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Abstract. In this work, we study the deterministic Cucker-Smale model with topological interaction. Focusing on the solutions of the corresponding Liouville equation, we show that propagation of chaos holds. Moreover, considering monokinetic solutions, we also obtain a rigorous derivation of the hydrodynamic description given by a pressureless Euler-type system.

Contents

1. Introduction

In recent decades, physics of complex systems has increasingly dealt with the description of groups of animals exhibiting collective behavior, such as flocks of birds, fish schools, locust swarms, and migrating cells ([18],[13],[26],[2],[21]).

From a modeling point of view, these systems offer new challenges and various models have been proposed to describe their interaction (see for example [38],[4],[39]). Among the first to be introduced, the Cucker-Smale model ([14]) describes a bird as a self-propelling particle interacting with its neighbors. In this case, the interaction is such that neighboring birds tend to align their velocities and the strength of the interaction is described through weights which depend on the metric distance between the agents.

Around 2008, a new type of interaction between agents called "topological interaction" was introduced. In [1, 10], the CoBBS group in Rome, after collecting 3D observational data for flocks of starlings, observed that regardless of the density of the flock, each agent interacts on average with its first 6-7

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neighbors. This suggests that the strength of the interaction between agents does not depend on the metric distance between them, but rather on the "topological" distance that takes into account the proximity rank of the latter with respect to the former (see also [7, 9, 20, 33, 35, 28, 36, 41]).

In the following years, deterministic and stochastic models with topological interactions were introduced in the mathematical physics literature. In [5, 6, 15, 16] kinetic models of Boltzmann type are derived for topological interaction models based on jump processes. As far as deterministic models are concerned, in [24] a Cucker-Smale model is introduced where the interaction, instead of being metric as usual, is topological. In [24], the kinetic mean-field and hydrodynamic equations of Euler type are also written and derived in the case of a smoothed version of the model. Indeed, from a mathematical point of view, topological interactions fall outside the case of two-body interaction and present various problems in the derivation of mean-field and hydrodynamic equations. Specifically, the continuity estimate à la Dobrushin (see [17]), valid in the metrical case of regular two-body interactions, does not work here since the solutions are not weakly continuous with respect to the initial data, as shown in [3]. In [3] the existence of the dynamics and the mean-field limit have been rigorously proved for this same model, considering solutions of the limit equation with bounded density and proving that, for positive times t, the Wasserstein distance between the limit solution and the empirical measure at time t tends to zero as the number of particles increases, if this holds at the initial time.

In the present work, we focus on the problems of propagation of chaos and derivation of the hydrodynamic equations for this model. The starting point is the Liouville equation verified by the N-particle system and the aim is to show that the marginals of the N-body distribution function converge to tensorial powers of solutions of a suitable kinetic equation of Vlasov type. With the same approach, we show the validity of the hydrodynamic equations which describe the evolution of monokinetic initial data. The analysis will follow the approach in [22] (see also [31], [30], [29] for numerical considerations and [34] for the case of agent systems).

Due to the aforementioned difficulties, the topological nature of the problem requires different techniques for the proof of these two results. Regarding propagation of chaos, the analysis makes use of another distance between probability measures, called discrepancy distance, in addition to the Wasserstein one. Furthermore, being the dynamics well-defined only for almost all initial data, we need to consider a regularization of the monokinetic initial datum in the proof of the hydrodynamic derivation, obtaining the result for any limiting point with respect to the regularization.

The outline of the paper is as follows: in Section 2 we recall the topological Cucker-Smale model and introduce the associated Liouville equation, as well as recalling the results obtained in [3] which will be useful later on. In Section 3 we provide a proof of the propagation of chaos, which will be a direct consequence of the validity of the law of large numbers. In Section 4 we focus on

the derivation of the hydrodynamic description given by Euler-type equations studying the so-called monokinetic solutions.

2. Model and general framework

A Cucker-Smale type model for the motion of N agents, in the mean-field scaling, is the system

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} p_{ij}(v_j(t) - v_i(t)), \quad i = 1, \dots, N \end{cases} \tag{2.1}$$

where $(x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d$ ($d = 1, 2, 3, \dots$) and the "communication weights" $\{p_{ij}\}_{i,j=1}^N$ are positive functions that take into account the interactions between agents. In classical models, p_{ij} depends only on the euclidean distance $|x_i - x_j|$ between the agents. In topological models the weights depend on the positions of the agents through their rank:

$$p_{ij} \coloneqq K\big(M(x_i, |x_i - x_j|)\big), \tag{2.2}$$

where $K\colon [0,1] \to \mathbb{R}^+$ is a positive Lipschitz continuous non-increasing function, and, for $r > 0$, the rank function

$$M(x_i, r) \coloneqq \frac{1}{N}\sum_{k=1}^{N} \mathcal{X}\{|x_k - x_i| \leqslant r\} \tag{2.3}$$

counts the number of agents at a distance less than or equal to r from x_i, normalized with N. Note that in this case p_{ij} is a piecewise-constant function of the positions of all the agents.

We indicate by $\mathcal{P}(\mathbb{R}^k)$ the space of probability measures on $\mathbb{R}^k$. In the mean-field limit $N \to +\infty$, the one-agent distribution function $f_t = f(t, x, v) \in \mathcal{P}(\mathbb{R}^{2d})$ is expected to verify the equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \nabla_v \cdot (W[Sf_t, f_t](x,v) f(t,x,v)) = 0, \tag{2.4}$$

where $Sf_t(x) \coloneqq \int f_t(x,v)\,\mathrm{d}v \in \mathcal{P}(\mathbb{R}^d)$ denotes the spatial distribution and, given a probability measure $f \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and a probability measure $\rho \in \mathcal{P}(\mathbb{R}^d)$, $W[\rho, f]$ is the mean-field interaction given by

$$W[\rho, f](x,v) \coloneqq \int_{\mathbb{R}^d \times \mathbb{R}^d} K\left(M[\rho](x, |x-y|)\right)\,(w - v) f(y,w)\,\mathrm{d}y\,\mathrm{d}w, \tag{2.5}$$

with

$$M[\rho](x,r) \coloneqq \int_{\bar{B}_r(x)} \mathrm{d}\rho. \tag{2.6}$$

Here and after, $\bar{B}_r(x)$ denotes the closed ball in $\mathbb{R}^d$ of center x and radius r. We also indicate by $\bar{B}_R$ the closed ball $\bar{B}_R(0)$. Note that $M[\rho](x,r) \leqslant 1$ for any $x \in \mathbb{R}^d$ and $r \geqslant 0$.

A weak formulation of equation (2.4) is given requiring that the solution f_t fulfills the identity

$$\int \alpha(x,v)\,\mathrm{d}f_t(x,v) = \int \alpha\left(x(t,x,v), v(t,x,v)\right)\mathrm{d}f_0(x,v) \tag{2.7}$$

for any $\alpha \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, where f_0 is the initial probability measure and $(x(t,x,v), v(t,x,v))$ is the solution of the following Cauchy problem

$$\begin{cases} \dot{x}(t,x,v) = v(t,x,v) \\ \dot{v}(t,x,v) = W[Sf_t, f_t](x(t,x,v), v(t,x,v)) \\ x(0,x,v) = x, \quad v(0,x,v) = v. \end{cases} \tag{2.8}$$

In other words, f_t is the push-forward of f_0 along the flow generated by the one-particle system (2.8), where the force W depends on f_t itself.

