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► **To cite this version:**

Dario Benedetto, Thierry Paul, Stefano Rossi. Propagation of chaos and hydrodynamic description for topological models. 2023. hal-04102006v2

HAL Id: hal-04102006

<https://hal.science/hal-04102006v2>

Preprint submitted on 1 Aug 2023 (v2), last revised 13 Jan 2024 (v3)

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PROPAGATION OF CHAOS AND HYDRODYNAMIC DESCRIPTION FOR TOPOLOGICAL MODELS

DARIO BENEDETTO, THIERRY PAUL, AND STEFANO ROSSI

ABSTRACT. In this work, we study the deterministic Cucker-Smale model with topological interaction. Focusing on the solutions of the corresponding Liouville equation, we show that propagation of chaos holds. Moreover, by looking at the monokinetic solutions, we also obtain a rigorous derivation of the hydrodynamic description given by a pressureless Euler-type system.

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1. INTRODUCTION

In recent decades, physics of complex systems has increasingly dealt with the description of groups of animals exhibiting collective behavior, such as flocks of birds, fish schools, locust swarms, and migrating cells ([16],[12],[24],[2],[19]).

From a modeling point of view, these systems offer new challenges and various models have been proposed to describe their interaction (see for example [34],[4],[35]). Among the first to be introduced, the Cucker-Smale model ([13]) describes a bird as a self-propelling particle interacting with its neighbors. In this case, the interaction is such that neighboring birds tend to align their velocities and the strength of the interaction is described through weights which depend on the metric distance between the agents.

Around 2008, a new type of interaction between agents called “topological interaction” was introduced. In [1, 9], the CoBBS group in Rome, after collecting 3D observational data for flocks of starlings, observed that regardless of the density of the flock, each agent interacts on average with its first 6-7

Date: July 25, 2023.

MSC2020 subject classifications. 35Q92, 35Q83, 82B40,

Key words and phrases. propagation of chaos, monokinetic solutions, topological interaction, Cucker-Smale model.

neighbors. This suggests that the strength of the interaction of an agent with another one does not depend on the metric distance between them, but rather on the “topological” distance that takes into account the proximity rank of the latter with respect to the former (see also [7, 8, 18, 30, 31, 26, 32, 36]).

From a mathematical point of view topological interactions fall outside the case of two-body interaction and present various problems in the derivation of mean-field and hydrodynamic equations. Among the works in the existing literature, topological models taken into consideration are either deterministic or stochastic. From the stochastic side, in [5, 6, 14, 15] kinetic models of Boltzmann type are derived for topological interaction models based on jump processes.

As far as deterministic models are concerned, in [22] the author has introduced a Cucker-Smale model where the interaction instead of being metric as usual is topological. In this work, the kinetic mean-field and hydrodynamic equations of Euler type are also written and formally derived from a smoothed version of the model.

In the present work, we focus on the problems of propagation of chaos and hydrodynamic behavior for the deterministic topological Cucker-Smale system, extending the analysis started in [3] where the existence of the dynamics and the mean-field limit have been rigorously proved for this same model.

The starting point is the Liouville equation verified by the N -particle system and the aim is to show that the marginals of the N -body distribution function converge to tensorial powers of solutions of suitable kinetic/hydrodynamic limiting equations of Vlasov/Euler type. The analysis will follow from a quantitative version of the law of large numbers, following the approach in [20] (see also [28],[27]).

The outline of the paper is as follows: in Section 2 we recall the topological Cucker-Smale model and introduce the associated Liouville equation, as well as recall the results obtained in [3] which will be useful later on. In Section 3, we provide a proof of the propagation of chaos, which will be a direct consequence of the validity of the law of large numbers. In Section 4, we focus on the derivation of the hydrodynamic description given by Euler-type equations studying the so-called monokinetic solutions.

2. MODEL AND GENERAL FRAMEWORK

A Cucker-Smale type model for the motion of N agents, in the mean-field scaling, is the system

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N p_{ij}(v_j(t) - v_i(t)), \quad i = 1, \dots, N \end{cases} \quad (2.1)$$

where $(x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d$ ($d = 1, 2, 3, \dots$) and the “communication weights” $\{p_{ij}\}_{i,j=1}^N$ are positive functions that take into account the interactions between

agents. In classical models, p_{ij} depends only on the distance¹ $|x_i - x_j|$ between the agents. In topological models the weights depend on the positions of the agents through their rank:

$$p_{ij} := K(M(x_i, |x_i - x_j|)),$$

where $K: [0, 1] \rightarrow \mathbb{R}^+$ is a positive Lipschitz continuous non-increasing function such that $\int_0^1 K(z) dz = \gamma$ and, for $r > 0$, the rank function

$$M(x_i, r) := \frac{1}{N} \sum_{k=1}^N \mathcal{X}\{|x_k - x_i| \leq r\} \quad (2.2)$$

counts the number of agents at a distance less than or equal to r from x_i , normalized with N . Note that in this case p_{ij} is a stepwise function of the positions of all the agents.

