Sobolev maps to manifolds

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Graduate course notes

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Abstract

Sobolev spaces $W^{s,p}(\Omega;\mathscr{N})$ of maps to compact Riemannian manifolds $\mathscr{N}$ do not enjoy the standard properties of scalar Sobolev spaces; for example, approximability with smooth $\mathscr{N}$-valued maps may fail. We present selected significant results in several directions, such as lifting, strong approximation property, or singularities, with special focus on the case where $0 < s < 1$. The text, in progress, will be updated and extended. By the very nature of the topics treated, these notes have a non-empty intersection with Van Schaftingen's 2019 lecture notes [74].

Contents

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0 Motivation. Program. Preliminary remarks

We discuss a few natural questions concerning the Sobolev spaces $W^{s,p}(\Omega;\mathscr{N})$, in the following setting:

a) $\mathscr{N}$ is a smooth connected closed Riemannian manifold isometrically embedded into some $\mathbb{R}^\ell$.

b) $0 < s < \infty$, $1 \le p < \infty$.

c) $\Omega \subset \mathbb{R}^N$ is "smooth" and bounded. In most cases, Ω is a ball or a cube.

d) $W^{s,p}(\Omega;\mathscr{N}) := \{u \in W^{s,p}(\Omega;\mathbb{R}^\ell);\ u(x) \in \mathscr{N} \text{ a.e.}\}$, with its natural norm or seminorms.

Here are two typical examples of maps in such spaces.

Lemma 0.1. Let Ω be the unit ball in $\mathbb{R}^N$ and $|\ |$ denote the Euclidean norm.

a) Fix some smooth non constant map $v : \mathbb{S}^{N-1} \to \mathscr{N}$ and let $u(x) := v(x/|x|)$, $\forall\, x \in \Omega$. Then $u \in W^{s,p}(\Omega;\mathscr{N}) \iff sp < N$.

b) Let $\varphi : \Omega \to \mathbb{R}$, $\alpha > 0$, $\varphi(x) := |x|^{-\alpha}$ and $u := e^{i\varphi}$. Then $u \in W^{s,p}(\Omega;\mathbb{S}^1) \iff$ [(i) either $s < 1$ and $(\alpha+1)sp < N$ or (ii) $s \ge 1$, $(\alpha+s)p < N$, and $(\alpha+1)sp < N$].

The proofs are postponed to Section 5. (Item a) is a special case of Lemma 5.5 c).)

Remark 0.2.

a) In many of the results we present, $\mathscr{N}$ could have a boundary. However, some kind of "compactness of $\mathscr{N}$ at infinity" is required; see Bousquet, Ponce, and Van Schaftingen [20, 21].

b) In most of the limiting examples arising from material sciences, $\mathscr{N}$ is a "vacuum manifold" and is indeed closed: (i) $\mathscr{N} = \mathbb{RP}^2$ in the Oseen-Frank theory of liquid crystals; (ii) $\mathscr{N}$ is the set of (normalized) uniaxial Q-tensors in the Landau-de Gennes theory; (iii) $\mathscr{N} = \mathbb{S}^1 = \mathbb{T}$ in the Ginzburg-Landau theories; (iv) $\mathscr{N}$ is a general closed manifold in the theory of harmonic maps. For more examples, see Bethuel and Chiron [9].

c) A natural question is whether the definition of the space $W^{s,p}(\Omega;\mathscr{N})$ depends on the embedding of $\mathscr{N}$. This is not the case when $\mathscr{N}$ is compact. For non-compact $\mathscr{N}$, $W^{s,p}(\Omega;\mathscr{N})$, if defined appropriately, is still independent of the embedding when $0 < s < 1$ or, by a limiting procedure [13], when $s = 1$ (see Chiron [38] for a thorough discussion on the case $s = 1$; see also Bousquet, Ponce, and Van Schaftingen [21, Proposition 2.1]). However, when $s > 1$, this need not be the case; see Convent and Van Schaftingen [39].

Real- (or even *vector-*)valued spaces of Sobolev maps have standard and well-known properties, e. g., approximability with smooth maps, extension properties (or, seen from another perspective, trace theory), etc. However, essentially all these properties *fail* for $\mathscr{N}$-valued maps, as shows the following result.

Proposition 0.3. (All purposes counterexample) Let $\Omega = \mathbb{D}$ (the unit disc in $\mathbb{R}^2$) and $u : \Omega \to \mathbb{S}^1$, $u(x) := \dfrac{x}{|x|}$, $\forall\, x \in \Omega$. Then:

a) (Failure of the strong approximation property) There is no sequence $(u_j) \subset C^\infty(\overline{\Omega};\mathbb{S}^1)$ such that $u_j \to u$ in $W^{1,1}$. (Despite the fact that $u \in W^{1,1}$.)

b) (Failure of the weak approximation property) If $1 < p < 2$, there is no sequence $(u_j) \subset C^\infty(\overline{\Omega};\mathbb{S}^1)$ such that $u_j \rightharpoonup u$ in $W^{1,p}$. (Despite the fact that $u \in W^{1,p}$.)

c) (Failure of the extension property) For $2 < p < 3$, there is no $U \in W^{1,p}(\Omega \times (0,1);\mathbb{S}^1)$ such that $\operatorname{tr} U = u$. (Despite the fact that $u \in W^{1-1/p,p}$.)

d) (Failure of the lifting property) There is no $\varphi \in W^{1,1}(\Omega;\mathbb{R})$ such that $u = e^{i\varphi}$. (Despite the fact that $u \in W^{1,1}$.)

Proof. The regularity properties of u follow from Lemma 0.1 a). For the negative properties in items a)–d), we always argue by contradiction. We rely on some basic properties of Sobolev spaces recalled in Section 4.

a) By Corollary 4.3, along a subsequence and for a.e. $0 < r < 1$, we have $u_j \to u$ strongly in $W^{1,1}(C_r(0);\mathbb{S}^1)$, and thus uniformly. Therefore, $\deg(u_j, C_r(0)) \to \deg(u, C_r(0))$ for any such r. However, $\deg(u, C_r(0)) = 1$, while, by a homotopy argument, $\deg(v, C_r(0)) = 0$ for every r and $v \in C(\mathbb{D};\mathbb{S}^1)$.

b) Let $0 < \rho < 1$ and $1 < q < \infty$ be such that $1 < \rho q < p$. By the Gagliardo-Nirenberg inequalities, the compactness of the embedding $W^{t,q}(\Omega) \hookrightarrow W^{t-\varepsilon,q}(\Omega)$, and slicing (Corollary 4.3), up to a subsequence, we have $u_j \to u$ in $W^{\rho,q}(C_r(0);\mathbb{S}^1)$ for a.e. $0 < r < 1$. For any such r, by the Sobolev embeddings, we have $u_j \to u$ uniformly on $C_r(0)$, and we reach a contradiction as above.

c) The argument in the previous item shows that, in the space $W^{1-1/p,p}(\Omega;\mathbb{S}^1)$, we cannot approximate u with smooth, or even continuous, maps. If U exists, then, as $\varepsilon \to 0$, we have (by trace theory) $U(\cdot,\varepsilon) \to u$ in $W^{1-1/p,p}(\Omega;\mathbb{S}^1)$ as $\varepsilon \to 0$. This is a contradiction, since, for a "generic" ε, we have $U(\cdot,\varepsilon) \in W^{1,p}(\Omega;\mathbb{S}^1) \hookrightarrow C(\overline{\Omega};\mathbb{S}^1)$.

d) On a "generic" r, we have $u = e^{i\varphi}$ on $C_r(0)$, with $\varphi \in W^{1,1}(C_r(0);\mathbb{R}) \hookrightarrow C(C_r(0);\mathbb{R}))$, and this cannot happen for our specific u. QED

Before proceeding further, let us note that a common feature of the above proofs is that the presence of topological invariants prevents the existence of extensions, or strongly approximating sequences, or other classical properties of scalar Sobolev spaces.

We now present a research program, in part initially developed by Bethuel in his groundbreaking contribution [6], motivated by the pathologies exhibited in Proposition 0.3.

General program

Strong density problems

i) Characterize $W^{s,p}(\Omega;\mathscr{N})$ having the strong density property ($C^\infty(\overline{\Omega};\mathscr{N})$ is strongly dense in $W^{s,p}(\Omega;\mathscr{N})$).

ii) If the density property fails, find a class $\mathscr{R}$ of maps u enquoteas smooth as possible dense in $W^{s,p}(\Omega;\mathscr{N})$.

iii) If the density property fails, characterize the closure of $C^\infty(\overline{\Omega};\mathscr{N})$ in $W^{s,p}(\Omega;\mathscr{N})$.

(Sequential) Weak density problems

i) Characterize $W^{s,p}(\Omega;\mathscr{N})$ having the (sequentially) weak density property ($C^\infty(\overline{\Omega};\mathscr{N})$ is (sequentially) weakly dense in $W^{s,p}(\Omega;\mathscr{N})$).

ii) If the (sequentially) weak density property fails, characterize the (sequentially) weak closure of $C^\infty(\overline{\Omega};\mathscr{N})$ in $W^{s,p}(\Omega;\mathscr{N})$.

Extension problems Here, we assume that s is not an integer. (We could also let s be an integer when $p=2$.)

i) Characterize $W^{s,p}(\Omega;\mathscr{N})$ having the extension property: $\forall\, u \in W^{s,p}(\Omega;\mathscr{N})$, $\exists\, U \in W^{s+1/p,p}(\Omega\times(0,1);\mathscr{N})$ sucht that $\operatorname{tr} U = u$.

ii) If the extension property fails, characterize $\operatorname{tr} W^{s+1/p,p}(\Omega\times(0,1);\mathscr{N})$.

Lifting problems Let $\pi:\mathscr{E}\to\mathscr{N}$ be a non-trivial (locally isometric) covering map, with $\mathscr{E}$ a smooth embedded manifold.

i) Characterize $W^{s,p}(\Omega;\mathscr{N})$ having the lifting property, in the sense that for every $u\in W^{s,p}(\Omega;\mathscr{N})$ there exists some $\varphi\in W^{s,p}(\Omega;\mathscr{E})$ such that $u=\pi\circ\varphi$.

ii) If the lifting property fails, characterize $\pi\circ W^{s,p}(\Omega;\mathscr{E})$.

In full generality, this program is still partly open (especially for the weak density problems, which are the new frontier). In what follows, I will present some of the main results in these directions, some basic tools and elements of proofs, and indicate additional results that are beyond the scope of these notes. Before proceeding, let us discard two cases: (i) the easy case where $W^{s,p}\hookrightarrow C^0$ (i. e., when $sp>N$ or $s=N$ and $p=1$); (ii) the relatively easy case where $sp=N$.

Proposition 0.4. Assume that $W^{s,p}\hookrightarrow C^0$. Then $W^{s,p}(\Omega;\mathscr{N})$ has the strong (and thus weak) density property, the extension property (provided s is not an integer), and the lifting property.

Proof. The proofs of all the properties are similar: they rely on smoothing and nearest point projection on $\mathscr{N}$. We detail the extension property. Let $\rho \in C_c^\infty(B_1(0); \mathbb{R}_+)$ be a standard mollifier, and set

$$V(x,\varepsilon) := u * \rho_\varepsilon(x),\ \forall\, \varepsilon > 0,\ \forall\, x \in \Omega_\varepsilon := \{x \in \Omega;\ d(x, \partial\Omega) < \varepsilon\}. \tag{0.1}$$

By standard trace theory (see, e. g., [35, Proof of Lemma 15.47] and Section 4.4), we have $V \in W^{s+1/p,p}$ on its domain. Let $\delta > 0$ be such that the nearest point projection Π on $\mathscr{N}$ is smooth on the set $\{y \in \mathbb{R}^\ell;\ d(y, \mathscr{N}) \le \delta\}$. Let ε_0 be such that

$$[0 < \varepsilon < \varepsilon_0,\ x \in \Omega_\varepsilon] \implies d(V(x,\varepsilon), \mathscr{N}) \le \delta. \tag{0.2}$$

(The existence of ε_0 follows from the embedding $W^{s,p} \hookrightarrow C^0$ and the definition of V.) Set

$$T(x,\varepsilon) := \Pi \circ V(x,\varepsilon),\ \forall\, 0 < \varepsilon < \varepsilon_0,\ \forall\, x \in \Omega_\varepsilon. \tag{0.3}$$

Then T is clearly $\mathscr{N}$-valued and belongs to $W^{s+1/p,p}$ (Theorems 4.13 and 4.14). Let us next note that T is defined on

$$W := \{(x,\varepsilon);\ 0 < \varepsilon < \varepsilon_0,\ x \in \Omega_\varepsilon\}. \tag{0.4}$$

Picking a diffeomorphism $\Psi : \overline{\Omega} \times [0,1] \to \overline{W}$ such that $\Psi(x,0) = (x,0), \forall\, x \in \overline{\Omega}$, $\Psi(\Omega_\varepsilon \times \{\varepsilon\}) = \Omega \times \{\varepsilon\}, \forall\, 0 \le \varepsilon \le \varepsilon_0$, we find that $U := T \circ \Psi$ belongs to $W^{s+1/p,p}(\Omega \times (0,1); \mathscr{N})$. Finally, since $U(x,\varepsilon) = \Pi \circ V \circ \Psi(x,\varepsilon) \to \Pi \circ u \circ \Psi(x,0) = u(x)$ as $\varepsilon \to 0$, $\forall\, x \in \Omega$, we find that U has all the required properties. QED

The limiting case $sp = N$ is slightly more involved, and requires additional ingredients: the embedding $W^{s,p} \hookrightarrow \mathrm{VMO}$ (see Theorem 4.6) combined with a remarkable property of smoothing of VMO maps, made popular by Brezis and Nirenberg [37] (see Lemma 0.6 below, with roots in Schoen and Uhlenbeck [69] and Boutet de Monvel and Gabber [22]).

Proposition 0.5. Assume that $sp = N$. Then $W^{s,p}(\Omega; \mathscr{N})$ has the strong (and thus weak) density property and the extension property (provided s is not an integer).

Proof. We may assume that $p > 1$, since for $p = 1$ we have $W^{N,1} \hookrightarrow C^0$. We consider only the extension property. The proof is similar to the previous one. The main novelty stems in the proof of the existence of ε_0 satisfying (0.2) (see Lemma 0.6), since one cannot invoke anymore the continuity of u. Granted the existence of ε_0, we construct U as above. To see that $\mathrm{tr}\, U = u$, we argue as follows. (i) We clearly have $\mathrm{tr}\, V \circ \Psi = u$. (Start by considering a smooth u, then pass to the limits, using trace theory.) (ii) Extend Π to a smooth compactly supported map, still denoted Π. By trace theory and Theorems 4.13 and 4.14, for every map $Y \in W^{s+1/p,p}(\Omega \times (0,1); \mathbb{R}^\ell)$, we have $\mathrm{tr}\, \Pi \circ Y = \Pi \circ (\mathrm{tr}\, Y)$. Applying (ii) to $Y = V \circ \Psi$ and using (i), we find that, for our specific u, we have indeed $\mathrm{tr}\, U = u$, as claimed. QED

Lemma 0.6. Let $u \in \mathrm{VMO}(\Omega; F)$, where $F \subset \mathbb{R}^\ell$. Let $\rho \in C_c(B_1(0); \mathbb{R}_+)$ be such that $\int \rho = 1$. Then

$$\lim_{\varepsilon \to 0} \sup_{x \in \Omega_\varepsilon} d(u * \rho_\varepsilon(x), F) = 0. \tag{0.5}$$

Proof. For $x \in \Omega_\varepsilon$, we have

$$\fint_{B_\varepsilon(x)} |u(y) - u * \rho_\varepsilon(x)|\ dy \leq \int_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)|\ |\rho_\varepsilon(x-z)|\, dydz$$
$$\leq \omega_N \|\rho\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)|\ dydz.$$

We find that there exists some $y_0 \in B_\varepsilon(x)$ such that

$$|u(y_0) - u * \rho_\varepsilon(x)| \leq \omega_N \|\rho\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)|\ dydz.$$

For such a y_0, we have

$$d(u * \rho_\varepsilon(x), F) \leq |u_\varepsilon(x) - u(y_0)|$$
$$\leq \omega_N \|\rho\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)|\ dydz. \tag{0.6}$$

We conclude using the definition of VMO. QED

In view of the above,

in what follows, we assume, unless specified otherwise, that $sp < N$. (0.7)

Also, in order to simplify the statements,

in what follows, we assume, unless specified otherwise, that $N \geq 2$. (0.8)

1 Lecture # 1. Lifting

Recall that the implicit assumptions in this section are $N \geq 2$ and $sp < N$.

Let $\pi \in C^\infty(\mathscr{E}, \mathscr{N})$ be a Riemannian covering. We assume that: (i) $\mathscr{E}$ is connected, embedded into some $\mathbb{R}^m$; (ii) π is locally isometric and non-trivial (i. e., $\pi^{-1}(z)$ contains at least two points, $\forall\, z \in \mathscr{N}$). A special important case is the one of the universal covering of a non-simply connected manifold $\mathscr{N}$. Here are three prototypical examples.

1. $\mathscr{N} = \mathbb{S}^1$, $\mathscr{E} = \mathbb{R}$, $\pi(t) = e^{\imath t}$.
2. $\mathscr{N} = \mathbb{RP}^k$, $\mathscr{E} = \mathbb{S}^k$ (with $k \geq 2$), $\pi(t) = \{t, -t\}$.
3. $\mathscr{N} = \mathbb{S}^1$, $\mathscr{E} = \mathbb{S}^1$ (viewed as subsets of $\mathbb{C}$), $\pi(t) = t^k$, with $k \in \mathbb{Z}$, $|k| \geq 2$.

The last two examples belong to the *compact case* where $\mathscr{E}$ is compact, while the first one belongs to the *non-compact case* ($\mathscr{E}$ is non-compact).

We next discuss the seminorm we consider on $W^{s,p}(\Omega;\mathscr{E})$ when $0<s<1$. Set

$$|\varphi|^p_{W^{s,p}}:=\int_\Omega\int_\Omega \frac{[d_{\mathscr{E}}(\varphi(x),\varphi(y))]^p}{|x-y|^{N+sp}}\,dxdy,$$

where $d_{\mathscr{E}}$ is the geodesic distance on $\mathscr{E}$. When $\mathscr{E}$ is compact, the above seminorm is equivalent to the one obtained by taking the Euclidean distance $|\varphi(x)-\varphi(y)|$ in $\mathbb{R}^m$. This need not be the case in general.

We now present an important condition devised by Detaille [41].

$$\begin{aligned}&\text{There exists some } \Phi\in C^\infty(\mathbb{R}^m,\mathscr{L}(\mathbb{R}^\ell,\mathbb{R}^m)) \text{ with bounded derivatives such that}\\ &\Phi(t)((d_t\pi)(\tau))=\tau,\ \forall\, t\in\mathscr{E},\ \forall\,\tau\in T_t(\mathscr{E}).\end{aligned}\tag{1.1}$$

This condition requires the global existence of a "controlled" left-inverse of the isometry $d_t\pi : T_t(\mathscr{E})\to T_{\pi(t)}(\mathscr{N})$. An explicit construction shows that this condition is automatically satisfied in the compact case [41], but a counterexample in [41] shows that it may not be satisfied in the non-compact case. Intuitively, (1.1) requires that the embedding of $\mathscr{E}$ "does not swirl too much". Of importance for us is that this condition is satisfied by the universal covering $\pi:\mathbb{R}\to\mathbb{S}^1$ (take $\Phi(t)(x_1,x_2):=(-\sin t)x_1+(\cos t)x_2$, $\forall\,(x_1,x_2)\in\mathbb{R}^2$).

We have the following results (see [12] for the universal covering of $\mathbb{S}^1$, Bethuel and Chiron [9] for the non-compact case and partial results in the compact case, [61] for the full result when $0<s<1$, and Detaille [41] for the role of the condition (1.1)).

Theorem 1.1. Let $\Omega=(0,1)^N$. Assume that $s\geq 1$.

a) The lifting property fails when $1\leq sp<2$.

b) In the non-compact case, when $s>1$ further assume that (1.1) holds. The lifting property holds when $sp\geq 2$.

Theorem 1.2. Let $\Omega=(0,1)^N$. Assume that $0<s<1$.

a) The lifting property holds when $sp<1$.

b) The lifting property fails when $1\leq sp<2$.

c) In the non-compact case, the lifting property fails when $1\leq sp<N$.

d) In the compact case, the lifting property holds when $sp\geq 2$.

Proofs. Step 1. The lifting property fails when $1\leq sp<2$. Fix some point $z_0\in\mathscr{N}$. Assume, for simplicity, that $z_0=0$.

We first explain a gluing construction, valid for each integer $2\leq k\leq N$. (In our specific case, we will take $k=2$.) Consider, for some $0<a<1$, the cone $C:=\{x'=(x_1,x'')\in\mathbb{R}^k;\ |x''|\leq ax_1\}$. Consider a sequence of maps $u_j=u_j(x')$, smooth in $\mathbb{R}^k\setminus\{0\}$, such that $u_j(x')=0$ if $x'\notin C$ and $u_j\in W^{s,p}(B_1(0))$. Write a point in $\mathbb{R}^N$ in the form $x=(x',y)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$. We will define inductively points $b_j\in(0,1)^k$ and maps w_j, such that:

(i) The truncated cones $(C+b_j)\cap[0,1]^k$ are mutually disjoint.

(ii) $\{1\} \times (0,1)^{k-1} \not\subset \cup_{i \le j}(C + b_i), \forall\, j.$

(iii) $w_j(x) = u_j(x' - b_j), \forall\, j, \forall\, x \in \Omega.$

(iv) $\|w_j\|_{W^{s,p}} \le 2^{-j}, \forall\, j \ge 2.$

Let $b_1 = 0$. Assume that we have chosen $b_1, \dots, b_j$. Pick some point $(1, x'') \in \{1\} \times (0,1)^{k-1} \setminus \cup_{i \le j}(C + b_i)$. Then, for sufficiently small ε, the vertex $b_j := (1 - \varepsilon, x'')$ has all the required properties.

Moreover, clearly, by construction, we also have

(v) $\Omega \setminus \cup_{j \ge 1}[(C + b_j) \times (0,1)^{N-k}]$ is connected.

(vi) The map $w := \begin{cases} w_j, & \text{in } (C + b_j) \times (0,1)^{N-k} \\ 0, & \text{in } \Omega \setminus \cup_{j \ge 1}[(C + b_j) \times (0,1)^{N-k}] \end{cases}$ belongs to $W^{s,p}$ and is smooth in $\Omega \setminus \cup_{j \ge 1}[\{b_j\} \times (0,1)^{N-k}]$.

We now specialize to our situation. Recall that we have assumed that $z_0 = 0$. Let $(t_j)_{j \ge 1}$ be an enumeration (possibly with repetitions) of $\pi^{-1}(0)$. Let $0 < b < \pi/4$. By Lemma 7.1, there exists some some $v_j \in C^\infty(\mathbb{R}; \mathscr{N})$ such that $v_j(\theta) = 0, \forall\, j, \forall\, |\theta| \ge b$, and there exists no lifting $\zeta \in C([-b, b]; \mathscr{E})$ of v_j on $[-b, b]$ such that $\zeta(-b) = \zeta(b) = t_j$. Set $u_j(re^{\imath\theta}) := v_j(\theta), \forall\, r > 0$, $\forall\, \theta \in \mathbb{R}$. Then, clearly, u_j satisfies the assumptions at the beginning of this step (the property $u_j \in W^{s,p}(B_1(0))$ following from Lemma 5.5 c)). Let w be as above. We claim that w has no lifting $\varphi \in W^{s,p}$. Argue by contradiction. By Lemma 8.2, φ is continuous in the connected set $U := \Omega \setminus \cup_{j \ge 1}[\{b_j\} \times (0,1)^{N-2}]$. Since $w = 0$ in U, there exists some j such that $\varphi = t_j$ in U. By continuity, $\varphi = t_j$ on the set $[(\partial C + b_j) \cap \Omega] \setminus [\{b_j\} \times (0,1)^{N-2}]$. In particular, for small $\varepsilon > 0$, $\varphi(b_j + \varepsilon e^{\pm \imath b}) = t_j$. Going back to the definition of w_j, we find that v_j has, on $[-b, b]$, a continuous lifting ζ (given by $\zeta(\theta) := \varphi(b_j + \varepsilon e^{\imath\theta})$) such that $\zeta(-b) = \zeta(b)$, a contradiction.

For pedagogical reasons, the case $s = 1$ is split into two sub-cases.

Step 2. The lifting property holds when $s = 1$ and $p \ge 2$: the compact case. Let first u be in the class $\mathscr{R}$ in (2.1), with $\ell := N - \lfloor p \rfloor - 1 \le N - 3$. With A as in (2.1), the open set $U := \Omega \setminus A$ is simply connected (Lemma 7.2). Therefore, u has a smooth lifting φ in U. Since $|\nabla \varphi| = |\nabla u|$ pointwise, we find that $\varphi \in W^{1,p}(\Omega; \mathscr{E})$ and $\|\nabla \varphi\|_p = \|\nabla u\|_p$. Moreover, $\mathscr{E}$ being compact, we have $\|\varphi\|_p \le C := \max\{|t|;\, t \in \mathscr{E}\}$. Let now $u \in W^{1,p}(\Omega; \mathscr{N})$. Consider a sequence $(u_j) \subset \mathscr{R}$ such that $u_j \to u$ in $W^{1,p}$ (*cf* Theorem 2.1). The corresponding sequence (φ_j) of liftings is bounded in $W^{1,p}(\Omega; \mathscr{E})$. If $\varphi \in W^{1,p}(\Omega; \mathscr{E})$ is such that, up to a subsequence, $\varphi_j \rightharpoonup \varphi$, then $\varphi \in W^{1,p}$ and φ is a lifting of u.

Step 3. The lifting property holds when $s = 1$ and $p \ge 2$: the non-compact case. The additional issue is passing to the weak limit the φ_j's, since boundedness in L^p is not guaranteed anymore. This is achieved by induction on the space dimension, *via* the following result. Let $N \ge 2$ and let $(u_j) \subset W^{1,p}(\Omega; \mathscr{N})$ be a *convergent* sequence. Then there exists a *bounded* sequence $(\varphi_j) \subset W^{1,p}(\Omega; \mathscr{E})$ such that $\pi \circ \varphi_j = u_j, \forall\, j$. (Actually, this result also holds when $\mathscr{E}$ is compact, but the proof in Step 2 allows us to avoid its use.)

Step 3.1. The induction process. Assume that the above property holds for $N - 1$. Let $u \in W^{1,p}(\Omega; \mathscr{N})$. Consider a sequence $(u_j) \subset \mathscr{R}$ such that $u_j \to u$ in $W^{1,p}$. Let A_j be the corresponding singular

sets as in (2.1). Possibly after passing to a subsequence, for a.e. $\theta \in (0,1)$, the partial map $u_{j,\theta} := u(\cdot,\theta)$ belongs to $W^{1,p}((0,1)^{N-1};\mathscr{N})$, the set $A_{j,\theta} := \{x' \in (0,1)^{N-1};\ (x',\theta) \in A_j\}$ is a finite union of $N - \lfloor p \rfloor - 2$ planes (this condition being empty when $N < \lfloor p \rfloor + 2$) and $u_{j,\theta} \to u(\cdot,\theta)$ in $W^{1,p}$. Pick such θ. Let $\psi_j \in W^{1,p}((0,1)^{N-1};\mathscr{E})$ be a bounded sequence of liftings of $u_{j,\theta}$ (*cf* the induction assumption). By Lemma 8.2, ψ_j is smooth in $(0,1)^{N-1} \setminus A_{j,\theta}$. Fix some $x_0' \in (0,1)^{N-1} \setminus A_{j,\theta}$ and let φ_j be the smooth lifting of u_j in $\Omega \setminus A$ such that $\varphi_j(x_0',\theta) = \psi_j(x_0')$. (The existence of φ_j follows from Lemma 7.2.) Since $|\nabla \varphi_j| = |\nabla u_j|$ in $\Omega \setminus A$, we find that $\varphi_j \in W^{1,p}(\Omega)$. On the other hand, we clearly have $\mathrm{tr}_{(0,1)^{N-1}\times\{\theta\}}\, \varphi_j = \psi_j$. By the induction assumption and the standard inequality

$$\|\varphi_j\|_p \lesssim \|\psi_j\|_p + \|\nabla \varphi_j\|_p,$$

we find that (φ_j) is bounded in $W^{1,p}$. As in Step 2, we obtain that u has a lifting $\varphi \in W^{1,p}$.

Step 3.2. The case $N = 2$. Let $(u_j) \subset W^{1,p}((0,1)^2;\mathscr{N})$ be a *convergent* sequence. We let, as in the proof of Proposition 0.4, $V_j(x,\varepsilon) := u_j * \rho_\varepsilon(x)$. We claim that there exists an ε_0 (depending on the sequence (u_j)) such that (0.2) holds *uniformly in* j. When $p > 2$, this is clear, by Morrey's embedding $W^{1,p} \hookrightarrow C^{0,1-2/p}$. When $p = 2$, the claim follows from an inspection of the proof of Lemma 0.6. Extend u_j by reflection across $\partial\Omega$. Continuing the calculation (0.6), we find, for small ε, that

$$\begin{aligned}
d(u_j * \rho_\varepsilon(x), F) \lesssim & \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u_j(y) - u_j(z)|\, dydz \\
\lesssim & \varepsilon^{-2N} \int_{B_{2\varepsilon}(0)} \int_{B_\varepsilon(x)} |u_j(z+h) - u_j(z)|\, dzdh \\
\lesssim & \varepsilon^{-2N} \int_{B_{2\varepsilon}(0)} \int_{B_{3\varepsilon}(x)} |h|\, |\nabla u_j(y)|\, dydh \\
\lesssim & \varepsilon^{1-N} \int_{B_{3\varepsilon}(x)} |\nabla u_j(y)|\, dy \lesssim \varepsilon^{1-N+N(p-1)/p} \|\nabla u_j\|_{L^p(B_{3\varepsilon}(x))}.
\end{aligned}$$

Recalling that, in our case, $N = 2$ and $p = 2$, the claim follows from the above calculation, the assumption that (∇u_j) converges in L^2, and Lebesgue's lemma.

Associate now to u_j the map T_j as in (0.3). Fix some $M < \infty$ such that $\pi(\overline{B}_M(0) \cap \mathscr{E}) = \mathscr{N}$. Fix some $x_0 \in \Omega$ and consider some $t_j \in \mathscr{E}$ such that $|t_j| \leq M$ and $\pi(t_j) = T_j(x_0,\varepsilon_0)$. By construction, T_j is smooth in $\Omega \times (0,\varepsilon_0)$, and Lipschitz (with a Lipschitz constant independent of j) on $\Omega \times \{\varepsilon_0\}$. In addition, we have

$$|\nabla_x T_j(\cdot,\varepsilon)|_p \lesssim \|\nabla u_j\|_p \tag{1.2}$$

(with constants independent of j and $0 < \varepsilon \leq \varepsilon_0$) and, by standard trace theory,

$$\|\nabla T_j\|_{L^p(\Omega\times(0,\varepsilon_0))} \lesssim |u_j|_{W^{1-1/p,p}} \lesssim \|\nabla u_j\|_p. \tag{1.3}$$

Let ζ_j be the smooth lifting of T_j on $\Omega \times (0,\varepsilon_0)$ with $\zeta_j(x_0,\varepsilon_0) = t_j$. By the above, $\zeta_j(\cdot,\varepsilon_0)$ is Lipschitz, with controlled Lipschitz constant, and uniformly bounded at x_0, and thus $\|\zeta_j(\cdot,\varepsilon_0)\|_p \lesssim 1$. This, together with the L^p bound (1.3), implies that $\|\zeta_j(\cdot,\varepsilon)\|_p \lesssim \|\nabla u_j\|_p + 1$, $\forall\, j$, $\forall\, 0 < \varepsilon < \varepsilon_0$.