Given

$$Z_N = (x_1, \dots, x_N, v_1, \dots, v_N) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN},$$

we define the empirical measure on $\mathbb{R}^d \times \mathbb{R}^d$ as

$$\mu_{Z_N} \coloneqq \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \otimes \delta_{v_i}. \tag{2.9}$$

It is easy to verify that if $Z_N(t) = (x_1(t), \dots, x_N(t), v_1(t), \dots, v_N(t))$ solves (2.1), (2.2), (2.3), then $\mu_{Z_N(t)}$ is a weak solution of (2.4). Namely, $M[S\mu_{Z_N}](x,r)$ is exactly $M(x,r)$ defined in equation (2.3) above. Thus we can rewrite the agent evolution for the Cucker-Smale model (2.1) with topological interactions (2.2), (2.3) as

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = W[S\mu_{Z_N(t)}, \mu_{Z_N(t)}](x_i(t), v_i(t)), \quad i = 1, \dots, N. \end{cases} \tag{2.10}$$

We indicate by $Z_N(t, Z_N) = (X_N(t, Z_N), V_N(t, Z_N))$ the solution of this system with initial datum $Z_N \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$.

In [3], in the framework of the mean-field theory, the rigorous derivation of (2.4) starting from (2.10) is obtained. More precisely, the following theorem is proved.

Theorem 0. [3, Theorems 3.4, 4.3 and 5.2]

Let $K\colon [0,1] \to \mathbb{R}^+$ in (2.5) be a positive Lipschitz continuous non-increasing function. It holds that:

i) except for a set of initial data $Z_N \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$ with Lebesgue measure zero, there exists a unique global in-time solution

$$(X_N(t, Z_N), V_N(t, Z_N)) \in C^1(\mathbb{R}^+, \mathbb{R}^{2dN}) \times C(\mathbb{R}^+, \mathbb{R}^{2dN})$$

of (2.10). Moreover, if $|x_i| \leqslant R_x$ and $|v_i| \leqslant R_v$ for any $i \in \{1, \dots N\}$, we have that

$$|x_i(t)| \leqslant R_x + tR_v, \ |v_i(t)| \leqslant R_v, \quad i \in \{1, \dots, N\}. \tag{2.11}$$

ii) Let $f_0(x,v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a probability density such that $\operatorname{supp}(f_0) \subset \bar{B}_{R_x} \times \bar{B}_{R_v}$. Given $T > 0$, there exists a unique solution, in the weak sense defined by (2.7) and (2.8), of the kinetic equation (2.4) with initial datum f_0, such that $f \in C_w\left([0,T]; L^\infty(\mathbb{R}^d \times \mathbb{R}^d)\right)$, i.e. f_t is a probability measure with bounded density for any $t \geqslant 0$, weakly continuous in t.

Moreover, for any $t \geqslant 0$,

$$\operatorname{supp}(f_t) \subset \bar{B}_{R_x + tR_v} \times \bar{B}_{R_v} \quad \text{and} \quad \|f_t\|_\infty \leqslant \|f_0\|_\infty \mathrm{e}^{d\gamma t}, \tag{2.12}$$

where $\gamma = \int_0^1 K(z)\,\mathrm{d}z$.

iii) If f_0 *is as in item ii), for any* $t > 0$ *the empirical measure* $\mu_{Z_N(t)}$ *associated to the system* (2.10) *weakly converges to* f_t *in the limit* $N \to +\infty$*, provided this is true at time zero and the initial data* Z_N *are chosen such that the dynamics exists, as in item i).*

In the present work, we focus on the statistical description of the dynamical system (2.10), considering, at time 0, N independently and identically distributed particles with law $F_N(0, Z_N) = f_0^{\otimes N}$. At time t the particles are distributed with the law $F_N(t, Z_N)$, weak solution of the N-body Liouville equation

$$\begin{aligned}&\partial_t F_N(t, Z_N) + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F_N(t, Z_N)\\ &+ \sum_{i=1}^N \nabla_{v_i} \cdot \Big(\frac{1}{N} \sum_{j=1}^N K\Big(M[S\mu_{Z_N}](x_i, |x_i - x_j|)\Big)(v_j - v_i) F_N(t, Z_N)\Big) = 0,\end{aligned} \tag{2.13}$$

in the sense that $F_N(t, Z_N)$ is the push-forward of $F_N(0, Z_N)$ along the flow $Z_N(t, Z_N)$. Note that $F_N(t, Z_N)$, for $t > 0$, is symmetric in the exchange of particles.

In the next section we show that, in the limit $N \to +\infty$, for any $s \geqslant 1$ the s-particles marginals of $F_N(t, Z_N)$ factorize, and the limit is described by the solution of the mean-field equation f_t with initial datum f_0. To quantify the convergence, we use the Wasserstein distance $\mathscr{W}_1$ defined as follows. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^k)$ be two probability measures with finite first moments,

$$\mathscr{W}_1(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \int_{\mathbb{R}^k \times \mathbb{R}^k} |x - y|\,\mathrm{d}\pi(x, y), \tag{2.14}$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings of μ and ν, *i.e.* probability measures on the product space with marginals μ and ν in the first and second variable, respectively.

3. Convergence of the marginals

In this section we prove the propagation of chaos for solutions of the N-body Liouville equation (2.13). We briefly explain the meaning of this expression and we refer to [11, 12] for a review, and to [32] for a propagation of chaos result in the case of non-topological Cucker-Smale models.

We introduce the s-particles marginals as follows, with an abuse of notation regarding the order of the space and velocity variables in $F_N(Z_N)$:

$$F_{N:s}(Z_s) = \int F_N(Z_s, z_{s+1}, \dots, z_N) dz_{s+1} \dots dz_N, \qquad s = 1, 2, \dots, N,$$

where $z_i = (x_i, v_i) \in \mathbb{R}^{2d}$. We expect that, if N is large, the details of the individual mutual interactions are negligible, and the description given by $F_{N:s}(t, Z_s)$ is similar to the one given by $f_t^{\otimes s}$, where f_t weakly solves (2.4).

Note that $f_t^{\otimes s}(Z_s)$ weakly solves

$$\partial_t f_t^{\otimes s}(Z_s) + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_t^{\otimes s}(Z_s) + \sum_{i=1}^{s} \nabla_{v_i} \cdot \Big(W[Sf_t, f_t](x_i, v_i) f_t^{\otimes s}(Z_s) \Big) = 0.$$

This last equation describes the law of the system when each particle evolves independently from the others, with the interaction given by the mean-field force defined in (2.5). Then propagation of chaos holds if $F_{N:s}(t, Z_s)$ converges to $f_t^{\otimes s}(Z_s)$ for any $s \geqslant 1$.

We will prove the following result.

