $B_m = B + H_1F_1(\bar{m}(\alpha(t)))G_1$. Thus, the result in Lemma 3 also applies to H, G , and r being replaced by H_c, G_c , and $r + 1$, respectively.

- Let us note that when $B_1 = B_2 = \dots = B_r$, by choosing $Q = I$, and without solving the LMIs in Equation (10), the parameter matrix M can be given as $M = (B^T B)^{-1} B^T$, since it was proven in [35] that this set is optimal in the sense that the Euclidean norm of the mismatched disturbances is minimized.

The SMC strategy aligns the system state trajectories with the predefined sliding manifolds, leading to satisfaction of the ideal sliding surface for the nonlinear discrete-time system. Specifically, this condition can be expressed by the following equation:

$$\Delta S(t) = S(t + 1) - S(t) \tag{11}$$

Then, we obtain

$$\Delta S(t) = MB_m u(t) + MD_m d(t) - MB_m K_m x(t) = 0 \tag{12}$$

Since MB_m is non-singular, the equivalent control can be obtained as follows:

$$u_{eq}(t) = K_m x(t) - MD_m d(t) \tag{13}$$

4. Design of the Event-Triggered Communication Scheme

In this work, the T-S fuzzy system dynamics (Equation (4)) with event-triggered control is proposed as follows:

$$x(t + 1) = A_m x(t) + A_{\tau m} x(t - \tau(t)) + B_m u(t_s) + D_m d(t) \tag{14}$$

for $t \in [t_s, t_{s+1})$, where t_s and t_{s+1} represent the triggering instants and the next triggering instants, respectively. Then, the following calculation of the next triggering instants with the event-triggered condition is proposed:

$$t_{s+1} = \inf\{t > t_s \mid h(t) > 0\} \tag{15}$$

such that

$$h(t) = e^T(t)\Lambda e(t) - \varepsilon x^T(t)\Lambda x(t)$$

where $e(t) = x(t_s) - x(t)$ is the transmission error, ε represents the threshold value, for which $\varepsilon \in [0, 1)$, and Λ is a positive definite weighting matrix with an undetermined value.

By referring to Equation (15), it is evident that no event will be triggered for all values of t belonging to the interval $[t_s, t_{s+1})$. Consequently, we can deduce the following outcome:

$$e^T(t)\Lambda e(t) \leq \varepsilon x^T(t)\Lambda x(t) \tag{16}$$

We can express the equivalent sliding mode controller, which takes into account the event-triggered control strategy, in a new form:

$$u_{eq}(t_s) = K_m x(t_s) - MD_m d(t) \tag{17}$$

By replacing Equation (14) with the equivalent SMC law (Equation (17)), we can derive the sliding mode dynamics:

$$x(t + 1) = A_m x(t) + A_{\tau m} x(t - \tau(t)) + B_m K_m x(t_s) + (I - B_m M)D_m d(t) \tag{18}$$

Based on Equations (15) and (17), Equation (18) can be rewritten as follows:

$$x(t + 1) = A_c x(t) + A_{\tau m} x(t - \tau(t)) + B_c e(t) + D_c d(t) \tag{19}$$

where

$$A_c = A_m + B_m K_m, B_c = B_m K_m, D_c = (I - B_m M) D_m$$

Theorem 1. *The closed loop of the system in Equation (19) is asymptotically stable with an \mathcal{H}_∞ performance λ if there exist $\tilde{P}, \tilde{\Lambda}, \tilde{Q}_l, \tilde{R}_l$ (with $l = 1, 2$), and $\tilde{K}_{xj} \in R^{n \times m}, j = 1, \dots, r$ such that the following matrix inequalities hold ($s = 1, 2$).*

Minimize λ , subject to

$$\begin{aligned} \tilde{\Xi}_{ii}^{(s)} &< 0, \quad i = 1, \dots, r \\ \tilde{\Xi}_{ij}^{(s)} + \tilde{\Xi}_{ji}^{(s)} &< 0, \quad 1 \leq i < j < r \end{aligned} \tag{20}$$

such that

$$\tilde{\Xi}_{ij}^{(s)} = \begin{bmatrix} \tilde{\Xi}_{1ij}^{(s)} & \tilde{\Xi}_{2ij}^{(s)} \\ * & -I \end{bmatrix} \tag{21}$$

where

$$\begin{aligned} \tilde{\Xi}_{1ij}^{(s)} = & \text{sym}(\tilde{\Psi}_{ij}) + Y_1^{(s)} \tilde{P} Y_1^{(s)T} - Y_2^{(s)} \tilde{P} Y_2^{(s)T} + Y_3 \tilde{Q} Y_3^T \\ & - Y_4 \tilde{\Omega}_1 Y_4^T - Y_5 \tilde{\mathbb{R}} Y_5^T + Y_6 \tilde{\Phi} Y_6^T + \epsilon q_2^T \tilde{\Lambda} q_2 - q_9^T \tilde{\Lambda} q_9 \\ & - q_{10}^T \lambda^2 q_{10} \end{aligned} \tag{22}$$

$$\tilde{\Xi}_{2ij} = q_2^T \tilde{X} C_i^T + q_4^T \tilde{X} C_{\tau i}^T$$

and

$$\tilde{\Psi} = q_1^T [\tilde{X} q_1 + \tilde{\mathbb{A}}_c q_2 + \tilde{\mathbb{A}}_\tau q_4 + \tilde{\mathbb{B}}_c q_9] \tag{23}$$

with

$$\begin{cases} Y_1^{(1)} = [q_1 & \tau_1 q_6 - q_3 & q_7 + \bar{\tau} q_8 - q_4 - q_5], \\ Y_1^{(2)} = [q_1 & \tau_1 q_6 - q_3 & \bar{\tau} q_7 + q_8 - q_4 - q_5], \\ Y_2^{(1)} = [q_2 & \tau_1 q_6 - q_2 & q_7 + \bar{\tau} q_8 - q_3 - q_4], \\ Y_2^{(2)} = [q_2 & \tau_1 q_6 - q_2 & \bar{\tau} q_7 + q_8 - q_3 - q_4], \\ Y_3 = [q_2 & q_3 & q_5], \\ Y_4 = [q_2 - q_3 & q_2 + q_3 - 2q_6], \\ Y_5 = [q_3 - q_4 & q_3 + q_4 - 2q_7 & q_4 - q_5 \\ & q_4 + q_5 - 2q_8], \\ Y_6 = [q_1 & q_2], \end{cases}$$