Combining this with (1.2), we find that $\zeta_j(\cdot,\varepsilon)$ is uniformly bounded in $W^{1,p}(\Omega)$. We obtain the desired conclusion by letting φ_j be any weak limit of a sequence of the form $(\zeta_j(\cdot,\varepsilon_k))$, with $\varepsilon_k \to 0$.

Step 4. The lifting property holds when $s > 1$ and $sp \geq 2$. Let $u \in W^{s,p}(\Omega;\mathscr{N}) \subset W^{1,sp}(\Omega;\mathscr{N})$ (Corollary 4.8). By Step 3, there exists some $\varphi \in W^{1,sp}(\Omega;\mathscr{E})$ such that $u = \pi \circ \varphi$. We find that $d_x u = d_{\varphi(x)}\pi d_x u$, fo a.e. $x \in \Omega$, and thus

$$D\varphi = \Phi \circ \varphi \, Du, \tag{1.4}$$

where Φ is as in (1.1). We complete this step by proving that

$$[s \geq 1,\ \varphi \in W^{1,sp},\ u \in W^{s,p} \cap L^\infty,\ \text{(1.4) holds}] \implies \varphi \in W^{s,p}. \tag{1.5}$$

Step 4.1. Proof of (1.5) when s is an integer. The proof is by induction on s, the case $s = 1$ being clear. The key fact is that (1.4) allows to express $D^s\varphi$ in terms of $D\varphi, \dots, D^{s-1}\varphi$. Let, e.g, $s = 2$. We claim that, if $u \in W^{2,p} \cap L^\infty$ and $v \in W^{1,2p} \cap L^\infty$, then $v\,Du \in W^{1,p}$. When u, v are smooth, this follows from the Gagliardo-Nirenberg embedding $W^{2,\infty} \cap L^\infty \hookrightarrow W^{1,2p}$ combined with the identity (with loose notation) $D(vDu) = DuDv + vD^2u$. The general case follows by a standard limiting procedure. Combining this with (1.4), we find that (1.5) holds when $s = 2$. Moreover, we find that $|D^2\varphi| \lesssim |D^2u| + |Du|^2$.

The general case is obtained by an obvious argument. Let $u \in W^{s,p} \cap L^\infty$. By differentiating (1.4) $(s-1)$-times, we find that

$$|D^s\varphi| \lesssim \sum_{j \geq 1} \sum_{\ell_1 + \dots + \ell_j = s} |D^{\ell_1}u| \cdots |D^{\ell_j}u|. \tag{1.6}$$

(again, first formally, then using a limiting procedure). In the process, we use the assumption (1.1). We conclude using the fact that, by the Gagliardo-Nirenberg embeddings $W^{s,p} \cap L^\infty \hookrightarrow W^{k,sp/k}$, $\forall\, 1 \leq k \leq s-1$, the right-hand side of (1.5) is in L^p.

Step 4.2. Proof of (1.5) when s is not an integer. Write $s = k + \sigma$, with $0 < \sigma < 1$. By Step 4.1 and the Gagliardo-Nirenberg embedding $W^{s,p} \cap L^\infty \hookrightarrow W^{k,sp/k}$, we have $\varphi \in W^{k,sp/k}$. By Theorem 4.15 and (1.1), we have $J \circ \varphi \in W^{k,sp/k} \cap L^\infty$, while, clearly, $Du \in W^{s-1,p} \cap L^{sp}$. By (1.4) and Lemma 4.19 (with $f := J \circ \varphi$, $g := Du$, $s - 1 := k$, $s_2 := s - 1$, $p_1 := sp/k$, $p_2 = p$, $r = sp$), we have $D\varphi \in W^{s-1,p}$, whence the conclusion.

Step 5. The lifting property fails in the non-compact case when $0 < s < 1$ and $1 \leq sp < N$. We first present the simple special case of the universal cover $\pi : \mathbb{R} \to \mathbb{S}^1$. Assume for simplicity that $\Omega = B_1(0)$. Let $\alpha > 0$ and set $\zeta(x) := |x|^{-\alpha}$ and $u := e^{i\zeta}$. By Lemmas 0.1 b) and 5.3, if

$$\frac{N - sp}{p} \leq \alpha < \frac{N - sp}{sp}, \tag{1.7}$$

then $\zeta \notin W^{s,p}$, but $u \in W^{s,p}$. Argue by contradiction and assume that u has a $W^{s,p}$-lifting φ. Then φ is continuous in $B_1(0) \setminus \{0\}$, and thus there exists some $k \in \mathbb{Z}$ such that $\varphi = \zeta + 2k\pi$, a contradiction with the fact that $\zeta \notin W^{s,p}$.

We next explain how to treat the general case. As in Step 1, we rely on a gluing construction. We explain here the idea and postpone the explicit construction to Section 5 (Lemma 5.13). Fix some $z \in \mathscr{N}$ and let $\pi^{-1}(\{z\}) = \{t_j;\ j \in J\}$. We will construct points $x_j \in \Omega$, radii $r_j > 0$, $j \in J$, and maps $\zeta_j : B_{2r_j}(x_j) \to \mathscr{E}$ such that:

(i) The balls $B_{3r_j(x_j)}$ are mutually disjoint and contained in Ω.

(ii) $\zeta_j \in C^\infty(\overline{B}_{2r_j}(x_j) \setminus \{x_j\})$.

(iii) $\zeta_j = t_j$ in $\overline{B}_{2r_j}(x_j) \setminus B_{r_j}(x_j)$.

(iv) $\zeta_j \notin W^{s,p}(B_{2r_j}(x_j))$.

(v) The map $u := \begin{cases} \pi \circ \zeta_j, & \text{in } B_{2r_j}(x_j) \\ z, & \text{in } \Omega \setminus \cup_j B_{2r_j}(x_j) \end{cases}$ belongs to $W^{s,p}$.

Granted the existence of such ζ, we argue as follows. Assume, by contradiction, that u has a lifting $\varphi \in W^{s,p}$. By Lemma 8.2, φ is continuous in the connected set $V := \Omega \setminus \cup_j \{x_j\}$. Let $U := \Omega \setminus \cup_j B_{2r_j}(x_j)$, which is a connected set contained in V. Note that $u = z$ in U. Let j be such that $\varphi^{-1}(t_j) \cap U$ is non-empty. By connectedness of U and continuity of φ in U, we have $\varphi = t_j$ in U, and, again, by continuity and connectedness, $\varphi = \zeta_j$ in $B_{2r_j}(x_j) \setminus \{x_j\}$. This contradicts item (iv).

Step 6. The lifting property holds in the compact case when $0 < s < 1$ *and* $sp \geq 2$. We argue by density. It suffices to find, for $u \in \mathscr{R}$, a lifting φ such that $|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}$. (Then we may pass to weak limits.) Set $\nu := N - \lfloor sp \rfloor - 1 \leq N - 3$. Let $u \in \mathscr{R}$. Then u is smooth in $U := \Omega \setminus A$, where A is a union of ν-planes parallel to the ν-coordinate planes. By Lemma 7.2, U is simply connected, and thus u has a smooth lifting φ in U. Note that the set $U_N := \{x' \in (0,1)^{N-1}; [\{x'\} \times (0,1)] \cap A \neq \emptyset\}$ is a null set, and the same holds for

$$U_j := \{(x', x'') \in (0,1)^j \times (0,1)^{N-j-1};\ [\{x'\} \times (0,1) \times \{x''\}] \cap A \neq \emptyset\}, 1 \leq j \leq N-1.$$

Consider the partial function $x_N \mapsto v := u(x', x_N)$, $x_N \mapsto \psi := \varphi(x', x_N)$, with $x' \in U_N$ and $x_N \in (0,1)$. Let $0 < \theta_1 < \theta_2 < 1$. Let $\rho := \mathrm{inj}(\mathscr{N}) > 0$ (the *injectivity radius* of $\mathscr{N}$). If $v(\theta) \in B_\rho(v(\theta_1)), \forall\, \theta_1 \leq \theta \leq \theta_2$, then

$$d_{\mathscr{E}}(\psi(\theta), \psi(\theta_1)) = d_{\mathscr{N}}(v(\theta), v(\theta_1)),\ \forall\, \theta_1 \leq \theta \leq \theta_2. \tag{1.8}$$

Combining (4.30) with the fact that φ is uniformly bounded (since $\mathscr{E}$ is compact), we find that, for every θ_1, θ_2, we have

$$d_{\mathscr{E}}(\psi(\theta_2), \psi(\theta_1)) \lesssim |v|_{W^{0,\infty}((\theta_1,\theta_2))} := \left\| v - \fint_{\theta_1}^{\theta_2} v \right\|_{L^\infty((\theta_1,\theta_2))}. \tag{1.9}$$

From (1.9) and Corollary 4.23, we obtain the *linear* estimate

$$|\psi|^p_{W^{s,p}((0,1))} \lesssim |v|^p_{W^{s,p}((0,1))},\ \forall\, x' \in U_N, \tag{1.10}$$

and similar estimates hold for each U_j.

Combining (1.10) with slicing (Theorem 4.1), we obtain the linear estimate $|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}$, which allows us to complete Step 6.

Step 7. The lifting property holds when $sp < 1$. We argue again by density. Consider a grid of size ε with faces parallel to the coordinate hyperplanes having the origin as an edge, and $\mathscr{N}$-valued

maps constant on each cube of the grid. By (the proof of) Theorem 2.1, the restrictions to Ω of such maps are dense in $W^{s,p}(\Omega;\mathscr{N})$. It suffices to obtain, for each such map, a $W^{s,p}$ lifting with a norm control. In order to further simplify the presentation, we assume that $\varepsilon = 2^{-J}$ for some integer J. (This is not relevant for the validity of the final result.) We may now formalize our program. For $k \geq 0$, let $\mathscr{P}_k$ denote the collection of dyadic cubes Q_k of size 2^{-k} in Ω. We let $\mathscr{F}_k$ denote the set of the (step) functions constant on each Q_k. We will complete Step 7 by proving the following: for every $J \geq 0$ and every $u : \Omega \to \mathscr{N}$, $u \in \mathscr{F}_J$, there exists some lifting $\varphi \in \mathscr{F}_J$ of u such that

$$\|\varphi\|_{W^{s,p}} \lesssim 1 + |u|_{W^{s,p}}. \tag{1.11}$$

The construction is relatively involved. We first construct approximations of u at the larger scales 2^{-k}, $0 \leq k < J$, as follows. Fix once for all some point $z_* \in \mathscr{N}$. Let $\delta > 0$ be such that the nearest point projection Π on $\mathscr{N}$ is well-defined and smooth in the δ-neighborhood $\mathscr{N}_\delta$ of $\mathscr{N}$. Let $E_k(x) := \fint_{Q_k} u$, $\forall\, Q_k \in \mathscr{P}_k$, $\forall\, x \in Q_k$, and set

$$u_k(x) := \begin{cases} \Pi(E_k(x)), & \text{if } d(E_k(x),\mathscr{N}) \leq \delta \\ z_*, & \text{if } d(E_k(x),\mathscr{N}) > \delta \end{cases}.$$

Note that E_k and u_k belong to $\mathscr{F}_k$, $\forall\, k$.

We next construct, inductively, a lifting φ_k of u_k, $0 \leq k \leq J$, and finally set $\varphi := \varphi_J$. The construction goes as follows. Fix once for all some $t_* \in \pi^{-1}(z_*)$. Let z_0 be the value of u_0 and let $\varphi_0 \in \mathscr{E}$ be a point in $\pi^{-1}(\{z_0\})$, nearest from t_*. Inductively, given $Q_{j+1} \in \mathscr{P}_{j+1}$, let $Q_j \in \mathscr{P}_j$ be such that $Q_{j+1} \subset Q_j$. If t_j is the value of φ_j on Q_j and z_{j+1} is the value of u_{j+1} on Q_{j+1}, then the value of φ_{j+1} on Q_{j+1} is a point in $\pi^{-1}(\{z_{j+1}\})$, nearest from t_j. Clearly, $\varphi_k \in \mathscr{F}_k$ and $\pi \circ \varphi_k = u_k$.

In order to estimate the $W^{s,p}$-norm of φ, we rely on the following inequalities:

$$d_{\mathscr{E}}(t, \pi^{-1}(\{z\})) \leq d_{\mathscr{N}}(\pi(t), z),\ \forall\, t \in \mathscr{E},\ \forall\, z \in \mathscr{N}, \tag{1.12}$$

$$d_{\mathscr{N}}(u_k(x), u_{k-1}(x)) \lesssim f_k(x) := |u(x) - E_k(x)| + |u(x) - E_{k-1}(x)|,\ \forall\, k \geq 1,\ \forall\, x, \tag{1.13}$$

$$d_{\mathscr{N}}(u_0(x), z_*) \lesssim f_0(x) := |z_* - E_0(x)| + |u(x) - E_0(x)| \lesssim 1 + |u(x) - E_0(x)|, \forall\, x. \tag{1.14}$$

The first property is clear, since a geodesic γ of length L from $\pi(t)$ to z lifts to a curve of length L from t to some point in $\pi^{-1}(\{z\})$. For the second one, if both $E_k(x)$ and $E_{k-1}(x)$ are in $\mathscr{N}_\delta$, then (1.13) holds, since

$$\begin{aligned} d_{\mathscr{N}}(u_k(x), u_{k-1}(x)) \lesssim & |u_k(x) - u_{k-1}(x)| = |\Pi(E_k(x)) - \Pi(E_{k-1}(x))| \leq |E_k(x) - E_{k-1}(x)| \\ \leq & |u(x) - E_k(x)| + |u(x) - E_{k-1}(x)| \end{aligned}$$

(the first inequality following from the fact that the geodesic distance and the Euclidean distance are equivalent on $\mathscr{N}$).

On the other hand, if, say, $E_k(x) \notin \mathscr{N}_\delta$, then $|u(x) - E_k(x)| \geq \delta$, so that the right-hand side of (1.13) is at least δ, while the left-hand side of (1.13) is dominated by $\sup\{d_{\mathscr{N}}(z,w);\ z,w \in \mathscr{N}\} < \infty$. Thus (1.13) holds in all cases.

The proof of (1.14) is similar to the one of (1.13).

Going back to the construction of φ, let us note that $f_k \in \mathscr{E}_k$, $\forall\, k$, and that, by combining (1.12)–(1.14) with the construction of the φ_k's we have, for every j, every $Q_j \in \mathscr{P}_j$, and every $x, y \in Q_j$:

$$d_{\mathscr{E}}(\varphi(x), \varphi(y)) = d_{\mathscr{E}}(\varphi_J(x), \varphi_J(y))$$

$$
\begin{aligned}
&\leq \sum_{1\leq k\leq J} [d_{\mathscr{E}}(\varphi_k(x),\varphi_{k-1}(x)) + d_{\mathscr{E}}(\varphi_k(y),\varphi_{k-1}(y))] \\
&= \sum_{j<k\leq J} [d_{\mathscr{E}}(\varphi_k(x),\varphi_{k-1}(x)) + d_{\mathscr{E}}(\varphi_k(y),\varphi_{k-1}(y))] \\
&:= \sum_{j<k\leq J} [g_k(x) + g_k(y)], \qquad (1.15)
\end{aligned}
$$

where

$$
g_k(x) := d_{\mathscr{E}}(\varphi_k(x),\varphi_{k-1}(x)) \lesssim d_{\mathscr{N}}(u_k(x),u_{k-1}(x)) \lesssim f_k(x),\ \forall\, k\geq 1. \qquad (1.16)
$$

Using (1.15) and (1.16), the fact that $g_k \in \mathscr{F}_k$, $\forall\, k$, the assumption $sp<1$, and, successively, Lemmas 4.25 and 4.24 in Section 4, we find that

$$
|\varphi|^p_{W^{s,p}} \lesssim \sum_{k\geq 1} 2^{spk}\|g_k\|^p_p \lesssim \sum_{k\geq 0} 2^{spk}\|u-E_k\|^p_p \lesssim |u|^p_{W^{s,p}}. \qquad (1.17)
$$

On the other hand, we have, by Hölder's inequality, the construction of the φ_k's, (1.12)–(1.14), (1.16), and Lemma 4.24,

$$
\begin{aligned}
\|\varphi\|^p_p \leq& \left(\|\varphi_0\|_p + \sum_{k\geq 0}\|\varphi_{k+1}-\varphi_k\|_p\right)^p \lesssim \|\varphi_0\|^p_p + \sum_{k\geq 0} 2^{spk}\|\varphi_{k+1}-\varphi_k\|^p_p \\
\lesssim& 1 + \sum_{k\geq 0} 2^{spk}\|u-E_k\|^p_p \lesssim 1 + |u|^p_{W^{s,p}}. \qquad (1.18)
\end{aligned}
$$

Step 7 follows from (1.17) and (1.18).

The proof of Theorem 1.2 is complete. QED

We next investigate the existence of lifting in the limiting case $sp=N$, which was left apart in the previous section. We also consider the case where $N=1$, which is of interest here. By Theorems 1.1 and 1.2, a $W^{s,p}$-lifting does exist when $s\geq 1$ or, in the compact case, when $N\geq 2$. We may thus assume that we are in the cases uncovered by the previous results, i. e.,

$$
\text{Assume that } 0<s<1 \text{ and, if } N\geq 2, \text{ that } \mathscr{E} \text{ is non-compact.} \qquad (1.19)
$$

Let $k\geq 1$ be the least integer such that $s+k/p\geq 1$. We make a second assumption.

$$
\text{If } s+k/p>1 \text{ and } \mathscr{E} \text{ is non-compact, then (1.1) holds.} \qquad (1.20)
$$

Proposition 1.3. Assume (1.19)–(1.20). Then the lifting property holds.

Proof. Let $u\in W^{s,p}(\Omega;\mathscr{N})$. We first construct successive extensions of u until we reach the framework of Theorem 1.1. This goes as follows. Arguing as in the proof of Proposition 0.5, u has a smooth extension $u_1\in W^{s+1/p,p}(\Omega\times(0,1);\mathscr{N})$. Continuing inductively as above, if $k\geq 1$ is the least integer such that $s+k/p\geq 1$, the final map u_k is smooth in $\Omega\times(0,1)^k$, and belongs to $W^{s+k/p,p}(\Omega\times(0,1)^k;\mathscr{N})$. Since $s+k/p\geq 1$ and $(s+k/p)p\geq 2$, we are in position to apply Theorem 1.1 (here, when $s+k/p>1$ we use the assumption (1.20)), and obtain that u_k has a phase

$\varphi_k \in W^{s+k/p,p}(\Omega \times (0,1)^k; \mathscr{E})$. By Lemma 8.2, φ_k is smooth. By trace theory adapted to $\mathscr{E}$-valued maps (see Lemma 4.26 and Chiron [38, Section 3.2]), we have the uniform estimate

$$\|\varphi_k(\cdot, x'')\|_{W^{s,p}(\Omega)} \lesssim \|\varphi_k\|_{W^{s+k/p,p}(\Omega\times(0,1)^k)}, \ \forall\, x'' \in (0,1)^k. \tag{1.21}$$

By (1.21), the obvious embedding $W^{s,p}(\Omega;\mathscr{E}) \hookrightarrow W^{s,p}(\Omega;\mathbb{R}^m)$, standard trace theory, and the compactness of the embedding $W^{s,p}(\Omega;\mathbb{R}^m) \hookrightarrow L^p(\Omega;\mathbb{R}^m)$, we obtain the existence of a sequence $x_j'' \to 0$ and a $\varphi \in W^{s,p}(\Omega;\mathscr{E})$ such that $\varphi_k(\cdot, x_j'') \to \varphi$ a.e. and φ is a lifting of u. QED

Finally, we prove that, in the proof of Proposition 1.3, the assumption (1.20) is just an artefact of the proof, and can be removed using a different approach, as in [60].

Proposition 1.4. Assume (1.19). Then the lifting property holds.

Proof. Let V, Ω_ε, and T be as in (0.1)–(0.4), and Ψ be a diffeomorphism as in the proof of Proposition 0.4. Let ε_0 be such that (0.2) holds. In the simply connected set W given by (0.4), T has a lifting Φ such that

$$|\nabla\Phi| = |\nabla T| \lesssim |\nabla V|. \tag{1.22}$$

In order to explain our proof, we first invoke the following local version of the theory of weighted Sobolev spaces (see [60, proof of Lemma 3.7] for a proof when $\Omega = \mathbb{T}^N$; the argument there can be adapted to any Lipschitz bounded domain)

$$|\Phi(\cdot,\theta)|^p_{W^{s,p}(\Omega)} \leq C(\varepsilon_0) \int_\Omega \int_0^{\varepsilon_0} \varepsilon^{p(1-s)-1} |\nabla\Phi(x,\varepsilon)|^p \, d\varepsilon dx, \forall\, 0 < \theta < \varepsilon_0. \tag{1.23}$$

We call the attention of the reader to the fact that, in (1.23), the $|\ |_{W^{s,p}}$ seminorm is calculated with respect to the Euclidean distance in $\mathbb{R}^m$. However, the proof of (1.23) is obtained starting from

$$\begin{aligned} |\Phi(x+h,\theta) - \Phi(x,\theta)| \leq & |\Phi(x+h,\theta) - \Phi(x+h/2,\theta+|h|/2)| \\ & + |\Phi(x+h/2,\theta+|h|/2) - \Phi(x,\theta)| \\ \leq & \int_0^1 \left| \frac{d}{d\tau} \Phi(x+h/2+\tau h/2, \theta+|h|/2-\tau|h|/2) \right| d\tau \\ & + \int_0^1 \left| \frac{d}{d\tau} \Phi(x+\tau h/2, \theta+\tau|h|/2) \right| d\tau, \\ & \forall\, (x,h) \in \Omega \times \mathbb{R}^N \text{ s. t. } |h| \leq \frac{\varepsilon_0}{2} \text{ and } [x, x+h] \subset \Omega. \end{aligned} \tag{1.24}$$

Clearly, we may replace, in (1.24), the Euclidean distance with the geodesic distance $d_{\mathscr{E}}$ on $\mathscr{E}$, and find that (1.23) still holds for the adapted $W^{s,p}$-seminorm in $W^{s,p}(\Omega;\mathscr{E})$.

By (1.22), (1.23), and standard inverse trace theory (see [60]), we have

$$|\Phi(\cdot,\theta)|^p_{W^{s,p}(\Omega)} \leq C(\varepsilon_0) \int_\Omega \int_0^{\varepsilon_0} \varepsilon^{p(1-s)-1} |\nabla T(x,\varepsilon)|^p \, d\varepsilon dx \lesssim |u|^s_{W^{s,p}}, \forall\, 0 < \theta < \varepsilon_0. \tag{1.25}$$

Using (1.25) and a standard limiting procedure, we find that $\Phi(\cdot,\theta)$ has a weak limit $\varphi \in W^{s,p}(\Omega;\mathscr{E})$ as $\varepsilon \to 0$, satisfying $\pi \circ \varphi = u$. QED

2 Lecture # 2. Strong density

Recall the implicit assumption $sp < N$. In this section, we let $\Omega = (0,1)^N$.

If $0 \leq \nu \leq N-1$ is an integer, let

$$\begin{aligned}\mathscr{R} = \mathscr{R}_\nu := \{u : \Omega \to \mathscr{N};\ \exists\, \varepsilon > 0,\ \exists \text{ a finite union } A \text{ of } \nu\text{-planes parallel} \\ \text{to the } \nu\text{-coordinate planes},\ \exists\, U \in C^\infty([-\varepsilon, 1+\varepsilon]^N \setminus A; \mathscr{N}) \\ \text{such that } u = U_{|(0,1)^N} \text{ and } |D^k U(x)| \leq C_k[d(x,A)]^{-k},\ \forall\, k \geq 0\}.\end{aligned} \tag{2.1}$$

The importance of the class $\mathscr{R}$, devised by Bethuel [6], is illustrated by the following result (see Bethuel [6] for $s = 1$, Bousquet, Ponce, and Van Schaftingen [19] for $s = 2, 3, \ldots$, [31] for $0 < s < 1$, and Detaille [42] for the remaining cases).

Theorem 2.1. Let $\nu := N - \lfloor sp \rfloor - 1$. The class $\mathscr{R}_\nu$ is dense in $W^{s,p}(\Omega; \mathscr{N})$.

Theorem 2.1 is complemented by the following result (same references as above).

Theorem 2.2. $C^\infty(\overline{\Omega}; \mathscr{N})$ is dense in $W^{s,p}(\Omega; \mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N})$ is trivial.

The full proofs of the above results require more than hundred pages. We will present here only four elements of proof:

1. The necessity of the assumption that $\pi_{\lfloor sp \rfloor}(\mathscr{N})$ is trivial in Theorem 2.2 (following essentially Schoen and Uhlenbeck [69]).
2. Approximation with homogeneous maps when $0 < s < 1$ and $1 \leq sp < N$ (following [31, Section 5]).
3. Smoothing of homogeneous maps when $s = 1$ (following essentially Hang and Lin [50, Sections 2.3 and 3]).
4. Bethuel's singularities removing technique in $W^{1,p}$ (following Bethuel [6] and the presentation in [50]).

Contents

2.1.0 Necessity of the condition $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \sim \{0\}$

Proposition 2.3. If $\pi_{\lfloor sp \rfloor}(\mathscr{N})$ is non-trivial, then $C^\infty(\overline{\Omega}; \mathscr{N})$ is not dense in $W^{s,p}(\Omega; \mathscr{N})$.

Proof. We actually establish the stronger result that $C^\infty(\Omega; \mathscr{N}) \cap W^{s,p}(\Omega; \mathscr{N})$ is not dense in $W^{s,p}(\Omega; \mathscr{N})$. For simplicity of the formulas, we work in $B_1(0)$ instead of $(0,1)^N$.

Let $k := \lfloor sp \rfloor \leq N-1$. Assume first that $k \geq 1$. Consider some $v \in C^\infty(\mathbb{S}^k; \mathscr{N})$ that is not null-homotopic. Let $u(x) := v((x_1, \dots, x_{k+1})/|(x_1, \dots, x_{k+1})|)$. By Lemma 5.5 c), we have $u \in W^{s,p}(\Omega; \mathscr{N})$. We claim that u cannot be approximated with smooth $\mathscr{N}$-valued maps. Argue by contradiction and let $(u_j) \subset C^\infty(\Omega; \mathscr{N}) \cap W^{s,p}(\Omega; \mathscr{N})$ be such that $u_j \to u$ in $W^{s,p}$. By slicing (Corollary 4.3), up to a subsequence and for some $0 < r < 1/2$ and $x'' \in \mathbb{R}^{N-k-1}$ such that $|x''| < 1/2$, we have $u_j(\cdot, x'') \to u(\cdot, x'')$ in $W^{s,p}(r\mathbb{S}^{k-1}; \mathscr{N})$. By Corollary 4.28, for large j, $u_j(\cdot, x'')$ and $u(\cdot, x'')$ are homotopic as continuous functions from $r\mathbb{S}^{k-1}$ to $\mathscr{N}$. However, on the one hand $u(\cdot, x'')$ is not null-homotopic (for otherwise, v would also be), while $u_j(\cdot, x'')$ is always null-homotopic (by a homotopy argument, since u_j is smooth in $B_1(0)$). The contradiction achieves the proof when $k \geq 1$.

Assume next that $k = 0$, i. e., that $sp < 1$. Since we work with connected $\mathscr{N}$'s, $\pi_0(\mathscr{N})$ is trivial. For the record, let us note that, if $\mathscr{N}$ is *not connected*, then density fails. To see this, let $\mathscr{C}_1, \dots, \mathscr{C}_k$, $k \geq 2$, be the components of $\mathscr{N}$. Let $a_m \in \mathscr{C}_m$, $m = 1, 2$. Consider a ball B with $\overline{B} \subset \Omega$. Let $u = \begin{cases} a_1, & \text{in } B \\ a_2, & \text{in } \Omega \setminus B \end{cases}$. Then $u \in W^{s,p}(\Omega; \mathscr{N})$ (Lemma 5.2). We claim that u cannot be approximated with a sequence (u_j) of smooth $\mathscr{N}$-valued maps. Argue by contradiction. Up to a subsequence, we have $u_j \to u$ a.e., and in particular, for a given $\varepsilon > 0$ and large j, the sets $\{x \in \Omega; |u_j(x) - a_m| < \varepsilon\}$ have positive measure. If we let, in particular, $\varepsilon < \min\{d(\mathscr{C}_i, \mathscr{C}_m),\ i \neq m\}$, we find that, for large j, u_j has to take values both in $\mathscr{C}_1$ and $\mathscr{C}_2$. However, this cannot happen, since the image of u_j is connected. QED

2.2.0 Approximation with homogeneous maps when $0 < s < 1$

In this section, $|\ |$ stands for the $\|\ \|_\infty$ norm.

We start by describing a procedure for constructing homogeneous maps on $\mathbb{R}^N$. Fix some $\varepsilon > 0$ and $t \in \mathbb{R}^N$. Consider the mesh $\mathscr{C}_N = \mathscr{C}_{N,t} = \mathscr{C}_{N,t,\varepsilon}$ of N-dimensional cubes (with vertices parallel to the coordinate axes) of side-length 2ε having t as one of the centers. (Thus, cubes in $\mathscr{C}_N$ are of the form $t + 2\varepsilon k + [-\varepsilon, \varepsilon]^N$, with $k \in \mathbb{Z}^N$.) Let $\mathscr{C}_{N-1} = \mathscr{C}_{N-1,t}$ be the $(N-1)$-dimensional skeleton associated with this mesh, i. e., $\mathscr{C}_{N-1}$ is the union of the boundaries of the cubes in $\mathscr{C}_N$. Let H_N be the mapping that associates with every $g : \mathscr{C}_{N-1} \to \mathbb{R}^\ell$ its homogeneous extension (on each cube of $\mathscr{C}_N$) to $\mathbb{R}^\ell$. Analytically, if $\mathfrak{C}$ is a cube in $\mathscr{C}_N$, of center u, then $H_N(g)(x) = g(u + \varepsilon(x-u)/|x-u|), \forall\, x \in \mathfrak{C}$. In order to keep notation reasonably simple, we will identify $\mathscr{C}_j$ with the union of its cubes, so that we write both $\mathfrak{C} \in \mathscr{C}_j$ and, if $x \in \mathfrak{C}$, $x \in \mathscr{C}_j$.

We next consider a more general situation. We start by defining the lower dimensional skeletons and cubes associated with $\mathscr{C}_N$. This is done by backward induction: $\mathscr{C}_{N-2} = \mathscr{C}_{N-2,t} = \mathscr{C}_{N-2,t,\varepsilon}$ is the union of the $(N-2)$-dimensional boundaries of the cubes in $\mathscr{C}_{N-1} = \mathscr{C}_{N-1,t} = \mathscr{C}_{N-1,t,\varepsilon}$, and so on. A cube in $\mathscr{C}_N$ is any cube of the mesh $\mathscr{C}_N$. A cube in $\mathscr{C}_{N-1}$ is any of the $2N$ faces of a cube in $\mathscr{C}_N$. For $j \leq N-2$, a cube in $\mathscr{C}_j$ is any of the $2(j+1)$ faces of any cube in $\mathscr{C}_{j+1}$.

For $g : \mathscr{C}_j \to \mathbb{R}^\ell$, let $H_{j+1}(g)$ be its homogeneous extension to $\mathscr{C}_{j+1}$.

Let $0 \le j < N$. For $\varepsilon > 0$ and $t \in \mathbb{R}^N$, we associate with each map $f : \mathbb{R}^N \to \mathbb{R}^\ell$ a map $f_t = f_{t,\varepsilon} = f_{t,\varepsilon,j} : \mathbb{R}^N \to \mathbb{R}^\ell$ through the formula

$$f_t = g_j := H_N \circ H_{N-1} \circ \cdots \circ H_{j+1} \circ g; \text{ here, we set } g := f|_{\mathscr{C}_j}. \tag{2.2}$$

More generally, given any map $g : \mathscr{C}_j \to \mathbb{R}^\ell$, the map g_j given by the right-hand side of (2.2) is referred to as a *j-homogeneous map* or the *j-homogeneous extension of g*.

Here is our main result in this section.