Theorem 1 (Propagation of chaos for the topological CS model)**.** *Assume that the interaction function K is as in Theorem 0, namely a positive Lipschitz continuous non-increasing function. Let $f \in C_w([0,T]; L^\infty(\mathbb{R}^{2d}))$ be a weak solution of the kinetic equation (2.4) with initial datum $f_0(x,v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\mathrm{supp}(f_0) \subset \bar{B}_{R_x} \times \bar{B}_{R_v}$. Consider $F_N(t)$ weak solution of (2.13) such that $F_N(0) = f_0^{\otimes N}$.*

For any integer $s \in \{1, \dots, N\}$, it holds that

$$\sup_{t\in[0,T]} \mathscr{W}_1(F_{N:s}(t), f^{\otimes s}(t)) \leqslant s \mathrm{e}^{\lambda(T)\mathcal{K}(1+\|f_0\|_\infty \mathrm{e}^{d\gamma T})} \sqrt{C_d(N)},$$

where $\gamma := \int_0^1 K(z)\,\mathrm{d}z$, $\lambda(T)$ is a constant depending on d, R_x, R_v and T, and $\mathcal{K} := \max(1, \mathrm{Lip}(K), \|K\|_\infty)$, where $\mathrm{Lip}(K)$ is the Lipschitz constant of K, while

$$C_d(N) := \begin{cases} N^{-1/2} & \text{if} \quad d = 1 \\ N^{-1/2}\log(N) & \text{if} \quad d = 2 \\ N^{-1/d} & \text{if} \quad d > 2. \end{cases} \tag{3.1}$$

Remark 1. *In the proof we will need a quantitative version of the law of large numbers, i.e. an estimate of $\mathbb{E}[\mathscr{W}_1(\rho, \mu_{X_N})]$ when $X_N = (x_1, \dots, x_N)$ are N independently and identically distributed $\mathbb{R}^d$-valued random variables with law $\rho \in L^\infty(\mathbb{R}^d)$. This is a widely studied problem in probability, in connection with topics of optimal transport, random matching and combinatorial optimization (see, for instance, the bibliographical notes to Chapter 6 of* [40]*). Here we use the explicit bounds reported by Fournier and Guillin in* [19]*, which imply that $\mathbb{E}[\mathscr{W}_1(\rho, \mu_{X_N})] \leqslant c C_d(N)$, where $C_d(N)$ is defined in (3.1) and c is a constant depending polynomially on the diameter of $\mathrm{supp}(\rho)$ and on d.*

Remark 2. *The time dependence of the constant $\lambda(T)$ in Theorem 1 comes from the growth of the support of the distribution, and therefore grows polynomially in time. In particular, it is also possible to prove the validity of the theorem for times T slowly growing with N.*

In the proof of Theorem 1 we will employ as a technical tool the discrepancy distance, defined as follows:

$$\mathscr{D}(\mu, \nu) := \sup_{x,r>0} \Big| \int_{\bar{B}_r(x)} \mathrm{d}\mu - \int_{\bar{B}_r(x)} \mathrm{d}\nu \Big|, \tag{3.2}$$

for μ, ν two probability measures on $\mathbb{R}^d$. We will need the following results concerning it.

Proposition 0. [3, Propositions 2.4 and 2.5]

i) Let ρ and ν be two probability measures on $\mathbb{R}^d$ with support in a ball $\bar{B}_R$ and such that $\rho \in L^\infty(\mathbb{R}^d)$. Then

$$\mathscr{D}(\nu,\rho) \leqslant c\sqrt{R^{d-1}\|\rho\|_\infty \mathscr{W}_1(\nu,\rho)}, \tag{3.3}$$

where c is a constant that depends on the dimension d.

ii) Given $X_N = (x_1,\dots,x_N)$, $Y_N = (y_1,\dots y_N)$, with $|x_i - y_i| \leqslant \delta$ for some δ and any $i = 1,\dots N$, consider the two empirical measures μ_{X_N} and μ_{Y_N}. Then, for any probability measure $\rho \in L^\infty(\mathbb{R}^d)$ supported on a ball $\bar{B}_R$,

$$\mathscr{D}(\mu_{X_N},\mu_{Y_N}) \leqslant cR^{d-1}\|\rho\|_\infty \delta + c\mathscr{D}(\mu_{Y_N},\rho). \tag{3.4}$$

In the following lemma, we summarize the technical details of the proofs concerning the Lipschitz estimates of the interaction function W defined in (2.5).

Lemma 1. *Given $r_x, r_v > 0$, let $f_1, f_2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be two probability measures supported in $\bar{B}_{r_x} \times \bar{B}_{r_v}$ and $\rho_1, \rho_2 \in \mathcal{P}(\mathbb{R}^d)$ two probability measures supported in $\bar{B}_{r_x}$ and such that $\|\rho_2\|_\infty < +\infty$. For any $\xi_1, \xi_2 \in \bar{B}_{r_x}$ and $\eta_1, \eta_2 \in \bar{B}_{r_v}$ it holds:*

$$\begin{aligned}|W[\rho_1,f_1](\xi_1,\eta_1) - W[\rho_2,f_2](\xi_2,\eta_2)| \leqslant 2r_v \mathrm{Lip}(K)\mathscr{D}(\rho_1,\rho_2) + \\ \left(c\mathrm{Lip}(K)\|\rho_2\|_\infty r_x^{d-1} r_v + \|K\|_\infty\right)\left(\mathscr{W}_1(f_1,f_2) + |\xi_1-\xi_2| + |\eta_1-\eta_2|\right),\end{aligned}$$

where c is a constant that depends only on d.

Proof of Lemma 1. By the triangle inequality, $|W[\rho_1,f_1](\xi_1,\eta_1) - W[\rho_2,f_2](\xi_2,\eta_2)|$ can be bounded by the sum of

$$\begin{aligned}A_1 &:= |W[\rho_1,f_1](\xi_1,\eta_1) - W[\rho_2,f_1](\xi_1,\eta_1)|,\\ A_2 &:= |W[\rho_2,f_1](\xi_1,\eta_1) - W[\rho_2,f_2](\xi_1,\eta_1)|,\\ A_3 &:= |W[\rho_2,f_2](\xi_1,\eta_1) - W[\rho_2,f_2](\xi_2,\eta_2)|.\end{aligned}$$

Estimate of A_1: by the definition of M in (2.6) and the definition of discrepancy distance in (3.2), we have for any $r \geqslant 0$

$$|M[\rho_1](\xi_1,r) - M[\rho_2](\xi_1,r)| \leqslant \mathscr{D}(\rho_1,\rho_2).$$

By the definition of W in (2.5) and the hypothesis on the supports, we get that A_1 is bounded by

$$A_1 \leqslant 2\mathrm{Lip}(K) r_v \mathscr{D}(\rho_1,\rho_2).$$

Estimates of A_2 and A_3: it is easy to prove that, given $x \in \mathbb{R}^d$ and $r_1, r_2 > 0$,

$$|M[\rho_2](x,r_1) - M[\rho_2](x,r_2)| \leqslant c\|\rho_2\|_\infty \left|r_1^d - r_2^d\right| \tag{3.5}$$

and, given $x_1, x_2 \in \mathbb{R}^d$ and $r > 0$,

$$|M[\rho_2](x_1,r) - M[\rho_2](x_2,r)| \leqslant cd\|\rho_2\|_\infty r^{d-1}|x_1 - x_2|. \tag{3.6}$$