$$\begin{cases} \tilde{Q} = \text{diag}\{ \tilde{Q}_2, \tilde{Q}_1 - \tilde{Q}_2, \tilde{Q}_1 \}, \\ \tilde{\Phi} = \bar{\tau}^2 \tilde{R}_1 + \tau_m^2 \tilde{R}_2; \quad \bar{\tau} = \tau_M - \tau_m \\ \tilde{\Omega}_1 = \text{diag}\{ \tilde{R}_2, 3 \left(\frac{\tau_m + 1}{\tau_m - 1} \right) \tilde{R}_2 \}, \\ \tilde{\Omega}_2 = \text{diag}\{ \tilde{R}_1, 3 \tilde{R}_1 \}, \\ \tilde{\mathbb{R}} := \begin{bmatrix} \tilde{\Omega}_2 & \tilde{N} \\ * & \tilde{\Omega}_2 \end{bmatrix} \end{cases}$$

such that $K_j = \tilde{K}_j \tilde{X}^{-1}, i, j = 1, \dots, r$.

Proof. For the sake of simplicity, we denote $\eta(t) = x(t + 1) - x(t)$. Let

$$\begin{aligned} \xi(t) &= [x^T(t + 1), x^T(t), x^T(t - \tau_m), x^T(t - \tau(t)), \\ &\quad x^T(t - \tau_M), \varphi_1^T(t), \varphi_2^T(t), \varphi_3^T(t), e^T(t), d^T(t)]^T \\ q_i &:= [\underbrace{0_{n \times n}, \dots, 0_{n \times n}}_{i-1} \ I_{n \times n} \ \underbrace{0_{n \times n}, \dots, 0_{n \times n}}_{10-i}]^T, \end{aligned}$$

where $i = 1, \dots, 10$ and

$$\begin{aligned} \varphi_1(t) &= \frac{1}{\tau_m + 1} \sum_{i=t-\tau_m}^t x(i), \\ \varphi_2(t) &= \frac{1}{\tau(t) - \tau_m + 1} \sum_{i=t-\tau(t)}^{t-\tau_m} x(i), \\ \varphi_3(t) &= \frac{1}{\tau_M - \tau(t) + 1} \sum_{i=t-\tau_M}^{t-\tau(t)} x(i) \end{aligned}$$

We choose the Lyapunov function as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{24}$$

where

$$\begin{aligned} V_1(t) &= \begin{bmatrix} x(t) \\ \sum_{i=t-\tau_m}^{t-1} x(i) \\ \sum_{i=t-\tau_M}^{t-\tau_m-1} x(i) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \sum_{i=t-\tau_m}^{t-1} x(i) \\ \sum_{i=t-\tau_M}^{t-\tau_m-1} x(i) \end{bmatrix} \\ V_2(t) &= \sum_{i=t-\tau_M}^{t-\tau_m-1} x^T(t) Q_1 x(t) + \sum_{i=t-\tau_m}^{t-1} x^T(t) Q_2 x(t) \\ V_3(t) &= \bar{\tau} \sum_{i=-\tau_M}^{-\tau_m-1} \sum_{h=t+i}^{t-1} \eta^T(t) R_1 \eta(t) \\ &\quad + \tau_m \sum_{i=-\tau_m}^{-1} \sum_{h=t+i}^{t-1} \eta^T(t) R_2 \eta(t) \end{aligned} \tag{25}$$

with $P = P^T$, Q_i^T , and R_i^T (with $i = 1, 2$) as the Lyapunov matrix.

Notice that

$$\tau_1 = \tau_m + 1, \quad \tau_2 = \tau(t) - \tau_m + 1, \quad \tau_3 = \tau_M - \tau(t) + 1$$

The finite difference of $\Delta V_1(t)$ is expressed as follows:

$$\begin{aligned} \Delta V_1(t) &= V_1(t + 1) - V_1(t) \\ &= \begin{bmatrix} x(t + 1) \\ \tau_1 \varphi_1(t) - x(t - \tau_m) \\ \tau_2 \varphi_2(t) + \tau_3 \varphi_3(t) - x(t - \tau(t)) - x(t - \tau_M) \end{bmatrix}^T \\ &\quad \times P(*) \\ &\quad - \begin{bmatrix} x(t) \\ \tau_1 \varphi_1(t) - x(t) \\ \tau_2 \varphi_2(t) + \tau_3 \varphi_3(t) - x(t - \tau(t)) - x(t - \tau_M) \end{bmatrix}^T \end{aligned} \tag{26}$$

$$\begin{aligned} &\times P(*) \\ &= \xi^T(t) \{ Y_1(\tau(t)) P Y_1^T(\tau(t)) - Y_2(\tau(t)) P \\ &\quad Y_2^T(\tau(t)) \} \xi(t) \end{aligned}$$

where

$$\begin{aligned} Y_1(\tau(t)) &= [q_1 \quad \tau_1 q_6 - q_3 \quad \tau_2 q_7 + \tau_3 q_8 - q_4 - q_5], \\ Y_2(\tau(t)) &= [q_2 \quad \tau_1 q_6 - q_2 \quad \tau_2 q_7 + \tau_3 q_8 - q_3 - q_4] \end{aligned}$$

For $\Delta V_2(t)$, we have

$$\begin{aligned} \Delta V_2(t) &= V_2(t+1) - V_2(t) \\ &= x^T(t) Q_2 x(t) - x^T(t - \tau_M) Q_1 x(t - \tau_M) \\ &\quad + x^T(t - \tau_m) (Q_1 - Q_2) x(t - \tau_m) \\ &= \xi^T(t) \{ Y_3 Q Y_3^T \} \xi(t) \end{aligned} \tag{27}$$

with

$$Q = \text{diag}\{Q_2, Q_1 - Q_2, Q_1\}, \quad Y_3 = [q_2 \quad q_3 \quad q_5]$$

Furthermore, $\Delta V_3(t)$ can be expressed as follows:

$$\begin{aligned} \Delta V_3(t) &= V_2(t+1) - V_2(t) \\ &= \eta^T(t) (\bar{\tau}^2 R_1 + \tau_m^2 R_2) \eta(t) \\ &\quad - \underbrace{\tau_m \sum_{i=t-\tau_m}^{t-1} \eta^T(i) R_2 \eta(i)}_{\psi_1} - \underbrace{\bar{\tau} \sum_{i=t-\tau_M}^{t-\tau_m-1} \eta^T(i) R_1 \eta(i)}_{\psi_2} \end{aligned} \tag{28}$$