Theorem 2.4. Let $0 \le j < N, 0 < s < 1, sp < j + 1$, and let $f \in W^{s,p}(\mathbb{R}^N; \mathbb{R}^\ell)$. Then there exist sequences $\varepsilon_k \to 0$ and $(t_k) \subset \mathbb{R}^N$ such that $f_{t_k,\varepsilon_k} \to f$ in $W^{s,p}(\mathbb{R}^N)$.

For the sake of simplicity, we prove Theorem 2.4 under the extra assumptions

$$1 < p < \infty, \ j \ge 1. \tag{2.3}$$

(Theorem 2.4 holds without these assumptions, but the treatment of the remaining cases is more involved.) Under these assumptions, we will obtain the following improvement of Theorem 2.4.

Theorem 2.5. Assume $1 \le j < N, 0 < s < 1, 1 < p < \infty$, and $sp < j + 1$. Let $f \in W^{s,p}(\mathbb{R}^N; \mathbb{R}^\ell)$. Then, for each $\varepsilon \in \mathbb{R}^N$, there exists some $t_\varepsilon \in \mathbb{R}^N$ such that $f_{t_\varepsilon,\varepsilon} \to f$ in $W^{s,p}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

Before proceeding to the proof of Theorem 2.5, let us note the following consequence of Theorem 2.4.

Corollary 2.6. Assume that $0 < s < 1$ and $sp < N$. Let $F \subset \mathbb{R}^\ell$ be an arbitrary set. Let $f \in W^{s,p}(\Omega; F)$. Then there exists a sequence of j-homogeneous maps $(f_k) \subset W^{s,p}(\mathbb{R}^N; \mathbb{R}^\ell)$ such that $f_k \to f$ in $W^{s,p}(\Omega)$ and f_k is F-valued in $(-1, 2)^N$.

Proof. Extend f, by reflexions, to a map $\widetilde{f} \in W^{s,p}((-2, 3)^N; F)$. Then extend $\widetilde{f}$ to a map $h \in W^{s,p}(\mathbb{R}^N; \mathbb{R}^\ell)$. (We do not claim that h is F-valued.) Finally, let $f_k := h_{t_k,\varepsilon_k}$, with ε_k, t_k as in Theorem 2.4 (applied to h). Then the f_k's have, for large k (and thus small ε_k), the desired properties. QED

Proof of Theorem 2.5. We start by introducing some useful notation. Set $Q_\varepsilon := [-\varepsilon, \varepsilon]^N$. In order to keep notation easier to follow, we will sometimes denote a point in Q_ε by x^N rather than x. We denote by x^{N-1} the radial projection (centered at 0) of x^N onto the $(N-1)$-skeleton (thus the boundary) of Q_ε ; this projection is defined except when $x^N = 0$. With an abuse of notation, x^{N-1} also denotes a "generic" point of ∂Q_ε. We next let x^{N-2} denote the radial projection of x^{N-1} onto the $(N-2)$-skeleton of Q_ε. The point x^{N-2} is obtained as follows: if $x^{N-1} \in \partial Q_\varepsilon$ belongs to an $(N-1)$-dimensional face F of ∂Q_ε, and if x^{N-1} is not the center C of F, then the radial projection (centered at C) of x^{N-1} on ∂F is well-defined, and yields x^{N-2}. By backward induction, we define x^j, $0 \le j \le N-1$, as the radial projection of x^{j+1} onto $\partial Q_\varepsilon \cap \mathscr{C}_{j,0}$; this is defined for all but a finite number of x^{j+1}'s. Again, with an abuse of notation, x^j is the "generic" point of $\partial Q_\varepsilon \cap \mathscr{C}_{j,0}$. Note that x^0 is one of the vertices of Q_ε.

When x^j is obtained starting from x^N, we will denote x^j as the radial projection of x^N (onto $\partial Q_\varepsilon \cap \mathscr{C}_{j,0}$). This projection is defined except on a set of finite $\mathscr{H}^{N-j-1}$ measure.

More generally, let $j < k \leq N$. We identify x^k with a "generic" point of $\partial Q_\varepsilon \cap \mathscr{C}_{k,0}$. Then x^j is the projection of x^k onto $\partial Q_\varepsilon \cap \mathscr{C}_{j,0}$ (except for a set of x_k's of finite $\mathscr{H}^{k-j-1}$ measure).

Let $k \in \mathbb{Z}^N$ and set $u = t + 2\varepsilon k$. Then the radial projection of $u + x^N$ onto $\mathscr{C}_j$ is $u + x^j$. If $j < k \leq N$, then for $\mathscr{H}^k$-a.e. $x^k \in \partial Q_\varepsilon \cap \mathscr{C}_{k,0}$, the projection of $u + x^k$ onto $\mathscr{C}_j$ is $u + x^j$.

With the above notation, formula (2.2) is equivalent to

$$f_t(t + 2\varepsilon k + x^N) = f(t + 2\varepsilon k + x^j),\ \forall\, k \in \mathbb{Z}^N,\ \text{for } \mathscr{H}^N - \text{a.e. } x^N \in Q_\varepsilon. \tag{2.4}$$

We now proceed to the proof of the theorem. Set

$$F_\varepsilon(f)(t,x) := f_{t,\varepsilon}(x), \forall\, t \in Q_\varepsilon,\ \forall\, x \in \mathbb{R}^N.$$

Step 1. An L^q-estimate for $F_\varepsilon(f)$. Let $1 \leq q < \infty$ and $f \in L^q(\mathbb{R}^N)$. We claim that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{Q_\varepsilon} \|f - f_{t,\varepsilon}\|_q^q \, dt = 0 \tag{2.5}$$

and

$$\|F_\varepsilon(f)\|_q \leq C\, \varepsilon^{N/q} \|f\|_q, \text{ with } C \text{ independent of } \varepsilon \text{ or } f. \tag{2.6}$$

(Here, we do not require $j \geq 1$.) Set $Q_\varepsilon(x) := x + Q_\varepsilon$, $\forall\, x \in \mathbb{R}^N$. Using the facts that: (i) $Q_\varepsilon(t + 2\varepsilon k))_{k \in \mathbb{Z}^n}$ is an a.e. partition of $\mathbb{R}^N$; (ii) $f_{t,\varepsilon} = f_{t+2\varepsilon k,\varepsilon}$ for $t \in \mathbb{R}^N$ and $k \in \mathbb{Z}^N$ (thanks to (2.4)), and (iii) the "change of variable" $t = x + y^N$, $\forall\, t \in Q_\varepsilon(x)$ (with $y^N \in Q_\varepsilon$), we have

$$\begin{aligned}
\frac{1}{\varepsilon^N} \int_{Q_\varepsilon} \|f - f_{t,\varepsilon}\|_q^q \, dt &= \frac{1}{\varepsilon^N} \int_{Q_\varepsilon} \sum_{k \in \mathbb{Z}^N} \int_{Q_\varepsilon(t+2\varepsilon k)} |f(x) - f_{t,\varepsilon}(x)|^q \, dx dt \\
&= \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \int_{Q_\varepsilon(x)} |f(x) - f_{t,\varepsilon}(x)|^q \, dt dx \\
&= \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \int_{Q_\varepsilon} \left|f(x) - f\left(x + y^N - y^j\right)\right|^q \, dy dx \\
&= \frac{1}{\varepsilon^N} \int_{Q_\varepsilon} \left\|f(\cdot) - f(\cdot + y^N - y^j)\right\|_q^q dy.
\end{aligned}$$

We next note that $y^N \in Q_\varepsilon \implies y^N - y^j \in Q_\varepsilon$. Therefore,

$$\frac{1}{\varepsilon^N} \int_{Q_\varepsilon} \|f - f_{t,\varepsilon}\|_q^q \, dt \leq 2^N \sup\{\|f(\cdot) - f(\cdot + z)\|_q^q; |z| \leq \varepsilon\}. \tag{2.7}$$

Finally, we note that (2.7) implies both (2.5) and (2.6).

Step 2. A $W^{1,r}$ estimate for $F_\varepsilon(f)$. Let $1 \leq j \leq N-1$, $1 \leq r < j+1$, and $f \in W^{1,r}(\mathbb{R}^N)$. We claim that

$$\|F_\varepsilon(f)\|_{L^r(Q_\varepsilon; W^{1,r}(\mathbb{R}^N))} \leq C\, \varepsilon^{N/r} \|f\|_{W^{1,r}}, \text{ with } C \text{ independent of } \varepsilon \text{ or } f. \tag{2.8}$$

In view of Step 1, in order to obtain (2.8) it suffices to establish, with $C = C(N,j,r)$, the estimate

$$\int_{Q_\varepsilon}\int_{\mathbb{R}^N} |\nabla f_{t,\varepsilon}(x)|^r dxdt \le C\varepsilon^N \int_{\mathbb{R}^N} |\nabla f(x)|^r\, dx. \tag{2.9}$$

We next observe that it suffices to prove (2.9) when $f \in C_c^\infty$. Indeed, assuming for the moment that (2.9) holds for such f, Step 1 combined with (2.9) for $f \in C_c^\infty$ and with a standard limiting argument implies that (2.9) holds for every $f \in W^{1,r}$.

We finally turn to the proof of (2.9) when $f \in C_c^\infty$. Let, for $t \in \mathbb{R}^N$ and $\varepsilon > 0$, $v := t+(\varepsilon,\dots,\varepsilon)$ and $\mathscr{C} := \mathscr{C}_{N-j-1,v,\varepsilon}$. Then (see Lemma 6.1) the projection

$$\mathbb{R}^N \setminus \mathscr{C} \ni t + 2\varepsilon k + x^N \overset{\Psi}{\mapsto} t + 2\varepsilon k + x^j \in \mathscr{C}_{j,t,\varepsilon},\ \forall\, k \in \mathbb{Z}^N,\ \forall\, x^N \in Q_\varepsilon$$

is well-defined, locally Lipschitz, and satisfies

$$|\nabla\Psi(x)| \lesssim \frac{\varepsilon}{d(x,\mathscr{C})}. \tag{2.10}$$

It follows from (2.10) and the fact that $\mathscr{C}$ is locally a finite union of $(N-j-1)$-planes that

$$|\nabla\Psi| \in L^r_{loc}(\mathbb{R}^N),\ \forall\, 1 \le r < j+1. \tag{2.11}$$

Combining (2.11), the fact that $f \in C_c^\infty$, Lemma 5.11, and the observation that $f_{t,\varepsilon} = f\circ\Psi$, we find that $f_{t,\varepsilon} \in W^{1,r}(\mathbb{R}^N)$, $\forall\, 1 \le r < j+1$, and $df_{t,\varepsilon} = [(df)\circ\Psi]\, d\Psi$ in the sense of distributions.

After these preliminary remarks, we proceed to the proof of (2.9). By symmetries of the formula defining Ψ, it suffices to establish (2.9) when $\mathbb{R}^N$ is replaced by

$$\begin{aligned}&\mathbb{R}^N_* := \cup_{k\in\mathbb{Z}^N}\,(t+2\varepsilon k + Q^*_\varepsilon),\\ &\qquad \text{with } Q^*_\varepsilon := \{x^N \in Q_\varepsilon;\ x_1 \ge \cdots \ge x_{N-j} \ge |x_m|,\ \forall\, m > N-j\}.\end{aligned}$$

We note that, when $x^N \in Q^*_\varepsilon \setminus \mathscr{C}$, we have, $\forall\, k \in \mathbb{Z}^N$,

$$\begin{aligned}\Psi(t+2\varepsilon k + x^N) =& t + 2\varepsilon k + z(x^N),\\ &\text{with } z(x^N) := (\varepsilon,\dots,\varepsilon,\varepsilon x_{N-j+1}/x_{N-j},\dots,\varepsilon x_N/x_{N-j}),\end{aligned} \tag{2.12}$$

and

$$|\nabla\Psi(t+2\varepsilon k + x^N)| \lesssim \frac{\varepsilon}{x_{N-j}},\ \forall\, x^N \in Q^*_\varepsilon. \tag{2.13}$$

Using (2.12) and (2.13), we find that

$$\begin{aligned}\int_{Q_\varepsilon}\int_{\mathbb{R}^N_*} |\nabla f_{t,\varepsilon}(x)|^r dxdt &= \int_{Q_\varepsilon}\sum_{k\in\mathbb{Z}^N}\int_{t+2\varepsilon k+Q^*_\varepsilon} |\nabla f_{t,\varepsilon}(x)|^r dxdt\\ &\lesssim \int_{Q_\varepsilon}\sum_{k\in\mathbb{Z}^N}\int_{t+2\varepsilon k+Q^*_\varepsilon} \frac{\varepsilon^r}{x^r_{N-j}} |\nabla f(t+2\varepsilon k + z(x)|^r\, dxdt\end{aligned}$$

$$= \|\nabla f\|_r^r \int_{Q_\varepsilon^*} \frac{\varepsilon^r}{x_{N-j}^r}\, dr \sim \varepsilon^N \|\nabla f\|_r^r,$$

where the last line uses the definition of Q_ε^* and the fact that $r < j+1$.

We find that (2.9) holds for $\mathbb{R}^N_*$ and thus, as explained above, for $\mathbb{R}^N$. Step 2 is now completed.

Step 3. Average estimate for $f - f_{t,\varepsilon}$ and conclusion. (Here, we use $1 < p < \infty$ and $j \geq 1$.) Let $0 < s < 1$, $1 < p < \infty$, and $1 \leq j \leq N-1$ be such that $sp < j+1$. We claim that there exist q, r such that

$$1 < q < \infty,\ 1 < r < j+1,\ \frac{1}{p} = \frac{s}{r} + \frac{1-s}{q}. \tag{2.14}$$

Indeed, the existence of r and q as in (2.14) is equivalent to

$$\frac{s}{j+1} + \frac{1-s}{\infty} < \frac{1}{p} < \frac{s}{1} + \frac{1-s}{1},$$

which clearly holds.

We next recall three classical interpolation results. Given two Banach spaces X and Y, we use the standard notation $[X,Y]_{s,p}$; see e. g. [70, Section 1.5]. First, when (2.14) holds we have [70, Section 2.4.2, Theorem 1 (a), eq. (2), p. 185]

$$[W^{1,r}, L^q]_{s,p} = W^{s,p}. \tag{2.15}$$

Next, if X and Y are Banach spaces and s, p, q, r are as above, then [70, Section 1.18.4, Theorem, eq. (3), p. 128]

$$[L^r(\Omega; X), L^q(\Omega; Y)]_{s,p} = L^p(\Omega; [X,Y]_{s,p}). \tag{2.16}$$

By (2.15) and (2.16),

$$\forall\, r, q \text{ as in (2.14)},\ [L^r(Q_\varepsilon; W^{1,r}(\mathbb{R}^N)), L^q(Q_\varepsilon; L^q(\mathbb{R}^N))]_{s,p} = L^p(Q_\varepsilon; W^{s,p}(\mathbb{R}^N)). \tag{2.17}$$

Final classical result. Let s, p, q, r, X, and Y be as above. Let F be a linear continuous operator from X into $L^r(\Omega; X)$ and from Y into $L^q(\Omega; Y)$. Then F is linear continuous from $[X,Y]_{s,p}$ into $L^p(\Omega; [X,Y]_{s,p})$ and satisfies the norm inequality

$$\|F\|_{\mathscr{L}([X,Y]_{s,p}; L^p(\Omega; [X,Y]_{s,p}))} \leq \|F\|^s_{\mathscr{L}(X; L^r(\Omega;X))}\, \|F\|^{1-s}_{\mathscr{L}(Y; L^q(\Omega;Y))}. \tag{2.18}$$

By (2.6), (2.8), and (2.18), we find that

$$\|F_\varepsilon(f)\|_{L^p(Q_\varepsilon; W^{s,p}(\mathbb{R}^N))} \leq C\, \varepsilon^{N/p} \|f\|_{W^{s,p}(\mathbb{R}^N)},\ \text{with } C \text{ independent of } \varepsilon. \tag{2.19}$$

(In principle, the constant C in (2.19) may depend on ε, since we apply the interpolation result (2.17) in an ε-dependent domain. The fact that C does not depend on ε is obtained by a straightforward scaling argument: we consider, instead of F_ε, the map

$$G_\varepsilon(f) : Q_1 \times \mathbb{R}^N \to \mathbb{R}^\ell,\ G_\varepsilon(f)(t,x) = f_{\varepsilon t, \varepsilon}(x).$$

We obtain (2.19) by applying (2.18) to $G_\varepsilon(f)$ in Q_1.)

A clear consequence of (2.19) is

$$\frac{1}{\varepsilon^N}\int_{Q_\varepsilon}\|f_{t,\varepsilon}-f\|^p_{W^{s,p}(\mathbb{R}^N)}\,dt\le C\|f\|^p_{W^{s,p}(\mathbb{R}^N)}. \tag{2.20}$$

Arguing as above and using (2.5) instead of (2.6), we improve (2.20) to

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_{Q_\varepsilon}\|f_{t,\varepsilon}-f\|^p_{W^{s,p}(\mathbb{R}^N)}\,dt=0. \tag{2.21}$$

Clearly, (2.21) and a mean-value argument yield the conclusion of Theorem 2.5. QED

2.3.0 Smoothing in $W^{1,p}$

We start by explaining how Theorem 2.5 is used in the proof of Theorem 2.1 (when $0<s<1$). Let $0<s<1$, $1<p<\infty$, $1\le j\le N-1$ be such that $sp<j+1$. (Here and in what follows, j is fixed.) Let $f\in W^{s,p}(\mathbb{R}^N;\mathbb{R}^\ell)$. It will be convenient here to consider f as a everywhere defined Borel map (rather than an equivalence class). By (2.21), for a.e. $t\in Q_\varepsilon$ we have

$$f_{t,\varepsilon,j}\in W^{s,p}. \tag{2.22}$$

On the other hand (by a "generalized slicing" argument, see e. g. [31, Lemma 6.1]) for a.e. $t\in Q_\varepsilon$ we have

$$f_{|\mathscr{C}_{m,t,\varepsilon}}\in W^{s,p}(\mathscr{C}_{m,t,\varepsilon}),\ \forall\, 0\le m\le N-1. \tag{2.23}$$

(The discussion here being rather informal, we do not give the precise definition of the space $W^{s,p}$ on a skeleton. We will be precise in the case $s=1$ detailed below.)

Moreover, when $sp>1$, we have, for a.e. $t\in Q_\varepsilon$ [31, Appendix E],

$$\mathrm{tr}(f_{|\mathscr{C}_{m,t,\varepsilon}})=f_{|\mathscr{C}_{m-1,t,\varepsilon}},\ \forall\, 1\le m\le N. \tag{2.24}$$

Note the assumption $sp>1$, which implies that trace theory makes sense in $W^{s,p}$. (The assumption $sp>1$ can be relaxed to $sp\ge 1$, provided we replace, when $sp=1$, the notion of trace with the one of good restriction; see [30, Appendix B, Appendix E].)

Combing these facts with (2.21), we find that there exists $t=t_\varepsilon\in Q_\varepsilon$ such that (2.22)–(2.24) hold and, in addition $f_{t,\varepsilon,j}\to f$ in $W^{s,p}$ as $\varepsilon\to 0$.

Assume now that we start from $u\in W^{s,p}(\Omega;\mathscr{N})$, that we first extend by reflexions to $(-2,3)^N$, next to a map $f\in W^{s,p}(\mathbb{R}^N;\mathbb{R}^\ell)$. Then, by construction, for small ε,

$$f_{t,\varepsilon,j}\text{ is }\mathscr{N}\text{-valued in }(-1,2)^N. \tag{2.25}$$

Up to now, the fact that $\mathscr{N}$ is a manifold was irrelevant. The next step consists of taking advantage of the smoothness of $\mathscr{N}$ and of the properties (2.22)–(2.25). More specifically, if: (i) $1\le j\le sp<j+1$; (ii) ε is fixed; (iii) $f\in W^{s,p}$ and t are such that (2.22)–(2.25) hold, then one may

prove that it is possible to approximate, in $W^{s,p}((-1/2,3/2)^N)$, $f_{t,\varepsilon,j}$ with a map v: (j) $\mathscr{N}$-valued; (jj) locally Lipschitz in $[-1/2,3/2]\setminus \mathscr{C}_{N-j-1,v,\varepsilon}$; (jjj) satisfying $|\nabla v(x)|\leq C(\varepsilon)/d(x,\mathscr{C}_{N-j-1,v,\varepsilon})$. By Lemma 5.7, we have $v\in W^{1,q}$, $\forall\, q<j+1$. Granted the existence of v as above, we thus obtain that: under the assumptions $0<s<1$, $1\leq sp<N$, and $1\leq q<\lfloor sp\rfloor+1$, each $u\in W^{s,p}((0,1)^N;\mathscr{N})$ can be approximated in $W^{s,p}$ with maps $v\in W^{1,q}((0,1)^N;\mathscr{N})$ such that each v is locally Lipschitz outside a finite union of $(N-\lfloor sp\rfloor-1)$-planes. Then a rather standard smoothing procedure (see, e. g., Brezis and Li [26, Proposition A.4] when $s=1$) allows to further smoothen the v's and obtain approximation with maps in the class $\mathscr{R}$.

To summarize, the heart of the transition from $f_{t,\varepsilon,j}$ to maps in the class $\mathscr{R}$ is the construction of v as above. When $0<s<1$, this is performed in [31, Section 7], following a scheme conceptually similar to the one of Hang and Lin for $s=1$ [50, Section 2.3, Section 3]. In order to keep the presentation technically simple but yet relevant concerning the main ideas, we present here the $s=1$ counterpart of the above, consistent with the schemes in [50, 31]. More specifically, we will prove the following.

Proposition 2.7. Let $\varepsilon=1/2$ and $t=0$. Let $M\in\mathbb{N}$ and $\Omega=(-M-1/2,M+1/2)^N$. Let $1\leq j<p<j+1\leq N$. Let $u\in W^{1,p}(\Omega;\mathscr{N})$ be such that

$$u_{|\mathscr{C}_{m,0,1/2}}\in W^{1,p}(\mathscr{C}_{m,0,1/2}),\ \forall\, 1\leq m\leq j, \tag{2.26}$$

$$\mathrm{tr}(u_{|\mathscr{C}_{m,0,1/2}})=u_{|\mathscr{C}_{m-1,0,1/2}},\ \forall\, 1\leq m\leq j. \tag{2.27}$$

Then, for every $\lambda>0$, there exists some Lipschitz map $g:\mathscr{C}_{j,0,1/2}\to\mathscr{N}$ such that the j-homogeneous extension $v=g_j$ of g (given by (2.2)) satisfies $\left\|u_{0,1/2,j}-v\right\|_{W^{1,p}}<\lambda$. [†]

Before proceeding to the proof of Proposition 2.7, let us precise some notation and assumptions. With an abuse of notation, $\mathscr{C}_j=\mathscr{C}_j\cap\mathscr{C}$, $0\leq j\leq N-1$, where $\mathscr{C}:=\cup_{k\in\mathbb{Z}^N,|k|\leq M}Q_{1/2}(k)$ is the part of the grid $\mathscr{C}_N=\mathscr{C}_{N,0,1/2}$ corresponding to Ω. $\mathfrak{C}$ denotes a generic cube in $\mathscr{C}_m$. The meaning of (2.26)–(2.27) is that, for each $1\leq m\leq j$, and each cube $\mathfrak{C}$ of $\mathscr{C}_m$, $u_{|\mathfrak{C}}$ belongs to $W^{1,p}(\mathfrak{C})$ and, in addition, the trace of $u_{|\mathfrak{C}}$ to $\partial\mathfrak{C}$ is $u_{|\partial\mathfrak{C}}$. (Recall that we consider everywhere defined maps.) We naturally define

$$\|u\|^p_{L^p(\mathscr{C}_m)}:=\sum_{\mathfrak{C}\in\mathscr{C}_m}\|u\|^p_{L^p(\mathfrak{C})},\ |u|^p_{W^{1,p}(\mathscr{C}_m)}=\|\nabla u\|^p_{L^p(\mathscr{C}_m)}:=\sum_{\mathfrak{C}\in\mathscr{C}_m}\|\nabla u\|^p_{L^p(\mathfrak{C})},$$

$$\|u\|_{W^{1,p}(\mathscr{C}_m)}:=\|u\|_{L^p(\mathscr{C}_m)}+|u|_{W^{1,p}(\mathscr{C}_m)}.$$

In view of Lemma 6.2, the conclusion of Proposition 2.7 follows from the following fact, that we will establish below: if $1\leq j<N$, $j<p<\infty$, and $g:\mathscr{C}_j\to\mathscr{N}$ satisfies

$$g_{|\mathscr{C}_{m,0,1/2}}\in W^{1,p}(\mathscr{C}_{m,0,1/2}),\ \forall\, 1\leq m\leq j, \tag{2.28}$$

$$\mathrm{tr}(g_{|\mathscr{C}_{m,0,1/2}})=g_{|\mathscr{C}_{m-1,0,1/2}},\ \forall\, 1\leq m\leq j, \tag{2.29}$$

then

$$\forall\,\lambda>0,\ \exists\,\widetilde{g}\in\mathrm{Lip}(\mathscr{C}_j;\mathscr{N})\ \text{s. t.}\ \|\widetilde{g}-g\|_{W^{1,p}(\mathscr{C}_j)}<\lambda. \tag{2.30}$$

(Note the wider range $j<p<\infty$ instead of $j<p<j+1$.)

[†]Proposition 2.7 still holds when $p=j$, but when $j\geq 2$ the case $p=j$ requires a separate argument (in the spirit of the proof of Proposition 0.5), since the embedding $W^{1,p}(\mathbb{R}^j)\hookrightarrow C^0$ holds for $p>j$, but fails for $p=j$. For simplicity, we do not consider here the case $p=j$.

Proof of Proposition 2.7. Step 1. Choice of a continuous representative. Assume that $1 \le j \le N$ and $p > j$. Assume that g satisfies (2.28)–(2.29). We claim that there exists a *continuous* function $\widetilde{g} : \mathscr{C}_j \to \mathscr{N}$ such that $\widetilde{g}_{|\mathscr{C}_m} = g_{|\mathscr{C}_m}$ $\mathscr{H}^m$-a. e., $\forall\, 0 \le j \le m$. The construction of $\widetilde{g}$ is performed successively on each $\mathscr{C}_m$, by induction on m. For $m = 0$, we simply let $\widetilde{g} = g$. Assuming $\widetilde{g}$ constructed on $\mathscr{C}_{m-1}$, with $1 \le m \le j$, we let, for $\mathfrak{C} \in \mathscr{C}_m$, $\widetilde{g}$ be the continuous representative of g on $\mathfrak{C}$. It is easy to see that $\widetilde{g}_{|\mathscr{C}_{m-1}}$ agrees with the map already constructed on $\mathscr{C}_{m-1}$, is continuous, and that the final map constructed on $\mathscr{C}_j$ has all the required properties. From now on, we assume that g is continuous.

Step 2. Reduction to almost $\mathscr{N}$-valued maps. Fix some small $\delta > 0$ such that the projection $\Pi : \mathscr{N}_\delta \to \mathscr{N}$ is well-defined and smooth in the δ-neighborhood $\mathscr{N}_\delta$ of $\mathscr{N}$. Assume that we are able to construct a Lipschitz map $\widetilde{g} : \mathscr{C}_j \to \mathscr{N}_\delta$ such that $\|\widetilde{g} - g\|_{W^{1,p}(\mathscr{C}_j)} < \lambda$. Then, clearly, $\Pi \circ \widetilde{g}$ is Lipschitz and, by Lemma 6.2, $\|(\Pi \circ \widetilde{g})_j - g\|_{W^{1,p}} < F(\lambda)$, for some function F such that $\lim_{\lambda \to 0} F(\lambda) = 0$. In conclusion, it suffices to prove (2.30) in the apparently weaker form

$$\forall\, \lambda > 0,\ \exists\, \widetilde{g} \in \mathrm{Lip}(\mathscr{C}_j; \mathscr{N}_\delta) \text{ s. t. } \|\widetilde{g} - g\|_{W^{1,p}(\mathscr{C}_j)} < \lambda. \tag{2.31}$$

Step 3. Approximation on a fixed cube. Let $g \in W^{1,p}(\mathscr{C}_j; \mathbb{R}^\ell)$. Let $\mathfrak{C}$ be a cube in $\mathscr{C}_j$, of center $0_{\mathfrak{C}}$. The projection of the point $0_{\mathfrak{C}} + x^j \in \mathfrak{C}$ on $\partial\mathfrak{C}$ is $0_{\mathfrak{C}} + x^{j-1}$, where $x^{j-1} = x^j/(2|x^j|)$.

We first define convenient approximations of g as follows. For $0 < \mu < 1$, we set, with the above notation,

$$g_\mu(0_{\mathfrak{C}} + x^j) = \begin{cases} g(0_{\mathfrak{C}} + x^{j-1}), & \text{if } |x^j| \ge (1-\mu)/2 \\ g\left(0_{\mathfrak{C}} + \dfrac{x^j}{1-\mu}\right), & \text{if } |x^j| < (1-\mu)/2 \end{cases}.$$

Note that g_μ is continuous on $\mathscr{C}_j$, $\mathscr{N}$-valued, and clearly satisfies (2.28)–(2.29). The following fact is straightforward.

$$[1 \le p < \infty,\ 1 \le j \le N,\ \mathfrak{C} \in \mathscr{C}_j,\ g \in W^{1,p}(\mathfrak{C})] \implies g_\mu \to g \text{ in } W^{1,p}(\mathfrak{C}) \text{ as } \mu \to 0. \tag{2.32}$$

Let $\rho \in C_c^\infty((-1/2, 1/2)^j)$ be a standard mollifier. Given $h \in L^1(\mathscr{C}_j; \mathbb{R}^\ell)$, the convolution $h * \rho_t$ is well-defined and smooth in the set $\cup_{\mathfrak{C} \in \mathscr{C}_j} \{0_{\mathfrak{C}} + x^j;\ |x^j| \le (1-t)/2\}$. (Here, we naturally identify each $\mathfrak{C}$ with a subset of $\mathbb{R}^j$.)

Fix some function $\eta \in C_c^\infty([0, 1/2); [0, 1])$ such that $\eta(\theta) = 1$ for small θ. For small t, the map

$$\mathfrak{C} \ni 0_{\mathfrak{C}} + x^j \mapsto g^t(0_{\mathfrak{C}} + x^j) := \eta(|x^j|)\, g * \rho_t(0_{\mathfrak{C}} + x^j)$$

is well-defined and smooth in $\mathfrak{C}$.

We also set

$$g^0(0_{\mathfrak{C}} + x^j) := \eta(|x^j|)\, g(0_{\mathfrak{C}} + x^j).$$

The following is straightforward.

$$[1 \le p < \infty,\ 1 \le j \le N,\ \mathfrak{C} \in \mathscr{C}_j,\ g \in W^{1,p}(\mathfrak{C})] \implies g^t \to g^0 \text{ in } W^{1,p}(\mathfrak{C}) \text{ as } t \to 0. \tag{2.33}$$

Given $f : \partial\mathfrak{C} \to \mathbb{R}^\ell$, set (with H_j the homogeneous extension from $\mathscr{C}_{j-1}$to $\mathscr{C}_j$)

$$T(f)(0_{\mathfrak{C}} + x^j) := (1 - \eta(|x^j|))\, H_j(f)(0_{\mathfrak{C}} + x^j),\ \forall\, 0_{\mathfrak{C}} + x^j) \in \mathfrak{C}. \tag{2.34}$$

We note the following consequence of (the proof of) Lemma 6.2.

$$\begin{aligned}&[1 \le p < \infty,\ 1 \le j \le N, \mathfrak{C} \in \mathscr{C}_j]\\ &\qquad\qquad \Longrightarrow\ [W^{1,p}(\partial\mathfrak{C}) \ni f \mapsto T(f) \in W^{1,p}(\mathfrak{C}) \text{ is continuous}].\end{aligned} \tag{2.35}$$

Finally, let us note the following consequence of the embedding $W^{1,p}(\mathfrak{C}) \hookrightarrow C^0$, valid when $p > j$.

$$\begin{aligned}&[1 \le j < p < \infty,\ 1 \le j \le N,\ \mathfrak{C} \in \mathscr{C}_j, g \in W^{1,p}(\mathfrak{C}; \mathscr{N})]\\ &\qquad \Longrightarrow\ \exists\, t_0 \text{ s. t. } [t \le t_0,\ |x^j| \le (1-t)/2\ \Longrightarrow\ g * \rho_t(0_{\mathfrak{C}} + x^j) \in \mathscr{N}_\delta].\end{aligned} \tag{2.36}$$

(The validity of (2.36) when $p = j$ requires a separate argument, relying on Lemma 0.6.)

