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ABSTRACT. In this work we extend the analysis carried out in [3], proving propagation of chaos, i.e. convergence of the marginals, for the deterministic Cucker-Smale model with topological interaction. By looking at the monokinetic solutions, we also obtain a rigorous derivation of the hydrodynamic description given by the corresponding Euler system.

CONTENTS

1. INTRODUCTION

In recent decades, physics of complex systems has increasingly dealt with the description of groups of animals exhibiting collective behavior, such as flocks of birds, schools of fish, or locust swarms ([19], [2], [25], [17]).

From a modeling point of view, these systems offer new challenges and various models have been proposed to describe their interaction (see [24] or [4]). Among the first to be introduced, the Cucker-Smale model [12] describes a bird as a self-propelling particle interacting with its neighbors. The interaction is such that neighboring birds tend to align their velocities.

After collecting observational data for bird flocks, in [1, 9] a new kind of interaction between agents was introduced called "topological interaction" (see also [7, 8, 16, 21, 22]) . In this case, the strength of the interaction of an agent with another one is a function of the proximity rank of the latter with respect to the former.

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From a mathematical point of view topological interactions come out of the case of two-body interaction and present various problems in the derivation of mean-field and hydrodynamic equations. Among the works in the existing literature [20, 23, 26], topological models taken into consideration were both deterministic and stochastic. From the stochastic side, in [5, 6, 13, 14] kinetic models of Boltzmann type are derived for topological interaction models based on jump processes.

As far as deterministic models are concerned, in [18] the author has introduced a Cucker-Smale model where the interaction instead of being metric as usual is topological. In this work, the kinetic mean-field and hydrodynamic equations of Euler type are also written and formally derived from a smoothed version of the model.

This work extend the analysis started in [3], where the existence of the dynamics and the mean field limit have been proved for the Cucker-Smale topological model.

Here, we focus instead on the problems of propagation of chaos and hydrodynamic behaviour for the topological Cucker-Smale system. The starting point will be the Liouville equation verified by the N-particle system.

The outline of the paper is as follows: in section (2) we recall the topological Cucker-Smale model and introduce the associated Liouville equation, as well as recall the results obtained in [3] which will be useful later on. In section (3), we provide a proof of the propagation of chaos, which will be a direct consequence of the validity of a law of large numbers. In section (4), we focus on the derivation of the hydrodynamic description given by Euler type equations studying the so-called monokinetic solutions.

2. Model and general framework

A Cucker-Smale type model for the motion of N agents, in the mean-field scaling, is the system

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} p_{ij}(v_j(t) - v_i(t)), \end{cases} \tag{2.1}$$

where $(x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d$ $(d = 1, 2, 3, \dots)$ and the "communication weights" $\{p_{ij}\}_{i,j=1}^N$ are positive functions that take into account the interactions between agents. In classical models, p_{ij} depends only on the distance[1] $|x_i - x_j|$ between the agents. In topological models the weights depend on the positions of the agents by their rank

$$p_{ij} \coloneqq K\big(M(x_i, |x_i - x_j|)\big), \tag{2.2}$$

[1]From now on, $|\cdot| = |\cdot|_d$ denotes the euclidean norm on $\mathbb{R}^d$.

where $K\colon [0,1] \to \mathbb{R}^+$ is a positive Lipschitz continuous non-increasing function such that $\int_0^1 K(z)\,\mathrm{d}z = \gamma$ and, for $r > 0$, the function

$$M(x_i, r) \coloneqq \frac{1}{N} \sum_{k=1}^{N} \mathcal{X}\{|x_k - x_i| \leqslant r\} \tag{2.3}$$

counts the number of agents at distance less than or equal to r from x_i, normalized with N. Note that in this case p_{ij} is a stepwise function of the positions of all the agents. In the sequel we assume that K

In the mean-field limit $N \to +\infty$, the one-agent distribution function $f_t = f(x, v, t)$ is expected to verify the equation

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (W[Sf_t, f_t](x, v) f_t) = 0, \tag{2.4}$$

where $Sf_t(x) \coloneqq \int f_t(x, v)\,\mathrm{d}v$ is the spatial distribution and where, given a probability density f in $\mathbb{R}^d \times \mathbb{R}^d$ and a probability density ρ in $\mathbb{R}^d$,

$$W[\rho, f](x, v) \coloneqq \int K\left(M[\rho](x, |x - y|)\right)\,(w - v) f(y, w)\,\mathrm{d}y\,\mathrm{d}w, \tag{2.5}$$

with

$$M[\rho](x, r) \coloneqq \int_{B_x(r)} \mathrm{d}\rho. \tag{2.6}$$

Here and after, $B_r(x)$ denotes the closed ball of center x and radius r in $\mathbb{R}^d$. We also indicate by B_R the closed ball $B_R(0)$.

A weak formulation of this equation is given requiring that the solution f_t fulfills

$$\int \alpha(x, v)\,\mathrm{d}f_t(x, v) = \int \alpha\left(x_t(x, v), v_t(x, v)\right)\mathrm{d}f_0(x, v) \tag{2.7}$$

for any $\alpha \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, where f_0 is the initial probability measure and $(x_t(x, v), v_t(x, v))$ is the flow defined by

$$\begin{cases} \dot{x}_t(x, v) = v_t(x, v) \\ \dot{v}_t(t, x, v) = W[Sf_t, f_t](x_t(x, v,), v_t(x, v)) \\ x_0(x, v) = x, \quad v_0(x, v) = v. \end{cases} \tag{2.8}$$

In other words, f_t is the push-forward of f_0 along the flow generated by the velocity field, determined by f_t itself.