Now, by applying Lemma 1, the expression ψ_1 can be bounded by

$$\begin{aligned} &-\tau_m \sum_{i=t-\tau_m}^{t-1} \eta^T(i) R_2 \eta(i) \\ &\leq - \left[\begin{array}{c} x(t) - x(t - \tau_m) \\ x(t) + x(t - \tau_m) - 2\varphi_1(t) \end{array} \right]^T \Omega_1(*) \end{aligned} \tag{29}$$

with

$$\begin{aligned} \Omega_1 &= \text{diag}\left\{ R_2, 3\left(\frac{\tau_m+1}{\tau_m-1}\right) R_2 \right\}, \\ Y_4 &= [q_2 - q_3 \quad q_2 + q_3 - 2q_6] \end{aligned}$$

On the contrary, for the sum ψ_2 , we have to split it into two terms as follows:

$$\begin{aligned}
 -\bar{\tau} \sum_{i=t-\tau_M}^{t-\tau_m-1} \eta^T(t)R_1\eta(t) &= -\bar{\tau} \underbrace{\sum_{i=t-\tau(t)}^{t-\tau_m-1} \eta^T(t)R_1\eta(t)}_{\psi_{21}} \\
 &\quad -\bar{\tau} \underbrace{\sum_{i=t-\tau_M}^{t-\tau(t)-1} \eta^T(t)R_1\eta(t)}_{\psi_{22}}
 \end{aligned} \tag{30}$$

Similarly, by using the Lemma 2, ψ_{21} is bounded as follows:

$$\begin{aligned}
 &-\bar{\tau} \sum_{i=t-\tau(t)}^{t-\tau_m-1} \eta^T(t)R_1\eta(t) \\
 &\leq -\frac{1}{\alpha(t)} \begin{bmatrix} x(t-\tau_m) - x(t-\tau(t)) \\ x(t-\tau_m) + x(t-\tau(t)) - 2\varphi_2(t) \end{bmatrix}^T \Omega_2(*)
 \end{aligned} \tag{31}$$

where $\Omega_2 = \text{diag}\{R_1, 3R_1\}$ and $\alpha(t) = \tau(t) - \tau_m/\bar{\tau}$.

In addition, ψ_{21} is bounded by

$$\begin{aligned}
 &-\bar{\tau} \sum_{i=t-\tau_M}^{t-\tau(t)-1} \eta^T(t)R_1\eta(t) \\
 &\leq -\frac{1}{\beta(t)} \begin{bmatrix} x(t-\tau(t)) - x(t-\tau_M) \\ x(t-\tau(t)) + x(t-\tau_M) - 2\varphi_3(t) \end{bmatrix}^T \Omega_2(*)
 \end{aligned} \tag{32}$$

such that $\beta(t) = 1 - \alpha(t)$.

We can utilize Lemma 3 of the reciprocally convex approach to handle the time-varying terms in Equations (31) and (32). By applying this approach to any matrix N , we obtain

$$\begin{aligned}
 &-\bar{\tau} \sum_{i=t-\tau(t)}^{t-\tau_m-1} \eta^T(t)R_1\eta(t) - \bar{\tau} \sum_{i=t-\tau_M}^{t-\tau(t)-1} \eta^T(t)R_1\eta(t) \\
 &\leq \begin{bmatrix} x(t-\tau_m) - x(t-\tau(t)) \\ x(t-\tau_m) + x(t-\tau(t)) - 2\varphi_2(t) \\ x(t-\tau(t)) - x(t-\tau_M) \\ x(t-\tau(t)) + x(t-\tau_M) - 2\varphi_3(t) \end{bmatrix}^T \mathbb{R}(\ast) \\
 &\leq -\xi^T(t) \{Y_5 \mathbb{R} Y_5^T\} \xi(t)
 \end{aligned} \tag{33}$$

where

$$\begin{cases} \mathbb{R} := \begin{bmatrix} \Omega_2 & N \\ * & \Omega_2 \end{bmatrix}, \\ Y_5 := \begin{bmatrix} q_3 - q_4 & q_3 + q_4 - 2q_7 & q_4 - q_5 \\ q_4 + q_5 - 2q_8 \end{bmatrix} \end{cases}$$

Then, we obtain

$$\Delta V_3 \leq \xi^T(t) \{Y_6 \Phi Y_6^T - Y_5 \mathbb{R} Y_5^T\} \xi(t) \tag{34}$$

such that $\Phi = \bar{\tau}^2 R_1 + \tau_m^2 R_2$ and $Y_6 = [q_1 \quad q_2]$.

Furthermore, for any matrix X with appropriate dimensions, we have

$$2\zeta^T(t)(Xq_1^T)[-q_1 + A_cq_2 + A_\tau q_4 + B_cq_9 + D_cq_{10}]\zeta(t) = 0 \tag{35}$$

Hence, $\Delta V(t)$ is given by

$$\begin{aligned} \Delta V(t) \leq & \zeta^T(t)(\text{sym}(\Psi) + Y_1(\tau(t))PY_1^T(\tau(t)) \\ & - Y_2(\tau(t))PY_2^T(\tau(t)) + Y_3QY_3^T - Y_4\Omega_1Y_4^T \\ & - Y_5RY_5^T + Y_6\Phi Y_6^T)\zeta(t) \end{aligned} \tag{36}$$

with

$$\Psi = q_1^T X[-q_1 + A_cq_2 + A_\tau q_4 + B_cq_9 + D_cq_{10}]$$

Consider the event-triggered conditions in Equation (15). For any $t \in [t_s, t_{s+1})$, it is satisfied that

$$\varepsilon x^T(t)\Lambda x(t) - e^T(t)\Lambda e(t) \geq 0 \tag{37}$$

Then, we have

$$\begin{aligned} \Delta V(t) \leq & \zeta^T(t)(\text{sym}(\Psi) + Y_1(\tau(t))PY_1^T(\tau(t)) \\ & - Y_2(\tau(t))PY_2^T(\tau(t)) + Y_3QY_3^T - Y_4\Omega_1Y_4^T \\ & - Y_5RY_5^T + Y_6\Phi Y_6^T + \varepsilon q_2^T \Lambda q_2 - q_9^T \Lambda q_9)\zeta(t) \end{aligned} \tag{38}$$

Our goal is to select the gain K_j ($j = 1, \dots, r$) such that the closed-loop system in Equation (19) is asymptotically stable while satisfying the \mathcal{H}_∞ criterion:

$$\|y(t)\|_{l_2} \leq \lambda \|d\|_{l_2} \tag{39}$$

where $\lambda > 0$ represents the level of disturbance attenuation that we aim to minimize.