Step 4. Proof of (2.31) *when* $j = 1$. By (2.32), it suffices to prove (2.31) when g is replaced by g_μ. Since $j = 1$ and thus $\mathscr{C}_0$ is a finite collection of points, we may thus assume that g is constant near each point in $\mathscr{C}_0$:

$$\exists\, 0 < \mu < 1 \text{ s. t. } [\mathfrak{C} \in \mathscr{C}_1,\ |x^1| \ge (1-\mu)/2]\ \Longrightarrow\ g(0_{\mathfrak{C}} + x^1) = g(0_{\mathfrak{C}} + x^0)]. \tag{2.37}$$

Let now $\eta \in C^\infty([0, 1/2); [0, 1])$ be such that

$$\eta(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta \le 1/2 - \mu/4 \\ 0, & \text{if } \theta > 1/2 - \mu/6 \end{cases}. \tag{2.38}$$

When $0 < t \le \mu/6$, the map

$$\mathscr{C}_1 \ni x = 0_{\mathfrak{C}} + x^1 \mapsto G^t(x) = \eta(|x^1|)\, g * \rho_t(0_{\mathfrak{C}} + x^1) + \big(1 - \eta(|x^1|)\big)\ g(0_{\mathfrak{C}} + x^0)$$

is well-defined everywhere on $\mathscr{C}_1$, and is Lipschitz. Moreover, by (2.33) and the choice of η, we clearly have $G^t \to g$ in $W^{1,p}$ as $t \to 0$.

It remains to prove that, for small t, we have

$$G^t(0_{\mathfrak{C}} + x^1) \in \mathscr{N}_\delta,\ \forall\, \mathfrak{C} \in \mathscr{C}_1,\ \forall\, 0_{\mathfrak{C}} + x^1 \in \mathfrak{C}. \tag{2.39}$$

By (2.36) and (2.38), property (2.39) holds when $|x^1| \le 1/2 - \mu/5$. Clearly, (2.39) holds also when $|x^1| \ge 1/2 - \mu/6$. Finally, when $1/2 - \mu/5 \le |x^1| \le 1/2 - \mu/6$ and $t \le \mu/6$, we have

$$G^t(0_{\mathfrak{C}} + x^1) = g * \rho_t(0_{\mathfrak{C}} + x^1) = g(0_{\mathfrak{C}} + x^0) \in \mathscr{N}.$$

This completes Step 3.

Step 5. Proof of Proposition 2.7 by induction on j. (Here, we use the assumptions (2.26)–(2.27) at all dimensions $1 \le m \le j$.) Let $2 \le j \le N$. Let f be the restriction of g to $\mathscr{C}_{j-1}$.

By (2.32), we may assume that there exists some $\mu \in (0,1)$ such that

$$g(0_{\mathfrak{C}} + x^j) = f(0_{\mathfrak{C}} + x^{j-1}),\ \forall\, \mathfrak{C} \in \mathscr{C}_j,\ \forall\, 0_{\mathfrak{C}} + x^j \in \mathfrak{C} \text{ s. t. } |x^j| \geq (1-\mu)/2. \tag{2.40}$$

By (2.28)–(2.29) and the induction hypothesis, the map f is the limit in $W^{1,p}$ of a sequence $(F^k) \subset \mathrm{Lip}(\mathscr{C}_{j-1}; \mathscr{N})$. With η as in (2.38) and $0 < t \leq \mu/6$, we define, everywhere on $\mathscr{C}_j$, the Lipschitz maps

$$\begin{aligned} \mathscr{C}_j \ni 0_{\mathfrak{C}} + x^j \mapsto G^{k,t}(0_{\mathfrak{C}} + x^j) :=& \eta(|x^j|)\, g * \rho_t(0_{\mathfrak{C}} + x^j) \\ &+ (1 - \eta(|x^j|))\, F^k(0_{\mathfrak{C}} + x^{j-1}). \end{aligned}$$

By (2.33) and (2.35), we have

$$\lim_{k\to\infty} \lim_{t \searrow 0} G^{k,t} = g \text{ in } W^{1,p}(\mathscr{C}_j).$$

In order to complete Step 5 and the proof of Proposition 2.7, it remains to prove that, for large k and sufficiently small t (possibly depending on k) we have

$$G^{k,t}(0_{\mathfrak{C}} + x^j) \in \mathscr{N}_\delta,\ \forall\, \mathfrak{C} \in \mathscr{C}_j,\ \forall\, 0_{\mathfrak{C}} + x^j \in \mathfrak{C}. \tag{2.41}$$

As in Step 4, (2.41) holds when $|x^j| \leq 1/2 - \mu/5$ or $|x^j| \geq 1/2 - \mu/6$. When $1/2 - \mu/5 \leq |x^j| \leq 1/2 - \mu/6$, we argue as follows. By the Sobolev embeddings, we have $F^k \to f$ uniformly. Let k_0 be such that

$$\left\| F^k - f \right\|_\infty \leq \delta,\ \forall\, k \geq k_0. \tag{2.42}$$

By (2.40) and the continuity of f, for every fixed k we have

$$\begin{aligned} \lim_{t \searrow 0} G^{k,t}(0_{\mathfrak{C}} + x^j) &= \eta(|x^j|)\, f(0_{\mathfrak{C}} + x^{j-1}) + (1 - \eta(|x^j|))\, F^k(0_{\mathfrak{C}} + x^{j-1}) \\ &\text{uniformly in the set } \bigcup_{\mathfrak{C} \in \mathscr{C}_j} \{0_{\mathfrak{C}} + x^j;\ 1/2 - \mu/5 \leq |x^j| \leq 1/2\mu/6\}. \end{aligned} \tag{2.43}$$

We complete the proof of (2.41) using (2.42) and (2.43). QED

2.4.0 Singularities removing technique in $W^{1,p}$

One of our purposes here is the proof of Theorem 2.2 when $0 < s < 1$, under the necessary condition that $\pi_{\lfloor sp \rfloor}(\mathscr{N})$ is trivial. We have seen in Sections 2.2 and 2.3 that maps of the form g_j, where $g \in \mathrm{Lip}(\mathscr{C}_j; \mathscr{N})$, are dense in $W^{s,p}(\Omega; \mathscr{N})$, at least when $1 \leq j < sp < j+1 \leq N$. We have already noted that g_j actually belongs to the space $W^{1,q}(\Omega; \mathscr{N})$, $\forall\, 1 \leq q < j+1$. We will prove below that g_j can be approximated, in $W^{1,q}(\Omega)$, $\forall\, 1 \leq q < j+1$, with Lipschitz $\mathscr{N}$-valued maps. This fact, combined with the Gagliardo-Nirenberg inequalities (Corollary 4.8) and a straightforward smoothing argument, implies Theorem 2.2 when $0 < s < 1$.

After these introductory remarks, we present and prove the main result of this section (see Bethuel [6], with roots in White [75, Section 6], for the main idea of the proof (Step 2 below) and, for the presentation we give here, also Hang and Lin [50, Section 6] and Bousquet, Ponce, and Van Schaftingen [19, Section 7]). The result is stated, with no loss of generality, in $\Omega = (-M - 1/2, M + 1/2)^N$.

Proposition 2.8. Let $1 \le j \le N-1$ and $1 \le q < j+1$. Assume that $\pi_j(\mathscr{N})$ is trivial. Then, for every $g \in \mathrm{Lip}(\mathscr{C}_j; \mathscr{N})$, the map g_j is strong limit in $W^{1,q}$ of maps in $\mathrm{Lip}(\Omega; \mathscr{N})$.

Proof. Step 1. Construction of a Lipschitz $\mathscr{N}$-valued extension h of g to $\mathscr{C}_{j+1}$. (Here, we use the assumption on $\pi_j(\mathscr{N})$.) Let $\mathfrak{C} \in \mathscr{C}_{j+1}$. Since $\partial\mathfrak{C}$ is bi-Lipschitz homeomorphic with $\mathbb{S}^j$, and, by assumption, $\pi_j(\mathscr{N})$ is trivial, there exists a homotopy $G_{\mathfrak{C}} : \partial\mathfrak{C} \times [0,1] \to \mathscr{N}$ such that $G_{\mathfrak{C}}(x,0) = g(x)$, $\forall\, x \in \partial\mathfrak{C}$, and $G_{\mathfrak{C}}(x,\theta) = b_{\mathfrak{C}}$, $\forall\, x \in \partial\mathfrak{C}$, $\forall\, \theta \ge 1/2$, for some constant $b_{\mathfrak{C}} \in \mathscr{N}$. Moreover, by a smoothing argument, we may assume that $G_{\mathfrak{C}}$ is Lipschitz. The map

$$\mathscr{C}_{j+1} \ni h(0_{\mathfrak{C}} + x^{j+1}) := \begin{cases} G_{\mathfrak{C}}(0_{\mathfrak{C}} + x^j, 1 - 2|x^{j+1}|), & \text{if } |x^{j+1}| \ge 1/4 \\ b_{\mathfrak{C}}, & \text{if } |x^{j+1} \le 1/4 \end{cases}$$

is a Lipschitz $\mathscr{N}$-valued extension of g to $\mathscr{C}_{j+1}$.

Step 2. Construction of a Lipschitz $\mathscr{N}$-valued extension k of g to $\mathscr{C}_N$. (Here, we use the existence of the map h from the previous step.) We rely on the following geometrically obvious fact (see Lemma 7.4 for a formal proof). There exists a Lipschitz homotopy $G = G(x,\theta) : \mathscr{C}_N \times [0,1] \to \mathscr{C}_N$ such that:

a) $G(x,0) = x$, $\forall\, x \in \mathscr{C}_N$.

b) $G(x,\theta) = a$, for some (fixed) point $a \in \mathscr{C}_{j+1}$, $\forall\, x \in \mathscr{C}_N$, $\forall\, \theta \ge 1/2$.

c) $G(x,\theta) \in \mathscr{C}_{j+1}$, $\forall\, x \in \mathscr{C}_j$, $\forall\, \theta$.

Granted the existence of G, and with h as in Step 1, we let, $\forall\, 0_{\mathfrak{C}} + x^N \in \mathfrak{C} \in \mathscr{C}_N$,

$$k(0_{\mathfrak{C}} + x^N) := \begin{cases} h(G(0_{\mathfrak{C}} + x^j, 2d(0_{\mathfrak{C}} + x^N, \mathscr{C}_j))), & \text{if } d(0_{\mathfrak{C}} + x^N, \mathscr{C}_j) \le 1/4 \\ h(a), & \text{if } d(0_{\mathfrak{C}} + x^N, \mathscr{C}_j) \ge 1/4 \end{cases}.$$

Clearly, k is a Lipschitz $\mathscr{N}$-valued extension of g to $\mathscr{C}_N$.

Step 3. Approximation of g_j. (Here, we use the assumption $q < j+1$.) For $0 < \mu < 1/2$, consider the following sets and functions:

$$\begin{aligned}
&U_\mu := \{x \in \mathscr{C}_N;\ d(x, \mathscr{C}_j) \le 1/2 - \mu\},\\
&V_\mu := \{x \in \mathscr{C}_N;\ 1/2 - \mu \le d(x, \mathscr{C}_j) \le 1/2 - \mu/2\},\\
&W_\mu := \{x \in \mathscr{C}_N;\ d(x, \mathscr{C}_j) \ge 1/2 - \mu/2\},\\
&f_1 : [1/2-\mu, 1/2] \to [0,1],\ f_1(\theta) := \begin{cases} 0, & \text{if } \theta \ge 1/2 - \mu/2 \\ (1-2\theta)/\mu - 1, & \text{if } 1/2 - \mu \le \theta \le 1/2 - \mu/2 \end{cases},\\
&f_2 : [1/2-\mu, 1/2] \to [0,1],\ f_2(\theta) := \begin{cases} 1, & \text{if } \theta \ge 1/2 - \mu/2 \\ -(1-2\theta)/\mu + 2, & \text{if } 1/2 - \mu \le \theta \le 1/2 - \mu/2 \end{cases},\\
&d_j(x) := d(x, \mathscr{C}_j).
\end{aligned}$$

We define the following approximation of g_j:

$$\mathscr{C}_N \ni x = 0_{\mathfrak{C}} + x^N \mapsto F_\mu(x) := \begin{cases} g_j(x) = g(0_{\mathfrak{C}} + x^j), & \text{if } x \in U_\mu, \\ k(f_1(d_j(x))(0_{\mathfrak{C}} + x^j) + f_2(d_j(x))x), & \text{if } x \in V_\mu \cup W_\mu \end{cases}.$$

We note that F_μ is well-defined, Lipschitz, $\mathscr{N}$-valued, equals g_j in U_μ, and is Lipschitz (with Lipschitz constant independent of μ) in W_μ. Since $|W_\mu| \to 0$ as $\mu \to 0$, in order to prove that $F_\mu \to g_j$ in $W^{1,q}(\Omega)$ as $\mu \to 0$ it remains to prove that $\|\nabla F_\mu\|_{L^q(V_\mu)} \to 0$ as $\mu \to 0$. In turn, this follows from the fact that, from the definition of F_μ and the fact that k is Lipschitz, we have

$$\int_{V_\mu} |\nabla F_\mu|^q \lesssim \frac{1}{\mu^q} |V_\mu| \sim \frac{\mu^{j+1}}{\mu^q} \to 0 \text{ as } \varepsilon \to 0. \qquad \text{QED}$$

3 Lecture # 3. Hearing singularities

Let us return to the all purposes counterexample in Proposition 0.3. It relies on the existence of a non-trivial topological invariant (in that case, the winding number of maps $f \in C^0(\mathbb{S}^1; \mathbb{S}^1)$) and on the construction of a map "carrying" the topological invariant around a singular point. This raises several questions: (a) what is a topological singularity? (b) can one detect such singularities? (c) do standard properties of Sobolev spaces hold for maps in $W^{s,p}(\Omega; \mathscr{N})$ without topological singularities? In full generality, the answer to question c) is negative (see Bethuel and Demengel [11], Bethuel [7], or [61] for examples of smooth maps with no extensions or liftings). Depending on the answer we choose to question a), the answer to question (b) could be positive. However, a full theory allowing to encode singularities and/or to clarify their role as only possible obstructions is, for the time being, out of reach. Let us mention several topological invariants that have been investigated so far in the literature: (i) Brouwer degree of maps $f : \mathbb{S}^k \to \mathbb{S}^k$ (starting with Brezis, Coron, and Lieb [25]); (ii) spherical homology (starting with Giaquinta, Modica, and Souček [48]); (iii) (Hopf) degree of maps $f : \mathbb{S}^3 \to \mathbb{S}^2$ (starting with Rivière [67]); (iv) higher homotopy groups of general manifolds, under restrictive assumptions on the lower homotopy groups (Pakzad and Rivière [65]); (v) rational homotopies (Hardt and Rivière [51]).

In this section, we discuss the best understood situation, the one of sphere-valued maps. In order to further simplify the presentation and focus on analytical (rather than geometrical measure theory) issues, we first assume that the space dimension N and the dimension k of the sphere are related by $N = k+1$. In Section 3.5, we provide a glimpse of the general case and of the additional difficulties it raises.

Contents

3.1.0 The distributional Jacobian

We let, in Sections 3.1–3.4, $u : \Omega \to \mathbb{S}^{N-1}$, where $\Omega = (0,1)^N$ and $N \geq 2$. Recall that we always assume that $sp < N$. If $sp < N-1$, then $C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})$ is dense in $W^{s,p}(\Omega; \mathbb{S}^{N-1})$ (by Theorem

2.2 and the fact that $\pi_j(\mathbb{S}^{N-1})$ is trivial when $j < N-1$). Therefore, the interesting range is

$$N - 1 \leq sp < N. \tag{3.1}$$

For such s and p, maps in the class $\mathscr{R} = \mathscr{R}_0$ (i. e., maps as in (2.1), with A a finite subset of Ω) are dense in $W^{s,p}(\Omega; \mathbb{S}^{N-1})$ (Theorem 2.1). When $u \in \mathscr{R}$, one can define the singular set simply as A. However, this is not a tractable definition, since it is not clear how to pass to the limits sets of points. The appropriate substitute is the distribution

$$Ju := C_N \sum_{a \in A} \deg(u, a) \delta_a \in \mathscr{D}'(\Omega), \tag{3.2}$$

where $C_N := |B_1(0)|$ (the volume of the unit ball in $\mathbb{R}^N$). Here, $\deg(u, a)$ is the (Brouwer) degree of the map $u_{|S_\varepsilon(a)} : S_\varepsilon(a) \to \mathbb{S}^{N-1}$, for small ε. Clearly, this integer does not depend on (small) ε.

The main result here is the following (see [15] for the full result, and, for special cases, Bethuel, Brezis, and Coron [8], Jerrard and Soner [52, 53], Hang and Lin [49]).

Theorem 3.1. Assume (3.1). Then the map

$$\mathscr{R} \ni u \overset{J}{\mapsto} Ju \in \mathscr{D}'(\Omega) \tag{3.3}$$

has a unique extension by continuity, still denoted J, to $W^{s,p}(\Omega; \mathbb{S}^{N-1})$.

In addition, Ju belongs to the space $[\mathrm{Lip}_0(\Omega)]^*$, the mapping $W^{s,p}(\Omega; \mathbb{S}^{N-1}) \ni u \mapsto Ju \in [\mathrm{Lip}_0(\Omega)]^*$ is continuous, and we have the estimate

$$\|Ju\|_{[\mathrm{Lip}_0(\Omega)]^*} \lesssim |u|^{(N-1)/s}_{W^{s,p}}. \tag{3.4}$$

Proof. Step 1. A first convenient formula for Ju. We first derive a tractable formula for Ju when $u \in \mathscr{R}$. This formula (which will explain the title of this section) appears in Brezis, Coron, and Lieb [25], with roots in Ball [4] and Morrey [62]. To start with, we note that $\mathscr{R} \subset W^{1,q}$, $\forall\, q < N$. (For this step, $\mathscr{R} \subset W^{1,N-1}$ suffices.) Let $\omega = \omega_{N-1}$ be the standard volume form on $\mathbb{S}^{N-1}$, given by

$$\omega_{N-1} := \sum_{j=1}^{N} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_N. \tag{3.5}$$

Denoting $u^\sharp \, \omega$ the pullback by u of ω, i. e.,

$$u^\sharp \, \omega = \sum_{j=1}^{N} (-1)^{j-1} u_j \, du_1 \wedge \ldots \wedge \widehat{du_j} \wedge \ldots \wedge du_N \in L^1(\Omega; \Lambda^{N-1}),$$

we claim that

$$Ju = \frac{1}{N} d(u^\sharp \, \omega) \text{ in } \mathscr{D}'(\Omega), \tag{3.6}$$

where, in (3.6), we have identified a scalar distribution (the left-hand side) with a N-form whose density is a distribution (the right-hand side).

To justify (3.6), a first important fact is that, when u is C^2 in some open set $V \subset \Omega$, we have, in V,

$$d(u^\sharp\,\omega) = N\,du_1 \wedge \cdots \wedge du_N = N\,(\operatorname{Jac} u)\,dx_1 \wedge \cdots \wedge dx_N = 0, \tag{3.7}$$

where Jac stands for the Jacobian determinant. The first equality is a clear consequence of the exterior calculus rules, and justifies the designation of $d(u^\sharp\omega)$ as (up to a constant factor N) *distributional Jacobian*. The last equality is justified by the fact that the N vectors $d_1u(x), \dots d_Nu(x)$, $x \in V$, are all in the $(N-1)$-dimensional tangent hyperplane $T_{\mathbb{S}^{N-1}}(x)$, and thus $\operatorname{Jac} u(x) = 0$, $\forall\, x \in V$.

A second important fact is *Kronecker's formula* (see, e. g., Dinca and Mawhin [43, Section 1.1, Section 1.2]): if S is any sphere in $\mathbb{R}^N$ (with the usual orientation), then

$$\deg(v, S) = \frac{1}{|\mathbb{S}^{N-1}|}\int_S v^\sharp\omega,\ \forall\, v \in C^1(S; \mathbb{S}^{N-1}). \tag{3.8}$$

Let now $u \in \mathscr{R}$. Combining: (i) the definition of the distributional derivative; (ii) the fact that $u \in W^{1,N-1}$; (iii) (3.7); (iv) the divergence theorem; (v) the fact that $u \in \mathscr{R}$; (vi) (3.8), we find that, $\forall\, \varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}
d(u^\sharp\,\omega)(\varphi) &= -\int_\Omega d\varphi \wedge (u^\sharp\omega) = -\lim_{\varepsilon\to 0}\int_{\Omega\setminus\cup_{a\in A}B_\varepsilon(a)} d\varphi \wedge (u^\sharp\omega)\\
&= -\lim_{\varepsilon\to 0}\int_{\Omega\setminus\cup_{a\in A}B_\varepsilon(a)} d[\varphi(u^\sharp\omega)] = \sum_{a\in A}\lim_{\varepsilon\to 0}\int_{S_\varepsilon(a)} \varphi(u^\sharp\omega)\\
&= \sum_{a\in A}\varphi(a)\lim_{\varepsilon\to 0}\int_{S_\varepsilon(a)} u^\sharp\omega = |\mathbb{S}^{N-1}|\sum_{a\in A}\varphi(a)\deg(u,a)\\
&= NC_N\sum_{a\in A}\varphi(a)\deg(u,a) = NJu(\varphi),
\end{aligned}$$

so that (3.6) holds.

For further use, let us note that we have proved that, when $u \in \mathscr{R}$, we have

$$Ju(\varphi) = -\frac{1}{N}\int_\Omega d\varphi \wedge (u^\sharp\omega),\ \forall\, \varphi \in C_c^\infty(\Omega), \tag{3.9}$$

and that, if $u \in C^2(\Omega; \mathbb{R}^N)$, we have

$$\int_\Omega (\operatorname{Jac} u)\varphi = -\frac{1}{N}\int_\Omega d\varphi \wedge (u^\sharp\omega),\ \forall\, \varphi \in C_c^\infty(\Omega; \mathbb{R}). \tag{3.10}$$

Step 2. The easy case $s \geq 1$. The right-hand side of (3.9) is clearly continuous (with respect to the u's satisfying $|u| \leq 1$) in $W^{1,N-1}$. We complete this step by noting that, when $s \geq 1$ and $sp \geq N-1$, we have $W^{s,p} \cap L^\infty \hookrightarrow W^{1,N-1}$ (Corollary 4.8).

In the remaining part of the proof, we assume that $0 < s < 1$.

Step 3. A second convenient formula for Ju. This step appears in [15], but was essentially known before (see Dunford and Schwartz [44, p. 467]).

Let $u \in C^2(\overline{\Omega};\mathbb{R}^N)$, respectively $\varphi \in C_c^\infty(\Omega;\mathbb{R})$. Let $W = W(u) \in C^2(\overline{\Omega}\times[0,1);\mathbb{R}^N)$ be an extension of u, respectively $\Phi = \Phi(\varphi) \in C_c^1(\Omega\times[0,1);\mathbb{R})$ be an extension of φ. Let, for each $j \in \{1,\dots,N+1\}$, E_j=$E_j(W)$ denote the determinant whose columns are the N partial derivatives $\partial_1 W,\dots,\widehat{\partial_j W},\dots,\partial_{N+1}W$. We claim that

$$\sum_{j=1}^{N+1}(-1)^{j-1}\partial_j E_j = 0. \tag{3.11}$$

Indeed, identifying $(E_1,\dots,E_{N+1})$ with the N-form $\zeta := dW_1\wedge\cdots\wedge dW_N$, (3.11) amounts to the trivial equality $d\zeta = 0$ (see [28, Lemma 1.3] for details).

Combining: (i) (3.11); (ii) the divergence theorem; (iii) the fact that, on $\Omega\times\{0\}$, we have $E_{N+1}\Phi = (\operatorname{Jac} u)\varphi$; (iv) (3.10), we find that

$$\begin{aligned}\int_{\Omega\times(0,1)}\sum_{j=1}^{N+1}(-1)^{N+j}E_j\,\partial_j\Phi &= \int_{\Omega\times(0,1)}\sum_{j=1}^{N+1}\partial_j((-1)^{N+j}E_j\,\Phi) = \int_{\Omega\times\{0\}}E_{N+1}\Phi\\ &= \int_\Omega(\operatorname{Jac} u)\varphi = -\frac{1}{N}\int_\Omega d\varphi\wedge(u^\sharp\omega),\end{aligned}$$

so that

$$\frac{1}{N}\int_\Omega d\varphi\wedge(u^\sharp\omega) = -\int_{\Omega\times(0,1)}\sum_{j=1}^{N+1}(-1)^{N+j}E_j\,\partial_j\Phi. \tag{3.12}$$

At this stage, we know that (3.12) holds when $u \in C^2$ and $W \in C^2$. By a straightforward argument, (3.12) still holds provided $\varphi\in\operatorname{Lip}_0(\Omega)$, $\Phi\in\operatorname{Lip}_0(\Omega\times[0,1))$ and

$$\begin{gathered}u\in W^{1,N-1}_{loc}(\Omega;\mathbb{R}^N)\cap L^\infty,\ W\in W^{1,N}_{loc}(\Omega\times(0,1);\mathbb{R}^N)\cap L^\infty,\\ W(\cdot,\varepsilon)\to u \text{ in } W^{1,N-1}_{loc}(\Omega) \text{ as } \varepsilon\to 0.\end{gathered} \tag{3.13}$$

Combining the first two steps, we find the useful identity

$$Ju(\varphi) = -\int_{\Omega\times(0,1)}\sum_{j=1}^{N+1}(-1)^{N+j}E_j\,\partial_j\Phi,\ \forall\, u\in\mathscr{R},\ \forall\, W \text{ as in (3.13)}, \tag{3.14}$$

and also the fact that the right-hand side of (3.14) is well-defined (under the assumption (3.14)) for $\Phi\in\operatorname{Lip}_0(\Omega\times[0,1))$.

The heart of the proof of Theorem 3.1 consists of proving the existence, for each $u\in\mathscr{R}$, of a convenient extension $W = W(u)$ such that the right-hand side of (3.14) is continuous in $W^{s,p}$ (with respect to u).

Step 4. The main geometric estimate. (Here, we do not use the assumption $sp \geq N-1$.) Consider a linear continuous operator $W^{s,p}\ni u = u(x)\mapsto U = U(x,\varepsilon)$, $x\in\Omega$, $0<\varepsilon\leq 1$ such that: (i) $U\in C^\infty$; (ii) $U\in W^{s+1/p,p}$; (iii) $\operatorname{tr} U = u$; (iv) $|U|_{W^{s+1/p,p}}\lesssim |u|_{W^{s,p}}$; (v) $\|U\|_\infty\lesssim\|u\|_\infty$ if $u\in L^\infty$; (vi) if $u\in W^{\sigma,q}$, then $U(\cdot,\varepsilon)\in W^{\sigma,q}$ and $|U(\cdot,\varepsilon)|_{W^{\sigma,q}}\lesssim |u|_{W^{\sigma,q}}$ (see Section 4.4). By slicing, trace

theory, and Sobolev embeddings, for a.e. $x \in \Omega$ we have (j) $\lim_{\varepsilon\to 0} U(x,\varepsilon) = u(x) \in \mathbb{S}^{N-1}$; (jj) $U(x,\cdot) \in W^{s+1/p,p}((0,1)) \hookrightarrow C^s([0,1])$. Define the function

$$d(x) := \inf\{0 < \varepsilon \leq 1; |U(x,\varepsilon)| \leq 1/2\},$$

with the convention that $\inf \emptyset = \infty$. We claim that

$$\int_\Omega \frac{1}{[d(x)]^{sp}}\, dx \lesssim |u|^p_{W^{s,p}}. \tag{3.15}$$

Indeed, if x is such (j) and (jj) hold and $d(x) < \infty$, then $d(x) > 0$ and $|U(x,d(x))| = 1/2$, and therefore

$$1/2 \leq |u(x) - U(x,d(x))| \lesssim [d(x)]^s |U(x,\cdot)|_{C^s([0,1])} \lesssim [d(x)]^s |U(x,\cdot)|_{W^{s+1/p,p}((0,1))}. \tag{3.16}$$

Using (3.16), slicing, and (iv), we find that

$$\int_\Omega \frac{1}{[d(x)]^{sp}}\, dx \lesssim \int_\Omega |U(x,\cdot)|^p_{W^{s+1/p,p}((0,1))}\, dx \lesssim |u|^p_{W^{s,p}},$$

so that (3.15) holds, as claimed.

Step 5. Construction of $W(u)$ and $\Phi(\varphi)$. Fix some $\zeta \in C^1_c([0,1);\mathbb{R})$ such that $\zeta(0) = 1$. Let $\Phi(\varphi)(x,\varepsilon) := \zeta(\varepsilon)\varphi(x)$, $\forall\, x \in \Omega$, $\forall\, \varepsilon \in [0,1]$. Clearly, $\varphi \mapsto \Phi$ is linear and continuous from $\mathrm{Lip}_0(\Omega)$ into $\mathrm{Lip}_0(\Omega \times [0,1))$.

Let $\Pi \in C^\infty(\mathbb{R}^N;\mathbb{R}^N)$ be such that $\Pi(x) = x/|x|$ when $|x| \geq 1/2$. (Here, $|\,|$ stands for the Euclidean norm.) Let U be as in the previous step. Set $W := \Pi \circ U$. If $u \in \mathscr{R}$, then $u \in W^{1,N-1}$ and, by the construction of U, we have $\|\nabla U(\cdot,\varepsilon)\|_{N-1} \lesssim \|\nabla u\|_{N-1}$. By the formula of W, we also have $\|\nabla W(\cdot,\varepsilon)\|_{N-1} \lesssim \|\nabla u\|_{N-1}$. On the other hand, we have $U(\cdot,\varepsilon) \to u$ in $W^{1,N-1}$ as $\varepsilon \to 0$ and $U \in L^\infty$. By Theorem 4.13 and the fact that $\Pi \circ u = u$, we find that $W(\cdot,\varepsilon) \to u$ in $W^{1,N-1}$ as $\varepsilon \to 0$. Therefore, (3.13) holds, and thus (3.14) holds when $u \in \mathscr{R}$ and $W = W(u)$.

Step 6. Conclusion. It remains to prove that the right-hand side of (3.14) is continuous from $\mathscr{R}$ (with the distance inherited from $W^{s,p}$) into $[\mathrm{Lip}_0(\Omega)]^*$. Consider the open set

$$\begin{aligned} V = V(u) :=& \{(x,\varepsilon) \in \Omega \times (0,1);\ |U(x,\varepsilon)| > 1/2\} \\ \subset& \{(x,\varepsilon) \in \Omega \times (0,1); 0 < \varepsilon < \min\{d(x),1\}\}. \end{aligned}$$

In V, we have $|W| = 1$, and thus $E_j = 0$, $\forall\, j$ (see the proof of (3.7)). On the other hand, for $(x,\varepsilon) \notin V$, we have, by the construction of U (see Section 4.4)

$$|\nabla W(x,\varepsilon)| \lesssim |\nabla U(x,\varepsilon)| \lesssim \frac{1}{\varepsilon}\|u\|_\infty \lesssim \frac{1}{\varepsilon}. \tag{3.17}$$

Using: (i) (3.14); (ii) (3.17); (iii) the fact that, when $(x,\varepsilon) \notin V$, we have $\varepsilon \geq d(x)$; (iv) (3.15), we find that

$$\begin{aligned} |Ju(\varphi)| \lesssim& \|\nabla\Phi\|_\infty \int_\Omega \int_{\varepsilon \geq d(x)} \frac{1}{\varepsilon^N}\, d\varepsilon dx \lesssim \|\nabla\varphi\|_\infty \int_\Omega \frac{1}{[d(x)]^{N-1}}\, dx \\ \lesssim& \|\nabla\varphi\|_\infty |u|^{(N-1)/s}_{W^{s,p}}, \end{aligned} \tag{3.18}$$

where the last line uses (3.15), Hölder's inequality, and the assumption $sp \geq N-1$.