By (3.5) and (3.6), the Lipschitz constants of the function

$$K\Big(M[\rho_2](x,|x-\xi|)\Big)(v-\eta)$$

in the variables $x, \xi \in \bar{B}_{r_x}$ and $v, \eta \in \bar{B}_{r_v}$ are bounded by $c\mathrm{Lip}(K)\|\rho_2\|_\infty r_x^{d-1} r_v$ and $\|K\|_\infty$, respectively. Then, by the definition (2.14) of Wasserstein distance,

$$A_2 \leqslant (c\mathrm{Lip}(K)\|\rho_2\|_\infty r_x^{d-1} r_v + \|K\|_\infty)\mathscr{W}_1(f_1, f_2),$$

and

$$A_3 \leqslant (c\mathrm{Lip}(K)\|\rho_2\|_\infty r_x^{d-1} r_v + \|K\|_\infty)(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|).$$

□

Proof of Theorem 1. For any $\Sigma_N = (y_1, \dots, y_N, w_1, \dots, w_N) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$, we consider

$$\Sigma_N(t, \Sigma_N) = (y_1(t), \dots, y_N(t), w_1(t), \dots, w_N(t)) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN},$$

where, for $i \in \{1, \dots, N\}$, $(y_i(t), w_i(t))$ has initial datum (y_i, w_i) and evolves independently with the mean-field interaction:

$$\begin{cases} \dot{y}_i(t) = w_i(t) \\ \dot{w}_i(t) = W[Sf_t, f_t](y_i(t), w_i(t)), \quad i = 1, \dots, N. \end{cases} \tag{3.7}$$

We associate to such $\Sigma_N(t, \Sigma_N)$ the empirical measure $\mu_{\Sigma_N(t)}$ as in (2.9).

We define a coupling π_t^N between $F_N(t)$ and $f_t^{\otimes N}$ in the following way: at time $t = 0$, it is given by

$$\pi_0^N(Z_N, \Sigma_N) := f_0^{\otimes N}(Z_N)\delta(Z_N - \Sigma_N).$$

For positive times, π_t^N is given by the push-forward of π_0^N along the product of the flows given by (2.10) and (3.7), *i.e.* for any $\varphi \in C_b(\mathbb{R}^{2dN} \times \mathbb{R}^{2dN})$

$$\int \varphi(Z_N, \Sigma_N)\, \mathrm{d}\pi_t^N(Z_N, \Sigma_N) = \int \varphi(Z_N(t, Z_N), \Sigma_N(t, Z_N))\, f_0^{\otimes N}(Z_N)\, \mathrm{d}Z_N.$$

Next, given $i \in \{1, \dots, N\}$, we introduce the quantity

$$D_N(t) := \int \Big(|x_i - y_i| + |v_i - w_i| \Big)\, \mathrm{d}\pi_t^N(Z_N, \Sigma_N),$$

which does not depend on i, thanks to the symmetry of the law. We prove the weak convergence of the s-marginals $F_{N:s}$ to $f^{\otimes s}$ by showing that $D_N(t) \to 0$: namely, using the symmetry of π_t^N,

$$\mathscr{W}_1(F_{N:s}(t), f_t^{\otimes s}) \leqslant \sum_{i=1}^{s} \int \Big(|x_i - y_i| + |v_i - w_i| \Big)\, \mathrm{d}\pi_t^N(Z_N, \Sigma_N) \leqslant s D_N(t),$$

where we used that $|Z_s - \Sigma_s| \leqslant \sum_{i=1}^s (|x_i - y_i|_d + |v_i - w_i|_d)$.

From the definition of π_t, we have, for any $i \in \{1, \dots, N\}$,

$$D_N(t) = \int \Big(|x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| \Big)\, \mathrm{d}f_0^{\otimes N}(Z_N),$$

where $(x_i(t), v_i(t))$ for $i = 1, \dots N$ solves (2.10) with initial datum Z_N, and $(y_i(t), w_i(t))$ for $i = 1, \dots N$ solves the decoupled system (3.7) with the same initial datum Z_N. It follows that

$$D_N(t) \leqslant \int \delta(t, Z_N)\, \mathrm{d}f_0^{\otimes N}(Z_N), \tag{3.8}$$

with $\delta(t, Z_N) := \max_{i=1,\dots,N}(|x_i(t) - y_i(t)| + |v_i(t) - w_i(t)|)$.

Since $(x_i(t), v_i(t))$ and $(y_i(t), w_i(t))$ have the same initial conditions, it holds that

$$|x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| \leqslant \int_0^t |v_i(\tau) - w_i(\tau)| \, \mathrm{d}\tau + \int_0^t \Big| W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau)) \Big| \, \mathrm{d}\tau.$$

By Lemma 1 with $r_v = R_v$, $r_x = R_x + \tau R_x$, we bound the last integrand by

$$2R_v \mathrm{Lip}(K) \mathscr{D}(S\mu_{Z_N(\tau)}, Sf_\tau) + (c\mathrm{Lip}(K)\|Sf_\tau\|_\infty (R_x + \tau R_v)^{d-1} R_v + \|K\|_\infty) \big(\mathscr{W}_1(\mu_{Z_N(\tau)}, f_\tau) + \delta(\tau, Z_N) \big).$$

By choosing the coupling $\pi = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(\tau)} \delta_{v_i(\tau)} \delta_{y_i(\tau)} \delta_{w_i(\tau)}$ in definition (2.14), we can estimate the Wasserstein distance between the two empirical measures $\mu_{Z_N(\tau)}$ and $\mu_{\Sigma_N(\tau)}$ by $\delta(\tau, Z_N)$, so that

$$\mathscr{W}_1(\mu_{Z_N(\tau)}, f_\tau) \leqslant \delta(\tau, Z_N) + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau).$$

By the triangle inequality

$$\mathscr{D}(S\mu_{Z_N(\tau)}, Sf_\tau) \leqslant \mathscr{D}(S\mu_{Z_N(\tau)}, S\mu_{\Sigma_N(\tau)}) + \mathscr{D}(S\mu_{\Sigma_N(\tau)}, Sf_\tau),$$

and, by (3.4) with $\rho = Sf_\tau$,

$$\mathscr{D}(S\mu_{\Sigma_N(\tau)}, S\mu_{Z_N(\tau)}) \leqslant c(R_x + \tau R_v)^{d-1} \|Sf_\tau\|_\infty \delta(\tau, Z_N) + c\mathscr{D}(Sf_\tau, S\mu_{\Sigma_N(\tau)}),$$

where, by (3.3),

$$\mathscr{D}(Sf_\tau, S\mu_{\Sigma_N(\tau)}) \leqslant c\sqrt{(R_x + \tau R_v)^{d-1} \|Sf_\tau\|_\infty \mathscr{W}_1(Sf_\tau, S\mu_{\Sigma_N(\tau)})}.$$

In the sequel we indicate by $\lambda(\tau)$ any positive, increasing polynomial function of τ, that depends on d, R_v, R_v, *i.e.* on the support of f_τ and Sf_τ, and with c any constant that depends at most on the dimension d.