In order to ensure that the expected performance (Equation (39)) of the designed ISMC is guaranteed, it is necessary to satisfy the following condition:

$$\Delta V(t) + y^T(t)y(t) - \lambda^2 d^T(t)d(t) \leq 0 \tag{40}$$

Based on Equation (3) and by applying the Schur complement, we obtain

$$\Delta V(t) \leq \zeta^T(t)\Xi(\tau(t))\zeta(t) \tag{41}$$

where

$$\Xi(\tau(t)) = \begin{bmatrix} \Xi_1(\tau(t)) & \Xi_2 \\ * & -I \end{bmatrix} \tag{42}$$

with

$$\begin{aligned} \Xi_1(\tau(t)) = & \text{sym}(\Psi) + Y_1(\tau(t))PY_1^T(\tau(t)) - Y_2(\tau(t))P \\ & Y_2^T(\tau(t)) + Y_3QY_3^T - Y_4\Omega_1Y_4^T - Y_5RY_5^T \\ & + Y_6\Phi Y_6^T + \varepsilon q_2^T \Lambda q_2 - q_9^T \Lambda q_9 - q_{10}^T \lambda^2 q_{10} \\ \Xi_2 = & q_2^T C_m^T + q_4^T C_{\tau m}^T \end{aligned}$$

Now, we can pre- and post-multiply Equation (36) by

$$\text{bloc} - \text{diag} \underbrace{\{\tilde{X}, \dots, \tilde{X}, I_n, I_n\}}_9$$

such that $\tilde{X} = X^{-1}$, as well as their transposes. Note that $\tilde{P}_{[kl]} = \tilde{X}P_{[kl]}\tilde{X}^T$ ($k, l = 1, 2, 3$), $\tilde{Q}_i = \tilde{X}Q_i\tilde{X}^T$, $\tilde{\Lambda} = \tilde{X}\Lambda\tilde{X}^T$, and $\tilde{R}_i = \tilde{X}R_i\tilde{X}^T$. We obtain the following inequality:

$$\Delta V(t) \leq \zeta^T(t)\tilde{\Xi}(\tau(t))\zeta(t) \tag{43}$$

where

$$\tilde{\Xi}(\tau(t)) = \begin{bmatrix} \tilde{\Xi}_1(\tau(t)) & \tilde{\Xi}_2 \\ * & -I \end{bmatrix} \tag{44}$$

where

$$\begin{aligned} \tilde{\Xi}_1(\tau(t)) &= \text{sym}(\tilde{\Psi}) + Y_1(\tau(t))\tilde{P}Y_1^T(\tau(t)) - Y_2(\tau(t)) \\ &\quad \tilde{P}Y_2^T(\tau(t)) + Y_3\tilde{Q}Y_3^T - Y_4\tilde{\Omega}_1Y_4^T - Y_5\tilde{R}Y_5^T \\ &\quad + Y_6\tilde{\Phi}Y_6^T + \varepsilon q_2^T\tilde{\Lambda}q_2 - q_9^T\tilde{\Lambda}q_9 - q_{10}^T\lambda^2q_{10} \\ \tilde{\Xi}_2 &= q_2^T\tilde{X}C_m^T + q_4^T\tilde{X}C_{\tau m}^T \end{aligned}$$

and

$$\tilde{\Psi} = q_1^T[\tilde{X}q_1 + \mathbb{A}_c q_2 + \mathbb{A}_\tau q_4 + \mathbb{B}_c q_9 + \mathbb{D}_c q_{10}]$$

such that $\mathbb{A}_c = A_m\tilde{X}^T + B_m\tilde{K}_m$, $\mathbb{B}_c = B_m\tilde{K}_m$, and $\mathbb{D}_c = (I - B_mM)D_m$.

Note that Equation (43) can also be written as

$$\tilde{\Xi} = \sum_{i=1}^r m_i^2 \tilde{\Xi}_{ii} + \sum_{i=1}^r \sum_{i < j} m_i m_j (\tilde{\Xi}_{ij} + \tilde{\Xi}_{ji}) < 0 \tag{45}$$

It is straightforward that from Equation (43), we obtain the constraint described in the theorem. \square

5. Sliding Mode Control Synthesis

In this part, ISMC is developed, and the analysis of the sliding mode stability is presented. In particular, the design of ISMC is based on an equivalent control law and a nonlinear switching control law in order to tolerate disturbances or parameter uncertainties. Hence, global ISMC is given by the theorem below:

Theorem 2. For the T-S fuzzy system described in Equation (1) as well as the sliding manifold designed in Equation (7), ET-ISMC is proposed as follows:

$$u(t) = Kx(t_s) - \sigma S(t) - \rho \times \text{sat}(t) \tag{46}$$

where $\rho = \bar{d}\|MD_m\|$ and

$$\text{sat}(t) = \begin{cases} S(t)/\varphi, & \|S(t)\| \leq \varphi \\ \text{sign}(S(t)), & \|S(t)\| > \varphi \end{cases}$$

In addition, the sliding manifold function attached to the event-triggered approach is developed as follows:

$$\begin{cases} S(t) = Mx(t_s) - Mx_0 - \beta(t_s) \\ \beta(t_s) = \beta(t_s - 1) - \delta(t_s - 1) \end{cases} \tag{47}$$

where

$$\delta(t_s) = M(A_m + B_mK_m - I)x(t_s) + MA_{\tau m}x(t_s - \tau(t))$$

Proof. The following Lyapunov function is selected:

$$V(t) = \frac{1}{2}S^T(t)S(t) \tag{48}$$

From the derived sliding mode surface function in Equation (47), we have

$$\begin{aligned} \Delta S(t) &= S(t+1) - S(t) \\ &= MB_mu(t) - MB_mK_mx(t) + MD_md(t) \\ &= MD_md(t) - \sigma S(t) - \rho sat(t) \end{aligned} \tag{49}$$

Then, by inferring the increment from $V(t)$ and considering the ET-SMC law in Equation (46) and $\Delta S(t)$, we obtain

$$\begin{aligned} \Delta V(t) &= V(t+1) - V(t) \\ &= S^T(t)\Delta S(t) + \frac{1}{2}\Delta S^T(t)\Delta S(t) \\ &= S^T(t)(MD_md(t) - \rho sat(t)) - \sigma S^T(t)S(t) \\ &\quad + \frac{1}{2}\Delta S^T(t)\Delta S(t) \\ &\leq -\sigma S^T(t)S(t) + \frac{1}{2}\Delta S^T(t)\Delta S(t) + \|S^T(t)\| \\ &\quad \|MD_m\| \|d(t)\| - \rho S^T(t) sat(t) \\ &\leq -\sigma S^T(t)S(t) + \frac{1}{2}\Delta S^T(t)\Delta S(t) + \|S^T(t)\| \rho \\ &\quad - \rho S^T(t) sat(t) \end{aligned} \tag{50}$$