In order to complete the proof, it suffices to prove the continuity of $\mathscr{R} \ni u \mapsto Ju(\varphi)$ for a fixed φ. Consider a sequence $(u_j) \subset \mathscr{R}$ converging in $W^{s,p}$ to some u. By: (i) trace theory; (ii) slicing; (iii) the converse to the dominated convergence theorem, there exist a subsequence, still denoted (u_j), and a function $F \in L^p(\Omega)$ such that $|U_j(x,\cdot)|_{W^{s+1/p,p}} \leq F(x)$ for each j and a. e. $x \in \Omega$. An inspection of the proof of (3.16) shows that, for each j and a. e. x, we have $d(x) \gtrsim [F(x)]^{-1/s}$, and therefore the corresponding sets $V(u_j)$ satisfy

$$[\Omega \times (0,1)] \setminus V(u_j) \subset Z := \{(x,\varepsilon) \in \Omega \times (0,1);\ \varepsilon \gtrsim [F(x)]^{-1/s}\}. \tag{3.19}$$

(Note that Z does not depend on the (sub)sequence (u_j).) Using: (i) (3.17); (ii) the fact that, clearly, $W(u_j)(x,\varepsilon) \to W(u)(x,\varepsilon)$, $\forall\, x, \forall\, \varepsilon$; (iii) (3.19); (iv) the fact that $(x,\varepsilon) \mapsto 1/\varepsilon^N \in L^{sp/(N-1)}(Z) \subset L^1(Z)$; (v) dominated convergence, we find that (possibly along a subsequence) $(J(u_j)(\varphi))$ converges to the right-hand side of (3.14) corresponding to u. Finally, the uniqueness of the limit implies that convergence holds for the full original sequence.

Moreover, using the above domination and the explicit construction of $\Phi(\varphi)$, the continuity of $W^{s,p}(\Omega;\mathbb{S}^{N-1}) \ni u \mapsto Ju \in [\mathrm{Lip}_0(\Omega)]^*$ is routine. The estimate (3.4) easily follows from (3.18) and a limiting argument. QED

3.2.0 The range of the distributional Jacobian

Recall that we consider maps $u : \Omega \subset \mathbb{R}^N \to \mathbb{S}^{N-1}$, with $N \geq 2$. The main result here is the following.

Theorem 3.2. Assume (3.1).

1. If $u \in W^{s,p}(\Omega;\mathbb{S}^{N-1})$, then there exist points $P_j, N_j \in \overline{\Omega}$, $j \geq 1$, such that

$$\sum_j |P_j - N_j| \lesssim |u|^{(N-1)/s}_{W^{s,p}}, \tag{3.20}$$

$$Ju = C_N \sum_j (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathscr{D}'(\Omega). \tag{3.21}$$

2. Conversely, given points $P_j, N_j \in \overline{\Omega}$ satisfying (3.21) and $\sum_j |P_j - N_j| < \infty$, there exists $u : \Omega \to \mathbb{S}^{N-1}$ such that, for every s, p satisfying $sp = N-1$: (i) $u \in W^{s,p}(\Omega;\mathbb{S}^{N-1})$; (ii) (3.21) holds; (iii)

$$|u|^p_{W^{s,p}} \lesssim \inf\left\{ \sum_k |\widetilde{P}_k - \widetilde{N}_k|;\ \sum_k (\delta_{\widetilde{P}_k} - \delta_{\widetilde{N}_k}) = \sum_j (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathscr{D}'(\Omega) \right\}. \tag{3.22}$$

See [15, 18] for the general case, and [24, 14, 36] for special cases. When $s = 1$ and $p = N-1$, the above theorem is a special case of the main result in Alberti, Baldo, and Orlandi [2, Theorem 5.6], but obtaining Theorem 3.2 from [2, Theorem 5.6] requires an additional argument. The proof involves two important ingredients: a *duality formula* and a *dipole construction*, both due to Brezis, Coron, and Lieb [25], complemented with a *dipole insertion technique* due to Bethuel [5].

Remark 3.3. Note the range $N-1 \le sp < N$ in item 1, and the range $sp = N-1$ in item 2. Thus, item 2 is not the exact converse of item 1. When $N=2$ (i. e., we consider $\mathbb{S}^1$-valued maps), the exact converse of item 1 is known (see Bousquet [16]), i. e., when $1 < sp < 2$, it is possible to characterize the set $\{Ju;\ u \in W^{s,p}(\Omega;\mathbb{S}^1)\}$. The counterpart of the result in [16] is not known when $N > 2$.

Elements of proof of Theorem 3.2. Step 1. A pseudometric and a duality formula. Set, for $P, N \in \overline{\Omega}$,

$$d(P,N) = \min\{|P-N|,\ \mathrm{dist}(P,\partial\Omega) + \mathrm{dist}(N,\partial\Omega)\}. \tag{3.23}$$

Clearly, d is a pseudometric, and, for each $P, N \in \overline{\Omega}$: (i) either $d(P,N) = |P-N|$ and the interior of the segment $[P,N]$ is completely contained in Ω, or: (ii) there exist points $P_1, N_1 \in \partial\Omega$ such that $|P-N_1| = d(P,\partial\Omega)$, $|P_1-N| = d(N,\partial\Omega)$, and $d(P,N) = |P-N_1|+|P_1-N|$. Moreover, in the latter case, if, for example, $P \in \Omega$, then the line segment $[P,N_1]$ is normal to $\partial\Omega$ at N_1, and its interior is completely contained in Ω.

Given $P_j, N_j \in \overline{\Omega}$, $1 \le j \le m$, set

$$L((P_j),(N_j)) := \min_{\sigma\in S_m} \sum_j d(P_j, N_{\sigma(j)}). \tag{3.24}$$

It is clear from the definitions (3.23) and (3.24) that, given initial collections (P_j), (N_j), $1 \le j \le m$, we may find new collections, still denoted, for simplicity, (P_k), (N_k) (containing, possibly, more points), such that

$$\sum_{\text{new points}} (\delta_{P_k} - \delta_{N_k}) = \sum_{\text{initial points}} (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathscr{D}'(\Omega), \tag{3.25}$$

$$L((P_j),(N_j)) = L((P_k),(N_k)) = \sum_k |P_k - N_k|, \tag{3.26}$$

$$\begin{aligned}&\text{for each } k, \text{ the points } P_k, N_k \text{ are distinct, at least one of them is in } \Omega,\\ &\quad \text{and, if } P_k \in \partial\Omega \text{ or } N_k \in \partial\Omega, \text{ the segment } [P_k, N_k] \text{ is normal to } \partial\Omega.\end{aligned} \tag{3.27}$$

Note that, if $\varphi \in \mathrm{Lip}_0(\Omega)$ and $P, N \in \overline{\Omega}$, then

$$\varphi(P) - \varphi(N) \le d(P,N)|\varphi|_{\mathrm{Lip}}, \tag{3.28}$$

and therefore

$$L((P_j),(N_j)) \ge \max\left\{\sum_j (\varphi(P_j) - \varphi(N_j));\ \varphi \in \mathrm{Lip}_0(\Omega),\ |\varphi|_{\mathrm{Lip}} \le 1\right\}. \tag{3.29}$$

Remarkably, we actually have equality in (3.29) (see [25] for the original result and, for other proofs, Brezis [23] and [33]):

$$L((P_j),(N_j)) = \max\left\{\sum_j (\varphi(P_j) - \varphi(N_j));\ \varphi \in \mathrm{Lip}_0(\Omega),\ |\varphi|_{\mathrm{Lip}} \le 1\right\}. \tag{3.30}$$

For further use, let us note the following consequence of (3.30) combined with (3.32) below and (3.25)–(3.28):

$$\|Ju\|_{[\mathrm{Lip}_0(\Omega)]^*} = C_N \min\left\{ \sum_k |P_k - N_k|;\ P_k, N_k \in \overline{\Omega}, Ju = C_N \sum (\delta_{P_k} - \delta_{N_k}) \right\}, \quad \forall\, u \in \mathscr{R}_0. \tag{3.31}$$

Step 2. Proof of item 1. We start with a preliminary remark. If $u \in \mathscr{R}$ then (in view of the definition (3.2) of Ju), possibly after adding fictitious points on $\partial\Omega$, we may always write

$$Ju = C_N \sum_j (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathscr{D}'(\Omega) \tag{3.32}$$

(where the sum contains a finite number of terms).

Let now $u \in W^{s,p}(\Omega; \mathbb{S}^{N-1})$. Let u_0 be any constant in $\mathbb{S}^{N-1}$. Consider a sequence $(u_i)_{i\geq 1} \subset \mathscr{R}$ such that $u_i \to u$ in $W^{s,p}$, $|u_1|_{W^{s,p}} \lesssim |u|_{W^{s,p}}$, and $\|Ju_{i+1} - Ju_i\|_{[\mathrm{Lip}_0(\Omega)]^*} \leq 2^{-i}|u|^{(N-1)/s}_{W^{s,p}}$, $\forall\, i \geq 1$ (see Theorem 3.1). Combining the observation (3.32) with (3.30), (3.25)–(3.26), and the estimate (3.4), we find that there exist sequences $(P_{k,i})_k$, $(N_{k,i})_k$ such that

$$Ju_{i+1} - Ju_i = C_N \sum_k (\delta_{P_{k,i}} - \delta_{N_{k,i}}) \text{ in } \mathscr{D}'(\Omega),\ \forall\, i \geq 0, \tag{3.33}$$

$$\sum_k |P_{k,0} - N_{k,0}| \lesssim |u|^{(N-1)/s}_{W^{s,p}}, \tag{3.34}$$

$$\sum_k |P_{k,i} - N_{k,i}| \lesssim 2^{-i}|u|^{(N-1)/s}_{W^{s,p}},\ \forall\, i \geq 1. \tag{3.35}$$

Combining (3.33)–(3.35) with the continuity of J, we find that (3.20)–(3.21) hold.

Step 3. Partial proof of item 2: setting and strategy. We present the proof of a weaker result: we let $N \geq 3$ and we *fix* $0 < s < N-1$ and $1 < p < \infty$ such that $sp = N-1$. (For an "all couples" $(s,p$ procedure, based on a diagonal process and Gagliardo-Nirenberg, in a similar context, see [18, proof of Theorem 1.3].) For such s, p, and N, and all sequences (P_j), (N_j) as in item 2, we prove the existence of a map u satisfying (3.22). For the proof of item 2 in full generality, we refer the reader to [18].

For pedagogical purposes, we *temporarily* assume that

$$\sum_k |P_k - N_k|^{1/p} < \infty; \tag{3.36}$$

we will remove this assumption in the final step.

It will be more convenient to work in the full space $\mathbb{R}^N$, $N \geq 3$. More precisely, given sequences $(P_k)_{k\geq 1}, (N_k)_{k\geq 1} \subset \mathbb{R}^N$ such that $\sum_k |P_k - N_k| < \infty$ and $P_k \neq N_k$, $\forall\, k$, and a point $a \in \mathbb{S}^{N-1}$, we will construct a map $u : \mathbb{R}^N \to \mathbb{S}^{N-1}$ such that:

a) $u - a \in W^{s,p}(\mathbb{R}^N)$.

b) $Ju = C_N \sum_k (\delta_{P_k} - \delta_{N_k})$ in $\mathscr{D}'(\mathbb{R}^N)$.

c) $\|u-a\|^p_{W^{s,p}} \lesssim \sum_k |P_k - N_k| < \infty$.

Clearly, the existence of such a map implies item 2 of the theorem.

The map u will be obtained as the limit of a sequence of maps, each iterative step consisting of *dipoles insertions*.

Step 4. The dipole construction. We fix a map $f \in C^\infty([0,1];[0,1])$ such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(\theta) > 0, \forall\, \theta \in (0,1)$.

Given a line segment S in $\mathbb{R}^N$, say, in order to simplify the statement, $S = [0, Le_N]$, and $0 < \varepsilon \leq L$, there exists a map $u_\varepsilon \in C^\infty(\mathbb{R}^N \setminus \{0, Le_N\}; \mathbb{S}^{N-1})$ such that

$$u_\varepsilon \in \mathscr{R}_0, \ \forall\, \varepsilon, \tag{3.37}$$

$$|u_\varepsilon|^q_{W^{\sigma,q}} \lesssim L, \ \forall\, \sigma, q \text{ such that } \sigma q = N-1, \forall\, \varepsilon, \tag{3.38}$$

$$Ju_\varepsilon = C_N(\delta_0 - \delta_{Le_N}) \text{ in } \mathscr{D}'(\mathbb{R}^N), \ \forall\, \varepsilon, \tag{3.39}$$

$$\operatorname{supp}(u_\varepsilon - a) \subset \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; 0 \leq x_N \leq L, |x'| \leq L\varepsilon f(x_N/L)\}, \ \forall\, \varepsilon \tag{3.40}$$

(see Lemmas 5.14 and 5.15). A similar conclusion holds for an arbitrary segment. A noticeable fact is that the estimate (3.38) involves the length of the segment.

Step 5. The iterative construction. We will construct a sequence (v_k) such that $v_0 = a$ and

$$Jv_m = C_N \sum_{n \leq m} (\delta_{P_n} - \delta_{N_n}) \text{ in } \mathscr{D}'(\mathbb{R}^N), \ \forall\, m, \tag{3.41}$$

$$\|v_m - v_{m-1}\|^p_{W^{s,p}} \lesssim |P_m - N_m|, \ \forall\, m \geq 1, \tag{3.42}$$

$$\|v_m\|^p_{W^{s,p}} - \|v_{m-1}\|^p_{W^{s,p}} \lesssim |P_m - N_m|, \ \forall\, m \geq 1. \tag{3.43}$$

Assuming (3.41)–(3.43) and using the temporary assumption (3.36), we find that the limiting map $u := a + \sum_{m \geq 1}(v_m - v_{m-1})$ has all the desired properties.

To start with, we let, as in the dipole construction, $v_1 \in C^\infty(\mathbb{R}^N \setminus \{P_1, N_1\}; \mathbb{S}^{N-1})$ satisfy $Jv_1 = C_N(\delta_{P_1} - \delta_{N_1})$ and the estimates (3.42)–(3.43). (This is possible, for sufficiently small ε, since $P_1 \neq N_1$.) Assume next that we were able to construct $v_1, \ldots, v_{k-1}$ such that (3.41)–(3.43) hold, and, in addition, there exists an increasing sequence of finite sets $A_m \subset \mathbb{R}^N$, $1 \leq m \leq k-1$, such that:

$$v_m \text{ is smooth in } \mathbb{R}^N \setminus A_m, \tag{3.44}$$

$$v_m \in \mathscr{R}_0, \tag{3.45}$$

$$\text{for each } x \in A_m, \text{ there exists a non-empty open conical cap } C_x \text{ with vertex } x \text{ such that } v_m(x) = a \text{ in } C_x. \tag{3.46}$$

Note that these assumptions are satisfied when $m = 1$, with $A_1 = \{P_1, N_1\}$. We next construct v_k according to the position of P_k and N_k with respect to the set A_{k-1}.

Case 1. $[P_k, N_k] \cap A_{k-1} = \emptyset$. We first modify v_{k-1} in a convenient small open neighborhood V of $[P_k, N_k]$, such that the modified map, still denoted v_{k-1}, continues to satisfy (3.41)–(3.43), and $v_{k-1} = a$ in V. (Intuitively, this is possible since a segment in $\mathbb{R}^N$, with $N \geq 3$, has zero $W^{s,p}$-capacity.) The rigorous existence of such a modified map is established in Lemma 5.16.

We next consider, with an abuse of notation, the map

$$v_k = \begin{cases} v_{k-1}, & \text{in } \mathbb{R}^N \setminus V \\ u_\varepsilon, & \text{in } V, \end{cases}$$

where u_ε is the map in the dipole construction corresponding to the singularities P_k, N_k. Clearly, in view of (3.37)–(3.40) and of the Brezis-Lieb type Lemma 4.20, for small ε the map v_k satisfies (3.41)–(3.46), with $A_m := A_{m-1} \cup \{P_k, N_k\}$.

Case 2. $[P_k, N_k] \cap A_{k-1} \neq \emptyset$. In this case, we may construct a finite chain $D = [Q_1, Q_2] \cup \ldots \cup [Q_{t-1}, Q_t]$ without self intersections and such that:

(i) If $x \in D \setminus \{P_k, N_k\}$, then $x \notin A_{k-1} \cup \{P_k, N_k\}$. In particular, $Q_2, \ldots, Q_{t-1} \notin A_{k-1} \cup \{P_k, N_k\}$.

(ii) If $P_k \in A_{k-1}$, then, near P_k, the segment $[Q_1, Q_2]$ is contained in C_{P_k}, where C_{P_k} is as in (3.46). Similarly for N_k.

(iii) $\sum_j |Q_{j+1} - Q_j| \lesssim |P_k - N_k|$.

We next modify v_{k-1} in a neighborhood of $D \setminus \{P_k, N_k\}$ such that: (j) (3.41)–(3.46) still hold; (jj) v_{k-1} equals a in a neighborhood of $D \setminus \{P_k, N_k\}$. The construction of the modified map and the corresponding estimates are established in Lemma 5.17. Finally, we insert $(t-1)$-dipoles u_{j,ε_j}, $1 \leq j \leq t-1$, satisfying $Ju_{j,\varepsilon_j} = C_N(\delta_{Q_{j+1}} - \delta_{Q_j})$. By the multi-sequences Brezis-Lieb lemma 4.21, for convenient small ε_j, the new map

$$v_k(x) = a + \begin{cases} u_{j,\varepsilon_j} - a, & \text{in } \operatorname{supp}(u_{j,\varepsilon_j} - a) \\ v_{k-1} - a, & \text{in } \mathbb{R}^N \setminus \cup_j \operatorname{supp}(u_{j,\varepsilon_j} - a) \end{cases}$$

has all the required properties, with $A_k = A_{k-1} \cup \{Q_1, \ldots, Q_t\}$.

Step 6. Removing the assumption (3.36). Let $S := \sum_k |P_k - N_k|$. We consider integers $1 = j_0 < j_1 < j_2 < \ldots$ such that $\sum_{j_{k-1} \leq j < j_k} |P_j - N_j| \lesssim 2^{-k} S$, $\forall\, k$. We let $v_0 = a$ and construct, as explained in Step 5, Case 2 (using several chains and the multi-sequences Brezis-Lieb Lemma 4.21), a sequence (v_k) such that

$$Jv_k = C_N \sum_{j < j_k} (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathscr{D}'(\mathbb{R}^N),\ \forall\, k,$$

$$\|v_k - v_{k-1}\|^p_{W^{s,p}} \lesssim \sum_{j_{k-1} \leq j < j_k} |P_j - N_j| \lesssim 2^{-k} S,\ \forall\, k \geq 1,$$

$$\|v_k\|^p_{W^{s,p}} - \|v_{k-1}\|^p_{W^{s,p}} \lesssim \sum_{j_{k-1} \leq j < j_k} |P_j - N_j| \lesssim 2^{-k} S,\ \forall\, k \geq 1,$$

v_k is smooth in $\mathbb{R}^N \setminus A_k$, for some finite A_k,

$v_k \in \mathscr{R}_0$,

for each $x \in A_k$, there exists a non-empty open conical cap C_x with vertex x such that $v_k(x) = a$ in C_x.

Then $(v_k - a)$ converges in $W^{s,p}$ to some map v with $v - a \in W^{s,p}$ and such that $Jv = C_N \sum_j (\delta_{P_j} - \delta_{N_j})$. QED

3.3.0 Inserting singularities

Recall that we consider maps $u : \Omega \subset \mathbb{R}^N \to \mathbb{S}^{N-1}$, with $N \geq 2$. The main result here is due to Bethuel when $s = 1$ and $N \geq 3$ [5].

Theorem 3.4. Let $u \in \mathscr{R}_0$.

1. Let $N \geq 3$. There exists some map $v \in \mathscr{R}_0$ such that $Jv = 0$ in $\mathscr{D}'(\Omega)$ and
$$|v-u|^p_{W^{s,p}} \lesssim \|Ju\|_{[\mathrm{Lip}_0(\Omega)]^*},\ \forall\, 0 < s < N-1,\ 1 < p < \infty \text{ s. t. } sp = N-1. \tag{3.47}$$
2. Let $N = 2$. There exists some map $v \in \mathscr{R}_0$ such that $Jv = 0$ in $\mathscr{D}'(\Omega)$ and
$$|v-u|^p_{W^{s,p}} \lesssim \|Ju\|_{[\mathrm{Lip}_0(\Omega)]^*},\ \forall\, 0 < s \leq 1,\ 1 \leq p < \infty \text{ s. t. } sp = 1. \tag{3.48}$$

Elements of proof. For simplicity, we consider only the case where $N \geq 3$, and we prove the estimate (3.47) for a *fixed* couple (s,p). (For $N = 2$, see [35, Proposition 1.1, Proposition 15.2].) The proof is very similar to the one of item 2 in Theorem 3.2. Let $u \in \mathscr{R}_0$ and the sets U, A as in (2.1). We may assume that $Ju \neq 0$. Write, as in (3.32), $Ju = C_N \sum_k (\delta_{P_k} - \delta_{N_k})$, where the points P_k, N_k satisfy (3.26)–(3.27).

Step 1. Modification of u near its singularities. The purpose of this step is to obtain a new map, $\widetilde{u} \in \mathscr{R}_0$ such that
$$J\widetilde{u} = Ju \text{ in } \mathscr{D}'(\Omega), \tag{3.49}$$
$$\|\widetilde{u} - u\|^p_{W^{s,p}} \lesssim \|Ju\|_{[\mathrm{Lip}_0(\Omega)]^*}, \tag{3.50}$$
$$\text{near each of its singularities in } \overline{\Omega},\ \widetilde{u} \text{ satisfies the assumption (i) of Lemma 5.17.} \tag{3.51}$$

This modification is performed in Lemma 5.18. For this step, we require $0 < s < N-1$ and we exclude the couple $(s,p) = (N-1,1)$. For the record, it is possible to extend the validity of Lemma 5.18 to this couple, if, for each singularity $x \in A$, we have $\deg(u,x) \neq 0$.

Step 2. Dipole insertion, and conclusion. By Step 1, we may assume that (3.51) holds for u (instead of $\widetilde{u}$). We next construct a map $v \in \mathscr{R}_0$ such that
$$Jv = 0 \text{ in } \mathscr{D}'(\Omega),$$
$$\|v-u\|^p_{W^{s,p}} \lesssim \sum_k |P_k - N_k|. \tag{3.52}$$

The construction is performed using the procedure explained in Steps 5 and 6 of the proof of Theorem 3.2, by inserting, at each singularity P_k (respectively N_k) a dipole of degree -1 (respectively $+1$).

We complete the proof by noting that (3.26), (3.31), and (3.52) imply (3.47). QED

3.4.0 Characterization of the closure of smooth maps

Recall that we consider maps $u : \Omega \subset \mathbb{R}^N \to \mathbb{S}^{N-1}$, with $N \geq 2$. The main result we present here is due to Demengel [40] when $s = 1$, $1 \leq p < 2$, and $N = 2$, Bethuel [5] when $s = 1$ and $p = N-1$, and has been announced, with indications of proof, by Bethuel, Coron, Demengel, and Coron [10] when $s = 1$ and $N - 1 < p < N$, respectively Mucci [63] when $0 < s < 1$. See also Ponce and Van Schaftingen [66].

Theorem 3.5. Let $N \geq 1$, $0 < s \leq 1$, $1 \leq p < \infty$ be such that $N - 1 \leq sp < N$. Then, for $u \in W^{s,p}(\Omega; \mathbb{S}^{N-1})$, we have

$$u \in \overline{C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})}^{W^{s,p}} \iff Ju = 0.$$

Elements of proof. The implication " $\Longrightarrow$ " follows from Theorem 3.1 and the fact that, when $u \in C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})$, we have $Ju = 0$. The remaining part of the proof is devoted to the reverse implication. In Step 2, we limit ourselves to the case $0 < s < 1$, since we rely on our constructive proof of Theorem 2.1 for $0 < s < 1$. However, as explained in [10], we could have completed Step 2 even for $s = 1$, by combining our argument with Bethuel's constructive proof of Theorem 2.1 when $s = 1$ [6].

Step 1. " $\Longleftarrow$ " holds when: (i) $sp = N - 1$; (ii') $0 < s < N - 1$ when $N \geq 3$; (ii") $0 < s \leq 1$ when $N = 2$. By Theorem 2.1, there exists a sequence $(u_i) \subset \mathscr{R}_0$ such that $u_i \to u$ in $W^{s,p}$. By Theorem 3.1 and the assumption $Ju = 0$, we have $Ju_i \to 0$ in $[\mathrm{Lip}_0(\Omega)]^*$ as $i \to \infty$. By Theorem 3.4, there exists a sequence $(v_i) \subset \mathscr{R}_0$ such that $Jv_i = 0$, $\forall\, i$, and $v_i \to u$ in $W^{s,p}$. In order to complete this step, it remains to prove that

$$[v \in \mathscr{R}_0,\ Jv = 0] \implies v \in \overline{C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})}^{W^{s,p}}. \tag{3.53}$$

In order to prove (3.53), we argue as in the proof of Lemma 5.18. Since $\deg(v, x_j) = 0$ near each singularity x_j of v, we find that the restriction of v to a small sphere $S_\delta(x_j)$ around x_j is homotopic to a fixed constant $a \in \mathbb{S}^{N-1}$, and then, for every $\mu > 0$, we may construct, as in the proof of Lemma 5.18, a smooth map $\widetilde{v} : \overline{\Omega} \to \mathbb{S}^{N-1}$ such that $\|\widetilde{v} - v\|_{W^{s,p}} \leq \mu$ and, near each x_j, $\widetilde{v} = a$. This construction completes Step 1.

Step 2. " $\Longleftarrow$ " holds when $0 < s < 1$ and $N - 1 < sp < N$. (Sketch of proof.) We work with $|x| := \|x\|_\infty$. Let $\Omega_\delta := \{x \in \Omega; d(x, \partial\Omega > \delta)\}$. We will prove that, for each $\delta > 0$, $u_{|\Omega_\delta}$ can be approximated in $W^{s,p}$ with maps in $C^\infty(\overline{\Omega_\delta}; \mathbb{S}^{N-1})$. The same conclusion on Ω will then follow by a standard argument based on domain diffeomorphisms. In order to simplify the formulas, we work in Ω rather than Ω_δ, and then we may assume that $u \in W^{s,p}(U; \mathbb{S}^{N-1})$ and $Ju = 0$ in $\mathscr{D}'(U)$, where U is the larger domain $\{x \in \mathbb{R}^N;\ d(x, \Omega) < \delta\}$. By Step 1, we have

$$u \in \overline{C^\infty(\overline{U}; \mathbb{S}^{N-1})}^{W^{\sigma,q}}, \forall\, 0 < \sigma < N - 1,\ 1 < q < \infty \text{ s. t. } \sigma q = N - 1. \tag{3.54}$$

Consider an extension of u, denoted f, to $\mathbb{R}^N$, such that $f \in W^{s,p}(\mathbb{R}^N; \mathbb{R}^N)$. (We do not claim that the extension is $\mathbb{S}^{N-1}$-valued.) Let $0 < \varepsilon < \delta/2$. Using the notation in the proof of Theorem 2.5, formula (2.21) holds for f (by Theorem 2.5). Moreover, as explained at the beginning of the Section 2.3, for a. e. $t \in Q_\varepsilon$, f satisfies (2.22) with $j = N - 1$ and (2.23) with $m = N - 1$. By Step 1, there exists a sequence $(u_i) \subset C^\infty(\overline{U}; \mathbb{S}^{N-1})$ such that $u_i \to u$ in $W^{(N-1)/p,p}(U; \mathbb{S}^{N-1})$. By slicing [31, Lemma 6.1], possibly after passing to a subsequence, still denoted (u_i), for a. e. $t \in Q_\varepsilon$, we have

$$u_{i|\mathscr{C}_{N-1,t,\varepsilon} \cap U} \to u_{|\mathscr{C}_{N-1,t,\varepsilon} \cap U} \text{ in } W^{s,p}. \tag{3.55}$$

Consider now a $t \in Q_\varepsilon$ such that (2.22), (2.23), and (3.55) hold, and any fixed cube $C \in \mathscr{C}_{N,t,\varepsilon}$ contained in U. Let $v_i := u_{i|\partial C}$, $v := u_{|\partial C}$. Note that v has a continuous representative (since (2.22) holds with $j = N - 1$). By the above and a homotopy argument, we have

$$0 = \lim_i \deg(u_i, S_\varepsilon(t)) = \deg(v, S_\varepsilon(t)), \tag{3.56}$$

where the second equality follows from the embedding $W^{s,p}(\partial C) \hookrightarrow C^0$ and the stability of the Brouwer degree under uniform convergence.

Consider now the smoothing process described in Section 2.3: on the cubes $C \in \mathscr{C}_{N,t,\varepsilon}$ contained in U and such that (2.22), (2.23), and (3.56) hold, we may approximate u with a $(N-1)$-homogeneous map w such that its restriction on the boundary of each C is Lipschitz and (by the above stability argument) has zero degree. By Lemma 5.19, we may approximate, in $W^{s,p}$, w (and thus u) with Lipschitz $\mathbb{S}^{N-1}$-valued maps. By an additional smoothing argument, we may approximate, in $W^{s,p}$, u with maps in $C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})$. QED

For the record, let us note that the second step above has little to do with sphere-valued maps. It reveals a more general scheme that we formalize in the next statement.

Proposition 3.6. Let:

(i) $N \geq 2$.

(ii) $0 < s \leq 1$, $1 \leq p < \infty$, $1 \leq j < N$ such that $j < sp < j+1$.

(iii) If $j = N-1$, Ω is any Lipschitz bounded domain. If $j < N-1$, we take $\Omega = (0,1)^N$.

Set $\sigma := j/p < s$. Let $u \in W^{s,p}(\Omega; \mathscr{N})$. Then

$$u \in \overline{C^\infty(\overline{\Omega}; \mathscr{N})}^{W^{s,p}} \iff u \in \overline{C^\infty(\overline{\Omega}; \mathscr{N})}^{W^{\sigma,p}}.$$

Sketch of proof. " $\Longrightarrow$ " is clear. For the reverse implication, we argue essentially as in Step 2 above, and let U as there. First, we introduce an *ad hoc* notation. Let $\widetilde{\mathscr{C}}_N = \widetilde{\mathscr{C}}_{t,N,\varepsilon} := \cup\{C \in \mathscr{C}_N;\ C \subset U\}$ and, for $0 \leq j \leq N-1$, $\widetilde{\mathscr{C}}_j := \mathscr{C}_j \cap \widetilde{\mathscr{C}}_N$.