In the mean-field limit $N \rightarrow +\infty$, the one-agent distribution function $f_t = f(x, v, t)$ is expected to verify the equation

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (W[Sf_t, f_t](x, v) f_t) = 0, \quad (2.3)$$

where, in equation (2.3), $Sf_t(x) := \int f_t(x, v) dv$ denotes the spatial distribution and, given a probability density f on $\mathbb{R}^d \times \mathbb{R}^d$ and a probability density ρ on \mathbb{R}^d , $W[\rho, f]$ is the mean-field interaction given by

$$W[\rho, f](x, v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} K(M[\rho](x, |x - y|)) (w - v) f(y, w) dy dw, \quad (2.4)$$

with

$$M[\rho](x, r) := \int_{B_r(x)} d\rho.$$

Here and after, $B_r(x)$ denotes the closed ball of center x and radius r in \mathbb{R}^d . We also indicate by B_R the closed ball $B_R(0)$.

A weak formulation of equation (2.3) is given requiring that the solution f_t fulfills the identity

$$\int \alpha(x, v) df_t(x, v) = \int \alpha(x_t(x, v), v_t(x, v)) df_0(x, v) \quad (2.5)$$

for any $\alpha \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, where f_0 is the initial probability measure and $(x_t(x, v), v_t(x, v))$ is the flow defined by

$$\begin{cases} \dot{x}_t(x, v) = v_t(x, v) \\ \dot{v}_t(t, x, v) = W[Sf_t, f_t](x_t(x, v), v_t(x, v)) \\ x_0(x, v) = x, \quad v_0(x, v) = v. \end{cases} \quad (2.6)$$

In other words, f_t is the push-forward of f_0 along the flow generated by the velocity field, determined by f_t itself.

Given

$$Z_N = (x_1, \dots, x_N, v_1, \dots, v_N) = (X_N, V_N) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN},$$

¹From now on, $|\cdot| = |\cdot|_d$ denotes the euclidean norm on \mathbb{R}^d .

we can now define the empirical measure on $\mathbb{R}^d \times \mathbb{R}^d$ as

$$\mu_{Z_N} := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \otimes \delta_{v_i}.$$

It is easy to verify that if $Z_N(t) = (x_1(t), \dots, x_N(t), v_1(t), \dots, v_N(t))$ solves (2.1), then $\mu_{Z_N(t)}$ is a weak solution of (2.3). Namely, $M[S\mu_{Z_N}](x, r)$ is exactly $M(x, r)$ defined in (2.2) (from now on we use the more complete notation $M[S\mu_{Z_N(t)}](x, r)$). Thus, recalling (2.4), we can rewrite the agent evolution in (2.1) as

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = W[S\mu_{Z_N(t)}, \mu_{Z_N(t)}](x_i(t), v_i(t)), \quad i = 1, \dots, N. \end{cases}$$

In [3], in the framework of the mean-field theory, the rigorous derivation of (2.3) starting from (2.1) is obtained. More precisely, the following theorem is proved.

Theorem 1. [3, Theorems 3.4, 4.3 and 5.2]

It holds that:

- i) *except for a set of Lebesgue measure zero, any $Z_N \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$ gives rise to a unique global solution*

$$(X_N(t, Z_N), V_N(t, Z_N)) \in C^1(\mathbb{R}^+, \mathbb{R}^{2dN}) \times C(\mathbb{R}^+, \mathbb{R}^{2dN})$$

of (2.1) with initial datum Z_N . Moreover, given $R_x > 0$ and $R_v > 0$, expect for a set of Lebesgue measure zero, we have that

$$|x_i(t)| \leq R_x + tR_v, \quad |v_i(t)| \leq R_v$$

for any i , if $|x_i| \leq R_x$ and $|v_i| \leq R_v$.

- ii) *Let $f_0(x, v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a probability density such that $\text{supp}(f_0) \subset B_{R_x} \times B_{R_v}$. Given $T > 0$, there exists a unique weak solution $f \in C([0, T]; L^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ of the kinetic equation (2.3). Moreover*

$$\text{supp}(f_t) \subset B_{R_x + tR_v} \times B_{R_v} \tag{2.7}$$

and $\|f_t\|_\infty \leq \|f_0\|_\infty e^{d\gamma t}$.

- iii) *In the hypothesis of the previous items, μ_t^N weakly converges to f_t , if this is true at time zero.*

In the present work, we focus on the statistical description of the dynamical system (2.1), considering particles independently and identically distributed with law $F_N(t = 0, Z_N) = f_0^{\otimes N}$, where f_0 is the initial datum for the limit kinetic equation (2.3). Then, the law at time t of the dynamical system (2.1) is given by $F_N(t, Z_N)$, weak solution of the N -body Liouville equation

$$\begin{aligned} & \partial_t F_N(t, Z_N) + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F_N(t, Z_N) \\ & + \sum_{i=1}^N \nabla_{v_i} \cdot \left(\frac{1}{N} \sum_{j=1}^N K\left(M[S\mu_{Z_N}](x_i, |x_i - x_j|)\right) (v_j - v_i) F_N(t, Z_N) \right) = 0. \end{aligned} \tag{2.8}$$

Note also that $F_N(t, Z_N)$, for $t > 0$, is symmetric in the exchange of particles.

To quantify the distance between probability measures, we will consider the Wasserstein distance of order 1:

$$\mathcal{W}_1(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y),$$

where μ and ν are two measures over \mathbb{R}^d with finite first moments and $\mathcal{C}(\mu, \nu)$ is the set of all couplings, *i.e.* measures on the product space with marginals respectively μ and ν in the first and second variables.