Based on the saturation function $sat(t)$, we derive that if $\|S^T(t)\| \leq \varphi$, with the designed controller in Equation (22), we obtain

$$\begin{aligned} \Delta V(t) &\leq -\sigma S^T(t)S(t) + \frac{1}{2}\Delta S^T(t)\Delta S(t) \\ &\quad + \rho \left(\|S(t)\| - \|S(t)\|^2 / \varphi \right) \\ &\leq -\sigma S^T(t)S(t) + \frac{1}{2}\Delta S^T(t)\Delta S(t) \end{aligned} \tag{51}$$

If $\|S^T(t)\| > \varphi$, with the designed controller in Equation (22), we obtain

$$\Delta V(t) \leq -\sigma S^T(t)S(t) + \frac{1}{2}\Delta S^T(t)\Delta S(t) \tag{52}$$

Therefore, σ can be chosen appropriately, which can ensure that $\Delta V(t) \leq 0$. **The proof is completed.** □

6. Simulation Results

To illustrate the effectiveness of the obtained results, we consider the problem of stabilizing an inverted pendulum on a cart. The nonlinear model of the pendulum is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ f_1 \\ x_4 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g_1 \\ 0 \\ g_2 \end{bmatrix} u(t) + Dd(t) \tag{53}$$

with

$$\begin{aligned} f_1 &= \frac{-g \sin(x_1) - amlx_2^2 \sin(2x_1)/2}{4l/3 - aml\cos^2(x_1)}, \quad g_1 = \frac{-a \cos(x_1)/2}{4l/3 - aml\cos^2(x_1)} \\ f_2 &= \frac{mag \sin(2x_1)/2 + 4amlx_2^2 \sin(x_1)/3}{4l/3 - aml\cos^2(x_1)}, \quad g_2 = \frac{4a/3}{4l/3 - aml\cos^2(x_1)} \end{aligned}$$

where $g = 9.8 \text{ m/s}^2$ is the gravity acceleration, $m = 2 \text{ kg}$ and $M = 8 \text{ kg}$ are the mass of the pendulum and the cart, respectively, $2l = 1 \text{ m}$ is the length of the pendulum, and $a = 1/(M + m)$.

The system is linearized around $x_1 = 0$ and $x_1 \pm \frac{\pi}{4}$, and a fuzzy model is obtained from [36] as follows.

Model 1: IF x_1 is 0 THEN

$$\begin{aligned} \dot{x}(t) &= A_1x(t) + B_1u(t) + Dd(t) \\ y(t) &= C_1x(t) \end{aligned} \tag{54}$$

Model 2: IF x_1 is $\pm \frac{\pi}{4}$ THEN

$$\begin{aligned} \dot{x}(t) &= A_2x(t) + B_2u(t) + Dd(t) \\ y(t) &= C_2x(t) \end{aligned} \tag{55}$$

such that

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -17.31 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1.7312 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -14.32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.716 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = [0 \ -0.1765 \ 0 \ 0.1176]^T, \quad B_2 = [0 \ -0.1147 \ 0 \ 0.1176]^T,$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T$$

The overall output of the T-S model is represented as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^2 m_i(A_ix(t) + B_iu(t) + Dd(t)) \\ y(t) &= \sum_{i=1}^2 m_iC_ix(t) \end{aligned} \tag{56}$$

where

$$\begin{aligned} m_1(t) &= \left(1 - \frac{1}{1+e^{(-14(x_1(t)-\frac{\pi}{8}))}}\right) \times \frac{1}{1+e^{(-14(x_1(t)+\frac{\pi}{8}))}}; \\ m_2(t) &= 1 - m_1(t) \end{aligned}$$

We assume that the pendulum is subject to delay. The global output becomes

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^2 m_i[(1-s)A_ix(t) + sA_ix(t-\tau(t)) + \\ &\quad B_iu(t) + D_id(t)] \\ y(t) &= \sum_{i=1}^2 m_i[(1-s)C_ix(t) + sC_ix(t-\tau(t))] \end{aligned} \tag{57}$$

We discretize Equation (57) with the sample time $T_e = 0.05s$ and $s = 0.1$, and we obtain

$$\begin{cases} x(t+1) = \sum_{i=1}^2 m_i[Ad_ix(t) + Ad_{\tau_i}x(t-\tau(t)) + \\ \quad Bd_iu(t) + Dd_id(t)] \\ y(t) = \sum_{i=1}^2 m_i[Cd_ix(t) + Cd_{\tau_i}x(t-\tau(t))] \end{cases} \tag{58}$$

where

$$Ad_1 = \begin{bmatrix} 0.9 & 0.045 & 0 & 0 \\ -0.7789 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0.045 \\ 0.0779 & 0 & 0 & 0.9 \end{bmatrix},$$

$$Ad_{\tau_1} = \begin{bmatrix} 0.1 & 0.005 & 0 & 0 \\ -0.0865 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0.005 \\ 0.0087 & 0 & 0 & 0.1 \end{bmatrix},$$

$$Bd_1 = [0 \ -0.0088 \ 0 \ 0.0059]^T,$$

$$Ad_2 = \begin{bmatrix} 0.9 & 0.045 & 0 & 0 \\ -0.6444 & 0.9 & 0 & 0 \\ 0 & 0 & 0.9 & 0.045 \\ 0.0322 & 0 & 0 & 0.9 \end{bmatrix},$$

$$Ad_{\tau_2} = \begin{bmatrix} 0.1 & 0.005 & 0 & 0 \\ -0.0716 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0.005 \\ 0.0036 & 0 & 0 & 0.1 \end{bmatrix},$$

$$Bd_2 = [0 \ -0.0057 \ 0 \ 0.0054]^T,$$

$$Cd_1 = Cd_2 = 10^{-1} \times \begin{bmatrix} 0.45 & 0 & 0 & 0.45 \\ 0 & 0.45 & 0.45 & 0.45 \end{bmatrix},$$

$$Cd_{\tau_1} = Cd_{\tau_2} = 10^{-2} \times \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0.5 \end{bmatrix},$$

$$Dd_1 = Dd_2 = 10^{-1} \times \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}^T$$

such that $x_0 = [-0.2 \ 0 \ 0.2 \ 0]^T$ for $t = -\tau_M, \dots, -\tau_m$ is the initial condition of the states. The external disturbance is given by

$$d(t) = \begin{bmatrix} 0.1e^{-0.1t} \\ 0.1e^{-0.1t} \end{bmatrix}$$

The time-varying delay is between $\tau_m = 2$ and $\tau_M = 6$.