Collecting the previous estimates, and using that $x^{1/2} \leqslant (1+x)/2$ for $x \geqslant 0$, we get, for a suitable $\lambda(\tau)$,

$$\delta(t, Z_N) \leqslant \int_0^t \mathcal{K}\lambda(\tau)(1 + \|Sf_\tau\|_\infty) \Big(\delta(\tau, Z_N) + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) + \sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} \Big) \, \mathrm{d}\tau,$$

where we used that $\mathcal{K} = \max(1, \mathrm{Lip}(K), \|K\|_\infty)$. Note that $\|Sf_\tau\|_\infty \leqslant cR_v^d \|f_\tau\|_\infty$ and, by Theorem 0, $\|f_\tau\|_\infty \leqslant \mathrm{e}^{d\gamma\tau} \|f_0\|_\infty$. By the Grönwall's lemma and the fact that $x < \mathrm{e}^x$,

$$\delta(t, Z_N) \leqslant \mathrm{e}^{\mathcal{K}\lambda(T)(1 + \|f_0\|_\infty \mathrm{e}^{d\gamma T})} \int_0^t \Big(\mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) + \sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} \Big) \, \mathrm{d}\tau. \tag{3.9}$$

To estimate (3.8), we have to evaluate the expected value of the integrand in (3.9) w.r.t. the probability measure $\mathrm{d}f_0^{\otimes N}(Z_N)$. Since the empirical measure $\mu_{\Sigma_N(\tau)}$ is supported in $\Sigma_N(t, Z_N)$, then by (2.7),

$$\int \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) \, \mathrm{d}f_0^{\otimes N}(Z_N) = \int \mathscr{W}_1(\mu_{Z_N}, f_\tau) \, \mathrm{d}f_\tau^{\otimes N}(Z_N) \leqslant \lambda(\tau) C_{2d}(N),$$

where the rate $C_d(N)$, as defined in (3.1), is obtained by using the Fournier and Guillin bound in [19], and $\lambda(\tau)$ grows polynomially in τ. By concavity we also obtain

$$\int \sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)}\,\mathrm{d}f_0^{\otimes N}(Z_N) = \int \sqrt{\mathscr{W}_1(S\mu_{Z_N}, Sf_\tau)}\,\mathrm{d}f_\tau^{\otimes N}(Z_N) \leqslant \lambda(\tau)\sqrt{C_d(N)}.$$

We finally arrive at

$$D_N(t) \leqslant \mathrm{e}^{\mathcal{K}\lambda(T)(1+\|f_0\|_\infty \mathrm{e}^{d\gamma T})}\left(\sqrt{C_d(N)} + C_{2d}(N)\right),$$

from which the thesis follows. □

4. Euler systems associated to monokinetic initial data

In this section we study the hydrodynamic description of the topological Cucker-Smale model, by considering the following pressureless Euler-type system for $(\rho(t,x), u(t,x)) : [0,T]\times\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^d$:

$$\begin{cases} \partial_t\rho(t,x) + \nabla_x\cdot(\rho(t,x)u(t,x)) = 0 \\ \partial_t u(t,x) + (u(t,x)\cdot\nabla_x)u(t,x) = \\ \qquad\qquad \displaystyle\int_{\mathbb{R}^d} K\Big(M[\rho(t)](x,|x-y|)\Big)(u(t,y)-u(t,x))\rho(t,y)\,\mathrm{d}y \\ (\rho(0,x), u(0,x)) = (\rho_0(x), u_0(x)), \end{cases} \tag{4.1}$$

where $K(M[\rho])$ is defined as in (2.5) and (2.6), and $(\rho_0(x), u_0(x))$ is a regular compactly supported initial datum.

In the following, given $T>0$, we assume that (4.1) admits a unique solution $(\rho(t,x), u(t,x))$ which is regular and compactly supported for $t\in[0,T]$. We leave this fact as a hypothesis, however see [23] for a proof in the case of the non-topological Cucker Smale model (see also [25, 37, 27]).

It is not difficult to show that, if $(\rho(t,x), u(t,x))$ is a regular solution of (4.1), then

$$f(t,x,v) := \rho(t,x)\delta(v-u(t,x)) \tag{4.2}$$

is a weak solution of the topological Vlasov equation in the sense of (2.7), (2.8) with measure-valued initial datum $f(0,x,v) = \rho_0(x)\delta(v-u_0(x))$: these are called monokinetic solutions. Note that in this case the field $W[Sf_t, f_t](x,v)$ is regular, so the flow in (2.8) is well-defined.

We will show how to obtain solutions of the Euler system (4.1) starting from solutions of the topological Liouville equation (2.13). The main obstacle is that, as stated in Theorem 0, the flow for the particle system is not defined for every initial datum, so in general it is not possible to consider a measure-valued solution of the Liouville equation (2.13). To overcome this problem we consider a regularization of the monokinetic initial datum:

$$f_0^\varepsilon(x,v) = (\rho_0(x)\delta(v-u_0(x)))\ast_{x,v}\eta_\varepsilon(x,v), \tag{4.3}$$

where η is a $C^\infty(\mathbb{R}^{2d})$ compactly supported mollifier and $\eta_\varepsilon(z) = \eta(z/\varepsilon)/\varepsilon^{2d}$.

Our next result shows that after relaxing the regularity of the initial monokinetic measure as in (4.3), in the limit $N \to +\infty$, uniformly in ε, the marginals of solutions of the Liouville equation tend, in weak sense, to the tensorial powers of a monokinetic measure (4.2), built up out of the solution of the associated Euler system.

Theorem 2 (From particles to Euler)**.** *Let us consider a regular solution $(\rho(t,x), u(t,x))$, $t \in [0,T]$, of the Euler system (4.1) with regular compactly supported initial data $(\rho_0(x,v), u_0(x,v))$. We suppose moreover that, for any $t \in [0,T]$ the supports of $\rho(t,x)$ and $u(t,x)$ are contained in $\bar{B}_{R_x+tR_v}$ and that $\|u(t,x)\|_\infty \leqslant R_v$, for some $R_x, R_v > 0$. Let $F_N^\varepsilon(t)$ be a weak solution of the Liouville equation (2.13) with initial datum $F_N^\varepsilon(0) := (f_0^\varepsilon)^{\otimes N}$, with f_0^ε as in (4.3).*

Then, for any $t \in [0,T]$ and any $s \in \{1,\dots,N\}$, setting $V_s = (v_1,\dots,v_s)$, we have

$$\mathscr{W}_1\left(\int_{\mathbb{R}^{ds}} F_{N:s}^\varepsilon(t)\,\mathrm{d}V_s, \rho(t)^{\otimes s}\right) \leqslant s\mathrm{e}^{\mathcal{K}\lambda(T)\int_0^T(1+\|\rho_\tau\|_\infty)\,\mathrm{d}\tau}\left(\sqrt{C_d(N)} + \sqrt[4]{\varepsilon}\right),$$

and, for any Lipschitz function $\Phi : \mathbb{R}^{sd} \times \mathbb{R}^{sd} \to \mathbb{R}$, setting $X_s = (x_1,\dots,x_s)$,

$$\left|\int \mathrm{d}X_s\,\mathrm{d}V_s \Phi(X_s,V_s) F_{N:s}^\varepsilon(t,X_s,V_s) - \int \mathrm{d}X_s \prod_{i=1}^s \rho(t,x_i)\Phi(X_s, u_1(t,x_1)\dots u_s(t,x_s))\right|$$
$$\leqslant \mathrm{Lip}(\Phi) s\mathrm{e}^{\mathcal{K}\lambda(T)\int_0^T(1+\|\rho_\tau\|_\infty)\,\mathrm{d}\tau}\left(\sqrt{C_d(N)} + \sqrt[4]{\varepsilon}\right),$$

where $\lambda(T)$ is a constant depending on d, R_x, R_v, T and $C_d(N)$ is defined in (3.1).