3. CONVERGENCE OF THE MARGINALS

In this section we prove the propagation of chaos for solutions of the N -body Liouville equation (2.8). We briefly explain the meaning of this expression and we refer to [10, 11] for a review and to [29] for a propagation of chaos result in the case of non-topological Cucker-Smale models.

To do this, we introduce the s -particle marginals

$$F_{N:s}(Z_s) = \int F_N(Z_s, z_{s+1}, \dots, z_N) dz_{s+1} \dots dz_N, \quad s = 1, 2, \dots, N \quad (3.1)$$

where $z_i = (x_i, v_i) \in \mathbb{R}^{2d}$. We expect that, if N is large, the details of the individual mutual interactions are negligible, and the description given by $F_{N:s}(t, Z_s)$ is similar to the one given by $f_t^{\otimes s}$, where f_t solves (2.3). Note that $f_t^{\otimes s}(Z_s)$ solves

$$\partial_t f_t^{\otimes s}(Z_s) + \sum_{i=1}^s v_i \cdot \nabla_{x_i} f_t^{\otimes s}(Z_s) + \sum_{i=1}^s \nabla_{v_i} \cdot \left(W[Sf_t, f_t](x_i, v_i) f_t^{\otimes s}(Z_s) \right) = 0.$$

This last equation describes the law of the system when each particle evolves independently from the others, with an interaction given by the mean-field force defined in (2.4). Then propagation of chaos holds if, for fixed s , the convergence of $F_{N:s}(Z_s; t)$ to $f_t^{\otimes s}(Z_s; t)$ is proved in some distance between probability distributions.

We will prove the following result.

Theorem 2. *Let $f \in C([0, T]; L^\infty(\mathbb{R}^{2d}))$ solution of the kinetic equation (2.3) with initial datum $f_0(x, v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\text{supp}(f_0) \subset B_{R_x} \times B_{R_v}$. Assume that the interaction function K is Lipschitz-continuous and consider $F_N(t)$ solution of (2.8) such that $F_N(0) = f_0^{\otimes N}$.*

If $F_{N:s}$ denotes the s -marginal as defined in (3.1), for $t \in [0, T]$ and $s \in \{1, \dots, N\}$, it holds that

$$\mathcal{W}_1(F_{N:s}(t), f_t^{\otimes s}(t)) \leq \lambda_T s e^{\beta T^d e^{d\gamma T}} \sqrt{C_d(N)}, \quad (3.2)$$

where λ_T is a constant depending on d, R_x, R_v and T ; β a constant specified in (3.10) depending on

$$\text{Lip}(K) := \sup_{x \neq y} \frac{|K(x) - K(y)|}{|x - y|},$$

on $\|f_0\|_\infty$ and on the support, while

$$C_d(N) = \begin{cases} N^{-1/2} & \text{if } d = 1 \\ N^{-1/2} \log(N) & \text{if } d = 2 \\ N^{-1/d} & \text{if } d > 2. \end{cases} \quad (3.3)$$

In the next proof of Theorem 2 we will employ as a technical tool the discrepancy distance, defined as follows:

$$\mathcal{D}(\mu, \nu) = \sup_{x, r > 0} \left| \int_{B_r(x)} d\mu - \int_{B_r(x)} d\nu \right|,$$

for μ, ν two probability measures on \mathbb{R}^d . We will need the following results concerning it.

Proposition 1. [3, Propositions 2.4 and 2.5]

i) Let ρ and ν be two probability measures on \mathbb{R}^d with support in a ball B_R and such that $\rho \in L^\infty(\mathbb{R}^d)$. Then

$$\mathcal{D}(\nu, \rho) \leq c \sqrt{R^{d-1} \|\rho\|_\infty \mathcal{W}_1(\nu, \rho)}, \quad (3.4)$$

where c is a constant that depends on the dimension d .

ii) Given $X_N = (x_1, \dots, x_N)$, $Y_N = (y_1, \dots, y_N)$, with $|x_i - y_i| \leq \delta$ for some δ and any $i = 1, \dots, N$, consider the two empirical measures μ_{X_N} and μ_{Y_N} . Then, for any probability measure $\rho \in L^\infty(\mathbb{R}^d)$ supported on a ball B_R ,

$$\mathcal{D}(\mu_{X_N}, \mu_{Y_N}) \leq c R^{d-1} \|\rho\|_\infty \delta + c \mathcal{D}(\mu_{Y_N}, \rho). \quad (3.5)$$

Proof of Theorem 2. We start the proof by introducing

$$\Sigma_N(t) = (y_1(t), \dots, y_N(t), w_1(t), \dots, w_N(t)) = (Y_N(t), W_N(t)) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN},$$

where (y_i, w_i) for all i evolve independently with the mean-field interaction:

$$\begin{cases} \dot{y}_i(t) = w_i(t) \\ \dot{w}_i(t) = W[Sf_t, f_t](y_i(t), w_i(t)). \end{cases} \quad (3.6)$$

We associate to such $\Sigma_N(t)$ the empirical measure $\mu_{\Sigma_N(t)}$.