The ensuing discussions aim to elucidate the distinctions between the existing method and our proposed strategy:

- The SMC scheme, which is presented in [19], is built on an assumption that the system $\left(A_i, \frac{1}{r} \sum_{i=1}^r B_i\right)$ can be expressed in the regular form $\left(\begin{bmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{bmatrix}, \begin{bmatrix} 0 \\ B \end{bmatrix}\right)$ (where $\det(B) \neq 0$) through suitable transformation. Nevertheless, for this transformation to take place, it is necessary that the input matrices of all linear models are identical. For this reason, we have developed an SMC scheme in this work that overcomes this constraint.
- In [20], the ISMC design for a class of nonlinear systems, which are depicted by T-S fuzzy models, was investigated. However, this suggested scheme is based on two highly restrictive assumptions. The first assumption is that all linear local models of the T-S fuzzy system share a common input matrix (i.e., $B_i \equiv B$ (where $i = 1, \dots, r$)). Secondly, there exists a matrix G with an appropriate dimension such that GB is non-singular. Furthermore, both of these assumptions greatly limit real-world applications, including the widely known inverted pendulum on a cart [22]. By removing this assumption, our proposed SMC method can robustly stabilize fuzzy systems with different input matrices.

By solving the LMI conditions of Lemma 4, we obtain

$$M = [0 \quad -85.8124 \quad 0 \quad 66.5105]$$

Some other parameters are given, such as the event-triggered parameter $\varepsilon = 0.1$. By solving Theorem 1, the following designed matrices are obtained:

$$\Lambda = \begin{bmatrix} 3.5949 & -0.0937 & -0.8182 & -1.3304 \\ -0.0937 & 0.0024 & 0.0213 & 0.0347 \\ -0.8182 & 0.0213 & 0.1865 & 0.3032 \\ -1.3304 & 0.0347 & 0.3032 & 0.4931 \end{bmatrix}$$

$$K_1 = [453.9662 \quad -11.8263 \quad -103.9457 \quad -169.0464]$$

$$K_2 = [442.9797 \quad -11.5434 \quad -100.3551 \quad -163.1753]$$

$$\lambda = 0.07$$

Furthermore, Table 1 displays the results of 400 independent simulation steps for different values of the event-triggering parameter ε , showing that the event-triggered ratio (i.e., the number of event-triggering release instants over the total number of simulation steps) decreased as the value of ε increased. Therefore, this also suggests that the proposed method has the ability to decrease the number of required data transmissions, thereby reducing the communication burden.

Table 1. Event-triggered rates with different ε values.

Triggering Parameter ε	0	0.05	0.1	0.3	0.6
Event-Triggered Rates	100%	61.5%	45%	40%	23.25%

The simulation results for the designed ISMC model based on the ET are presented in Figure 1, where $\varepsilon = 0.1$ (and in Figure 2, where $\varepsilon = 0.6$). The evolution of the states vector of the nonlinear system is given in Figure 1a. The control input is plotted in Figure 1b, and this shows that the control remained constants until an event was triggered to update the control signal. The sliding function is shown in Figure 1c, and the evolution of the event-triggered signal transmission is given in Figure 1d.

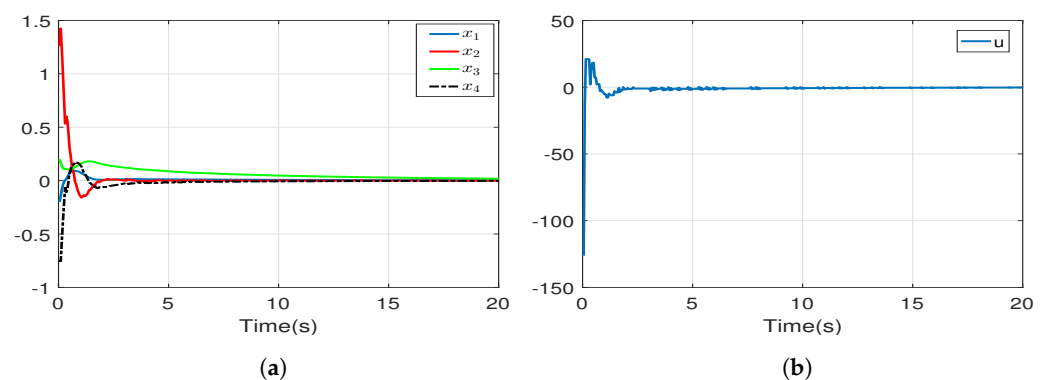


Figure 1. Cont.

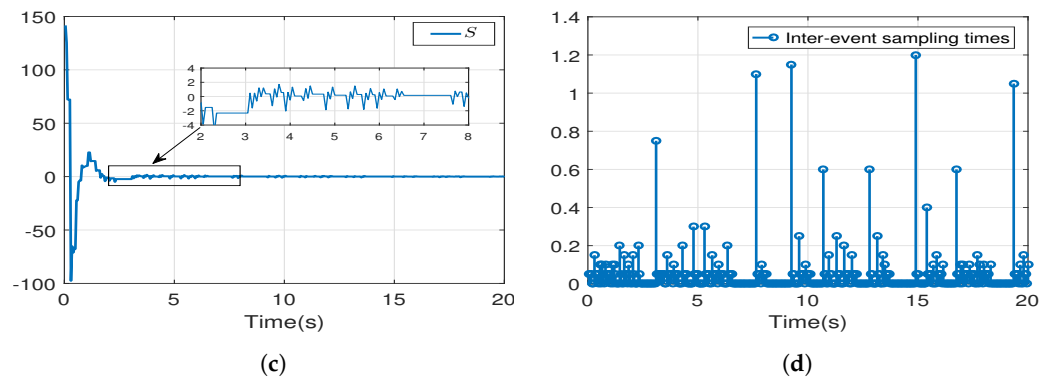


Figure 1. The simulation results with $(\epsilon = 0.1)$. (a) The states vector. (b) The control law. (c) The sliding function. (d) The inter-event time.