Remark 3. *Note that the statement doesn't involve the Vlasov equation and that ε and N are independent, so the convergence holds also for $N \to \infty$ and any sequence of limiting points as $\varepsilon \to 0$ of the distributions $\{F_N^\varepsilon(t)\}_{N\in\mathbb{N}}$. In particular the rate of convergence is $O(\sqrt{C_d(N)})$ along any sequence with $\varepsilon = \varepsilon_N = O(C_d(N))^2$. Note the algebraic behavior in N, in contrast with the logarithmic one, when the dynamics is mollified as, e.g., in* [34, 8].

Remark 4. *Along the same lines, we can also generalize the content of Theorem 2, proving a convergence result for marginals $F_{N:s}^\varepsilon$ of the Liouville equation with general initial conditions $(g_0^\varepsilon)^{\otimes N}$ such that $\mathscr{W}_1\left(g_0^\varepsilon, \rho_0(x)\delta(v-u_0(x))\right) \to 0$ as $\varepsilon \to 0$.*

To prove Theorem 2, we need the following stability result for solutions of the Vlasov system.

Proposition 1. *Given $T > 0$, for $i \in \{1,2\}$, let $f^i \in C_w([0,T], \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d))$ be two weak solutions of the topological Vlasov equation (2.4) with initial data $f_0^i \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ such that Sf_t^i are well-defined and belong to $L^\infty(\mathbb{R}^d)$. Assume moreover that $\mathrm{supp}(f_t^i) \subset \bar{B}_{R_x+R_v t} \times \bar{B}_{R_v}$ for $t > 0$ and $R_x, R_v > 0$.*

Then, for $t \in [0,T]$,

$$\mathscr{W}_1\left(f_t^1, f_t^2\right) \leqslant \mathrm{e}^{\mathcal{K}\lambda(T)\min_{i=1,2}\int_0^T(1+\|Sf_\tau^i\|_\infty)\,\mathrm{d}\tau} \max\left\{\mathscr{W}_1(f_0^1,f_0^2), \sqrt{\mathscr{W}_1(f_0^1,f_0^2)}\right\}, \tag{4.4}$$

where $\mathcal{K} := \max(1, \mathrm{Lip}(K), \|K\|_\infty)$ *and* $\lambda(T)$ *is a constant depending on* d, R_x, R_v *and* T.

The proof, which requires a nontrivial extension of inequality (3.4), is given in the Appendix.

Proof of Theorem 2. To prove the thesis, it is sufficient to establish the following estimate for $t \in [0, T]$:

$$\mathscr{W}_1(F^\varepsilon_{N:s}(t), f_t^{\otimes s}) \leqslant s\mathrm{e}^{\mathcal{K}\lambda(T)\int_0^T (1+\|\rho(\tau)\|_\infty)\,\mathrm{d}\tau}\left(\sqrt{C_d(N)} + \sqrt[4]{\varepsilon}\right), \tag{4.5}$$

where f is the monokinetic solution (4.2) of the Vlasov equation associated to the solution $(\rho(t), u(t))$ of the Euler system (4.1) and $\lambda(T)$ is a constant that depends on d, R_v, $R_x + TR_v$.

Let $f^\varepsilon(t)$ be the solution of the topological Vlasov equation (2.4) with initial datum f_0^ε. By the triangle inequality, we have

$$\mathscr{W}_1\Big(F^\varepsilon_{N:s}(t), f_t^{\otimes s}\Big) \leqslant \mathscr{W}_1\Big(F^\varepsilon_{N:s}(t), (f_t^\varepsilon)^{\otimes s}\Big) + s\mathscr{W}_1\Big(f_t^\varepsilon, f_t\Big),$$

where we are using that $\mathscr{W}_1((f_t^\varepsilon)^{\otimes s}, f_t^{\otimes s}) \leqslant s\mathscr{W}_1(f_t^\varepsilon, f_t)$.

The second term is managed by the stability estimate (4.4) in Proposition 1, which gives

$$\mathscr{W}_1\Big(f_t^\varepsilon, f_t\Big) \leqslant \mathrm{e}^{\mathcal{K}\lambda(T)\int_0^T (1+\|\rho(\tau)\|_\infty)\,\mathrm{d}\tau} \max\left\{\mathscr{W}_1(f_0^\varepsilon, f_0), \sqrt{\mathscr{W}_1(f_0^\varepsilon, f_0)}\right\}. \tag{4.6}$$

By choosing $\pi(\mathrm{d}z, \mathrm{d}z') = \varepsilon^{-2d}\eta((z-z')/\varepsilon) f_0(z)\,\mathrm{d}z\,\mathrm{d}z'$ in the definition (2.14) of the Wasserstein distance, we can estimate $\mathscr{W}_1(f_0^\varepsilon, f_0)$ with $\varepsilon\int |z|\eta(z)\,\mathrm{d}z$. Then for $\varepsilon < 1$, we estimate the maximum in (4.6) by $c\sqrt{\varepsilon}$, where c depends only on η.

The estimate of the first term is similar to the one given in the proof of Theorem 1. In this case we have

$$\mathscr{W}_1\Big(F^\varepsilon_{N:s}(t), (f_t^\varepsilon)^{\otimes s}\Big) \leqslant s\int_{\mathbb{R}^{2dN}} \delta^\varepsilon(t, Z_N)\,\mathrm{d}(f_0^\varepsilon)^{\otimes N}(Z_N),$$

where $\delta^\varepsilon(t, Z_N) := \max_{i=1,\dots,N}(|x_i(t) - y_i^\varepsilon(t)| + |v_i(t) - w_i^\varepsilon(t)|)$, with the independent flow $\Sigma^\varepsilon(t) := (y_i^\varepsilon(t), w_i^\varepsilon(t)) = (y_i^\varepsilon(t, Z_N), w_i^\varepsilon(t, Z_N))$ solving

$$\begin{cases} \dot{y}_i^\varepsilon(t) = w_i^\varepsilon(t) \\ \dot{w}_i^\varepsilon(t) = W[Sf_t^\varepsilon, f_t^\varepsilon](y_i^\varepsilon(t), w_i^\varepsilon(t)), \quad i = 1, \dots, N. \end{cases}$$

This time we get

$$\begin{aligned} |x_i(t) - y_i^\varepsilon(t)| + |v_i(t) - w_i^\varepsilon(t)| &\leqslant \int_0^t |v_i(\tau) - w_i^\varepsilon(\tau)|\,\mathrm{d}\tau \\ &+ \int_0^t \Big|W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau^\varepsilon, f_\tau^\varepsilon](y_i^\varepsilon(\tau), w_i^\varepsilon(\tau))\Big|\,\mathrm{d}\tau. \end{aligned}$$