We define a coupling $\pi^N(t)$ between $F_N(t)$ and $f_t^{\otimes N}$ in the following way: at time $t = 0$, it is given by

$$\pi^N(0) = f_0^{\otimes N}(Z_N) \delta(Z_N - \Sigma_N).$$

For positive times, $\pi^N(t)$ is given by the push-forward of $\pi^N(0)$ along the product flow given by (2.1) and (3.6). Then, from now on, the initial datum of $\Sigma_N(t)$ and $Z_N(t)$ is the same, indicated by Z_N .

Next, for any $i \in \{1, \dots, N\}$, we introduce the quantity

$$D_N(t) = \int (|x_i - y_i| + |v_i - w_i|) d\pi^N(t),$$

which does not depend on i , thanks to the symmetry of the law.

The weak convergence of the s -marginals $F_{N:s}$ to $f^{\otimes s}$ is a consequence of the fact that $D_N(t) \rightarrow 0$: namely, using the symmetry of π^N ,

$$\mathscr{W}_1(F_{N:s}(t), f^{\otimes s}(t)) \leq \sum_{i=1}^s \int \left(|x_i - y_i| + |v_i - w_i| \right) d\pi^N(Z_s, \Sigma_s) \leq sD_N(t),$$

(here we are using that $|Z_s - \Sigma_s|_{2ds} \leq \sum_{i=1}^s (|x_i - y_i|_d + |v_i - w_i|_d)$).

We have, for any $i \in \{1, \dots, N\}$,

$$D_N(t) = \int \left(|x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| \right) df_0^{\otimes N}(Z_N),$$

where $(x_i(t), y_i(t)) \equiv (x_i(t, Z_N), y_i(t, Z_N))$ and $(y_i(t), w_i(t)) = (y_i(t, Z_N), w_i(t, Z_N))$. It follows that

$$D_N(t) \leq \int \delta(t, Z_N) df_0^{\otimes N}(Z_N), \quad (3.7)$$

with $\delta(t, Z_N) = \max_{i=1, \dots, N} |x_i(t) - y_i(t)| + |v_i(t) - w_i(t)|$. Since $(x_i(t), v_i(t))$ and $(y_i(t), w_i(t))$ have the same initial conditions, it holds that

$$\begin{aligned} |x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| &\leq \int_0^t |v_i(\tau) - w_i(\tau)| d\tau \\ &+ \int_0^t \left| W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau)) \right| d\tau. \end{aligned}$$

In order to estimate $\delta(t)$, we need to evaluate the last integrand. To do this we bound this quantity by the sum of four terms:

- (a) $|W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau))|,$
- (b) $|W[Sf_\tau, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, \mu_{\Sigma_N(\tau)}](x_i(\tau), v_i(\tau))|,$
- (c) $|W[Sf_\tau, \mu_{\Sigma_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, \mu_{\Sigma_N(\tau)}](y_i(\tau), w_i(\tau))|,$
- (d) $|W[Sf_\tau, \mu_{\Sigma_N(\tau)}](y_i(\tau), w_i(\tau)) - W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau))|.$

Since

$$|M[\mu_{Z_N(\tau)}](x, r) - M[Sf_\tau](x, r)| \leq \mathscr{D}(Sf_\tau, S\mu_{Z_N(\tau)}),$$

by (2.7) we have that (a) is bounded by

$$2\text{Lip}(K)R_v\mathscr{D}(Sf_\tau, S\mu_{Z_N(\tau)}).$$

Let $\rho \in L^\infty(\mathbb{R}^d)$ be a probability density. Then, it's not difficult to prove that given $r_1, r_2 > 0$,

$$|M[\rho](x, r_1) - M[\rho](x, r_2)| \leq c\|\rho\|_\infty |r_1^d - r_2^d| \quad (3.8)$$

and, given $x_1, x_2 \in \mathbb{R}^d$ and $r > 0$,

$$|M[\rho](x_1, r) - M[\rho](x_2, r)| \leq c\|\rho\|_\infty r^{d-1} |x_1 - x_2|. \quad (3.9)$$

By (3.8) and (3.9), $K(M[Sf_\tau](x, |x - y|))$ is Lipschitz in both the x and y variables, by the triangle inequality it follows that (b) and (c) are estimated by $\alpha_1(\tau)\delta(\tau, Z_N)$, where $\alpha_1(\tau) = c\text{Lip}(K)\|Sf_\tau\|_\infty R_x^{d-1}(\tau)R_v + \|K\|_\infty$.

Note that $\|Sf_\tau\|_\infty \leq cR_v^d\|f_\tau\|_\infty$ and by Theorem (1), $\|f_\tau\|_\infty \leq e^{d\gamma\tau}\|f_0\|_\infty$. Hence

$$\alpha_1(\tau) \leq c\text{Lip}(K)\|f_0\|_\infty (R_x + \tau R_v)^{d-1} R_v^{d+1} e^{d\gamma\tau}.$$

The last term (d) is easily treatable since, by (3.8), the interaction term $K(M[Sf_\tau](x, |x - y|))$ is Lipschitz in y . We obtain

$$|W[Sf_\tau, \mu_{\Sigma_N(\tau)}](x, v) - W[Sf_\tau, f_\tau](x, v)| \leq \alpha_1(\tau) \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau).$$