Given

$$Z_N = (x_1, \ldots, x_N, v_1, \ldots, v_N) = (X_N, V_N).$$

we define the empirical measure as

$$\mu_{Z_N} \coloneqq \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(\mathrm{d}x)\,\delta_{v_i}(\mathrm{d}v).$$

It is easy to verify that if $Z_N(t)$ solves (2.1), then $\mu_{Z_N(t)}$ is a weak solution of (2.4). Namely, $M[S\mu_{Z_N}(x, r)$ is exactly $M(x, r)$ defined in (2.3) (from now on we use the more complete notation $M[S\mu_{Z_N(t)}, \mu_{Z_N(t)}](x, r))$. Thus, we can rewrite the agent evolution in (2.1) as

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = W[S\mu_{Z_N(t)}, \mu_{Z_N(t)}](x_i(t), v_i(t)). \end{cases} \tag{2.9}$$

In [3], in the framework of the mean-field theory, the rigorous derivation of (2.4) starting from (2.1) is obtained. More precisely, the following theorem is proved.

Theorem 1. [3, Theorems 3.4, 4.3 and 5.2]
It holds that:

i) except for a set of measure zero, given $Z_N \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$, there exists a unique global solution

$$(X_N(t, Z_N), V_N(t, Z_N)) \in C^1(\mathbb{R}^+, \mathbb{R}^{2dN}) \times C(\mathbb{R}^+, \mathbb{R}^{2dN})$$

of (2.1), with initial datum Z_N. Moreover, given $R_x > 0$ and $R_v > 0$, we have that

$$|x_i(t)| \leqslant R_x + tR_v, \ |v_i(t)| \leqslant R_v$$

for any i, if $|x_i| \leqslant R_x$ and $|v_i| \leqslant R_v$.

ii) Let $f_0(x, v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a probability density such that $supp(f_0) \subset B_{R_x} \times B_{R_v}$. Given $T > 0$, there exists a unique weak solution $f \in C\left([0, T]; L^\infty(\mathbb{R}^d \times \mathbb{R}^d)\right)$ of the kinetic equation (2.4). Moreover

$$supp(f_t) \subset B_{R_x + tR_v} \times B_{R_v} \tag{2.10}$$

and $\|f_t\|_\infty \leqslant \|f_0\|_\infty \mathrm{e}^{d\gamma t}$.

iii) In the hypotesis of the previous items, μ_t^N weakly converges to f_t, if this is true at time zero.

In this work, we focus on the statistical description of the dynamical system (2.1), considering particles independently and identically distributed with law $F^N(t = 0, Z_N) = f_0^{\otimes N}$, where f_0 is the initial datum for the limit kinetic equation (2.4). Then, the law at time t of the dynamical system (2.1) is given by $F^N(t, Z_N)$, weak solution of the N-body Liouville equation

$$\begin{aligned} &\partial_t F^N(t, Z_N) + \sum_{i=1}^N v_i \cdot \nabla_{x_i} F^N(t, Z_N) \\ &+ \sum_{i=1}^N \nabla_{v_i} \cdot \Big(\frac{1}{N} \sum_{j=1}^N K(M[S\mu_{Z_N}, \mu_{Z_N}])(x_i, |x_i - x_j|)(v_j - v_i) F^N(t, Z_N) \Big) = 0. \end{aligned} \tag{2.11}$$

Note also that $F^N(t, Z_N)$, for $t > 0$, is symmetric in the exchange of particles.

To quantify the distance between probability measures, we will consider the Wasserstein distances of order 1:

$$\mathscr{W}_1(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \, \mathrm{d}\pi(x, y),$$

where μ and ν are two measures over $\mathbb{R}^d$ and $\mathcal{C}(\mu, \nu)$ is the set of all couplings, *i.e.* measures on the product space with marginals respectively μ and ν in the first and second variables. As a technical tool, we also use the following discrepancy distance:

$$\mathscr{D}(\mu, \nu) = \sup_{x, r > 0} \Big| \int_{B_r(x)} \mathrm{d}\mu - \int_{B_r(x)} \mathrm{d}\nu \Big|.$$

3. Convergence of the marginals

In this section we prove the propagation of chaos for solutions of the N-body Liouville equation (2.11). We briefly explain the meaning of this term and we refer to [10, 11] for a review. To do this, we introduce the s-particle marginals

$$F_s^N(Z_s) = \int F^N(Z_s, z_{s+1} \cdots z_N) dz_{s+1} \cdots dz_N, \qquad s = 1, 2 \cdots N \tag{3.1}$$

where $z_i = (x_i, v_i)$. We expect that, if N is large, the details of the individual mutual interactions are negligible, and that the description given by $F_s^N(t, Z_N)$ is similar to the one given by $f_t^{\otimes s}$, where f_t solves (2.4). Note that$f_t^{\otimes s}(Z_s)$, solves

$$\partial_t f_t^{\otimes s}(Z_s) + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_t^{\otimes s}(Z_s) + \sum_{i=1}^{s} \nabla_{v_i} \cdot \Big(W[Sf_t, f_t](x_i, v_i) f_t^{\otimes s}(Z_s) \Big) = 0. \tag{3.2}$$

This last equation describes the law of the system when each particle evolves independently from the others, with an interaction given by the mean-field force defined in (2.5). Then propagation of chaos holds if, for fixed s, the convergence of $F_s^N(Z_N; t)$ to $f^{\otimes s}(Z_N; t)$ is proved in some distance between probability distributions.