In order to avoid terms in $\|Sf_\tau^\varepsilon\|_\infty$, which could diverge for $\varepsilon \to 0$, we use carefully the triangle inequality in the last integrand, which we bound by

$$\begin{aligned} &|W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[\rho_\tau, \mu_{\Sigma^\varepsilon_N(\tau)}](x_i(\tau), v_i(\tau))| + \\ &|W[\rho_\tau, \mu_{\Sigma^\varepsilon_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau^\varepsilon, f_\tau^\varepsilon](y_i^\varepsilon(\tau), w_i^\varepsilon(\tau))|. \end{aligned}$$

By using that $\mathscr{W}_1(\mu_{Z_N(\tau)}, \mu_{\Sigma^\varepsilon_N(\tau)}) \leqslant \delta^\varepsilon(\tau, Z_N)$ and $\mathcal{K} = \max\{1, \mathrm{Lip}(K), \|K\|_\infty\}$, by Lemma 1, we estimate the first term with

$$2R_v\mathcal{K}\mathscr{D}(S\mu_{Z_N(\tau)}, \rho_\tau) + \mathcal{K}\lambda(\tau)\delta^\varepsilon(\tau, Z_N),$$

and the second one with

$$2R_v\mathscr{D}(\rho_\tau, Sf^\varepsilon_\tau) + \mathcal{K}\lambda(\tau)\big(\mathscr{W}_1(\mu_{\Sigma^\varepsilon_N(\tau)}, f^\varepsilon_\tau) + \delta^\varepsilon(\tau, Z_N)\big). \tag{4.7}$$

We now estimate the discrepancies in the previous two equations. By the triangle inequality

$$\mathscr{D}(S\mu_{Z_N(\tau)}, \rho_\tau) \leqslant \mathscr{D}(S\mu_{Z_N(\tau)}, S\mu_{\Sigma^\varepsilon_N(\tau)}) + \mathscr{D}(S\mu_{\Sigma^\varepsilon_N(\tau)}, \rho_\tau),$$

in which, by inequality (3.4),

$$\mathscr{D}(S\mu_{Z_N(\tau)}, S\mu_{\Sigma^\varepsilon_N(\tau)}) \leqslant \lambda(\tau)\|\rho_\tau\|_\infty\delta^\varepsilon(\tau, Z_N) + c\mathscr{D}(S\mu_{\Sigma^\varepsilon_N(\tau)}, \rho_\tau).$$

We bound the last term by using inequality (3.3):

$$\mathscr{D}(S\mu_{\Sigma^\varepsilon_N(\tau)}, \rho_\tau) \leqslant \lambda(\tau)(1 + \|\rho_\tau\|_\infty)\Big(\sqrt{\mathscr{W}_1(S\mu_{\Sigma^\varepsilon_N(\tau)}, Sf^\varepsilon_\tau)} + \sqrt{\mathscr{W}_1(Sf^\varepsilon_\tau, \rho_\tau)}\Big).$$

Again by inequality (3.3), we estimate the discrepancy in (4.7) with

$$\mathscr{D}(\rho_\tau, Sf^\varepsilon_\tau) \leqslant \lambda(\tau)(1 + \|\rho_\tau\|_\infty)\sqrt{\mathscr{W}_1(Sf^\varepsilon_\tau, \rho_\tau)}.$$

Note that, by (4.6), $\mathscr{W}_1(Sf^\varepsilon_\tau, \rho_\tau)$ is of order $\sqrt{\varepsilon}$.

Collecting these estimates together, and using Grönwall's lemma, we arrive at

$$\begin{aligned}\delta^\varepsilon(t, Z_N) &\leqslant \mathrm{e}^{\mathcal{K}\lambda(T)\int_0^T(1+\|\rho_\tau\|_\infty)\,\mathrm{d}\tau} \\ &\quad\times\Big(\sqrt[4]{\varepsilon} + \int_0^T \mathrm{d}\tau \int \Big(\sqrt{\mathscr{W}_1(Sf^\varepsilon_\tau, S\mu_{\Sigma^\varepsilon_N(\tau)})} + \mathscr{W}_1(\mu_{\Sigma^\varepsilon_N(\tau)}, f^\varepsilon_\tau)\Big)\,\mathrm{d}f_0^{\otimes N}(Z_N)\Big).\end{aligned}$$

Using again the Fournier and Guillin bound in (3.1) we get the estimate (4.5), from which the thesis follows.

□

Appendix: proof of Proposition 1

We consider, for $t \in [0, T]$, two weak solutions f^1_t and f^2_t of (2.4) in the weak sense specified by (2.7) and (2.8), with support on $\bar{B}_{R_x+tR_v} \times \bar{B}_{R_v}$. We indicate by $Z^i(t, z)$, $i = 1, 2$, the corresponding flows and we assume that $\min_{i=1,2}\int_0^T \|Sf^i_\tau\|_\infty\,\mathrm{d}\tau$ is attained for $i = 1$. We define the intermediate dynamics $(\tilde{f}(t, z), \tilde{Z}(t, z))$, where $\tilde{f}(t)$ is the push-forward of f^2_0 by the flow $\tilde{Z}(t, z) = (\tilde{x}(t, x, v), \tilde{y}(t, x, v))$ which solves

$$\begin{cases}\dot{\tilde{x}}(t, x, v) = \tilde{v}(t, x, v) \\ \dot{\tilde{v}}(t, x, v) = W[Sf^1_t, \tilde{f}_t](\tilde{x}(t, x, v), \tilde{v}(t, x, v)) \\ \tilde{x}(0, x, v) = x, \quad \tilde{v}(0, x, v) = v.\end{cases}$$

From the hypothesis, for $i = 1, 2$, the spatial density Sf^i_t is bounded, and then, by (3.5) and (3.6), the field $K\,(M[Sf^i(t)](x, |x - y|))$ is locally Lipschitz in x and y. Hence $(Z^i(t, z))$, $i = 1, 2$, exists for any $t \in [0, T]$, and also the couple $(\tilde{f}(t, z), \tilde{Z}(t, z))$ is well-defined.

As in previous sections, we indicate by $\lambda(\tau)$ any constant that depends on d, R_v, $R_x + \tau R_v$. We have

$$\mathscr{W}_1\left(f_t^1, f_t^2\right) \leqslant \mathscr{W}_1\left(f_t^1, \tilde{f}_t\right) + \mathscr{W}_1\left(\tilde{f}_t, f_t^2\right). \tag{4.8}$$

The first term is under control since the continuity estimate à la Dobrushin holds. Namely, from Lemma 1,

$$|W[Sf_\tau^1, f_\tau^1](x,v) - W[Sf_\tau^1, \tilde{f}_\tau](x,v)| \leqslant \mathcal{K}\lambda(\tau)(1 + \|Sf_\tau^1\|_\infty)\mathscr{W}_1(f_\tau^1, \tilde{f}_\tau),$$

from which

$$|Z^1(t,z) - \tilde{Z}(t,\tilde{z})| \leqslant |z - \tilde{z}| + \int_0^t \mathcal{K}\lambda(\tau)(1 + \|Sf_\tau^1\|_\infty)\big(\mathscr{W}_1(f_\tau^1, \tilde{f}_\tau) + |Z^1(\tau,z) - \tilde{Z}(\tau,\tilde{z})|\big)\,\mathrm{d}\tau,$$

which allows us to obtain

$$\mathscr{W}_1\left(f_t^1, \tilde{f}_t\right) \leqslant \mathrm{e}^{\lambda(T)\mathcal{K}\int_0^t (1+\|Sf_\tau^1\|_\infty)\,\mathrm{d}\tau}\mathscr{W}_1\left(f_0^1, f_0^2\right), \quad t \in [0,T]. \tag{4.9}$$