Let $t \in Q_\varepsilon$ be such that (2.22)–(2.24) hold and

$$u_{i|\widetilde{\mathscr{C}}_j} \to u_{|\widetilde{\mathscr{C}}_j} \text{ in } W^{s,p},\ \forall\, 0 \leq j \leq N-1. \tag{3.57}$$

By the stability argument leading to (3.56), for every face $C \in \widetilde{\mathscr{C}}_{j+1}$, $u_{|\partial C} : \partial C \to \mathscr{N}$ is null homotopic. By the smoothing process described in Section 2.3 and the multi-sequences Brezis-Lieb Lemma 4.21, we may approximate in $W^{s,p}$, on $\widetilde{\mathscr{C}}_j$, $u_{|\widetilde{\mathscr{C}}_j}$ with Lipschitz maps, null homotopic on each ∂C with C as above. We now invoke Lemma 5.19 and approximate, on each C, the homogeneous extension of $u_{|\partial C}$ to C with a Lipschitz map. Then apply again Lemma 4.21 to obtain a global $W^{s,p}$-approximation. To summarize, we have sketched the argument of the fact that, for a "generic" $t \in Q_\varepsilon$, the homogeneous extension H_{j+1} of $u_{|\widetilde{\mathscr{C}}_j}$ to $\widetilde{\mathscr{C}}_{j+1}$ can be approximated, in $W^{s,p}$, with Lipschitz maps. If $j = N-1$, then we found that u itself may be approximated with Lipschitz (and then smooth) $\mathscr{N}$-valued maps, and we are done. When $j < N-1$, we use the fact that $\widetilde{\mathscr{C}}_N$ is, up to an affine transformation, a cube, and apply the singularities removing technique described in Section 2.4 to approximate, in $W^{1,q}$ (and thus in $W^{s,p}$, by Gagliardo-Nirenberg), $H_N \circ \cdots \circ H_{j+1}(u_{|\widetilde{\mathscr{C}}_j})$ with Lipschitz $\mathscr{N}$-valued maps and thus, finally, u with smooth $\mathscr{N}$-valued maps. QED

3.5.0 Overview of the higher co-dimensional case

We let here $N > k \geq 1$ and consider maps $u : \Omega \subset \mathbb{R}^N \to \mathbb{S}^k$. The case where $N = k+1$ corresponds to Sections 3.1–3.4; here, we rather focus on the case where $N \geq k+2$. For the exposition in this section and beyond, we refer the reader to Alberti [1] and [35, Chapter 4].

Contents

3.5.1 Jacobian and singularities

We will consider a slightly different route from the one in Section 3.1. We start with the analytical definition of the distributional Jacobian (analogue of (3.6)) rather than its geometric definition (analogue of (3.2)). For $u \in W^{1,k}(\Omega; \mathbb{S}^k)$, set

$$u^\sharp \, \omega_k := \sum_{j=1}^{k+1} (-1)^{j-1} u_j \, du_1 \wedge \ldots \wedge \widehat{du_j} \wedge \ldots \wedge du_{k+1} \in L^1(\Omega; \Lambda^k), \tag{3.58}$$

where ω_k is the standard volume form ω_k on $\mathbb{S}^k$ (ω_k corresponds to the choice $N = k+1$ in (3.5)). Note that, in particular, the definition (3.58) makes sense for $u \in \mathscr{R}_{N-k-1}$.

Then define

$$Ju := \frac{1}{k+1} \, d(u^\sharp \omega_k) \in \mathscr{D}'(\Omega; \Lambda^{k+1}). \tag{3.59}$$

The above definitions are consistent with the ones in Section 3.1. Let us note that, in the previous sections, it turned out to be more convenient to identify the N-form Ju with its density (a scalar distribution). The same situation occurs in any dimension: it will be more convenient to work with the $(N-k-1)$-form $*Ju$ (where $*$ stands for the Hodge operator) rather than with the $(k+1)$-form Ju. In this perspective, and with the notation of the present section, we should have written, on the left-hand side of (3.9), $*Ju(\varphi)$ rather than $Ju(\varphi)$.

We start with a fundamental example connecting Jacobians and singularities; see Jerrard and Soner [53, Section 5] (and, also, Bousquet [17]) for the first item, and Alberti, Baldo, and Orlandi [2, Theorem 5.6] for the second one.

Theorem 3.7.

1. Let Γ be a smooth connected oriented $(N-k-1)$-submanifold Γ without boundary (in Ω). Let $u \in W^{1,k}(\Omega; \mathbb{S}^k) \cap C(\Omega \setminus \Gamma)$. Then, with $m := \deg(u, \Gamma)$, we have

$$*Ju = C_{k+1} \, m \, \Gamma \text{ in } \mathscr{D}'(\Omega; \Lambda^{N-k-1}), \tag{3.60}$$

where C_{k+1} is the volume of the unit ball in $\mathbb{R}^{k+1}$ and Γ is identified, as usual, with an $(N-k-1)$-current.

2. Given any connected Γ of the form $\Gamma = \widetilde{\Gamma} \cap \Omega$, with $\widetilde{\Gamma} \subset \mathbb{R}^N$ a smooth closed oriented $(N-k-1)$-manifold, and any $m \in \mathbb{Z}$, there exists $u \in W^{1,k}(\Omega; \mathbb{S}^k)$ such that (3.60) holds.

Remark 3.8. Some comments are in order concerning the statement of Theorem 3.7.

a) The requirement that Γ is oriented is important only when $N-k-1 \geq 2$. Indeed, points and curves are always orientable, but not, for example, surfaces in $\mathbb{R}^4$.

b) Given an oriented submanifold Γ in $\mathbb{R}^N$, we may always choose a coherent orientation of the normal spaces: if $\{e_1, \dots, e_{N-k-1}\}$ is a direct basis of $T_x\Gamma$, then a basis $\{e_{N-k}, \dots, e_N\}$ of $N_x\Gamma$ is direct if $\{e_1, \dots, e_N\}$ is a direct basis of $\mathbb{R}^N$.

c) The degree $m = \deg(u, \Gamma)$ is defined as follows. Let $x \in \Gamma$. Consider a (small) k-dimensional (hyper)sphere, denoted $S_\varepsilon(x)$, of radius ε, in the (geometric) normal $(k+1)$-plane to Γ at x. The map $u_{|S_\varepsilon(x)} : S_\varepsilon(x) \to \mathbb{S}^k$ is continuous. By a homotopy argument, its Brouwer degree, denoted here m, does not depend on x or (small) ε.

d) The meaning of (3.60) is the following:

$$\int_\Omega d\varphi \wedge (u^\sharp \omega_k) = (-1)^{N-k}(k+1)C_{k+1}\, m \int_\Gamma \varphi, \ \forall\, \varphi \in \mathrm{Lip}_0(\Omega; \Lambda^{N-k-1}). \tag{3.61}$$

e) Item 2 cannot be an exact converse to item 1, in the sense that the condition that Γ has no boundary in Ω is not sufficient for the existence of u as in item 2. Here is an example. Let $\Omega := B_{10}(0) \setminus \overline{B}_1(0)$ and let Γ be the oriented segment from $(0,0,1)$ to $(0,0,10)$. Then there exists no $u \in C(\Omega \setminus \Gamma; \mathbb{S}^1)$ such that $\deg(u, \Gamma) \neq 0$ (and thus, in particular, for this Γ and for $m = 1$, the conclusion of item 2 does not hold). Indeed, argue by contradiction, and consider some $u \in C(\Omega \setminus \Gamma; \mathbb{S}^1)$, such that $\deg(u, \Gamma) = m \neq 0$. We note that every circle $C_\varepsilon(x)$ as in the definition of $\deg(u, \Gamma)$ is homotopic, in $\Omega \setminus \Gamma$, to the circle $\begin{cases} x_1^2 + x_2^2 = 1 \\ x_3 = -2 \end{cases}$, which in turn is homotopic to a point. Via a homotopy argument, we find that $m = 0$ – a contradiction.

We next present a limitation of the use of the Jacobian as a "singularities detector" (see [35, Section 4.3.1]).

Proposition 3.9. Let $u \in W^{1,k}(\Omega; \mathbb{S}^k) \cap C(\Omega \setminus \Gamma)$, where $\Gamma \subset \Omega$ is a closed set such that $\mathscr{H}^{N-k-1}(\Gamma) = 0$. Then $Ju = 0$.

Remark 3.10. Proposition 3.9 implies that, if $u(x) = H(x/|x|)$, where $x \in \Omega \subset \mathbb{R}^4$, $0 \in \Omega$, and $H \in C^1(\mathbb{S}^3; \mathbb{S}^2)$, then $Ju = 0$ in $\mathscr{D}'(\Omega)$. (Note that this u belongs to $W^{1,p}(\Omega; \mathbb{S}^2)$, $\forall\, p < 4$, by Lemma 0.1 a).) This implies that Ju does not detect lower dimensional "topological singularities," since H may carry a non-trivial Hopf degree.

On the other hand, if H is topologically non-trivial, then, by the argument in Section 2.1, u cannot be strongly approximated, in $W^{1,p}(\Omega; \mathbb{S}^2)$, $3 \leq p < 4$, with smooth maps. This implies that the condition $Ju = 0$ is not sufficient for approximability with smooth maps.

3.5.2 The distributional Jacobian. Disintegration (slicing)

We present here the higher-dimensional counterpart of Section 3.1. We assume that

$$0 < s < \infty,\ 1 \le p < \infty, k \le sp < k+1. \tag{3.62}$$

The main result here is the following (see [18]).

Theorem 3.11. Assume (3.62). Then the map

$$\mathscr{R}_{N-k-1} \ni u \overset{J}{\mapsto} Ju \in \mathscr{D}'(\Omega; \Lambda^{k+1}) \tag{3.63}$$

has a unique extension by continuity, still denoted J, to $W^{s,p}(\Omega; \mathbb{S}^k)$.

In addition, Ju belongs to the space $[\mathrm{Lip}_0(\Omega; \Lambda^{N-k-1})]^*$, the mapping $W^{s,p}(\Omega; \mathbb{S}^k) \ni u \mapsto Ju \in [\mathrm{Lip}_0(\Omega; \Lambda^{N-k-1})]^*$ is continuous, and we have the estimate

$$\|Ju\|_{[\mathrm{Lip}_0(\Omega;\Lambda^{N-k-1})]^*} \lesssim |u|^{k/s}_{W^{s,p}}. \tag{3.64}$$

The proof follows essentially Steps 2–5 of the proof of Theorem 3.1.

We next connect the distributional Jacobian defined above with the definition in Theorem 3.1. For simplicity, we let $\Omega = (0,1)^N$.

Set

$$I(N-k-1, N) := \{\alpha \subset \{1, \dots, N\};\ \mathrm{Card}\, \alpha = N-k-1\}.$$

For $\alpha \in I(N-k-1, N)$, set $\overline{\alpha} := \{1, \dots, N\} \setminus \alpha \in I(k+1, N)$. Let $\varphi \in C^\infty_c(\Omega; \Lambda^{N-k-1})$. Then we may write

$$\varphi = \sum_{\alpha \in I(N-k-1,N)} \varphi^\alpha \, dx^\alpha = \sum_{\alpha \in I(N-k-1,N)} (\varphi^\alpha)_{x_\alpha}(x_{\overline{\alpha}})\, dx^\alpha.$$

Here, dx^α denotes the canonical $(N-k-1)$-form induced by the coordinates x_j, $j \in \alpha$, and $(\varphi^\alpha)_{x_\alpha}(x_{\overline{\alpha}}) := \varphi^\alpha(x_\alpha, x_{\overline{\alpha}})$ belongs to $C^\infty_c((0,1)^{N-k-1}; \mathbb{R})$ (for fixed x_α).

Given $u \in W^{s,p}(\Omega; \mathbb{S}^k)$, by slicing (Corollary 4.2), for a. e. $x_\alpha \in (0,1)^{N-k-1}$ the partial map $(u^\alpha)_{x_\alpha} := x_{\overline{\alpha}} \mapsto u(x_\alpha, x_{\overline{\alpha}})$ belongs to $W^{s,p}((0,1)^{k+1}; \mathbb{S}^k)$. Assuming that s, p satisfy (3.62), for such x_α the distributional Jacobian $J(u^\alpha)_{x_\alpha}$ (or rather, as we have explained, $*J(u^\alpha)_{x_\alpha}$) is well-defined (*via* Theorem 3.1) as an element of $\mathscr{D}'(\Omega)$.

We have the following *disintegration* result.

Proposition 3.12. Assume (3.62). Let $\Omega = (0,1)^N$ and $u \in W^{s,p}(\Omega; \mathbb{S}^k)$. Then, with appropriate $\varepsilon(\alpha) \in \{-1, 1\}$ depending only on k, N, and $\alpha \in I(N-k-1, N)$, we have

$$*Ju(\varphi) = \sum_{\alpha \in I(N-k-1,N)} \varepsilon(\alpha) \int_{(0,1)^{N-k-1}} *J(u^\alpha)_{x_\alpha}((\varphi^\alpha)_{x_\alpha})\, dx^\alpha. \tag{3.65}$$

When $s \ge 1$, (3.65) follows from the Fubini theorem. The case where $0 < s < 1$ is more delicate. See [60, Lemma 3.12] for a proof when $s = 1/p$ and $k = 1$. (The argument there works in the general case.)

3.5.3 The range of the distributional Jacobian

I will try not to appeal here to the language of geometric measure theory. For the same story (moderately) using this language, see [35, Chapter 2, Chapter 4]. Let $n := N - k \leq N - 1$. Consider a C^1 oriented n-dimensional submanifold Σ of Ω, and a Borel set $A \subset \Sigma$ such that $\mathscr{H}^n(A) < \infty$. Then A acts by integration on smooth compactly supported n-forms, through the formula

$$A(\zeta) := \int_A \zeta, \ \forall \zeta \in C_c^\infty(\Omega; \Lambda^n). \tag{3.66}$$

Note that the integral makes sense since A is oriented.

This allows us to identify A with a linear object (a distribution, or rather a *current*), and thus allow operations as (infinite) sums.

Consider the set

$$\mathscr{F}^n := \Big\{ T = \sum_j A_j; A_j \subset \Omega \text{ is a Borel subset of a } C^1 \text{ oriented } n\text{-dimensional} \\ \text{submanifold } \Sigma_j \text{ of } \Omega, \sum_j \mathscr{H}^n(A_j) < \infty \Big\}. \tag{3.67}$$

Given $T \in \mathscr{F}^n$ (or, more generally, a distribution acting on smooth compactly supported n-forms), we define the *boundary* ∂T of T through the formula

$$\partial T(\varphi) := T(d\varphi), \ \forall \varphi \in C_c^\infty(\Omega; \Lambda^{N-1}). \tag{3.68}$$

The terminology is justified by the fact that, when T is the integration over a compact oriented manifold with boundary, ∂T is the integration over the geometric boundary ∂T of T. (In this case, (3.68) is simply the Stokes theorem.)

We may now present the higher co-dimensional counterpart of Theorem 3.2.

Theorem 3.13. Assume (3.62).

1. If $u \in W^{s,p}(\Omega; \mathbb{S}^k)$, then there exists some $T = \sum_j A_j \in \mathscr{F}^{N-k}$ such that

$$\sum_j \mathscr{H}^{N-k}(A_j) \lesssim |u|^{k/s}_{W^{s,p}}, \tag{3.69}$$

$$*Ju = C_{k+1} \partial T \text{ in } \mathscr{D}'(\Omega; \Lambda^{N-k-1}). \tag{3.70}$$

2. Conversely, given $T = \sum_j A_j \in \mathscr{F}^{N-k}$, there exists $u : \Omega \to \mathbb{S}^k$ such that, for every s, p satisfying $sp = N - 1$: (i) $u \in W^{s,p}(\Omega; \mathbb{S}^k)$; (ii) (3.70) holds; (iii)

$$|u|^p_{W^{s,p}} \lesssim \inf \left\{ \sum_k \mathscr{H}^{N-k}(\widetilde{A}_k); \ \partial \sum_k \widetilde{A}_k = \partial \sum_j A_j \text{ in } \mathscr{D}'(\Omega; \Lambda^{N-k-1}) \right\}. \tag{3.71}$$

Already the fact that Theorem 3.2 is the special case of Theorem 3.13 with $N = k + 1$ is not obvious (but not difficult to prove). When $s = 1$, item 1 is due to Alberti, Baldo, and Orlandi [2, Theorem 3.7], who extended to general maps an argument relying on the *co-area formula* devised by Almgren, Browder, and Lieb [3]. This main idea is illustrated, when u is sufficiently smooth, in [35, Section 3.3]. Item 2 with $s = 1$ is also due to Alberti, Baldo, and Orlandi [2, Theorem 5.6]. It relies on a delicate dipole insertion technique, reminiscent of the one in Brezis, Coron, and Lieb [25], but technically much more involved. The general case (arbitrary s, p) was obtained in [18].

3.5.4 Characterization of the closure of smooth maps

We have the following counterpart of Theorem 3.5 (with the same references as for Theorem 3.5).

Theorem 3.14. Assume that $0 < s \leq 1$, $1 \leq p < \infty$, $k \leq sp < k + 1$. Moreover, when $k < sp < k + 1$ and $N > k + 1$, assume that $\Omega = (0, 1)^N$. Then, for $u \in W^{s,p}(\Omega; \mathbb{S}^k)$, we have

$$u \in \overline{C^\infty(\overline{\Omega}; \mathbb{S}^k)}^{W^{s,p}} \iff Ju = 0.$$

4 Appendix # 1. Standard & less standard properties of Sobolev spaces

We present here, mostly without proofs, some of the basic properties of Sobolev maps that we use in the main text. For the full proofs, some useful general references are Triebel [71, 72], Runst and Sickel [68], Maz'ya [56], Leoni [54]; an elementary, but partial, account can be find in [58]. See also the specific references indicated below.

In what follows, $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded domain. Occasionally, it could be $\mathbb{R}^N$ or a half space. Unless specified otherwise, the Sobolev spaces $W^{s,p}$ and the corresponding norms are considered with respect to Ω.

Contents

4.1.0 Slicing and characterization via differences

For simplicity, we only consider the case of one-dimensional slices, i. e., given a map $u = u(x_1, \dots, x_N)$, we connect its regularity with the one of its one-dimensional slices $u(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_N)$, $k = 1, \dots, N$, but similar results are available for ℓ-dimensional slices.

Theorem 4.1. Let $s > 0$ be non-integer. Then

$$\|u\|^p_{W^{s,p}} \sim \|u\|^p_p + \sum_{k=1}^{N} \int_{-1}^{1} \left(\int \frac{|\delta^M_{te_k} u(x)|^p}{|t|^{1+sp}}\, dx \right) dt. \tag{4.1}$$

Here:

1. t is a one-dimensional variable.
2. M is any integer satisfying $M > s$.
3. The integral in x is computed over the set $\{x \in \Omega;\ [x, x + Mte_k] \subset \Omega\}$.
4. $\delta_h u(x) := u(x + h) - u(x)$, and $\delta^M_h := \underbrace{\delta_h \circ \dots \circ \delta_h}_{M \text{ times}}$.

For a proof, see [72, Section 3.5.3], [68, Section 2.3.1].

An immediate consequence of (4.1) is the following

Corollary 4.2. Let $s > 0$ be non-integer and $1 \le p < \infty$. Then

$$\|u\|^p_{W^{s,p}(\mathbb{R}^N)} \sim \sum_{k=1}^{N} \int \|u(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_N)\|^p_{W^{s,p}(\mathbb{R})}\, d\widehat{x_k}. \tag{4.2}$$

In particular, for a.e. $(x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, we have $u(x_1, \dots, x_{N-1}, \cdot) \in W^{s,p}(\mathbb{R})$.

Here, $d\widehat{x_j} := dx_1 \dots dx_{k-1} dx_{k+1} \dots dx_N$.

A straightforward consequence of (4.2) and of the "converse" to the dominated convergence theorem is the following Fubini type convergence result.

Corollary 4.3. Let $s > 0$ be a non-integer and $1 \le p < \infty$. Assume that $u_j \to u$ in $W^{s,p}(\mathbb{R}^N)$. Then, possibly up to a subsequence, we have $u_j(x_1, \dots, x_{N-1}, \cdot) \to u(x_1, \dots, x_{N-1}, \cdot)$ in $W^{s,p}(\mathbb{R})$ for a.e. $(x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$.

A variant of (4.2) holds for both fractional and integer Sobolev spaces. Let $u : \mathbb{R}^N \to \mathbb{R}$. If $\omega \in \mathbb{S}^{N-1}$, let $\omega^\perp$ denote the hyperplane orthogonal to ω, and consider the partial functions

$$\omega^\perp \ni x' \mapsto u^{x'}_\omega, \text{ with } u^{x'}_\omega(t) := u(x' + t\,\omega),\ \forall\, t \in \mathbb{R}.$$

Then we have the following

Proposition 4.4. Let $s \ge 0$ and $1 \le p < \infty$. Let $u : \mathbb{R}^N \to \mathbb{R}$. Then

$$\|u\|^p_{W^{s,p}(\mathbb{R}^N)} \sim \int_{\mathbb{S}^{N-1}} \int_{\omega^\perp} \|u^{x'}_\omega\|^p_{W^{s,p}(\mathbb{R})}\, dx' d\mathscr{H}^{N-1}(\omega). \tag{4.3}$$

For a proof, see [58, Lemma 22].

4.2.0 Sobolev embeddings

Optimal Sobolev embeddings are of the form $W^{s,p}(\Omega) \hookrightarrow W^{r,q}(\Omega)$, where s, r, p, q satisfy

$$s > r \geq 0,\ 1 \leq p < q \leq \infty,\ s - \frac{N}{p} = r - \frac{N}{q}. \tag{4.4}$$

Note that we allow the value $q = \infty$.

The following result incorporates the classical Sobolev, Morrey, and Gagliardo-Nirenberg embeddings.

Theorem 4.5. Let s, r, p, q, N satisfy (4.4). Then we have

$$W^{s,p}(\Omega) \hookrightarrow W^{r,q}(\Omega) \tag{4.5}$$

with the following exceptions, when (4.5) does not hold.

(a) When

$$N = 1,\ s \text{ is an integer } \geq 1,\ p = 1,\ 1 < q < \infty, \text{ and } r = s - 1 + 1/q, \tag{4.6}$$

we have

$$W^{s,1}(\Omega) \not\hookrightarrow W^{s-1+1/q,q}(\Omega). \tag{4.7}$$

In particular, we have

$$W^{1,1}((0,1)) \not\hookrightarrow W^{1/q,q}((0,1)),\ 1 < q < \infty. \tag{4.8}$$

(b) When

$$N \geq 1,\ 1 < p < \infty,\ q = \infty, \text{ and } s - \frac{N}{p} = r \text{ is an integer}, \tag{4.9}$$

we have

$$W^{s,p}(\Omega) \not\hookrightarrow W^{r,\infty}(\Omega). \tag{4.10}$$

For a proof, see, for example, [34, Appendix].

In the limiting case $sp = N$, we have the following substitute of the Sobolev non-embedding $W^{s,p} \hookrightarrow L^\infty$ (Brezis and Nirenberg [37], with roots in Boutet de Monvel and Gabber [22]).

Theorem 4.6. Assume that $sp \geq N$. Then $W^{s,p} \hookrightarrow \mathrm{VMO}$, i. e.,

$$u \in W^{s,p} \implies \limsup_{\varepsilon \to 0} \sup_{x \in \Omega_\varepsilon} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)|\, dydz = 0. \tag{4.11}$$

For a proof, see, e. g., [35, Proof of Lemma 15.20].

4.3.0 Gagliardo-Nirenberg inequalities

Consider the estimate

$$\|u\|_{W^{s,p}} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}} \|u\|^{1-\theta}_{W^{s_2,p_2}}, \ \forall\, u \in W^{s_1,p_1} \cap W^{s_2,p_2}, \tag{4.12}$$

where $s_1, s_2, s \geq 0$ and $1 \leq p_1, p_2, p \leq \infty$ are related by

$$\begin{cases} s = \theta s_1 + (1-\theta) s_2 \\ \dfrac{1}{p} = \dfrac{\theta}{p_1} + \dfrac{1-\theta}{p_2} \end{cases} \qquad \text{for some } \theta \in (0,1). \tag{4.13}$$

With no loss of generality, we may assume that

$$s_1 \leq s_2. \tag{4.14}$$

The following condition plays an essential role in the validity of (4.12):

$$s_2 \text{ is an integer } \geq 1,\ p_2 = 1 \text{ and } s_2 - s_1 \leq 1 - \frac{1}{p_1}. \tag{4.15}$$

(The latter condition can also be written in the more symmetric form $s_1 - 1/p_1 \geq s_2 - 1/p_2$.)

We have the following result [32] incorporating the classical Gagliardo-Nirenberg inequalities [47, 64].

Theorem 4.7. Let s, s_1, s_2, p, p_1, and p_2 satisfy (4.13) and (4.14). Then, (4.12) holds if and only if (4.15) fails.

More precisely, we have

1. If (4.15) fails, then, for every $\theta \in (0,1)$,

$$\|u\|_{W^{s,p}} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}} \|u\|^{1-\theta}_{W^{s_2,p_2}}, \ \forall\, u \in W^{s_1,p_1} \cap W^{s_2,p_2}. \tag{4.16}$$

Moreover, if $s_1 < s < s_2$, then we have (in a bounded domain) the estimate

$$|u|_{W^{s,p}} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}} |u|^{1-\theta}_{W^{s_2,p_2}}, \ \forall\, u \in W^{s_1,p_1} \cap W^{s_2,p_2}. \tag{4.17}$$

2. If (4.15) holds, there exists some $u \in W^{s_1,p_1} \cap W^{s_2,p_2}$ such that $u \notin W^{s,p}$, $\forall\, \theta \in (0,1)$.

Here is the special case of the above theorem we use the most often in this text.

Corollary 4.8. The embedding

$$W^{s,p} \cap L^{\infty} \hookrightarrow W^{\theta s, p/\theta}, \ \forall\, 0 < \theta < 1,$$

and the corresponding estimate (4.12) hold, $\forall\, s > 0$, $\forall\, 1 \leq p \leq \infty$, *except when* $s = p = 1$.

4.4.0 Trace theories

We consider, to simplify the presentation, only maps $u : \mathbb{R}^N \to \mathbb{R}$. In order to obtain the corresponding results on domains, either we extend maps from domains to $\mathbb{R}^N$, or we work directly in domains and then deform the domain $\{(x, \varepsilon);\ 0 < \varepsilon < \varepsilon_0,\ x \in \Omega,\ d(x, \partial\Omega) > \varepsilon\}$ to the cylinder $\Omega \times (0, \varepsilon_0)$ (as explained in the proof of Proposition 0.4). The classical results presented here are due to several authors, including Gagliardo [46] and Uspenskiĭ [73]. For the proofs, see, e. g., [18, Section 7], [59], [35, Lemma 15.47].

Let $u : \mathbb{R}^N \to \mathbb{R}$, ρ be a standard mollifier, and set $U(x, \varepsilon) := u * \rho_\varepsilon(x), \forall\, x \in \mathbb{R}^N, \forall\, \varepsilon > 0$.

We first present trace theory in weighted Sobolev spaces.

Theorem 4.9. (Inverse trace theory in weighted Sobolev spaces) Let $0 < s < m$, with s is non-integer and m integer. We have, with $C = C(m, s, p, N)$,

$$\sum_{|\alpha|=m} \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p(m-s)-1} |\partial^\alpha U(x, \varepsilon)|^p \, d\varepsilon dx \le C|u|^p_{W^{s,p}},\ \forall\, u : \mathbb{R}^N \to \mathbb{R}. \tag{4.18}$$

Theorem 4.10. (Direct trace theory in weighted Sobolev spaces) Let $0 < s < m$, with s is non-integer and m integer. Let $V \in C^\infty(\mathbb{R}^N \times (0, \infty); \mathbb{R})$. We have, with $C = C(m, s, p, N)$, $\forall\, 0 < \varepsilon < 1/2$,

$$\|V(\cdot, \varepsilon)\|^p_{W^{s,p}} \le C \sum_{|\alpha|=m} \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p(m-s)-1} |\partial^\alpha V(x, \varepsilon)|^p \, d\varepsilon dx + C\|V\|^p_{L^p(\mathbb{R}^N \times (0,1))}. \tag{4.19}$$

Moreover, if the right-hand side of (4.19) is finite, then the limit $u := \lim_{\varepsilon \to 0} V(\cdot, \varepsilon)$ exists in $W^{s,p}$ and satisfies

$$\|u\|^p_{W^{s,p}} \le C \sum_{|\alpha|=m} \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p(m-s)-1} |\partial^\alpha V(x, \varepsilon)|^p \, d\varepsilon dx + C\|V\|^p_{L^p(\mathbb{R}^N \times (0,1))}. \tag{4.20}$$

We next present trace theory in fractional Sobolev spaces. (The two theories coincide when $s + 1/p$ is an integer.)

Theorem 4.11. (Inverse trace theory in fractional Sobolev spaces) Let $s > 0$ be non-integer. We have, with $C = C(s, p, N)$,

$$|U|_{W^{s+1/p,p}} \le C|u|_{W^{s,p}},\ \forall\, u : \mathbb{R}^N \to \mathbb{R}. \tag{4.21}$$

Theorem 4.12. (Direct trace theory in fractional Sobolev spaces) Let $s > 0$ be non-integer. Let $V \in C^\infty(\mathbb{R}^N \times (0, \infty); \mathbb{R})$. We have, with $C = C(s, p, N)$, $\forall\, 0 < \varepsilon < 1/2$,

$$\|V(\cdot, \varepsilon)\|_{W^{s,p}} \le C|V|_{W^{s+1/p,p}} + C\|V\|_{L^p(\mathbb{R}^N \times (0,1))}. \tag{4.22}$$

Moreover, if the right-hand side of (4.22) is finite, then the limit $u := \lim_{\varepsilon \to 0} V(\cdot, \varepsilon)$ exists in $W^{s,p}$ and satisfies

$$\|u\|_{W^{s,p}} \le C|V|_{W^{s+1/p,p}} + C\|V\|_{L^p(\mathbb{R}^N \times (0,1))}. \tag{4.23}$$

4.5.0 Superposition operators

Given $\Phi : \mathbb{R} \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ (or $\Phi : \mathbb{R}^M \to \mathbb{R}^K$ and $u : \Omega \to \mathbb{R}^M$), set

$$F(u) := \Phi \circ u. \tag{4.24}$$

Here are the results on the continuity of F we rely on.

Theorem 4.13. Assume that Φ is Lipschitz. Let $0 < s \leq 1$, $1 \leq p < \infty$. Then F (given by (4.24)) is continuous from $W^{s,p}$ in itself.

The non-trivial part of the above result is continuity. For a proof when $s = 1$, see Marcus and Mizel [55]. For the case $0 < s < 1$, see, e. g., [14, proof of (5.43)]).

Theorem 4.14. Let $s > 1$ and $1 \leq p < \infty$. Let m denote the first integer $\geq s$. Let $\Phi \in C^m$. Then F (given by (4.24)) maps $W^{s,p} \cap L^\infty$ into $W^{s,p}$ and is continuous in the following sense:

$$\text{if } u_n \to u \text{ in } W^{s,p} \text{ and } \|u_n\|_{L^\infty} \leq C, \text{ then } \Phi(u_n) \to \Phi(u) \text{ in } W^{s,p}.$$

For an elementary proof of the above result for arbitrary s, see Escobedo [45].

Theorem 4.15. Let $s > 1$ and let $1 \leq p < \infty$. Let m be the least integer $\geq s$. Let $\Phi \in C^m(\mathbb{R}^M; \mathbb{R}^K)$ have bounded derivatives of order $\leq m$. Then, for every $u \in W^{s,p} \cap W^{1,sp}(\Omega; \mathbb{R}^M)$, we have $F(u) \in W^{s,p}(\Omega; \mathbb{R}^K)$. In addition, F (given by (4.24)) is continuous from $W^{s,p} \cap W^{1,sp}$ to $W^{s,p}$.

For a proof, see [29]. (The result there is stated for $p > 1$, but exactly the same proof applies when $p = 1$.) See also Maz'ya and Shaposhnikova [57].

Corollary 4.16. Let $0 < s < \infty$, $1 \leq p < \infty$. If either $s \leq 1$ or $sp \geq N$, then $\varphi \mapsto e^{\imath\varphi}$ acts continuously from $W^{s,p}(\Omega; \mathbb{R})$ to $W^{s,p}(\Omega; \mathbb{S}^1)$.

Proof. When $s \leq 1$, this is a special case of Theorem 4.13. When $s > 1$, by assumption we have $sp \geq N$, and in this case we rely on Theorem 4.15 and on the Sobolev embedding $W^{s,p} \hookrightarrow W^{1,sp}$. QED

4.6.0 Products

The most used product property of the Sobolev spaces is that $W^{s,p} \cap L^\infty$ is an algebra, in the following sense.