We want to prove the following result.

Theorem 2. *Let $f \in C([0,T]; L^\infty(\mathbb{R}^{2d}))$ solution of the kinetic equation (2.4) with initial datum $f_0(x,v) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $supp(f_0) \subset B_{R_x} \times B_{R_v}$. Assume that the interaction function K is Lipschitz-continuous and consider $F^N(t)$ solution of (2.11) such that $F^N(0) = f_0^{\otimes N}$.*

If F_s^N denotes the s-marginal as defined in (3.1), for $t \in [0,T]$ and $s \in \{1, \dots, N\}$, it holds that

$$\mathscr{W}_1(F_s^N(t), f^{\otimes s}(t)) \leqslant s \mathrm{e}^{\beta T^d \mathrm{e}^{d\gamma T}} \sqrt{C_{2d}(N)}, \tag{3.3}$$

where β is a constant specified in (3.16) depending on $Lip(K)$, $\|f_0\|_\infty$ and on the support, while

$$C_d(N) = \begin{cases} N^{-1/2} & \text{if} \quad d = 1 \\ N^{-1/2} \log(N) & \text{if} \quad d = 2 \\ N^{-1/d} & \text{if} \quad d > 2. \end{cases} \tag{3.4}$$

We start the proof by introducing

$$\Sigma_N(t) = (y_1(t), \dots, y_N(t), w_1(t), \dots, w_N(t)) = (Y_N(t), W_N(t)) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN},$$

where (y_i, w_i) for all i evolve indipendently with the mean-field interaction:

$$\begin{cases} \dot{y}_i(t) = w_i(t) \\ \dot{w}_i(t) = W[Sf_t, f_t](y_i(t), w_i(t)). \end{cases} \tag{3.5}$$

We associate to such $\Sigma_N(t)$ the empirical measure $\mu_{\Sigma_N(t)}$.

We define a coupling $\pi^N(t)$ between $F^N(t)$ and $f_t^{\otimes N}$ in the following way: at time $t = 0$, it is given by

$$\pi^N(0) = f_0^{\otimes N}(Z_N) \delta(Z_N - \Sigma_N). \tag{3.6}$$

For positive times, $\pi^N(t)$ is given by the push-forward of $\pi^N(0)$ along the product flow given by (2.1) and (3.5). Then, from now on, the initial datum of $\Sigma_N(t)$ and $Z_N(t)$ is the same, indicated by Z_N.

Next, for any $i \in \{1, \dots, N\}$, we introduce the quantity

$$D_N(t) = \int \Big(|x_i - y_i| + |v_i - w_i| \Big) \, \mathrm{d}\pi^N(t),$$

which does not depend on i, thanks to the symmetry of the law.

The weak convergence of the of the s-marginals F_s^N to $f^{\otimes s}$ is a consequence of the fact that $D_N(t) \to 0$: namely, using the symmetry of π^N,

$$\mathscr{W}_1(F_s^N(t), f^{\otimes s}(t)) \leqslant \sum_{i=1}^{s} \int \Big(|x_i - y_i| + |v_i - w_i| \Big) \, \mathrm{d}\pi^N(Z_s, \Sigma_s) \leqslant s D_N(t),$$

(here we are using that $|Z_s - \Sigma_s|_{2ds} \leqslant \sum_{i=1}^{s} (|x_i - y_i|_d + |v_i - w_i|_d)$).

In the proof we will employ the following results about the discrepancy distance.

Proposition 1. [3, Propositions 2.4 and 2.5]

i) Let ρ and ν be two probability measures on $\mathbb{R}^d$ with support in a ball B_R and such that $\rho \in L^\infty(\mathbb{R}^d)$. Then

$$\mathscr{D}(\nu, \rho) \leqslant c \sqrt{R^{d-1} \|\rho\|_\infty \mathscr{W}_1(\nu, \rho)}, \tag{3.7}$$

where c is a constant that depends on the dimension d.

ii) Given $X_N = (x_1, \dots, x_N)$, $Y_N = (y_1, \dots y_N)$, with $|x_i - y_i| \leqslant \delta$ for some δ and any $i = 1, \dots N$, consider the two empirical measure μ_{X_N} and μ_{Y_N}. Then, for any probability measure $\rho \in L^\infty(\mathbb{R}^d)$ supported on a ball B_R,

$$\mathscr{D}(\mu_{X_N}, \mu_{Y_N}) \leqslant c R^{d-1} \|\rho\|_\infty \delta + c \mathscr{D}(\mu_{Y_N}, \rho). \tag{3.8}$$