Collecting all the estimates, we get

$$\delta(t, Z_N) \leq \int_0^t \left[\alpha_1(\tau) \left(\delta(\tau) + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) \right) + c\text{Lip}(K)R_v \mathscr{D}(Sf_\tau, S\mu_{Z_N(\tau)}) \right] d\tau.$$

By (3.5) with $\rho = Sf_\tau$,

$$\mathscr{D}(S\mu_{\Sigma_N(\tau)}, S\mu_{Z_N(\tau)}) \leq C_1(\tau)\delta(\tau, Z_N) + c\mathscr{D}(Sf_\tau, S\mu_{\Sigma_N(\tau)}).$$

Using (3.4), we conclude that

$$\delta(t, Z_N) \leq \int_0^t \alpha_1(\tau) \left(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) + \delta(\tau, Z_N) \right) d\tau.$$

By the Gronwall's lemma, for $0 \leq t \leq T$, we arrive at

$$\delta(t, Z_N) \leq \alpha_2(T) \int_0^t \left(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) \right) d\tau,$$

where $\alpha_2(T) = \alpha_1(T) \exp(\int_0^T c\alpha_1(\tau) d\tau)$.

Noting that, since the initial datum of $\Sigma_N(t)$ is Z_N , thanks to (2.5),

$$\int \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) df_0^{\otimes N}(Z_N) = \int \mathscr{W}_1(\mu_{Z_N}, f_\tau) df_\tau^{\otimes N}(Z_N) \leq \lambda_T C_{2d}(N),$$

where the rate $C_d(N)$, as defined in (3.3), is obtained by using the Fournier and Guillin bound (see [17, Theorem 1]), which is a quantitative version of the law of large numbers. Here λ_T is a constant depending on d, R_x, R_v and T . By concavity we obtain also

$$\int \sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} df_0^{\otimes N}(Z_N) = \int \sqrt{\mathscr{W}_1(S\mu_{Z_N}, Sf_\tau)} df_\tau^{\otimes N}(Z_N) \leq \sqrt{\lambda_T C_d(N)}.$$

Using (3.7), we finally arrive at

$$D_N(t) \leq \lambda_T \alpha_2(T) \left(\sqrt{C_d(N)} + C_{2d}(N) \right).$$

Then (3.2) follows after denoting

$$\beta = c\text{Lip}(K) \|f_0\|_\infty \max\{R_x, R_v\}^{2d}. \quad (3.10)$$

□

4. EULER SYSTEMS ASSOCIATED TO MONOKINETIC INITIAL DATA

In this section we study the hydrodynamic description of the topological Cucker-Smale model, by considering the following pressureless Euler-type system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + (u \cdot \nabla) u = \int_{\mathbb{R}^d} K \left(M[\rho(t)](x, |x - y|) \right) (u(t, y) - u(t, x)) \rho(t, y) dy \end{cases} \quad (4.1)$$

with a regular compactly supported initial datum $(\rho_0(x), u_0(x))$.

In the following, we assume that (4.1) admits a unique solution $(\rho(t, x), u(t, x))$ which is regular and compactly supported for $t \in [0, T]$, for a fixed $T > 0$. We leave this fact as a hypothesis, however see [21] for a proof in the case of the non-topological Cucker Smale model (see also [23, 33, 25]).

It is not difficult to show that

$$f(t, x, v) = \rho(t, x)\delta(v - u(t, x)) \quad (4.2)$$

is a weak solution of the topological Vlasov equation in the sense of (2.5), (2.6): these are called monokinetic solutions. Note that in this case the field $W[Sf_t, f_t](x, v)$ is regular, so the flow in (2.6) is well-defined.

We will show how to obtain (4.2) from solutions of the Liouville equation.

Since the particle flow is not everywhere defined, in general it is not possible to consider a measure-valued solution of the corresponding Liouville equation. To overcome this problem we consider a regularization of the monokinetic initial datum:

$$f_0^\varepsilon(x, v) = (\rho_0(x)\delta(v - u_0(x))) *_{x,v} \eta_\varepsilon(x, v), \quad (4.3)$$

where η is a C^∞ compactly supported mollifier and $\eta_\varepsilon(z) = \eta(z/\varepsilon)/\varepsilon^{2d}$.

Our next result shows that after relaxing the regularization of the initial monokinetic measure and let the number of particles to infinity, the marginals of solutions of the Liouville equation tend, in Wasserstein topology, to tensorial powers of a monokinetic measure built up out of the solution of the associated Euler system.

Theorem 3 (from particles to Euler). *Let us consider a regular solution $(\rho(t), u(t)), t \in [0, T]$ of the Euler system (4.1) with regular compactly supported initial datum (ρ_0, u_0) and let $F_N^\varepsilon(t)$ be solution of the Liouville equation (2.8) with initial datum $F_N^\varepsilon(0) := (f_0^\varepsilon)^{\otimes N}$, with f_0^ε as in (4.3). Let assume that $\text{supp}\{(\rho_0, u_0)\} \subset B_{R_x}$ and $\text{Im}(u_0) \subset B_{R_v}$, for $R_x, R_v > 0$.*