Analysis of Figure 1 reveals that within the time interval of $[0, 20\text{ s}]$, a total of 180 sampled signals were transferred, which led to a transmission rate of 45%. This means that a significant 55% of communication information and resources can be saved.

Similarly, based on Figure 2, we observe that 90 sampled signals were transferred in the time interval $[0, 20\text{ s}]$, resulting in a transmission rate of 23.25%. This means that a significant 76.75% of communication information and resources can be saved.

The simulation results indicate that the proposed event trigger-based integral sliding mode control law is a practical and effective solution for the considered nonlinear system with time-varying delay. Therefore, it can be concluded that this advanced control strategy can significantly improve the system’s performance and reduce the resource utilization, providing valuable insights into the development of control techniques for complex and dynamic systems.

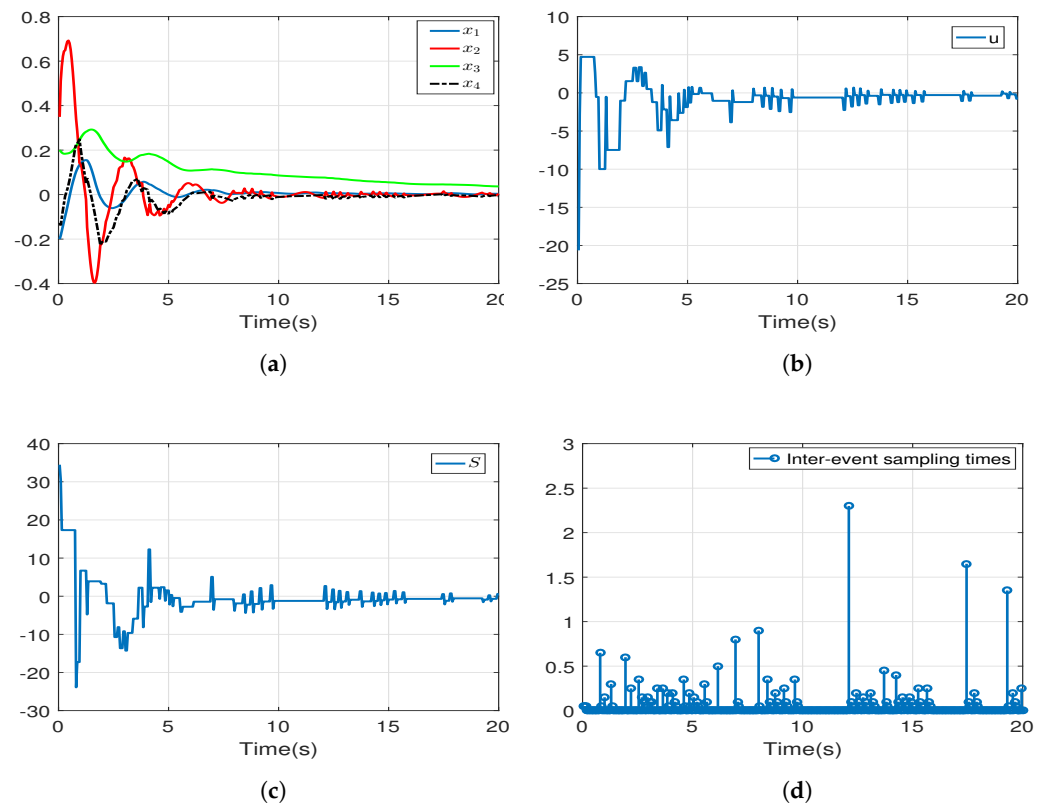


Figure 2. The simulation results with $\epsilon = 0.6$. (a) The states vector. (b) The control law. (c) The sliding function. (d) The inter-event time.

7. Conclusions

This paper presents a design for an event-triggered integral sliding mode control (ISMC) model specifically developed for T-S fuzzy systems with time-varying delay and measurement noise. The proposed approach introduces an integral sliding function for a class of nonlinear systems which removes the restrictive assumptions of many existing fuzzy SMC approaches. This paper formulates sufficient conditions in terms of linear matrix inequalities (LMIs) to guarantee the achievement and maintenance of sliding motion, as well as asymptotic stability of the residual system. Additionally, the ISMC design approach is improved by integrating an event-triggered scheme. This scheme ensures that data are transmitted only when the specified triggering condition is met, which helps to avoid the transmission of redundant information, conserve valuable network resources, and minimize the chattering phenomenon. Finally, a verification example is presented using a pendulum system to illustrate the effectiveness and potential of the theoretical results obtained.

As a future work, we propose studying the design of an interval observer based on the novel integral sliding mode control (SMC) analyzed in this paper in combination with an event-triggered control approach.

Author Contributions: Software, Y.B.; Writing—original draft, Y.B.; Writing—review & editing, Y.B., M.F. and A.S.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Takagi, T.; Sugeno, M. Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst. Man Cybern.* **1985**, *SMC-15*, 116–132.
2. Munakata, T.; Jani, Y. Fuzzy systems: An overview. *Commun. ACM* **1994**, *37*, 69–77.
3. Sugeno, M.; Kang, G. Structure identification of fuzzy model. *Fuzzy Sets Syst.* **1988**, *28*, 15–33.
4. Tanaka, K.; Hori, T.; Wang, H.O. A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Trans. Fuzzy Syst.* **2003**, *11*, 582–589.
5. Chen, P.; Chen, C.; Chiang, W. Linear matrix inequality conditions of nonlinear systems by genetic algorithm-based H_∞ adaptive fuzzy sliding mode controller. *J. Vib. Control* **2011**, *17*, 163–173.
6. Bresch-Pietri, D.; Chauvin, J.; Petit, N. Adaptive backstepping controller for uncertain systems with unknown input time-delay. application to SI engines. In Proceedings of the 49th IEEE Conference on Decision and Control (CDC), Atlanta, GA, USA, 15–17 December 2010, pp. 3680–3687.
7. Khelifa, H.; Ellouze, I. Adaptive robust state stabilisation of uncertain non linear time-varying systems with delayed perturbation. *Int. J. Syst. Sci.* **2021**, *52*, 3241–3253.
8. Pandey, S.; Tadepalli, S.K.; Leite, V.J.; Nigam, R.; Bhusnur, S. Stability of 2-D Discrete Systems in the Presence of Saturation Function and Delays. *Int. J. Control. Autom. Syst.* **2023**, *21*, 788–799.
9. Sobhanipour, H.; Rezaie, B. Enhanced Exponential Stability Analysis for Switched Linear Time-Varying Delay Systems Under Admissible Edge-Dependent Average Dwell-Time Strategy. *IEEE Trans. Syst. Man Cybern. Syst.* **2023**. <http://doi.org/10.1109/TSMC.2023.3266492>.
10. Graef, J.R.; Tunç, C.; Tunç, O. Stability of time-delay systems via the Razumikhin method. *Boletín de la Sociedad Matemática Mexicana* **2022**, *28*, 26.
11. Ye, Z.; Zhang, W.A.; Zhang, D.; Ni, H. On design of robust sliding mode observer for nonlinear networked time-delay systems with communication constraints. *ISA Trans.* **2020**, *124*, 260–270. <http://doi.org/10.1016/j.isatra.2020.05.014>.
12. Sakthivel, R.; Shobana, N.; Priyanka, S.; Kwon, O. State observer-based predictive proportional-integral tracking control for fuzzy input time-delay systems. *Int. J. Robust Nonlinear Control.* **2023**. <https://doi.org/10.1002/rnc.6681>.
13. Ben Atia, S.; Messaoud, A.; Ben Abdennour, R. Robust multiobserver design for discrete uncertain nonlinear systems with time-varying delay. *Trans. Inst. Meas. Control* **2018**, *40*, 191–201.
14. Ghotb, H.; Ataei, M.; Siah, M.; Moarefianpour, A. Design of developed distributed observers for nonlinear time-delay systems. *Trans. Inst. Meas. Control.* **2023**, 01423312221138893. <https://doi.org/10.1177/01423312221138893>.
15. Tognetti, E.S.; de Oliveira, G.A. Robust state feedback-based design of PID controllers for high-order systems with time-delay and parametric uncertainties. *J. Control. Autom. Electr. Syst.* **2022**, *33*, 382–392.