Proof of Theorem 2. We have, for any $i \in \{1, \dots, N\}$,

$$D_N(t) = \int \Big(|x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| \Big) \, \mathrm{d} f_0^{\otimes N}(Z_N),$$

where $(x_i(t), y_i(t)) \equiv (x_i(t, Z_N), v_i(t, Z_N))$ and $(y_i(t), w_i(t)) = (y_i(t, Z_N), w_i(t, Z_N))$. It follows that

$$D_N(t) \leqslant \int \delta(t, Z_N) \, \mathrm{d} f_0^{\otimes N}(Z_N), \tag{3.9}$$

with $\delta(t, Z_N) = \max_{i=1,\dots,N} |x_i(t) - y_i(t)| + |v_i(t) - w_i(t)|$. Since $(x_i(t), v_i(t))$ and $(y_i(t), w_i(t))$ have the same initial conditions, it holds that

$$\begin{aligned} |x_i(t) - y_i(t)| + |v_i(t) - w_i(t)| \leqslant & \int_0^t |v_i(\tau) - w_i(\tau)| \, \mathrm{d}\tau \\ & + \int_0^t \Big| W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau)) \Big| \, \mathrm{d}\tau. \end{aligned}$$

In order to estimate $\delta(t)$, we need to evaluate the last integrand. To do this we bound this quantity by the sum of four terms:

(a) $|W[S\mu_{Z_N(\tau)}, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau))|$,

(b) $|W[Sf_\tau, \mu_{Z_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, \mu_{\Sigma_N(\tau)}](x_i(\tau), v_i(\tau))|$,

(c) $|W[Sf_\tau, \mu_{\Sigma_N(\tau)}](x_i(\tau), v_i(\tau)) - W[Sf_\tau, \mu_{\Sigma_N(\tau)}](y_i(\tau), w_i(\tau))|$,

(d) $|W[Sf_\tau, \mu_{\Sigma_N(\tau)}](y_i(\tau), w_i(\tau)) - W[Sf_\tau, f_\tau](y_i(\tau), w_i(\tau))|$.

Since

$$|M[\mu_{Z_N(\tau)}](x, r) - M[Sf_\tau](x, r)| \leqslant \mathscr{D}(Sf_\tau, S\mu_{Z_N(\tau)}),$$

by (2.10) we have that *(a)* is bounded by

$$2\mathrm{Lip}(K)R_v\mathscr{D}(Sf_\tau, S\mu_{Z_N(\tau)}). \tag{3.10}$$

Let $\rho \in L^\infty(\mathbb{R}^d)$ be a probability density. Then, it's not difficult to prove that given $r_1, r_2 > 0$,

$$|M[\rho](x, r_1) - M[\rho](x, r_2)| \leqslant c\|\rho\|_\infty \left|r_1^d - r_2^d\right| \tag{3.11}$$

and, given $x_1, x_2 \in \mathbb{R}^d$ and $r > 0$,

$$|M[\rho](x_1, r) - M[\rho](x_2, r)| \leqslant c\|\rho\|_\infty r^{d-1}|x_1 - x_2|. \tag{3.12}$$

By (3.11) and (3.12), $K\left(M[Sf_\tau](x, |x - y|)\right)$ is Lipschitz in both the x and y variables, by the triangle inequality it follows that *(b)* and *(c)* are estimated by $\alpha_1(\tau)\delta(\tau, Z_N)$, where $\alpha_1(\tau) = c\,\mathrm{Lip}(K)\|Sf_\tau\|_\infty R_x^{d-1}(\tau)R_v + \|K\|_\infty$.

Note that $\|Sf_\tau\|_\infty \leqslant cR_v^d\|f_\tau\|_\infty$ and by Theorem (1), $\|f_\tau\|_\infty \leqslant \mathrm{e}^{d\gamma\tau}\|f_0\|_\infty$. Hence

$$\alpha_1(\tau) \leqslant c\mathrm{Lip}(K)\|f_0\|_\infty(R_x + \tau R_v)^{d-1}R_v^{d+1}\mathrm{e}^{d\gamma\tau}. \tag{3.13}$$

The last term *(d)* is easily treatable since, by (3.11), the interaction term $K(M[Sf_\tau](x, |x - y|))$ is Lipschitz in y. We obtain

$$|W[Sf_\tau, \mu_{\Sigma_N(\tau)}](x, v) - W[Sf_\tau, f_\tau](x, v)| \leqslant \alpha_1(\tau)\mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau). \tag{3.14}$$

Collecting all the estimates, we get

$$\delta(t, Z_N) \leqslant \int_0^t \left[\alpha_1(\tau)\Big(\delta(\tau) + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau)\Big) + c\mathrm{Lip}(K)R_v\mathscr{D}(Sf_\tau, S\mu_{Z_N(\tau)})\right]\mathrm{d}\tau.$$

By (3.8) with $\rho = Sf_\tau$,

$$\mathscr{D}(S\mu_{\Sigma_N(\tau)}, S\mu_{Z_N(\tau)}) \leqslant C_1(\tau)\delta(\tau, Z_N) + c\mathscr{D}(Sf_\tau, S\mu_{\Sigma_N(\tau)}).$$

Using (3.7), we conclude that

$$\delta(t, Z_N) \leqslant \int_0^t \alpha_1(\tau)\Big(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau) + \delta(\tau, Z_N)\Big)\,\mathrm{d}\tau.$$

By the Gronwall's lemma, for $0 \leqslant t \leqslant T$, we arrive at

$$\delta(t, Z_N) \leqslant \alpha_2(T)\int_0^t \left(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)} + \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau)\right)\mathrm{d}\tau, \tag{3.15}$$

where $\alpha_2(T) = \alpha_1(T)\exp(\int_0^T c\alpha_1(\tau)\,\mathrm{d}\tau)$.