Then, for any $t \in [0, T]$ and any positive integer $s < N$, setting $V_s^N = (v_1, \dots, v_s)$, we have

$$\mathcal{W}_1 \left(\int_{\mathbb{R}^{ds}} F_{N:s}^\varepsilon(t) dV_s^N, \rho(t)^{\otimes s} \right) \leq \lambda_T s e^{\bar{\beta}(t)} \left(\sqrt{C_d(N)} + \varepsilon \right),$$

and, for any Lipschitz function $\phi : \mathbb{R}^{ds} \rightarrow \mathbb{R}$,

$$\left| \left\langle \phi, \int_{\mathbb{R}^{ds}} F_{N:s}^\varepsilon(t) v_1^{(j)} \dots v_s^{(j)} dV_s^N - (\rho(t)u^{(j)}(t))^{\otimes s} \right\rangle \right| \leq \text{Lip}(\phi) R_v^{s-1} \lambda_T s e^{\bar{\beta}(t)} \left(\sqrt{C_d(N)} + \varepsilon \right),$$

where λ_T is a constant depending on d, R_x, R_v and T ; $C_d(N)$ is defined in (3.3), while

$$\bar{\beta}(t) = c \text{Lip}(K) (R_x + R_v t)^{d-1} R_v \int_0^t \|\rho(\tau)\|_\infty d\tau,$$

and $v_i^{(j)}$ denotes the j -component of $v_i \in \mathbb{R}^d$.

Remark 1. Note that the statement doesn't involve the Vlasov equation and that ε and N are independent, so the result is true for any sequence of limiting points (in ε) of the distributions $\{F_N^\varepsilon\}_{N \in \mathbb{N}}(t)$.

Remark 2. Along the same lines, we can also generalize the content of Theorem 3, proving a convergence result for marginals $F_{N:s}^\varepsilon$ of the Liouville equation with general initial conditions $(g_0^\varepsilon)^{\otimes N}$ such that $\mathcal{W}_1(g_0^\varepsilon, \rho_0(x)\delta(v - u_0(x))) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

To prove Theorem 3, we need the following stability result for solutions of the Vlasov system.

Proposition 2. Let f_t^i with $i = 1, 2$ be two probability measures, solutions of the topological Vlasov equation with initial data f_0^i such that $Sf^i \in L^\infty(\mathbb{R}^d)$. Suppose moreover that $\text{supp}(f_t^i) \subset B_{R_x + R_v t} \times B_{R_v}$, then

$$\mathcal{W}_1(f_t^1, f_t^2) \leq e^{\bar{\beta}_\wedge(t)} \max \left\{ \mathcal{W}_1(f_0^1, f_0^2), \sqrt{\mathcal{W}_1(f_0^1, f_0^2)} \right\}, \quad (4.4)$$

where

$$\bar{\beta}_\wedge(t) = c \text{Lip}(K)(R_x + R_v t)^{d-1} R_v \min_i \int_0^t \|Sf_\tau^i\|_\infty d\tau.$$

The proof is similar to the one of Theorem 1 in [3] and is given in the Appendix.

Proof of Theorem 3. To prove the thesis, it is sufficient to establish the following estimate:

$$\mathcal{W}_1(F_{N:s}^\varepsilon(t), (\rho(t, \cdot)\delta(v - u(t, \cdot)))^{\otimes s}) \leq \lambda_T s e^{\bar{\beta}(t)} \left(\sqrt{C_{2d}(N)} + \varepsilon \right). \quad (4.5)$$

Let $f^\varepsilon(t)$ be the solution of the topological Vlasov equation with initial datum f_0^ε . By the triangle inequality, we have

$$\mathcal{W}_1\left(F_{N:s}^\varepsilon(t), f^{\otimes s}(t)\right) \leq \mathcal{W}_1\left(F_{N:s}^\varepsilon(t), (f^\varepsilon)^{\otimes s}(t)\right) + s \mathcal{W}_1\left(f^\varepsilon(t), f(t)\right),$$

where we are using that $\mathcal{W}_1((f^\varepsilon)^{\otimes s}, f^{\otimes s}) \leq s \mathcal{W}_1(f^\varepsilon, f)$.

We only need to estimate the first term since the second term is managed by the stability estimate (4.4) in Proposition 2. Indeed, we get

$$\mathcal{W}_1\left(f^\varepsilon(t), f(t)\right) \leq e^{\bar{\beta}(t)} \max \left\{ \mathcal{W}_1(f_0^\varepsilon, f_0), \sqrt{\mathcal{W}_1(f_0^\varepsilon, f_0)} \right\} \leq \varepsilon e^{\bar{\beta}(t)} \int_{\mathbb{R}^d} |y| \eta(y) dy,$$

where the last inequality follows from a straightforward computation using test functions.

The remaining estimate is similar to the one given in the proof of Theorem (2). In this case we have

$$\mathcal{W}_1\left(F_{N:s}^\varepsilon(t), (f^\varepsilon)^{\otimes s}(t)\right) \leq s \int \delta(t, Z_N) d(f_0^\varepsilon)^{\otimes N}(Z_N), \quad (4.6)$$

where $\delta(t, Z_N)$ is defined as in (3.7) but with the independent flow $(y_i(t), w_i(t)) = (y_i(t, Z_N), w_i(t, Z_N))$ in (3.6) defined using $f^\varepsilon(t)$ instead of $f(t)$ (we keep the same notation for this flow without specifying the dependence on ε).