Lemma 4.17. Let $s > 0$, $1 \leq p < \infty$. If $u, v \in W^{s,p} \cap L^\infty$, then $uv \in W^{s,p}$ and

$$\|uv\|_{W^{s,p}} \leq C(\|u\|_{L^\infty}\|v\|_{W^{s,p}} + \|v\|_{L^\infty}\|u\|_{W^{s,p}}). \tag{4.25}$$

In addition, the map $(u, v) \mapsto uv$ is continuous in the following sense: if $u_n \to u$ and $v_n \to v$ in $W^{s,p}$, and if $\|u_n\|_{L^\infty}, \|v_n\|_{L^\infty} \leq C$, then $u_n v_n \to uv$ in $W^{s,p}$.

For a proof, see, e. g., Runst and Sickel [68].

When $s > 1$, the above result can be strengthened as follows.

Lemma 4.18. Let $1 < s < \infty, 1 \le p < \infty$. Let $u, v \in W^{s,p} \cap L^\infty$. Then $uDv \in W^{s-1,p}$.

For a proof, see, e. g., [35, Appendix 15.12].

A refinement of Lemma 4.18 is provided by the following result [29, Lemma 6.1].[†]

Lemma 4.19. A couple (s, p), with $s > 0$ and $1 \le p < \infty$, is *regular* if either $p > 1$ or $p = 1$ and s is not an integer.

Let (s_1, p_1), (s_2, p_2) be two regular couples such that $s_1 > s_2$ and $s_1 p_1 > s_2 p_2$. Let $1 < r < \infty$ be defined by

$$\frac{1}{r} = \frac{1}{p_2} - \frac{s_2}{s_1 p_1}.$$

If $f \in W^{s_1,p_1} \cap L^\infty(\Omega)$ and $g \in L^{s_2,p_2} \cap L^r(\Omega)$, then $fg \in W^{s_2,p_2}(\Omega)$.

4.7.0 Gluing

We have the following straightforward versions of the Brezis-Lieb lemma [27].

Lemma 4.20. Let $f \in W^{s,p}(\Omega)$. Consider a bounded sequence of maps $f_j \in W^{s,p}(\Omega)$ such that

$$\lim_{j\to\infty} |\cup_{k\ge j} \operatorname{supp} f_k| = 0.$$

Then

$$|f + f_j|^p_{W^{s,p}} = |f|^p_{W^{s,p}} + |f_j|^p_{W^{s,p}} + o(1) \text{ as } j \to \infty,$$
$$\|f + f_j\|^p_p = \|f\|^p_p + \|f_j\|^p_p + o(1) \text{ as } j \to \infty.$$

Lemma 4.21. (Multi-sequences Brezis-Lieb lemma) Let $f \in W^{s,p}(\Omega)$. Consider m bounded sequences $(f_{j,1}), \dots (f_{j,m}) \subset W^{s,p}(\Omega)$ such that

$$\lim_{j\to\infty} |\cup_{k\ge j} \operatorname{supp} f_{k,i}| = 0, \ \forall\, i,$$
$$|f_{j,i}|^p_{W^{s,p}} \le C_i, \ \forall\, j, \ \forall\, i,$$
$$\|f_{j,i}\|^p_p \le D_i, \ \forall\, j, \ \forall\, i.$$

Then there exist $j_1, \dots, j_m$ such that

$$|f + f_{j_1,1} + \dots + f_{j_m,m}|^p_{W^{s,p}} \le |f|^p_{W^{s,p}} + 2(C_1 + \dots + C_m),$$
$$|f_{j_1,1} + \dots + f_{j_m,m}|^p_{W^{s,p}} \le |f|^p_{W^{s,p}} + 2(C_1 + \dots + C_m),$$
$$\|f + f_{j_1,1} + \dots + f_{j_m,m}\|^p_p \le \|f\|^p_p + 2(D_1 + \dots + D_m),$$
$$\|f_{j_1,1} + \dots + f_{j_m,m}\|^p_p \le 2(D_1 + \dots + D_m).$$

In view of the applications, let us note that, in Lemma 4.21, the indices j_k may be chosen inductively with respect to k.

[†]The result in [29] is stated for $1 < p_1, p_2 < \infty$, but the proof there is valid for all regular couples.

4.8.0 Quantitative suboptimal Sobolev embeddings

In view of the applications we consider, we work here in the unit ball Ω (for some norm $|\,|$ in $\mathbb{R}^N$) and with less than one derivative, but what follows can be adapted to any domain and to higher order spaces.

Consider parameters satisfying

$$0 \le \alpha < s < 1,\ 1 \le p < \infty,\ 1 \le q \le \infty, \alpha - \frac{N}{q} < s - \frac{N}{p},\ p \le t \le \infty. \tag{4.26}$$

When $\alpha = 0$, set

$$|u|_{W^{0,q}(B_r(x))} := \left\| u - \fint_{B_r(x)} u \right\|_{L^q(B_r(x))}.$$

Lemma 4.22. Assume (4.26). Set

$$\beta := s - \alpha - N\left(\frac{1}{p} - \frac{1}{q}\right) > 0, \tag{4.27}$$

$$U := \{(x,r);\ x \in \Omega,\ r > 0,\ B_r(x) \subset \Omega\}. \tag{4.28}$$

Then (with the obvious modification when $t = \infty$)

$$\left(\int_U \frac{|u|^t_{W^{\alpha,q}(B_r(x))}}{r^{\beta t}} \frac{dxdr}{r^{N+1}} \right)^{1/t} \lesssim |u|_{W^{s,p}},\ \forall\, u : \Omega \to \mathbb{R}. \tag{4.29}$$

Proof. Let $\alpha < \sigma \le s$ be such that

$$\gamma := \sigma - \alpha - N\left(\frac{1}{p} - \frac{1}{q}\right) > 0.$$

By Sobolev's embedding, Poincaré's inequality, and scaling, we have

$$|u|_{W^{\alpha,q}(B_r(x))} \lesssim r^\gamma |u|_{W^{\sigma,p}(B_r(x))},\ \forall\, (x,r) \in U. \tag{4.30}$$

The choice $\sigma = s$ yields (4.29) for $t = \infty$. It thus suffices to obtain (4.29) when $t = p$; the general case will follow *via* Hölder's inequality. For this purpose, we note that, with $\alpha < \sigma < s$ as above and $\delta := \beta - \gamma = s - \sigma > 0$, we have (using (4.30))

$$\begin{aligned}
\int_U \frac{|u|^p_{W^{\alpha,q}(B_r(x))}}{r^{\beta p}} \frac{dxdr}{r^{N+1}} &\lesssim \int_U \frac{1}{r^{\delta p + N + 1}} \iint_{B_r(x)} \frac{|u(w) - u(z)|^p}{|w - z|^{N + \sigma p}}\, dwdz\, dxdr \\
&= \iint_{\Omega^2} \frac{|u(w) - u(z)|^p}{|w - z|^{N+\sigma p}} \iint_{\{(x,r) \in U;\, z,w \in B_r(x)\}} \frac{1}{r^{\delta p + N + 1}}\, dxdr\, dwdz \\
&\le \iint_{\Omega^2} \frac{|u(w) - u(z)|^p}{|w - z|^{N+\sigma p}} \int_{|w-z|/2}^\infty \int_{B_r(w)} dx \frac{1}{r^{\delta p + N + 1}} dr\, dwdz
\end{aligned}$$

$$\sim \iint_{\Omega^2} \frac{|u(w)-u(z)|^p}{|w-z|^{N+\sigma p}} \int_{|w-z|/2}^{\infty} \frac{1}{r^{\delta p+1}} dr\, dwdz$$
$$\sim \iint_{\Omega^2} \frac{|u(w)-u(z)|^p}{|w-z|^{N+\sigma p+\delta p}}\, dwdz = |u|^p_{W^{s,p}}. \qquad \text{QED}$$

We will use the following special case of Lemma 4.22 [61, Section 3.1].

Corollary 4.23. Assume that $0 < s < 1$, $1 \le p < \infty$, and $sp > 1$. Then

$$\int_0^1 \int_0^1 \frac{|u|^p_{W^{0,\infty}((x,y))}}{|y-x|^{1+sp}}\, dxdy \lesssim |u|^p_{W^{s,p}},\ \forall\, u : (0,1) \to \mathbb{R}. \tag{4.31}$$

Proof. Setting $r := |y-x|/2$, $z := (y+x)/2$, we see that the left-hand side I of (4.31) satisfies

$$I \lesssim \int_0^1 \int_{\{r;\, B_r(z) \subset (0,1)\}} \frac{|u|^p_{W^{0,\infty}(B_r(z))}}{r^{1+sp}} dr dz.$$

We conclude *via* (4.29) applied with $\alpha = 0$, $q = \infty$, $t = p$, and $N = 1$. QED

4.9.0 Martingales and Sobolev spaces

The framework is the one of Step 7 of the proof of Theorem 1.2. We work in $\Omega = [0,1)^N$ and with the $\| \ \|_\infty$ norm, denoted for simplicity $| \ |$. For $k \ge 0$, let $\mathscr{P}_k$ denote the collection of dyadic cubes of size 2^{-k} in Ω. Q_k denotes a generic cube in $\mathscr{P}_k$, and, for $x \in \Omega$, $Q_k(x)$ is the only cube in $\mathscr{P}_k$ containing x. We let $\mathscr{F}_k$ denote the set of the (step) functions constant on each Q_k. If $u : \Omega \to \mathbb{R}^\ell$, let $E_k(u) \in \mathscr{F}_k$ be defined by $E_k(u)(x) := \fint_{Q_k(x)} u$.

For the next result, see [12, Proof of Theorem A.1, Step 3].

Lemma 4.24. Let $0 < s < 1$. Then, for each $f \in L^p(\Omega; \mathbb{R}^\ell)$,

$$\sum_{k \ge 0} 2^{spk} \|f - E_k(f)\|_p^p \lesssim |f|^p_{W^{s,p}}.$$

We next present a variant of [12, Theorem A.1] adapted to our purposes.

Lemma 4.25. Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < 1$. Let $\Phi : \Omega^2 \to [0,\infty)$ be measurable and let, for each $k \ge 1$, $g_k \in \mathscr{F}_k$, $g_k \ge 0$. If

$$[x, y \in Q_j] \implies \Phi(x,y) \le \sum_{k>j} (g_k(x) + g_k(y)),\ \forall\, j \ge 0,\ \forall\, Q_j \in \mathscr{P}_j, \tag{4.32}$$

then

$$\int_\Omega \int_\Omega \frac{\Phi^p(x,y)}{|x-y|^{N+sp}}\, dxdy \lesssim \sum_{k \ge 1} 2^{spk} \|g_k\|_p^p. \tag{4.33}$$

Proof. Step 1. Reduction to inequalities for step functions. Given $x, y \in \Omega$, $x \neq y$, set

$$s(x,y) := \max\{k;\ Q_k(x) = Q_k(y)\},\ t(x,y) := \min\{k;\ |x-y| > 2^{-t}\}.$$

We note the following obvious facts: (i) s and t are symmetric; (ii) if $k > s(x,y)$, then $y \notin Q_k(x)$; (iii) $t(x,y) > s(x,y)$; (iv) $2^{-t(x,y)} < |x-y| \leq 2^{1-t(x,y)}$.

In view of (4.32), in order to obtain (4.33) it suffices to prove the inequality

$$\int_\Omega \int_\Omega \Big(\sum_{k > s(x,y)} g_k(x) \Big)^p \frac{dxdy}{|x-y|^{N+sp}} \lesssim K := \sum_{k \geq 1} 2^{spk} \|g_k\|_p^p. \tag{4.34}$$

Since, by Hölder's inequality, we have

$$\Big(\sum_{k > s(x,y)} (g_k(x)) \Big)^p \lesssim \sum_{k \geq t(x,y)} (k - t(x,y) + 1)^p g_k^p(x) + \sum_{s(x,y) < k < t(x,y)} (t(x,y) - k)^p g_k^p(x),$$

it suffices to establish the inequalities

$$I := \int_\Omega \int_\Omega \sum_{k \geq t(x,y)} (k - t(x,y) + 1)^p g_k^p(x) \frac{dxdy}{|x-y|^{N+sp}} \lesssim K, \tag{4.35}$$

$$J := \int_\Omega \int_\Omega \sum_{s(x,y) < k < t(x,y)} (t(x,y) - k)^p g_k^p(x) \frac{dxdy}{|x-y|^{N+sp}} \lesssim K. \tag{4.36}$$

Step 2. (4.35) *holds for* $0 < s < 1$ *and measurable* g_k*'s (without the assumptions* $sp < 1$ *or* $g_k \in \mathscr{F}_k$*).* We have

$$\begin{aligned} I &= \sum_{k \geq 1} \sum_{1 \leq j \leq k} (k - j + 1)^p \int_\Omega g_k^p(x) \int_{\{y \in \Omega;\ t(x,y) = j\}} \frac{dy}{|x-y|^{N+sp}} dx \\ &\lesssim \sum_{k \geq 1} \sum_{1 \leq j \leq k} (k - j + 1)^p \int_\Omega g_k^p(x)\, 2^{spj}\, dx \lesssim \sum_{k \geq 1} 2^{skp} \int_\Omega g_k^p(x)\, dx = K. \end{aligned}$$

Step 3. An auxiliary estimate. Let Q be a(ny) cube of size 2^{-k} in $\mathbb{R}^N$. If $sp < 1$, then

$$I_k := \sum_{j > k} (j-k)^p \iint_{\{(x,y);\ x \in Q,\ y \notin Q;\ t(x,y) = j\}} \frac{dxdy}{|x-y|^{N+sp}} \lesssim 2^{(sp-N)k}. \tag{4.37}$$

Indeed, the left-hand side of (4.37) does not depend on the center of the cube, and, by a scaling argument, we have $I_k = 2^{(sp-N)k} I_0$. It therefore suffices to prove that, with $Q := (-1/2, 1/2)^N$, we have

$$I_0 = \sum_{j > 0} j^p \iint_{\{(x,y);\ x \in Q,\ y \notin Q;\ t(x,y) = j\}} \frac{dxdy}{|x-y|^{N+sp}} < \infty. \tag{4.38}$$

For this purpose, let us note that

$$|\{x \in Q;\ |x| \geq 2^{-1} - \varepsilon\}| \lesssim \varepsilon,\ \forall\, \varepsilon > 0, \tag{4.39}$$

$$[x \in Q, y \notin Q, t(x,y) = j] \implies [2^{-j} < |x-y| \leq 2^{1-j} \text{ and } |x| \geq 1/2 - 2^{1-j}]. \tag{4.40}$$

Combining (4.38)–(4.40) and using, at the end, the assumption $sp < 1$, we find that

$$\begin{aligned} I_0 &\leq \sum_{j>0} j^p \iint_{\{(x,y);\, x\in Q,\, |x|\geq 1/2-2^{1-j},\, 2^{-j}<|x-y|\leq 2^{1-j}\}} \frac{dxdy}{|x-y|^{N+sp}} \\ &\lesssim \sum_{j>0} j^p \int_{\{x;\, x\in Q,\, |x|\geq 1/2-2^{1-j}\}} 2^{spj}\, dx \lesssim \sum_{j>0} j^p 2^{spj} 2^{1-j} = 2\sum_{j>0} j^p 2^{-(1-sp)j} < \infty. \end{aligned}$$

Step 4. (4.36) *holds when* $sp < 1$ *and* $g_k \in \mathscr{F}_k$. For $Q_k \in \mathscr{P}_k$, let $g_k(Q_k)$ denote the value of g_k on Q_k. For $k > s(x,y)$, we have $y \notin Q_k(x)$. Combining this observation with (4.37), we find that

$$\begin{aligned} J &= \sum_{k\geq 1} \sum_{Q_k\in\mathscr{P}_k} \sum_{j>k} (j-k)^p g_k^p(Q_k) \iint_{\{(x,y);\, x\in Q_k,\, y\notin Q_k;\, t(x,y)=j\}} \frac{dxdy}{|x-y|^{N+sp}} \\ &\lesssim \sum_{k\geq 1} \sum_{Q_k\in\mathscr{P}_k} g_k^p(Q_k)\, 2^{(sp-N)k} = \sum_{k\geq 1} 2^{spk} \|g_k\|_p^p = K. \end{aligned}$$

QED

4.10.0 Adapted trace theory

The following result is presented, with a sketch of proof, in Chiron [38, Section 3.2]. In the statement below, we impose an extra smoothness assumption on ζ that makes the arguments in [38] essentially complete.

Lemma 4.26. Let $0 < s < 1$. Let $k \geq 1$ be the smallest integer such that $s + k/p \geq 1$. Let $\zeta \in W^{s+k/p,p}((0,1)^{N+k}; \mathscr{E}) \cap C^\infty$. Then, for every $x'' \in (0,1)^k$,

$$\|\zeta(\cdot, x'')\|_{W^{s,p}((0,1)^N;\mathscr{E})} \lesssim \|\zeta\|_{W^{s+k/p,p}((0,1)^{N+k})}.$$

4.11.0 Homotopy and VMO

We consider two closed embedded Riemannian manifolds, $\mathscr{M} \subset \mathbb{R}^k$ and $\mathscr{N} \subset \mathbb{R}^\ell$.

Lemma 4.27. Let $u_j, u \in C(\mathscr{M}; \mathscr{N})$ be such that $u_j \to u$ in BMO and L^1. Then, for large j, u_j and u are homotopic in $C(\mathscr{M}; \mathscr{N})$.

Proof. Given a continuous map $v : \mathscr{M} \to \mathbb{R}^\ell$, set $v_0 := v$ and, for small $\varepsilon > 0$,

$$v_\varepsilon(x) := \fint_{B_\varepsilon(x)} v(y)\, dy.$$

By the proof of Lemma 0.6, we have

$$d(v_\varepsilon(x), \mathscr{N}) \leq \sup_{y\in\mathscr{M}} \fint_{B_\varepsilon(y)} \fint_{B_\varepsilon(y)} |v(w) - v(z)|\, dwdz. \tag{4.41}$$

Let $\delta > 0$ be such that the nearest projection on $\mathscr{N}$ is continuous on the δ-neighborhood $\mathscr{N}_\delta$ of $\mathscr{N}$. By (4.41) and the continuity of u, there exists some ε_0 such that

$$d(u_\varepsilon(x), \mathscr{N}) < \delta/2, \ \forall\, x \in \mathscr{M}, \ \forall\, \varepsilon \leq \varepsilon_0. \tag{4.42}$$

On the other hand, since $u_j \to u$ in BMO, there exists some j_0 such that

$$\sup_{y \in \mathscr{M}} \fint_{B_\varepsilon(y)} \fint_{B_\varepsilon(y)} |v(w) - v(z)|\, dwdz < \delta/2, \ \forall\, j \geq j_0, \ \forall\, y \in \mathscr{M}, \ \forall\, \varepsilon \leq \varepsilon_0. \tag{4.43}$$

Combining (4.41)–(4.43), we find that, for $j \geq j_0$ and $0 < \varepsilon \leq \varepsilon_0$, $(u_j)_\varepsilon$ and u_ε take values into $\mathscr{N}_\delta$.

Let $\Pi : \mathscr{N}_\delta \to \mathscr{N}$ be the nearest point projection. Then, clearly, the map

$$[0, \varepsilon_0] \ni \varepsilon \mapsto \Pi \circ [(u_j)_\varepsilon] \in C(\mathscr{M}; \mathscr{N})$$

is continuous, and therefore u_j and $\Pi \circ [(u_j)_{\varepsilon_0}]$ are homotopic. A similar conclusion holds for u. On the other hand, since $u_j \to u$ in L^1, we clearly have $\Pi \circ [(u_j)_{\varepsilon_0}] \to \Pi \circ [u_{\varepsilon_0}]$ uniformly, and thus, for large j, we have

$$u_j \sim \Pi \circ [(u_j)_{\varepsilon_0}] \sim \Pi \circ [u_{\varepsilon_0}] \sim u. \qquad \text{QED}$$

Corollary 4.28. Let s, p be such that $sp \geq \dim \mathscr{M}$. If $u_j, u \in C^\infty(\mathscr{M}; \mathscr{N})$ and $u_j \to u$ in $W^{s,p}$ then, for large j, u_j and u are homotopic in $C(\mathscr{M}; \mathscr{N})$.

Proof. It suffices to combine Lemma 4.27 with the continuous embeddings $W^{s,p} \hookrightarrow \mathrm{VMO}$ (Theorem 4.6) and $W^{s,p} \hookrightarrow L^1$. QED

5 Appendix # 2. Standard examples of maps in Sobolev spaces

Contents

5.1.0 Characteristic functions

Lemma 5.1. Let $0 < s < 1$ and $1 \leq p < \infty$ be such that $sp < 1$. Let $Q := (0,1)^N$ and $\Omega := (0,1)^{N-1} \times (-1,1)$. Then $\varphi_N := \chi_Q$ belongs to $W^{s,p}(\Omega)$.

Proof. In view of Theorem 4.1 (applied with $M = 1$), the conclusion of the lemma is equivalent to the straightforward fact that

$$\int_{-1}^{0}\int_{0}^{1} \frac{1}{|x-y|^{1+\sigma}}\, dxdy < \infty,\ \forall\, 0 < \sigma < 1. \tag{5.1}$$

Lemma 5.2. Let $s > 0$ and $1 \le p < \infty$. Let ω be a non-empty smooth relatively compact domain such that $\overline{\omega} \subset \Omega$. Then $\varphi := \chi_\omega \in W^{s,p}(\Omega) \iff sp < 1$.

Proof. If we straighten the coordinates around a point of $\partial\omega$, we find that, up to a constant factor, the left-hand side of (5.1), with $\sigma := sp$, is a lower bound for $|\varphi|^p_{W^{s,p}}$. When $sp \ge 1$, this implies that $\varphi \notin W^{s,p}$.

Assume next that $sp < 1$. Let (U_j) be a finite covering of $\overline{\Omega}$ such that, for each j, we have either $\varphi = 0$ in U_j, or there exists a bi-Lipschitz diffeomorphism Φ_j of $(0,1)^{N-1} \times (-1,1)$ onto U_j such that $\varphi \circ \Phi_j = \varphi_N$ (with φ_N as in the previous lemma). For any such j, we have $\varphi \circ \Phi_j \in W^{s,p}((0,1)^{N-1} \times (-1,1))$ (and thus $\varphi \in W^{s,p}$ in U_j). Finally, we have $\varphi \in W^{s,p}(U_j)$ for each j, which implies that $\varphi \in W^{s,p}(\Omega)$. QED

5.2.0 Power-type functions

Lemma 5.3. Let $\Omega := B_1(0) \subset \mathbb{R}^N$. Let $\alpha > 0$ and set $\varphi(x) := \dfrac{1}{|x|^\alpha}, \forall\, x \in \Omega \setminus \{0\}$. Then

$$\varphi \in W^{s,p}(\Omega) \iff (\alpha + s)p < N. \tag{5.2}$$

Proof. If s is an integer, the conclusion is clear. We may therefore assume that s is not an integer and that $\alpha < N$ (the latter condition is equivalent to $u \in L^1$). We rely on Theorem 4.1. Before going further, let us note that, since the specific φ we consider is smooth outside the origin, we have $\varphi \in W^{s,p}$ if and only if the double integral in (4.1) considered over the larger set $[-1,1] \times \Omega$ is convergent.

We use (4.1) with an integer M satisfying, in addition to $M > s$, the condition

$$(\alpha + M)p > N. \tag{5.3}$$

For simplicity, we drop the subscript j in e_j. Using the homogeneity of φ, we find that $\delta^M_{te}\varphi(x) = \dfrac{1}{|t|^\alpha}\delta^M_e\varphi(x/t)$. Hence,

$$I := \int_{-1}^{1}\int_{B_1(0)} \frac{|\delta^M_{te}\varphi(x)|^p}{t^{1+sp}}\, dxdt = 2\int_0^1 \int_{B_{1/t}(0)} t^{N-(\alpha+s)p-1}|\delta^M_e\varphi(y)|^p\, dydt. \tag{5.4}$$

" $\Longleftarrow$ " Assume that $(\alpha + s)p < N$. In this case, we show that $I < \infty$. Indeed, we have

$$I \le 2\int_0^1 t^{N-(\alpha+s)p-1}\, dt \int_{\mathbb{R}^N} |\delta^M_e\varphi(y)|^p\, dy.$$

Note that the assumption on α implies that $\int_0^1 t^{N-(\alpha+s)p-1}\,dt < \infty$. On the other hand, we have $\int_{\mathbb{R}^N} |\delta_e^M \varphi(y)|^p\,dy < \infty$, since $|\delta_e^M \varphi(y)| \sim \dfrac{1}{|y|^{\alpha+M}}$ at infinity and $|\delta_e^M \varphi|^p$ has integrable singularities. (These singularities, located at $y = 0, -e, ..., -Me$, behave like $|x|^{-\alpha p}$.) This proves " $\Longleftarrow$ ".

" $\Longrightarrow$ " We note that (5.4) implies that

$$I \geq 2 \int_0^1 \int_{B_1(0)} t^{N-(\alpha+s)p-1} |\delta_e^M \varphi(y)|^p\,dydt = \infty.$$

Here,we use the fact that $\displaystyle\int_{B_1(0)} |\delta_e^M \varphi(y)|^p\,dy > 0$ (since φ is not a polynomial in the e direction). QED

Remark 5.4. For a different approach, see the proof below of Lemma 0.1 b).

5.3.0 Homogeneous maps

By repeating the proof of Lemma 5.3, we obtain the following result.

Lemma 5.5.

a) Let $u(x) := \dfrac{x}{|x|}, \forall\, x \in \mathbb{R}^2 \setminus \{0\}$. Let $s > 0, 1 \leq p < \infty$. Then, $u \in W^{s,p}(B_1(0))$ if and only if $sp < 2$.

b) More generally, let $k \in \{1, \ldots, N\}$ and let

$$u : \mathbb{R}^N \to \mathbb{R}^k,\ u(x_1, \ldots, x_N) := \frac{(x_1, \ldots, x_k)}{|(x_1, \ldots, x_k)|},\ \forall\, x \in \mathbb{R}^N \setminus (\{0\} \times \mathbb{R}^{N-k}).$$

Then, $u \in W^{s,p}(B_1(0))$ if and only if $sp < k$.

c) Even more generally, let $v : \mathbb{S}^{k-1} \to \mathbb{R}^\ell$ be a smooth non constant map and

$$u : \mathbb{R}^N \to \mathbb{R}^k,\ u(x_1, \ldots, x_N) := v\left(\frac{(x_1, \ldots, x_k)}{|(x_1, \ldots, x_k)|}\right),\ \forall\, x \in \mathbb{R}^N \setminus (\{0\} \times \mathbb{R}^{N-k}).$$

Then, $u \in W^{s,p}(B_1(0))$ if and only if $sp < k$. In particular, Lemma 0.1 a) holds.

A more general result is the following

Lemma 5.6. Let U be a neighborhood of $\overline{\Omega}$ and let S be a d-dimensional submanifold of U, with $d \leq N-1$. Let $u \in C^\infty(\overline{U} \setminus S)$ satisfy

$$|D^\ell u(x)| \leq C_\ell [\text{dist}\,(x, S)]^{-\ell},\ \ell = 0, \ldots, N-d,\ \forall\, x \in \overline{U} \setminus S. \tag{5.5}$$

Then, $u_{|\Omega} \in W^{s,p}(\Omega)$ provided $sp < N - d$.

Proof. We consider only the more complicated case where $d \leq N-2$. Without loss of generality, we may assume that $sp > N-d-1$. Furthermore, we may also suppose that $s > N-d-1$, and thus $N-d-1 < s < N-d$ (since $sp < N-d$). Indeed, if $s \leq N-d-1$, fix any $\theta \in (0,1)$ such that

$$\frac{1}{p} < \theta < \frac{s}{N-d-1} \leq 1,$$

and let $r := s/\theta > N-d-1$, and $q := p\theta > 1$. From the Gagliardo–Nirenberg embedding $W^{r,q} \cap L^\infty \hookrightarrow W^{s,p}$ (see Corollary 4.8), and the fact that $u_{|\Omega} \in L^\infty$ (by (5.5) with $\ell = 0$), it suffices to prove that $u_{|\Omega} \in W^{r,q}$. Thus, as claimed, it suffices to prove the lemma under the extra assumption $s > N-d-1$, which implies that $N-d-1 < s < N-d$.

Let $m := N-d-1$ and write $s = m+\sigma$, with $0 < \sigma < 1$. The assumption $sp < N-d$ reads $(m+\sigma)p < N-d$.

We set $d(x) :=$ dist (x, S). Since S is d-dimensional, we find that

$$\int_\Omega \frac{1}{d^q(x)}\, dx < \infty, \ \forall\, 1 \leq q < N-d. \tag{5.6}$$

We next invoke the following well-known result.

Lemma 5.7. Let S be a d-submanifold of the open set $\Omega \subset \mathbb{R}^N$, with $d \leq N-2$. Let $u \in C^1(\Omega \setminus S)$ be such that $\nabla u \in L^1_{loc}(\Omega)$. Then $u \in W^{1,1}_{loc}(\Omega)$.

Proof of Lemma 5.6 continued. Combining (5.5) with (5.6) and Lemma 5.7, we find that $D^j u \in L^p(\Omega)$, $j = 0, \ldots, m$. It remains to check that

$$\int_\Omega \int_\Omega \frac{|D^m u(x) - D^m u(y)|^p}{|x-y|^{N+\sigma p}}\, dxdy < \infty. \tag{5.7}$$

For this purpose, we note that, with constants depending on u but not on x or y, we have:

$$|D^m u(x) - D^m u(y)| \leq C \max\left\{ \frac{|x-y|}{d^{m+1}(x)}, \frac{|x-y|}{d^{m+1}(y)} \right\}, \tag{5.8}$$

$$|D^m u(x) - D^m u(y)| \leq C \max\left\{ \frac{1}{d^m(x)}, \frac{1}{d^m(y)} \right\}. \tag{5.9}$$

We split

$$\iint \frac{|D^m u(x) - D^m u(y)|^p}{|x-y|^{N+\sigma p}}\, dx\, dy = 2(I+J),$$

where

$$I := \iint_{|x-y| < d(y) \leq d(x)} \ldots, \ J := \iint_{d(y) \leq \min\{d(x),\, |x-y|\}} \ldots$$

Using (5.8) (respectively (5.9)), in order to estimate I (respectively J), we find that

$$\begin{aligned}\iint \frac{|D^m u(x) - D^m u(y)|^p}{|x-y|^{N+\sigma p}}\,dx\,dy &\lesssim \iint_{|x-y|<d(y)\le d(x)} \frac{|x-y|^{p-N-\sigma p}}{d^{(m+1)p}(y)}\,dx\,dy \\ &\quad + \iint_{d(y)\le \min\{d(x),\,|x-y|\}} \frac{dx\,dy}{|x-y|^{N+\sigma p} d^{mp}(y)} \\ &\lesssim \int \frac{1}{d^{mp+\sigma p}(y)}\,dy < \infty,\end{aligned}$$

by (5.6) (since $mp + \sigma p < N - d$). QED

Remark 5.8. In Lemma 5.6, when $d \le N-2$, the assumptions on u can be weakened to $u \in C^{N-d}(\Omega \setminus S)$ and $|D^{N-d}u(x)| \le C[\text{dist}\,(x,S)]^{d-N}$.

Remark 5.9. The conclusion of Lemma 5.6 is still valid under the weaker assumption that S is a finite union of d-dimensional submanifolds. This has the following important (for us) consequence: the class $\mathscr{R}_\ell$ defined in (2.1) satisfies $\mathscr{R}_\ell \hookrightarrow W^{r,q}(\Omega;\mathscr{N}), \forall\, r, q$ such that $\lfloor rq \rfloor \le N-\ell-1$.

Remark 5.10. When $s = 1$, the above result (or versions of it) can be proved using more elementary arguments; see, e. g., [30, Lemma 2.15] for the following version of Lemma 5.6.

Lemma 5.11. Let $1 \le r < \infty$. Assume that $N \ge 2$ and let $U \subset \mathbb{R}^N$ be an open set. Let K be a closed subset of U such that $\mathscr{H}^{N-1}(K) = 0$. Let $u \in W^{1,1}_{loc}(U \setminus K)$ be such that $\int_{U\setminus K} |\nabla u|^r < \infty$. Then $u \in W^{1,r}_{loc}(U)$ and the Sobolev gradient of u is the Sobolev gradient of $u_{|U\setminus K}$.