Noting that, since the initial datum of $\Sigma_N(t)$ is Z_N, thanks to (2.7),

$$\int \mathscr{W}_1(\mu_{\Sigma_N(\tau)}, f_\tau)\,\mathrm{d}f_0^{\otimes N}(Z_N) = \int \mathscr{W}_1(\mu_{Z_N}, f_\tau)\,\mathrm{d}f_\tau^{\otimes N}(Z_N) \leqslant C_{2d}(N),$$

where the rate $C_d(N)$, as defined in (3.4), is obtained by using the Fournier and Guillin bound (see [15, Theorem 1]), which is a quantitative version of the law of large numbers. By concavity we obtain also

$$\int \sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)}, Sf_\tau)}\,\mathrm{d}f_0^{\otimes N}(Z_N) = \int \sqrt{\mathscr{W}_1(S\mu_{Z_N}, Sf_\tau)}\,\mathrm{d}f_\tau^{\otimes N}(Z_N) \leqslant \sqrt{C_d(N)},$$

Using (3.9), we finally arrive at

$$D_N(t) \leqslant T\alpha_2(T)\left(\sqrt{C_d(N)} + C_{2d}(N)\right).$$

Then (3.3) follows after denoting

$$\beta = c\mathrm{Lip}(K)\|f_0\|_\infty \max\{R_x, R_v\}^{2d}. \tag{3.16}$$

□

4. Monokinetic solutions

Suppose that for $t \in [0,T]$, $(\rho(t,x), u(t,x))$ is a regular solution of the Euler system

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \\ \partial_t u + (u\cdot\nabla)u = \displaystyle\int_{\mathbb{R}^d} K\Big(M[\rho(t)](x,|x-y|)\Big)(u(t,y)-u(t,x))\rho(t,y)\,\mathrm{d}y \end{cases} \tag{4.1}$$

with initial datum $(\rho_0(x), u_0(x))$.

It is not difficult to show that

$$f(t,x,v) = \rho(t,x)\delta(v-u(t,x)) \tag{4.2}$$

is a weak solution of the topological Vlasov equation in the sense of (2.7), (2.8), these are called monokinetic solutions. Note that the field $W[Sf_t, f_t](x,v)$ is regular, so the flow in (2.8) is well-defined.

We will show how to obtain (4.2) from solutions of the Liouville equation.

Since the particle flow is not defined everywhere, it is delicate to consider a measure-valued solution of the corresponding Liouville equation. To overcome this problem we consider a regularization of the monokinetic initial datum:

$$f_0^\varepsilon(x,v) = (\rho_0(x)\delta(v-u_0(x))) *_{x,v} \eta_\varepsilon(x,v),$$

where η is a compactly supported mollifier and $\eta_\varepsilon(z) = \eta(z/\varepsilon)$.

In the same way, it doesn't seem clear if we can relax a mollification of a monokinetic measure initial condition. Nevertheless, our next result shows that after relaxing the regularization of the initial monokinetic measure and let the number of particles to infinity, the marginals of solutions of the Liouville equation tend, in Wasserstein topology, to tensorial powers of a monokinetic measure built up out of the solution of the associated Euler system.

Theorem 3 (from particles to Euler)**.** *Let us consider a regular solution* $(\rho(t), u(t)), t \in [0,T]$ *of the Euler system with initial data* (ρ_0, u_0) *and let* $F^{N,\varepsilon}(t)$ *be the solution of Liouville equation* (2.11) *with initial datum* $F^{N,\varepsilon}(0) := (f_0^\varepsilon)^{\otimes N}$*. Then, for any* $t \in [0,T]$ *and* $s \in \mathbb{N}$*,*

$$\mathscr{W}_1(F_s^{N,\varepsilon}(t), (\rho(t,\cdot)\delta(v-u(t,\cdot))^{\otimes s}) \tag{4.3}$$

$$\leqslant s\mathrm{e}^{\bar\beta(t)}\left(\sqrt{C_{2d}(N)} + \max\left\{\mathscr{W}_1(f_0, f_0^\varepsilon), \sqrt{\mathscr{W}_1(f_0, f_0^\varepsilon)}\right\}\right), \tag{4.4}$$

where

$$\bar\beta(t) = cLip(K)(R_x + R_v t)^{d-1} R_v \int_0^t \|\rho(t)\|_\infty \,\mathrm{d}\tau. \tag{4.5}$$

and $f_0(x,v) = \rho_0(x)\delta(v-u_0(x))$*.*

Remark 1. *Note that the statement doesn't involve the Vlasov equation and that* ε *and* N *are independent, so the result is true for any sequence of limiting point (in* ε*) of the distributions* $\{F^N\}_{N\in\mathbb{N}}(t)$*.*