This time we get

$$|x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| \leq \int_0^t |v_i(\tau) - w_i(\tau)| d\tau + \int_0^t \left| W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau^\varepsilon, f_\tau^\varepsilon](y_i(\tau), w_i(\tau)) \right| d\tau.$$

In the last integrand we use the triangle inequality, so we need to bound the three terms

$$\begin{aligned} (a') & |W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau))|, \\ (b') & |W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau)) - W[Sf_\tau, f_\tau^\varepsilon](y_i(\tau), w_i(\tau))|, \\ (c') & |W[Sf_\tau, f_\tau^\varepsilon](y_i(\tau), w_i(\tau)) - W[Sf_\tau^\varepsilon, f_\tau^\varepsilon](y_i(\tau), w_i(\tau))|. \end{aligned}$$

The term (a) is studied as in the proof of Theorem (2), getting

$$(a') \leq \alpha_1(\tau) \left(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) + \delta(\tau, Z_N) \right).$$

While (b') is studied as in (d), obtaining

$$(b') \leq \alpha_1(\tau) \mathscr{W}_1(f_\tau^\varepsilon, f_\tau),$$

and (c') as in (a), obtaining

$$(c') \leq c\text{Lip}(K)R_v\mathscr{D}(Sf_\tau^\varepsilon, Sf_\tau).$$

Collecting these three estimates together and using the triangle inequality, we arrive at

$$\begin{aligned} \delta(t, Z_N) & \leq \bar{\alpha}(t) \left(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau^\varepsilon)} + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau^\varepsilon) + \delta(\tau, Z_N) \right) \\ & + \bar{\alpha}(t) \left(\sqrt{\mathscr{W}_1(Sf_\tau, Sf_\tau^\varepsilon)} + \mathscr{W}_1(f_\tau, f_\tau^\varepsilon) \right). \end{aligned}$$

By Gronwall's lemma and using again the Fournier and Guillin bound in (4.6) we get the estimate (4.5), from which the thesis follows. \square

APPENDIX: PROOF OF PROPOSITION 2

Let $Z^1(t, z)$ be the flow related to the field $(v, W[Sf^1, f^1](t, x, v))$ and analogously for $Z^2(t, z)$.

We define an intermediate dynamics $(\tilde{f}(t, z), \tilde{Z}(t, z))$, where $\tilde{f}(t)$ is the push-forward of f_0^2 by the flow $\tilde{Z}(t, z)$ related to the field $(v, W[Sf^1, \tilde{f}](t, x, v))$.

Note that the couple $(Z^1(t, \cdot), f^1(t, \cdot))$ exists for $t \in [0, T]$ since $Sf^1(t)$ is bounded and analogously for the couple $(Z^2(t, \cdot), f^2(t, \cdot))$, while $(\tilde{Z}(t, \cdot), \tilde{f}(t, \cdot))$ exists since $K(M[Sf^1(t)](x, |x - y|))$ is locally Lipschitz in x and y .

Proceeding as in [3], we have

$$\mathscr{W}_1(f_t^1, f_t^2) \leq \mathscr{W}_1(f_t^1, \tilde{f}_t) + \mathscr{W}_1(\tilde{f}_t, f_t^2). \quad (4.7)$$

The first term is under control since the density dependence of the field is the same and then

$$|W[Sf_t^1, f_t^1](x, v) - W[Sf_t^1, \tilde{f}_t^1](x, v)| \leq \bar{\alpha}(t) \mathcal{W}_1(f_t^1, \tilde{f}_t^1)$$

where $\bar{\alpha}_1(t) = c \text{Lip}(K) \|Sf_t^1\|_\infty (R_x + tR_v)^{d-1} R_v$. Then it is straightforward to prove that

$$\mathcal{W}_1(f_t^1, \tilde{f}_t^1) \leq e^{\bar{\beta}_1(t)} \mathcal{W}_1(f_0^1, f_0^2), \quad (4.8)$$

where

$$\bar{\beta}_1(t) = c \text{Lip}(K) (R_x + R_v t)^{d-1} R_v \int_0^t \|Sf_\tau^1\|_\infty d\tau.$$

Concerning the second term in the inequality (4.7), we have

$$\mathcal{W}_1(\tilde{f}(t), f^2(t)) \leq \int |Z^2(t, z) - \tilde{Z}(t, z)| df_0^2(z) \leq \delta(t) := \sup_z |Z^2(t, z) - \tilde{Z}(t, z)|.$$

In order to estimate $\delta(t)$, we have

$$|Z^2(t, z) - \tilde{Z}(t, z)| \leq \int_0^t \delta(\tau) + \left| W[Sf_\tau^2, f_\tau^2](Z^2(\tau, z)) - W[Sf_\tau^1, \tilde{f}_\tau^1](\tilde{Z}(\tau, z)) \right| d\tau.$$

By the triangle inequality, we bound the last term in the integrand by the sum of three quantities:

- (A) $|W[Sf_\tau^2, f_\tau^2](Z^2(\tau, z)) - W[Sf_\tau^1, f_\tau^2](Z^2(\tau, z))|,$
- (B) $|W[Sf_\tau^1, f_\tau^2](Z^2(\tau, z)) - W[Sf_\tau^1, \tilde{f}_\tau^1](Z^2(\tau, z))|,$
- (C) $|W[Sf_\tau^1, \tilde{f}_\tau^1](Z^2(\tau, z)) - W[Sf_\tau^1, \tilde{f}_\tau^1](\tilde{Z}(\tau, z))|.$