Concerning the second term in the inequality (4.8), we have

$$\mathscr{W}_1(\tilde{f}_t, f_t^2) \leqslant \int_{\mathbb{R}^{2d}} |Z^2(t,z) - \tilde{Z}(t,z)|\,\mathrm{d}f_0^2(z) \leqslant \delta(t) \coloneqq \sup_{z \in \mathrm{supp}(f_0^2)} |Z^2(t,z) - \tilde{Z}(t,z)|.$$

To estimate $\delta(t)$, we note that

$$|Z^2(t,z) - \tilde{Z}(t,z)| \leqslant \int_0^t \left(\delta(\tau) + \left|W[Sf_\tau^2, f_\tau^2](Z^2(\tau,z)) - W[Sf_\tau^1, \tilde{f}_\tau](\tilde{Z}(\tau,z))\right|\right)\mathrm{d}\tau.$$

Using Lemma 1, we estimate the last term in the integrand by

$$2R_v\mathcal{K}\mathscr{D}(Sf_\tau^2, Sf_\tau^1) + \mathcal{K}\lambda(\tau)(1 + \|Sf_\tau^1\|_\infty)(\mathscr{W}_1(f_\tau^2, \tilde{f}_\tau) + \delta(\tau)). \tag{4.10}$$

Hence we need only to estimate $\mathscr{D}(Sf_\tau^2, Sf_\tau^1)$. By the triangle inequality

$$\mathscr{D}(Sf_\tau^2, Sf_\tau^1) \leqslant \mathscr{D}(Sf_\tau^2, S\tilde{f}_\tau) + \mathscr{D}(S\tilde{f}_\tau, Sf_\tau^1).$$

The second term can be bounded by using inequality (3.3), obtaining that

$$\mathscr{D}(S\tilde{f}_\tau, Sf_\tau^1) \leqslant c\sqrt{\|Sf_\tau^1\|_\infty (R_x + \tau R_v)^{d-1}\mathscr{W}_1(S\tilde{f}_\tau, Sf_\tau^1)}. \tag{4.11}$$

We now establish an estimate for $\mathscr{D}(Sf_\tau^2, S\tilde{f}_\tau)$ of the type of inequality (3.4). For any $z_0, z \in \mathbb{R}^{2d}$, by definition of $\delta(t)$,

$$|z_0 - Z^2(t,z)| - \delta(t) \leqslant |z_0 - \tilde{Z}(t,z)| \leqslant |z_0 - Z^2(t,z)| + \delta(t).$$

Let X be the set of functions $\phi \in C([0,+\infty), \mathbb{R})$, with first derivative continuous up to a finite number of jumps. Given $\phi \in X$ and $\delta > 0$, let $\phi_\delta(r) \coloneqq \phi^+(r+\delta) - \phi^-(r-\delta)$, where, denoting by $\tilde{\phi}$ the function $\tilde{\phi}(r) \coloneqq \int_0^r |\phi'(s)|\,\mathrm{d}s$, we have defined

$$\phi^\pm(r) \coloneqq \begin{cases} \dfrac{1}{2}(\tilde{\phi}(r) \pm \phi(r)), & \text{if } r \geqslant 0, \\ \pm\dfrac{1}{2}\phi(0), & \text{if } r < 0. \end{cases}$$

It can be proved (see [3, Lemma 2.2]) that

$$\mathscr{D}(\rho_1,\rho_2)=\sup_{\phi\in X:\,\|\phi\|_X\leqslant 1}\sup_x\int\phi\big(|x-y|\big)\big(\mathrm{d}\rho_1(y)-\mathrm{d}\rho_2(y)\big),$$

where $\|\phi\|_X\coloneqq\int_0^{+\infty}|\phi'(r)|\,\mathrm{d}r$.

Fixed $x_0\in\mathbb{R}^d$, we define $\Phi(x)=\phi(|x-x_0|)$ and Φ_δ in the same way. Then, it is not difficult to see that (see [3, Lemma 2.3])

$$\Phi\left(X^2(t,z)\right)\leqslant\Phi_\delta\left(\tilde{X}(t,z)\right).$$

Hence

$$\begin{aligned}
&\int\Phi\left(\mathrm{d}Sf^2_\tau-\mathrm{d}S\tilde{f}_\tau\right)=\int\Phi(x)\left(\mathrm{d}f^2(z)-\mathrm{d}\tilde{f}(z)\right)\\
&=\int\Big(\Phi(X^2(\tau,z))-\Phi(\tilde{X}(\tau,z))\Big)\,\mathrm{d}f^2_0(z)\\
&\leqslant\int\Big(\Phi_\delta(\tilde{X}(\tau,z))-\Phi(\tilde{X}(\tau,z))\Big)\,\mathrm{d}f^2_0(z)=\int(\Phi_\delta(x)-\Phi(x))\,\mathrm{d}S\tilde{f}_\tau(x)\\
&=\int(\Phi_\delta-\Phi)\left(\mathrm{d}S\tilde{f}_\tau-\mathrm{d}Sf^1_\tau\right)+\int(\Phi_\delta-\Phi)\,\mathrm{d}Sf^1_\tau.
\end{aligned}$$

Since $\Phi_\delta-\Phi\in X$, the first term is bounded by $c\mathscr{D}(S\tilde{f}_\tau,Sf^1_\tau)$, while the second can be easily bounded by $c(R_x+\tau R_v)^{d-1}\delta(\tau)\|Sf^1_\tau\|_\infty$. We conclude that

$$\mathscr{D}(Sf^2_\tau,S\tilde{f}_\tau)\leqslant\lambda(\tau)\delta(\tau)\|Sf^1_\tau\|_\infty+c\mathscr{D}(S\tilde{f}_\tau,Sf^1_\tau).\tag{4.12}$$

Collecting estimates (4.10), (4.11) and (4.12), we arrive at

$$\delta(t)\leqslant\int_0^t\lambda(\tau)\mathcal{K}(1+\|Sf^1_\tau\|_\infty)\Big(\delta(\tau)+\sqrt{\mathscr{W}_1(Sf^1_\tau,S\tilde{f}_\tau)}\Big)\,\mathrm{d}\tau.$$

Using the Grönwall's lemma and that $x\leqslant\mathrm{e}^x$, for $t\in[0,T]$,

$$\begin{aligned}
\mathscr{W}_1\left(\tilde{f}_t,f^2_t\right)\leqslant\delta(t)&\leqslant\mathrm{e}^{\lambda(T)\mathcal{K}\int_0^T(1+\|Sf^1_\tau\|_\infty)\,\mathrm{d}\tau}\\
&\quad\times\int_0^T\sqrt{\mathscr{W}_1(Sf^1_\tau,S\tilde{f}_\tau)}\,\mathrm{d}\tau.
\end{aligned}$$

The thesis follows from the last inequality, together with (4.8) and (4.9).

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