Remark 2. *Using estimate* (4.3)*, it can be easily recovered the convergence of the velocity moments of the marginals to the hydrodynamic variables.Setting* $V_s^N = (v_1, \dots, v_s)$*, as* $\varepsilon \to 0$ *and* $N \to \infty$

$$\mathscr{W}_1\left(\int_{\mathbb{R}^d} F_s^{N,\varepsilon}(t)\,\mathrm{d}V_s^N, \rho(t)^{\otimes s}\right) \longrightarrow 0$$

and, weakly,

$$\int_{\mathbb{R}^d} F_s^{N,\varepsilon}(t) V_s^N \,\mathrm{d}V_s^N \longrightarrow (\rho(t)u(t))^{\otimes s}.$$

Remark 3. *Along the same lines, we can also generalize the content of Theorem 3, proving a convergence result for marginals* $F_s^{N,\varepsilon}$ *of the Liouville equation with general initial conditions* $(g_0^\varepsilon)^{\otimes N}$ *such that* $\mathscr{W}_1\left(g_0^\varepsilon, \rho_0(x)\delta(v-u_0(x))\right) \to 0$ *as* $\varepsilon \to 0$*.*

To prove Theorem 3, we need the following stability result for solutions of the Vlasov system.

Proposition 2. *Let* f_t^i *with* $i = 1,2$ *be two probability measures, solutions of the topological Vlasov equation with initial data* f_0^i *such that* $Sf^i \in L^\infty(\mathbb{R}^d)$*. Suppose moreover that* $supp(f_t^i) \subset B_{R_x+R_v t} \times B_{R_v}$*, then*

$$\mathscr{W}_1\left(f_t^1, f_t^2\right) \leqslant \mathrm{e}^{\bar\beta(t)} \max\left\{\mathscr{W}_1(f_0^1, f_0^2), \sqrt{\mathscr{W}_1(f_0^1, f_0^2)}\right\}, \tag{4.6}$$

where $\bar\beta(t)$ *is defined as in* (4.5)*.*

The proof is similar to the one of Theorem 1 in [3] and is given in the Appendix.

Proof of Theorem 3. Let $f^\varepsilon(t)$ be the solution of the topological Vlasov equation with initial datum f_0^ε. By the triangle inequality, we have

$$\mathscr{W}_1\Big(F_s^{\varepsilon,N}(t), f^{\otimes s}(t)\Big) \leqslant \mathscr{W}_1\Big(F_s^{\varepsilon,N}(t), (f^\varepsilon)^{\otimes s}(t)\Big) + s\mathscr{W}_1\Big(f^\varepsilon(t), f(t)\Big),$$

where we are using that $\mathscr{W}_1((f^\varepsilon)^{\otimes s}, f^{\otimes s}) \leqslant s\mathscr{W}_1(f^\varepsilon, f)$.

We only need to estimate the first term since the second term is managed by the stability estimate (4.6) in Proposition (2).

The remaining estimate is similar to the one given in the proof of Theorem (2). In this case we have

$$\mathscr{W}_1\Big(F_s^{\varepsilon,N}(t),(f^\varepsilon)^{\otimes s}(t)\Big)\leqslant s\int\delta(t,Z_N)\,\mathrm{d}(f_0^\varepsilon)^{\otimes N}(Z_N),\tag{4.7}$$

where $\delta(t,Z_N)$ is defined as in (3.9) but with the indipendent flow $(y_i(t),w_i(t))=(y_i(t,Z_N),w_i(t,Z_N))$ in (3.5) defined by means of $f^\varepsilon(t)$ instead of $f(t)$ (we keep the same notation for this flow without specifying the dependence on ε).

This time we get

$$\begin{aligned}|x_i(t)-y_i(t)|+|v_i(t)-w_i(t)|&\leqslant\int_0^t|v_i(\tau)-w_i(\tau)|\,\mathrm{d}\tau\\&+\int_0^t\Big|W[S\mu_{Z_N(\tau)},\mu_{Z_N(\tau)}](x_i(\tau),v_i(\tau))-W[Sf_\tau^\varepsilon,f_\tau^\varepsilon](y_i(\tau),w_i(\tau))\Big|\,\mathrm{d}\tau.\end{aligned}$$

In the last integrand we use the triangle inequality, so that we need to bound the three terms

$$\begin{aligned}&(a')\quad|W[S\mu_{Z_N(\tau)},\mu_{Z_N(\tau)}](x_i(\tau),v_i(\tau))-W[Sf_\tau,f_\tau](y_i(\tau),w_i(\tau))|,\\&(b')\quad|W[Sf_\tau,f_\tau](y_i(\tau),w_i(\tau))-W[Sf_\tau,f_\tau^\varepsilon](y_i(\tau),w_i(\tau))|,\\&(c')\quad|W[Sf_\tau,f_\tau^\varepsilon](y_i(\tau),w_i(\tau))-W[Sf_\tau^\varepsilon,f_\tau^\varepsilon](y_i(\tau),w_i(\tau))|.\end{aligned}$$

The term (a) is studied as in the proof of Theorem (2), getting

$$\text{(a')}\leqslant\alpha_1(\tau)\Big(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)},Sf_\tau)}+\mathscr{W}_1(\mu_{\Sigma_N(\tau)},f_\tau)+\delta(\tau,Z_N)\Big).$$