Reasoning as in the proof of Theorem (2), we get

$$\begin{aligned} (A) &\leq c \text{Lip}(K) R_v \mathcal{D}(Sf_\tau^2, Sf_\tau^1), \\ (B) &\leq \bar{\alpha}_1(\tau) \mathcal{W}_1(f_\tau^2, \tilde{f}_\tau^1) \leq \bar{\alpha}_1(\tau) \delta(\tau), \\ (C) &\leq \bar{\alpha}_1(\tau) \delta(\tau). \end{aligned} \quad (4.9)$$

Hence we need only to estimate $\mathcal{D}(Sf_\tau^2, Sf_\tau^1)$. By the triangular inequality

$$\mathcal{D}(Sf_\tau^2, Sf_\tau^1) \leq \mathcal{D}(Sf_\tau^2, S\tilde{f}_\tau^1) + \mathcal{D}(S\tilde{f}_\tau^1, Sf_\tau^1).$$

The second term can be bounded by using inequality (3.4) (see also Proposition 2.4 in [3]) obtaining that

$$\mathcal{D}(S\tilde{f}_\tau^1, Sf_\tau^1) \leq c \sqrt{\|Sf_\tau^1\|_\infty (R_x + \tau R_v)^{d-1} \mathcal{W}_1(S\tilde{f}_\tau^1, Sf_\tau^1)}. \quad (4.10)$$

In order to estimate $\mathcal{D}(Sf_\tau^2, S\tilde{f}_\tau^1)$ we proceed as in the proof of (3.5) (see Proposition 2.5 of [3]). We briefly recall it.

For any $z_0, z \in \mathbb{R}^{2d}$, by definition of $\delta(t)$,

$$|z_0 - Z^2(t, z)| - \delta(t) \leq |z_0 - \tilde{Z}(t, z)| \leq |z_0 - Z^2(t, z)| + \delta(t).$$

Let X be the set of functions $\phi \in C([0, +\infty), \mathbb{R})$, with first derivative continuous up to a finite number of jumps. Given $\phi \in X$, let $\phi_\delta(r) :=$

$\phi^+(r + \delta) - \phi^-(r - \delta)$, where, denoting by $\tilde{\phi}$ the function $\tilde{\phi}(r) := \int_0^r |\phi'(s)| ds$, we have defined

$$\phi^\pm(r) := \begin{cases} \frac{1}{2}(\tilde{\phi}(r) \pm \phi(r)), & \text{if } r \geq 0, \\ \pm \frac{1}{2}\phi(0), & \text{if } r < 0. \end{cases}$$

It can be proved (see Lemma 2.1 in [3]) that

$$\mathcal{D}(\rho_1, \rho_2) = \sup_{\phi \in X: \|\phi\|_X \leq 1} \sup_x \int \phi(|x - y|) (d\rho_1(y) - d\rho_2(y))$$

where $\|\phi\|_X := \int_0^{+\infty} |\phi'(r)| dr$.

Fixed $x_0 \in \mathbb{R}^d$, we define $\Phi(x) = \phi(|x - x_0|)$ and Φ_δ in the same way. Then, it is not difficult to see that (see Proposition 2.3 in [3])

$$\Phi(X^2(t, z)) \leq \Phi_\delta(\tilde{X}(t, z)).$$

Hence

$$\begin{aligned} \int \Phi(dSf_\tau^2 - dS\tilde{f}_\tau) &= \int \Phi(x) (df^2(z) - d\tilde{f}(z)) \\ &= \int (\Phi(X^2(\tau, z)) - \Phi(\tilde{X}(\tau, z))) df_0^2(z) \\ &\leq \int (\Phi_\delta(\tilde{X}(\tau, z)) - \Phi(\tilde{X}(\tau, z))) df_0^2(z) = \int (\Phi_\delta(x) - \Phi(x)) dS\tilde{f}_\tau(x) \\ &= \int (\Phi_\delta - \Phi) (dS\tilde{f}_\tau - dSf_\tau^1) + \int (\Phi_\delta - \Phi) dSf_\tau^1. \end{aligned}$$

Since $\Phi_\delta - \Phi \in X$, the first term is bounded by $c\mathcal{D}(S\tilde{f}_\tau, Sf_\tau^1)$, while the second can be easily bounded by $cR_x(\tau)^{d-1}\delta(\tau)\|Sf_\tau^1\|_\infty$. We conclude that

$$\mathcal{D}(Sf_\tau^2, S\tilde{f}_\tau) \leq c\delta(\tau)(R_x + R_v\tau)^{d-1}\|Sf_\tau^1\|_\infty + c\mathcal{D}(S\tilde{f}_\tau, Sf_\tau^1). \quad (4.11)$$

Collecting estimates (4.9), (4.10) and (4.11) and using Gronwall's lemma, we arrive at

$$\mathcal{W}_1(\tilde{f}(t), f^2(t)) \leq e^{\bar{\beta}_1(t)} \int_0^t \bar{\alpha}_1(\tau) \sqrt{\mathcal{W}_1(Sf_\tau^1, S\tilde{f}_\tau)} d\tau.$$

The thesis follows from the last inequality together with (4.7) and (4.8).

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