16. Bhuvaneshwari, G.; Prakash, M.; Rakkiyappan, R.; Manivannan, A. Stability and stabilization analysis of TS fuzzy systems with distributed time-delay using state-feedback control. *Math. Comput. Simul.* **2023**, *205*, 778–793.
17. Estrada-Manzo, V.; Lendek, Z.; Guerra, T. An alternative LMI static output feedback control design for discrete-time nonlinear systems represented by Takagi-Sugeno models. *ISA Trans.* **2019**, *84*, 104–110.
18. Choi, H.H. Robust stabilization of uncertain fuzzy-time-delay systems using sliding-mode-control approach. *IEEE Trans. Fuzzy Syst.* **2010**, *18*, 979–984.
19. Xue, Y.; Zheng, B.C.; Yu, X. Robust sliding mode control for TS fuzzy systems via quantized state feedback. *IEEE Trans. Fuzzy Syst.* **2017**, *26*, 2261–2272.
20. Subramaniam, R.; Song, D.; Joo, Y.H. TS fuzzy-based sliding mode controller design for discrete-time nonlinear model and its applications. *Inf. Sci.* **2020**, *519*, 183–199.
21. Zhang, J.; Zhu, F.; Karimi, H.R.; Wang, F. Observer-based sliding mode control for T-S fuzzy descriptor systems with time delay. *IEEE Trans. Fuzzy Syst.* **2019**, *27*, 2009–2023.
22. Chen, M.; Feng, G.; Ma, H.; Chen, G. Delay-Dependent H_∞ Filter Design for Discrete-Time Fuzzy Systems With Time-Varying Delays. *IEEE Trans. Fuzzy Syst.* **2008**, *17*, 604–616.
23. Zong, G.; Sun, X.; Yang, D.; Su, S.F.; Shi, K. Finite-time H_∞ control for switched fuzzy systems: A dynamic adaptive event-triggered control approach. *Fuzzy Sets Syst.* **2023**, *464*, 108475.
24. Tan, Y.; Yuan, Y.; Xie, X.; Tian, E.; Liu, J. Observer-based event-triggered control for interval type-2 fuzzy networked system with network attacks. *IEEE Trans. Fuzzy Syst.* **2023**. <https://doi.org/10.1109/TFUZZ.2023.3237846>.
25. Liu, Y.; Zhao, X.; Park, J.H.; Fang, F. Fault-tolerant control for TS fuzzy systems with an aperiodic adaptive event-triggered sampling. *Fuzzy Sets Syst.* **2023**, *452*, 23–41.
26. Su, X.; Wen, Y.; Shi, P.; Wang, S.; Assawinchaichote, W. Event-triggered fuzzy control for nonlinear systems via sliding mode approach. *IEEE Trans. Fuzzy Syst.* **2019**, *29*, 336–344.
27. Echreshavi, Z.; Farbood, M.; Shasadeghi, M. Fuzzy event-triggered integral sliding mode control of nonlinear continuous-time systems. *IEEE Trans. Fuzzy Syst.* **2021**, *30*, 2347–2359.
28. Yue, D.; Tian, E.; Han, Q.L. A delay system method for designing event-triggered controllers of networked control systems. *IEEE Trans. Autom. Control* **2012**, *58*, 475–481.
29. Zhivoglyadov, P.V.; Middleton, R.H. Networked control design for linear systems. *Automatica* **2003**, *39*, 743–750.
30. Yue, D.; Han, Q.L.; Lam, J. Network-based robust H_∞ control of systems with uncertainty. *Automatica* **2005**, *41*, 999–1007.
31. Cao, S.G.; Rees, N.W. Identification of dynamic fuzzy models. *Fuzzy Sets Syst.* **1995**, *74*, 307–320.
32. Seuret, A.; Gouaisbaut, F.; Fridman, E. Stability of discrete-time systems with time-varying delays via a novel summation inequality. *IEEE Trans. Autom. Control* **2015**, *60*, 2740–2745.
33. Park, P.; Ko, J.W.; Jeong, C. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica* **2011**, *47*, 235–238.
34. Choi, H.H. Robust stabilization of uncertain fuzzy systems using variable structure system approach. *IEEE Trans. Fuzzy Syst.* **2008**, *16*, 715–724.
35. Castaños, F.; Fridman, L. Analysis and design of integral sliding manifolds for systems with unmatched perturbations. *IEEE Trans. Autom. Control* **2006**, *51*, 853–858.
36. Zhu, B.; Wang, Y.; Zhang, H.; Xie, X. Fuzzy functional observer-based finite-time adaptive sliding-mode control for nonlinear systems with matched uncertainties. *IEEE Trans. Fuzzy Syst.* **2021**, *30*, 918–932.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.