While (b') is studied as in (d), obtaining

$$\text{(b')}\leqslant\alpha_1(\tau)\mathscr{W}_1(f_\tau^\varepsilon,f_\tau),$$

and (c') as in (a), obtaining

$$\text{(c')}\leqslant c\mathrm{Lip}(K)R_v\mathscr{D}(Sf_\tau^\varepsilon,Sf_\tau).$$

Collecting these three estimates together and using the triangle inequality, we arrive at

$$\begin{aligned}\delta(t,Z_N)&\leqslant\bar\alpha(t)\Big(\sqrt{\mathscr{W}_1(S\mu_{\Sigma_N(\tau)},Sf_\tau^\varepsilon)}+\mathscr{W}_1(\mu_{\Sigma_N(\tau)},f_\tau^\varepsilon)+\delta(\tau,Z_N)\Big)\\&+\bar\alpha(t)\Big(\sqrt{\mathscr{W}_1(Sf_\tau,Sf_\tau^\varepsilon)}+\mathscr{W}_1(f_\tau,f_\tau^\varepsilon)\Big)\end{aligned}$$

By the Gronwall's lemma and using again the Fournier and Guillin bound in (4.7) we get the thesis.

□

Appendix: proof of Proposition 2

Let $Z^1(t,z)$ be the flow related to the field $(v, W[Sf^1, f^1](t,x,v))$ and analogously for $Z^2(t,z)$.

We define an intermediate dynamics $(\tilde f(t,z), \tilde Z(t,z))$, where $\tilde f(t)$ is the push-forward of f_0^2 by the flow $\tilde Z(t,z)$ related to the field $(v, W[Sf^1, \tilde f](t,x,v))$.

Note that the couple $(Z^1(t,\cdot), f^1(t,\cdot))$ exists for $t \in [0,T]$ since $Sf^1(t)$ is bounded and analogously for the couple $(Z^2(t,\cdot), f^2(t,\cdot))$, while $(\tilde Z(t,\cdot), \tilde f(t,\cdot))$ exists for the standard Dobrushin theory, since $K\left(M[Sf^1(t)](x,|x-y|)\right)$ is locally Lipschitz in x and y.

Proceeding as in [3], we have

$$\mathscr{W}_1\left(f_t^1, f_t^2\right) \leqslant \mathscr{W}_1\left(f_t^1, \tilde f_t\right) + \mathscr{W}_1\left(\tilde f_t, f_t^2\right). \tag{4.8}$$

The first term is under control since the density dependence of the field is the same and then

$$|W[Sf_t^1, f_t^1](x,v) - W[Sf_t^1, \tilde f_t](x,v)| \leqslant \bar\alpha(t)\mathscr{W}_1(f_t^1, \tilde f_t)$$

where $\bar\alpha(t) = c\mathrm{Lip}(K)\|Sf_t\|_\infty (R_x + tR_v)^{d-1}R_v$. Then it is straightforward to prove that

$$\mathscr{W}_1\left(f_t^1, \tilde f_t\right) \leqslant \mathrm{e}^{\bar\beta(t)}\mathscr{W}_1\left(f_0^1, f_0^2\right), \tag{4.9}$$

where

$$\bar\beta(t) = c\mathrm{Lip}(K)(R_x + R_v t)^{d-1}R_v \int_0^t \|Sf_\tau^1\|_\infty \,\mathrm{d}\tau.$$

Concerning the second term in the inequality (4.8), we have

$$\mathscr{W}_1(\tilde f(t), f^2(t)) \leqslant \int |Z^2(t,z) - \tilde Z(t,z)|\,\mathrm{d}f_0^2(z) \leqslant \delta(t) \coloneqq \sup_z |Z^2(t,z) - \tilde Z(t,z)|.$$

In order to estimate $\delta(t)$, we have

$$|Z^2(t,z) - \tilde Z(t,z)| \leqslant \int_0^t \delta(\tau) + \left|W[Sf_\tau^2, f_\tau^2](Z^2(\tau,z)) - W[Sf_\tau^1, \tilde f_\tau](\tilde Z(\tau,z))\right| \mathrm{d}\tau,$$

By the triangle inequality, we bound the last term in the integrand by the sum of three quantities:

$$\begin{aligned}
&(A)\ \ |W[Sf_\tau^2, f_\tau^2](Z^2(\tau,z)) - W[Sf_\tau^1, f_\tau^2](Z^2(\tau,z))|,\\
&(B)\ \ |W[Sf_\tau^1, f_\tau^2](Z^2(\tau,z)) - W[Sf_\tau^1, \tilde f_\tau](Z^2(\tau,z))|,\\
&(C)\ \ |W[Sf_\tau^1, \tilde f_\tau](Z^2(\tau,z)) - W[Sf_\tau^1, \tilde f_\tau](\tilde Z(\tau,z))|.
\end{aligned}$$

Reasoning as in the proof of Theorem (2), we get

$$\begin{aligned}
&(A) \leqslant c\mathrm{Lip}(K)R_v\mathscr{D}(Sf_\tau^2, Sf_\tau^1),\\
&(B) \leqslant \bar\alpha(\tau)\mathscr{W}_1(f_\tau^2, \tilde f_\tau) \leqslant \bar\alpha(\tau)\delta(\tau),\\
&(C) \leqslant \bar\alpha(\tau)\delta(\tau).
\end{aligned} \tag{4.10}$$

Hence we need only to estimate $\mathscr{D}(Sf_\tau^2, Sf_\tau^1)$. By the triangular inequality

$$\mathscr{D}(Sf_\tau^2, Sf_\tau^1) \leqslant \mathscr{D}(Sf_\tau^2, S\tilde f_\tau) + \mathscr{D}(S\tilde f_\tau, Sf_\tau^1).$$

The second term can be bounded by using inequality (3.7) (see also Proposition 2.4 in [3]) obtaining that

$$\mathscr{D}(S\tilde{f}_\tau, Sf^1_\tau) \leqslant c\sqrt{\|Sf^1_\tau\|_\infty (R_x + \tau R_v)^{d-1} \mathscr{W}_1(S\tilde{f}_\tau, Sf^1_\tau)}. \tag{4.11}$$

In order to estimate $\mathscr{D}(Sf^2_\tau, S\tilde{f}_\tau)$ we proceed as in the proof of (3.8) (see Proposition 2.5 of [3]). We briefly recall it.

For any $z_0, z \in \mathbb{R}^{2d}$, by definition of $\delta(t)$,

$$|z_0 - Z^2(t,z)| - \delta(t) \leqslant |z_0 - \tilde{Z}(t,z)| \leqslant |z_0 - Z^2(t,z)| + \delta(t).$$

Let X be the set of functions $\phi \in C([0,+\infty), \mathbb{R})$, with first derivative continuous up to a finite number of jumps. Given $\phi \in X$, let $\phi_\delta(r) \coloneqq \phi^+(r+\delta) - \phi^-(r-\delta)$, where, denoting by $\tilde{\phi}$ the function $\tilde{\phi}(r) \coloneqq \int_0^r |\phi'(s)|\,\mathrm{d}s$, we have defined

$$\phi^\pm(r) \coloneqq \begin{cases} \dfrac{1}{2}(\tilde{\phi}(r) \pm \phi(r)), & \text{if } r \geqslant 0, \\ \pm\dfrac{1}{2}\phi(0), & \text{if } r < 0. \end{cases}$$

It can be proved (see Lemma 2.1 in [3]) that

$$\mathscr{D}(\rho_1, \rho_2) = \sup_{\phi \in X:\, \|\phi\|_X \leqslant 1} \sup_x \int \phi\big(|x-y|\big)\big(\mathrm{d}\rho_1(y) - \mathrm{d}\rho_2(y)\big)$$

where $\|\phi\|_X \coloneqq \int_0^{+\infty} |\phi'(r)|\,\mathrm{d}r$.

Fixed $x_0 \in \mathbb{R}^d$, we define $\Phi(x) = \phi(|x - x_0|)$ and Φ_δ in the same way. Then, it is not difficult to see that (see Proposition 2.3 in [3])

$$\Phi\left(X^2(t,z)\right) \leqslant \Phi_\delta\left(\tilde{X}(t,z)\right).$$

Hence

$$\begin{aligned}
&\int \Phi\left(\mathrm{d}Sf^2_\tau - \mathrm{d}S\tilde{f}_\tau\right) = \int \Phi(x)\left(\mathrm{d}f^2(z) - \mathrm{d}\tilde{f}(z)\right) \\
&= \int \Big(\Phi(X^2(\tau,z)) - \Phi(\tilde{X}(\tau,z))\Big)\,\mathrm{d}f^2_0(z) \\
&\leqslant \int \Big(\Phi_\delta(\tilde{X}(\tau,z)) - \Phi(\tilde{X}(\tau,z))\Big)\,\mathrm{d}f^2_0(z) = \int (\Phi_\delta(x) - \Phi(x))\,\mathrm{d}S\tilde{f}_\tau(x) \\
&= \int (\Phi_\delta - \Phi)\left(\mathrm{d}S\tilde{f}_\tau - \mathrm{d}Sf^1_\tau\right) + \int (\Phi_\delta - \Phi)\,\mathrm{d}Sf^1_\tau.
\end{aligned}$$

Since $\Phi_\delta - \Phi \in X$, the first term is bounded by $c\mathscr{D}(S\tilde{f}_\tau, Sf^1_\tau)$, while the second can be easily bounded by $cR_x(\tau)^{d-1}\delta(\tau)\|Sf^1_\tau\|_\infty$. We conclude that

$$\mathscr{D}(Sf^2_\tau, S\tilde{f}_\tau) \leqslant c\delta(\tau)(R_x + R_v\tau)^{d-1}\|Sf^1_\tau\|_\infty + c\mathscr{D}(S\tilde{f}_\tau, Sf^1_\tau). \tag{4.12}$$

Collecting estimates (4.10), (4.11) and (4.12) and using Gronwall's lemma, we arrive at

$$\mathscr{W}_1\left(\tilde{f}(t), f^2(t)\right) \leqslant \mathrm{e}^{\bar{\beta}(t)} \int_0^t \bar{\alpha}(\tau)\sqrt{\mathscr{W}_1(Sf^1_\tau, S\tilde{f}_\tau)}\,\mathrm{d}\tau.$$

The thesis follows from the last inequality together with (4.8) and (4.9).

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