Proof of Lemma 0.1 b).

" $\Longleftarrow$ " *Case* $s < 1$. By Lemma 5.3, we have $\psi \in W^{1,sp+\varepsilon}$, and thus also $u \in W^{1,sp+\varepsilon}$, for small $\varepsilon > 0$. By Corollary 4.8, we find that $u \in W^{s,p}$.

Case $s = 1$. The conclusion follows directly from Lemma 5.3.

Case $s > 1$. By Lemma 5.3, we have $\varphi \in W^{s,p} \cap W^{1,sp}$. We conclude *via* Theorem 4.15.

" $\Longrightarrow$ " *Case* $s = 1$. We have $|\nabla u| = |\nabla \varphi| \sim |x|^{\alpha-1}$ in $\Omega \setminus \{0\}$, and the conclusion is clear.

Case $s > 1$. By Corollary 4.8, we have $u \in W^{1,sp}$, and thus (as in the case $s = 1$) we find that $(\alpha+1)sp < N$. On the other hand, by differentiating the equality $u = e^{\imath\varphi}$, we find that $D\varphi = -\imath\bar{u}Du$. Lemma 4.18 implies that $D\varphi \in W^{s-1,p}$, and thus $\varphi \in W^{s,p}$. Lemma 5.3 implies the second condition, $(\alpha+s)p < N$.

Case $0 < s < 1$. This case is more involved. Setting

$$I(r_1, r_2, \beta) := \int_{S(0,r_1)} \int_{S(0,r_2)} \frac{d\mathscr{H}(x) d\mathscr{H}(y)}{|x-y|^\beta}, \tag{5.10}$$

we have, with $\beta := N + sp$, using the changes of variables $t = r_1^{-\alpha}$, $\tau = r_2^{-\alpha}$, Lemma 5.12 below, and the fact that $N - \beta + p > 0$,

$$|u|^p_{W^{s,p}} = 2\int_0^1 \int_0^{r_1} \left|e^{\imath/r_2^\alpha} - e^{\imath/r_1^\alpha}\right|^p I(r_1, r_2, \beta)\,dr_2 dr_1$$

$$
\begin{aligned}
&\sim \int_1^\infty \int_t^\infty |e^{\imath\tau} - e^{\imath t}|^p I(t^{-1/\alpha}, \tau^{-1/\alpha}, \beta) \tau^{-1/\alpha-1} t^{-1/\alpha-1}\, d\tau dt \\
&\gtrsim \int_1^\infty \int_t^{t+1} |e^{\imath\tau} - e^{\imath t}|^p t^{-(N-1)/\alpha} (t^{-1/\alpha} - \tau^{-1/\alpha})^{N-\beta-1} \tau^{-1/\alpha-1} t^{-1/\alpha-1}\, d\tau dt \\
&\sim \int_1^\infty \int_t^{t+1} (\tau - t)^p t^{-2-(N+1)/\alpha} (t^{-1/\alpha} - \tau^{-1/\alpha})^{N-\beta-1}\, d\tau dt \\
&\gtrsim \int_1^\infty \int_t^{t+1} (\tau - t)^p t^{-2-(N+1)/\alpha} (\tau - t)^{N-\beta-1} t^{-(N-\beta-1)(1/\alpha+1)}\, d\tau dt \\
&= \int_1^\infty \int_t^{t+1} (\tau - t)^{N-\beta+p-1} t^{-(N-\beta+1)-(2N-\beta)/\alpha}\, d\tau dt \\
&\gtrsim \int_1^\infty t^{-(N-\beta+1)-(2N-\beta)/\alpha}\, dt.
\end{aligned}
$$

Therefore, the exponent of t in the last integral above has to be < -1, and this amounts to $(\alpha+1)sp < N$. QED

Lemma 5.12. Fix $\beta > 0$. For $r_2 < r_1 \le 2r_1$, we have

$$
I(r_1, r_2, \beta) \gtrsim r_1^{N-1} (r_1 - r_2)^{N-\beta-1}.
$$

Proof. Write $r_1 = (1+t)r_2$, with $0 < t \le 1$. By scaling and invariance with respect to isometries, we have

$$
I(r_1, r_2, \beta) = \sigma_N r_1^{N-1} r_2^{N-\beta-1} J(t, \beta), \tag{5.11}
$$

where

$$
J(t, \beta) := \int_{\mathbb{S}^{N-1}} \frac{d\mathscr{H}^{N-1}(x)}{|x - (1+t)e_N|^\beta}.
$$

Write $x = (x', x_N)$. We have, using the facts that $t \le 1$ and $1 - \sqrt{1-s^2} \le t$ when $0 \le s \le t \le 1$, and the change of variable $x' = ty'$,

$$
\begin{aligned}
J(t, \beta) &\ge \int_{|x'|\le 1} \frac{dx'}{|(x', 1 + t - \sqrt{1 - x^2})|^\beta} \ge \int_{|x'|\le t} \frac{dx'}{|(x', 1 + t - \sqrt{1 - x^2})|^\beta} \\
&\ge \int_{|x'|\le t} \frac{dx'}{|(x', 2t)|^\beta} = t^{N-\beta-1} \int_{|y'|\le 1} \frac{dy'}{|(y', 2)|^\beta},
\end{aligned}
$$

so that, by (5.11),

$$
I(r_1, r_2, \beta) \gtrsim r_1^{N-1} r_2^{N-\beta-1} t^{N-\beta-1} = r_1^{N-1} (r_1 - r_2)^{N-\beta-1}.
$$
QED

5.4.0 Gluing maps

Lemma 5.13. Let $\pi : \mathscr{E} \to \mathscr{N}$ be a covering. Fix some $z \in \mathscr{N}$ and let J be at most countable such that $\pi^{-1}(\{z\}) = \{t_j;\, j \in J\}$. Then there exist: points $x_j \in \Omega$, radii $r_j > 0$, $j \in J$, and maps $\zeta_j : B_{2r_j}(x_j) \to \mathscr{E}$ such that:

(i) The balls $B_{3r_j(x_j)}$ are mutually disjoint and contained in Ω.

(ii) $\zeta_j \in C^\infty(\overline{B}_{2r_j}(x_j) \setminus \{x_j\})$.

(iii) $\zeta_j = t_j$ in $\overline{B}_{2r_j}(x_j) \setminus B_{r_j}(x_j)$.

(iv) $\zeta_j \notin W^{s,p}(B_{2r_j}(x_j))$.

(v) The map $u := \begin{cases} \pi \circ \zeta_j, & \text{in } B_{2r_j}(x_j) \\ z, & \text{in } \Omega \setminus \cup_j B_{2r_j}(x_j) \end{cases}$ belongs to $W^{s,p}$.

Proof. We consider only the non-compact case where $J = \{1, 2, \ldots\}$. We will construct inductively a sequence of maps (u_j) such that its weak limits has all the required properties. Start with $u_0 :\equiv z$. Assume that we have constructed ζ_k satisfying the above properties for $J = \{1, \ldots, k\}$, we construct ζ_{k+1}. Consider a point $x_{k+1} \in \Omega \setminus \cup_{j \le k} \overline{B}_{2r_j}(x_j)$. Let $\gamma = \gamma_{t_{k+1}}$ be as in Lemma 7.3. We fix $\lambda : [0, \infty) \to [0, \infty)$ such that $\lambda(\theta) = 0$ if $\theta \le 1$ and $\lambda(\theta) = 1$ if $\theta \ge 2$. Let α be as in (1.7). Let $0 < \varepsilon < 1$ be be fixed later, and set

$$\psi_\varepsilon(x) := \gamma \circ \lambda((\varepsilon/|x - x_{k+1}|)^\alpha), \forall\, x \in B_{2\varepsilon}(x_{k+1}),$$

extended with the value t_{k+1} outside $B_{2\varepsilon}(x_{k+1})$. Clearly, ψ_ε is smooth in $\mathbb{R}^N \setminus \{x_{k+1}\}$. If we set $v_\varepsilon := \pi \circ \psi_\varepsilon$, then $v_\varepsilon - z \in C^\infty(\mathbb{R}^N \setminus \{x_{k+1}\})$ and is supported in $B_\varepsilon(x_{k+1})$.

We claim the following:

$$\psi_\varepsilon \notin W^{s,p}(B_{2\varepsilon}(x_{k+1})),\ \forall\, \varepsilon > 0, \tag{5.12}$$

$$\|v_\varepsilon - z\|_{W^{s,p}} \to 0 \text{ as } \varepsilon \to 0. \tag{5.13}$$

Taking temporarily for granted the two above properties, we conclude as follows. By (5.13) and Lemma 4.20, for sufficiently small ε we have $\|u_k + v_\varepsilon - z\|_{W^{s,p}} \le \|u_k\|_{W^{s,p}} + 2^{-k}$. We complete the proof *via* a straightforward inductive process, by setting $u_{k+1} := u_k + v_\varepsilon - z$.

Proof of (5.12). Assume, for simplicity, that $x_{k+1} = 0$. If $|x| \le 2^{-1/\alpha}\varepsilon$, then $\lambda((\varepsilon/|x|)^\alpha) = 1$. Therefore, with $r := 2^{-1/\alpha}\varepsilon$, we have

$$\begin{aligned} |\psi_\varepsilon|^p_{W^{s,p}} &\ge \int_{B_r(0)} \int_{B_r(0)} \frac{d_{\mathscr{E}}(\gamma((\varepsilon/|x|)^\alpha), \gamma((\varepsilon/|y|)^\alpha))^p}{|x - y|^{N+sp}}\, dxdy \\ &= \int_{B_r(0)} \int_{B_r(0)} \frac{|(\varepsilon/|x|)^\alpha - (\varepsilon/|y|)^\alpha)|^p}{|x - y|^{N+sp}}\, dxdy = \infty, \end{aligned}$$

where we have used successively Lemma 7.3, (1.7), and Lemma 5.3.

Proof of (5.13). By construction, γ is 1-Lipschitz. We find that $|\nabla \psi_\varepsilon(x)| \lesssim \varepsilon^\alpha/|x|^{\alpha+1}$, and therefore $\|\nabla \psi_\varepsilon\|_{sp} \to 0$ as $\varepsilon \to 0$. Therefore, $\pi \circ \psi_\varepsilon - z \to 0$ in $W^{1,sp}(\mathbb{R}^N)$ as $\varepsilon \to 0$. Since the ψ_ε's are uniformly bounded, we conclude *via* the Gagliardo-Nirenberg inequality (4.12). QED

5.5.0 The dipole construction

We describe and analyze here the dipole construction of Brezis, Coron, and Lieb [25], in a form and functional setting adapted to our purposes.

Lemma 5.14. (Dipole construction, unscaled version) Let:

(i) $N \geq 2$.

(ii) $a \in \mathbb{S}^{N-1}$.

(iii) $f \in C^\infty([0,1];[0,1])$ be such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(\theta) > 0$, $\forall\, \theta \in (0,1)$.

(iv) $v \in C^\infty(\mathbb{R}^{N-1};\mathbb{S}^{N-1})$ be such that $v(x) = a$ if $|x| \geq 1/2$ and $\deg v = 1$.

Set, for $0 < \varepsilon \leq 1$,

$$u_\varepsilon(x',\theta) := \begin{cases} v\left(\dfrac{x'}{\varepsilon f(\theta)}\right), & \text{if } \theta \in (0,1) \\ a, & \text{if } \theta \notin (0,1) \end{cases}.$$

Then

$$u_\varepsilon \in \mathscr{R}_0,\ \forall\, \varepsilon, \tag{5.14}$$

$$|u_\varepsilon|^p_{W^{s,p}} \lesssim 1,\ \forall\, 0 < s < \infty,\ 1 \leq p < \infty \text{ such that } sp = N-1,\ \forall\, \varepsilon, \tag{5.15}$$

$$Ju_\varepsilon = C_N(\delta_0 - \delta_{e_N}) \text{ in } \mathscr{D}'(\mathbb{R}^N),\ \forall\, \varepsilon, \tag{5.16}$$

$$\operatorname{supp}(u_\varepsilon - a) \subset \{(x',\theta) \in \mathbb{R}^N; 0 \leq \theta \leq 1, |x'| \leq \varepsilon f(\theta)\},\ \forall\, \varepsilon. \tag{5.17}$$

Proof when $N \geq 3$. We present the proof when $N \geq 3$. As we will see, it relies on a Gagliardo-Nirenberg embedding that fails when $N = 2$. For a full proof (including the case $N = 2$), see [14, 18].

Step 1. Preliminary remarks. Property (5.17) is clear. It is also clear that u_ε is smooth except at 0 and e_N. Moreover, we claim that $\deg(u_\varepsilon, 0) = 1$ (and, similarly, $\deg(u_\varepsilon, e_N) = -1$). Indeed, let $\delta > 0$ be small. By a homotopy argument, $\deg(u_\varepsilon, S_\delta(0)) = \deg(u_\varepsilon, \{x \in \mathbb{R}^N; \|x\|_\infty = \delta\})$, and the latter number is 1, by the definitions of v and u_ε.

The claim, combined with (3.2) and the fact that, as we will see, $u_\varepsilon \in \mathscr{R}_0$, implies (5.16).

Step 2. The main estimate and conclusion. Set $V_\varepsilon := \{(x',\theta);\ 0 < \theta < 1, |x'| \leq \varepsilon f(\theta)\}$. By a tedious calculation, using the fact that $|x'/(\varepsilon f(\theta))| \leq 1$ when $x \in V_\varepsilon$, we find that

$$\left|\partial^\alpha_{x'} \partial^\beta_\theta u_\varepsilon(x',\theta)\right| \lesssim \frac{1}{\varepsilon^{|\alpha|} f^{|\alpha|+\beta}(\theta)} \leq \frac{1}{\varepsilon^{|\alpha|+\beta} f^{|\alpha|+\beta}(\theta)},\ \forall\, (x',\theta) \in V_\varepsilon. \tag{5.18}$$

On the one hand, using (5.18) and the assumptions on f, we find that $u_\varepsilon \in \mathscr{R}_0$. On the other hand, (5.18) implies that, for $1 \leq k \leq N-1$,

$$\int_{\mathbb{R}^N} |D^k u_\varepsilon(x)|\, dx \lesssim \int_{V_\varepsilon} \frac{1}{\varepsilon^k f^k(\theta)}\, dx' d\theta \sim \varepsilon^{N-1-k} \int_0^1 f^{N-1-k}(\theta)\, d\theta \lesssim 1. \tag{5.19}$$

In particular, (5.15) holds when $s = N-1$. By the Gagliardo-Nirenberg inequality (Corollary 4.8) and the fact that $N \geq 3$, we find that (5.15) still holds in the full range given in (5.15). QED

By scaling, Lemma 5.14 implies the following result.

Lemma 5.15. (Dipole construction, scaled version) Let N, a, f, v, s, p be as above. Let $A, B \in \mathbb{R}^N$. Set $\xi := (B-A)/|B-A|$ and $H := \xi^\perp$, oriented by an orthonormal basis $(e_1, \dots, e_{N-1})$ such that $(e_1, \dots, e_{N-1}, \xi)$ is a direct basis of $\mathbb{R}^N$. Let R be an orientation preserving linear isometry from H to $\mathbb{R}^{N-1}$. Write a point x in $\mathbb{R}^N$ as $x = x' + \theta\xi$, with $x' \in H$ and $\theta \in \mathbb{R}$. Let $L := |B-A|$ and set, for $0 < \varepsilon \leq L$,

$$u_\varepsilon(x'+\theta\xi) := \begin{cases} v\left(\dfrac{Rx'}{L\varepsilon f(\theta/L)}\right), & \text{if } \theta \in (0,L) \\ a, & \text{if } \theta \notin (0,L) \end{cases}.$$

Then

$$u_\varepsilon \in \mathscr{R}_0, \ \forall\, \varepsilon, \tag{5.20}$$

$$|u_\varepsilon|^p_{W^{s,p}} \lesssim L, \ \forall\, 0 < s < \infty, \ 1 \leq p < \infty \text{ such that } sp = N-1, \ \forall\, \varepsilon, \tag{5.21}$$

$$Ju_\varepsilon = C_N(\delta_A - \delta_B) \text{ in } \mathscr{D}'(\mathbb{R}^N), \ \forall\, \varepsilon, \tag{5.22}$$

$$\operatorname{supp}(u_\varepsilon - a) \subset \{(x'+\theta\xi) \in \mathbb{R}^N; 0 \leq \theta \leq L, |x'| \leq L\varepsilon f(\theta/L)\}, \ \forall\, \varepsilon. \tag{5.23}$$

We next explain how to "make room" for inserting a dipole into an already existent map $u : \Omega \to \mathbb{S}^{N-1}$.

Lemma 5.16. (Making room when u is locally smooth) Let $N \geq 3$. Let $0 < s < N-1$ and $1 < p < \infty$ be such that $sp = N-1$. Let $a \in \mathbb{S}^{N-1}$. Let γ be a smooth simple compact curve in $\mathbb{R}^N$. Let $u : \mathbb{R}^N \to \mathbb{S}^{N-1}$ be smooth in an open neighborhood U of γ. Fix $\delta, \mu > 0$. Then, for small $\varepsilon > 0$, there exists a map $\widetilde{u} : \mathbb{R}^N \to \mathbb{S}^{N-1}$ such that:

a) $\widetilde{u}(x) = u(x)$ if $d(x,\gamma) \geq \delta$.

b) $\widetilde{u}(x) = a$ if $d(x,\gamma) \leq \varepsilon^3$.

c) $\widetilde{u} \in C^\infty(U)$.

d) $|\widetilde{u} - u|_{W^{s,p}} \leq \mu$.

Similarly in the neighborhood of a finite union of smooth simple compact curves.

Proof. Step 1. Making u constant near its endpoints. Assume, for simplicity, that the origin is an endpoint of γ. Let $\varphi \in C^\infty(\mathbb{R}; [0,1])$ be such $\varphi(\theta) = 0$ if $\theta \leq 1/2$ and $\varphi(\theta) = 1$ if $\theta \geq 1$. Letting $\widetilde{u}(x) := u(\varphi(|x|/\varepsilon)x)$, it is easy to see (using the assumption that u is smooth) that, for small ε, $\widetilde{u}$ satisfies a)–c) with γ replaced by 0, and, in addition, $\left\|D^{N-1}(\widetilde{u}-u)\right\|_1 \to 0$ as $\varepsilon \to 0$. By the Gagliardo-Nirenberg inequality Corollary 4.8 (recall that $N \geq 3$ and that u is bounded), we find that d) holds as well.

Therefore, in what follows, we may assume that u is constant near the endpoints of γ.

Step 2. Construction of $\widetilde{u}$ and conclusion. Since $N \geq 3$, the set $u(\gamma) \cup \{a\}$ is contained in the interior of a closed spherical cap $\Sigma \neq \mathbb{S}^{N-1}$. By a standard argument, there exists a smooth map $H : \Sigma \times \mathbb{R} \to \Sigma$ such that

(i) $H(x,\theta) = a, \forall\, x \in \Sigma, \forall\, \theta \le -3.$

(ii) $H(x,\theta) = x, \forall\, x \in \Sigma, \forall\, \theta \ge -2.$

(iii) $H(a,\theta) = a, \forall\, \theta \in \mathbb{R}.$

Set, for small ε,

$$\widetilde{u}(x) = \begin{cases} u(x), & \text{if } d(x,\gamma) \ge \varepsilon^2 \\ H(u(x), \ln(d(x,\gamma))/|\ln \varepsilon|), & \text{if } d(x,\gamma) \le \varepsilon \end{cases}. \tag{5.24}$$

Clearly: (j) the definition is consistent when $\varepsilon^2 \le d(x,\gamma) \le \varepsilon$, thanks to the property (ii) of H; (jj) a)–c) hold. Noting that $\widetilde{u} - u \to 0$ in L^q as $\varepsilon \to 0, \forall\, q < \infty$, in order to verify d), it suffices to prove that, with $V_\varepsilon := \{x \in \mathbb{R}^N;\ \varepsilon^3 \le d(x,\gamma) \le \varepsilon^2\}$, we have

$$\|\widetilde{u}\|_{W^{N-1,1}(V_\varepsilon)} \le C(u) \text{ (i. e., } \left\|D^k \widetilde{u}\right\|_{L^1(V_\varepsilon)} \le C(u),\ \forall\, 0 < \varepsilon \le 1,\ \forall\, 1 \le k \le N-1), \tag{5.25}$$

$$\|\widetilde{u}\|_{W^{j,(N-1)/j}(V_\varepsilon)} \to 0 \text{ (i. e., } \left\|D^k \widetilde{u}\right\|_{L^{(N-1)/j}(V_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0,\ \forall\, 1 \le k \le j \le N-1), \tag{5.26}$$

then conclude *via* Gagliardo-Nirenberg. In order to prove (5.25)–(5.26), we note that: (i) thanks to the assumption that u is constant near the endpoints of A and B of γ, we may replace V_ε with the smaller set

$$\widetilde{V}_\varepsilon := \{x \in V_\varepsilon;\ |x - A| \ge \lambda,\ |x - B| \ge \lambda\},$$

for a fixed small constant λ; (ii) in $\widetilde{V}_\varepsilon$, we have $|D^k(d(x,\gamma))| \lesssim [d(x,\gamma)]^{1-k}, \forall\, k \ge 0$. Combining these facts with the definition (5.24), we find that

$$|D^k \widetilde{u}(x)| \lesssim \frac{1}{|\ln \varepsilon| [d(x,\gamma)]^k}\ \forall\, x \in \widetilde{V}_\varepsilon,\ \forall\, k \ge 1. \tag{5.27}$$

Integrating (5.27), we find, with the help of the coarea formula, that

$$\left\|D^k \widetilde{u}\right\|^q_{L^q(\widetilde{V}_\varepsilon)} \lesssim \begin{cases} \varepsilon^{N-1-qk}/|\ln \varepsilon|^q, & \text{if } qk < N-1 \\ 1/|\ln \varepsilon|^{q-1}, & \text{if } qk = N-1 \end{cases},$$

whence (5.25)–(5.26). QED

Lemma 5.17. (Making room near a singular endpoint (1)) Let $u : \mathbb{R}^N \to \mathbb{S}^{N-1}$. Let:

(i) $N \ge 3$.

(ii) $0 < s < N-1, 1 < p < \infty$ such that $sp = N-1$.

Let γ be a smooth simple compact curve in $\mathbb{R}^N$. Assume that 0 is one of the endpoints of γ, and that $\gamma'(0) = e_N$.

Set, for $\alpha, \beta > 0$,

$$W_{\alpha,\beta} := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R};\ 0 \le x_N \le \alpha, |x'| \le \beta x_N\}. \tag{5.28}$$

Assume that:

(i) There exist $\alpha > 0, \beta > 0$, and $a \in \mathbb{S}^{N-1}$ such that

$$u(x) = a, \ \forall\, x \in W_{\alpha,\beta}.$$

(ii) u is smooth in an open neighborhood $U \subset \mathbb{R}^N \setminus \{0\}$ of $\gamma \setminus \{0\}$.

Fix $\delta, \mu > 0$. Then, for small $\varepsilon > 0$, there exists a map $\widetilde{u} : \mathbb{R}^N \to \mathbb{S}^{N-1}$ such that:

a) $\widetilde{u}(x) = u(x)$ if $d(x, \gamma) \geq \delta$.

b) $\widetilde{u}(x) = a$ if $x \in W_{\alpha/2,\beta/2}$.

c) $\widetilde{u}(x) = a$ if $x \notin W_{\alpha/2,\beta/2}$ and $d(x, \gamma) \leq \varepsilon^2$.

d) $\widetilde{u} \in C^\infty(U)$.

e) $|\widetilde{u} - u|_{W^{s,p}} \leq \mu$.

Sketch of proof. The proof is essentially the same as the one of Lemma 5.16, with, as additional ingredient, the use of property (iii) of the homotopy H. The definition (5.24) has to be modified to

$$\widetilde{u}(x) = \begin{cases} u(x), & \text{if } d(x,\gamma) \geq \varepsilon^2 \text{ and } x \notin W_{\alpha/3,\beta/3} \\ H(u(x), \ln(d(x,\gamma))/|\ln \varepsilon|), & \text{if } d(x,\gamma) \leq \varepsilon \text{ and } x \notin W_{\alpha/3,\beta/3} \\ a, & \text{if } x \in W_{\alpha/2,\beta/2} \end{cases}.$$

Details are left to the reader. QED

Lemma 5.18. (Making room near a singular endpoint (2)) Let:

(i) $N \geq 3$ and $a \in \mathbb{S}^{N-1}$.

(ii) $0 < s \leq N - 1, 1 \leq p < \infty$ such that $sp = N - 1$.

(iii) $\mu > 0$.

(iv) $u : \mathbb{R}^N \to \mathbb{S}^{N-1}$.

Assume that, in the unit ball B, u belongs to $\mathscr{R}_0$. Let $0 < \delta < 1$ be such that u is smooth in $\overline{B}_\delta(0) \setminus \{0\}$. Then there exists some map $\widetilde{u} : \mathbb{R}^N \to \mathbb{S}^{N-1}$ such that:

a) $\widetilde{u} = u$ in $\mathbb{R}^N \setminus \overline{B}_\delta(0)$.

b) $\widetilde{u}$ is smooth in $\overline{B}_\delta(0) \setminus \{0\}$.

c) $\deg(\widetilde{u}, 0) = \deg(u, 0)$.

d) There exist $\alpha, \beta > 0$ such that $u(x) = a, \forall\, x \in W_{\alpha,\beta}$ (with $W_{\alpha,\beta}$ as in (5.28)).

e) $\|\widetilde{u} - u\|^p_{W^{s,p}} \leq \mu$.

Similarly for $\mathscr{N}$-valued maps, provided we replace conclusion c) with "$\widetilde{u}$ and u are homotopic on small spheres around the origin".

Proof. We write points in $\mathbb{R}^N \setminus \{0\}$ in the form $x = r\sigma$, $r > 0$, $\sigma \in \mathbb{S}^{N-1}$. Consider a map $w \in C^\infty(\mathbb{S}^{N-1}; \mathbb{S}^{N-1})$ of degree $\deg(u, 0)$ and such that $w = a$ near the North Pole e_N. Let $H \in C^\infty(\mathbb{S}^{N-1} \times (-\infty, \delta]; \mathbb{S}^{N-1})$ be such that:

(i) $H(\sigma, r) = u(r\sigma)$ if $\delta/2 \leq r \leq \delta$.

(ii) $H(\sigma, r) = w(\sigma)$ if $r \leq \delta/4$.

Let, for small ε,

$$\widetilde{u}(x) = \widetilde{u}(r\sigma) = \begin{cases} u(x), & \text{if } r \geq \varepsilon\delta/2 \\ H(\sigma, r/\varepsilon), & \text{if } r \leq \varepsilon\delta \end{cases}.$$

Note that: (j) the definition is consistent when $\varepsilon\delta/2 \leq |x| \leq \varepsilon\delta$; (jj) a)–d) hold. Next, we note that

$$|D^k(\widetilde{u} - u)(x)| \lesssim \frac{1}{|x|^k}, \text{ when } k \geq 1 \text{ and } |x| \leq \varepsilon\delta/2. \tag{5.29}$$

Integrating (5.29), we find that $\|\widetilde{u} - u\|_{W^{N-1,1}} \to 0$ as $\varepsilon \to 0$. We conclude *via* the Gagliardo-Nirenberg inequality. QED

A similar argument leads to the following

Lemma 5.19. Let:

(i) $\mathscr{C}_N$ a finite grid of cubes in $\mathbb{R}^N$.

(ii) $0 < s \leq 1$, $1 \leq p < \infty$ such that $sp < N$.

(iii) $g : \mathscr{C}_{N-1} \to \mathscr{N}$ a Lipschitz function.

Assume that, for each cube $C \in \mathscr{C}_N$, $g_{|\partial C} : \partial C \to \mathscr{N}$ is null homotopic. Let $f : \mathscr{C}_N \to \mathscr{N}$ be the $(N-1)$-homogenous extension of g. Then there exists a sequence $(f_i) \subset \mathrm{Lip}(\mathscr{C}_N; \mathscr{N})$ such that $f_i \to f$ in $W^{s,p}$.

Sketch of proof. We work with $|x| := \|x\|_\infty$. Let δ be the size of the cubes in $\mathscr{C}_N$. Fix some point $a \in \mathscr{N}$. Consider, for each cube $C_j \in \mathscr{C}_N$ of center X_j, a homotopy $H_j \in \mathrm{Lip}(S_1(0) \times (-\infty, \delta]; \mathscr{N})$ such that:

(i) $H_j(\sigma, r) = g(X_j + \delta\sigma/2)$ if $\delta/2 \leq r \leq \delta$.

(ii) $H_j(\sigma, r) = a$ if $r \leq \delta/4$.

Let, with ε_j small and $x \in C_j \in \mathscr{C}_N$,

$$\widetilde{f}(X_j + r\sigma) = \begin{cases} g(X_j + \delta\sigma/2), & \text{if } \varepsilon\delta/2 \leq r \leq \delta \\ H_j(\sigma, r/\varepsilon), & \text{if } r \leq \varepsilon\delta \end{cases}.$$

The fact that, for fixed arbitrarily small $\mu > 0$, may choose the ε_j's such that $\|\widetilde{f} - f\|_{W^{s,p}} \leq \mu$ follows from the multi-sequences Brezis-Lieb Lemma 4.21. QED

6 Appendix # 3. More on homogeneous maps

For the next result, we use the notation in Section 2.2. Let $\Psi = \Psi_{j,t,\varepsilon} : \mathbb{R}^N \to \mathscr{C}_j = \mathscr{C}_{j,t,\varepsilon}$ be the projection on the j-skeleton of size 2ε of $\mathbb{R}^N$ obtained from $\mathscr{C}_{N,t,\varepsilon}$. Set $v := t + (\varepsilon, \dots, \varepsilon)$.

Lemma 6.1. The mapping $\Psi : \mathbb{R}^N \setminus \mathscr{C}_{N-j-1,v,\varepsilon} \to \mathscr{C}_{j,t,\varepsilon}$ is locally Lipschitz, and satisfies, with C independent of ε and t,

$$|\nabla\Psi(x)| \leq \frac{C\varepsilon}{d(x, \mathscr{C}_{N-j-1,v,\varepsilon})} \text{ for a. e. } x \in \mathbb{R}^N. \tag{6.1}$$

Sketch of proof. By scaling, we may assume that $\varepsilon = 1$ and $t = 0$. In this case, $\mathscr{C} := \mathscr{C}_{N-j-1,v,1}$ is given by

$$\mathscr{C} = \{x \in \mathbb{R}^N;\ \text{at least } (j+1) \text{ coordinates of } x \text{ belong to } 2\mathbb{Z}^N\}.$$

For further use, let us also note that, if $x^N \in Q_1$, with $|x_1| \geq \cdots |x_{N-j}| \geq |x_m|, \forall\, m > N-j$, then

$$d(x^N + 2k, \mathscr{C}) = \min\{|x_m|;\ m \geq N-j\} \leq |x_{N-j}|,\ \forall\, k \in 2\mathbb{Z}^N. \tag{6.2}$$

Let $\sigma \in S_{N-j,N} := \{f : \{1, \dots, N-j\} \to \{1, \dots, N\},\ f \text{ into}\}$ and $\tau \in \{-1, 1\}^{N-j}$. Set

$$S_{\sigma,\tau} := \{x^N \in Q_1; |x_{\sigma(1)}| \geq \cdots \geq |x_{\sigma(N-j)}| \geq |x_m|,\ \forall\, m \notin \sigma(\{1, \dots, N-j\}),\\ \text{and } \tau_m x_m \geq 0,\ \forall\, 1 \leq m \leq N-j\}.$$

If $x^N \in S_{\mathrm{id},\tau} \setminus \mathscr{C}$ and $k \in \mathbb{Z}^N$, then

$$\Psi(x^N + 2k) = 2k + (\tau_1, \dots, \tau_{N-j}, x_{N-j+1}/x_{N-j}, \dots, x_N/x_{N-j}), \tag{6.3}$$

and a similar formula holds for a general $\sigma \in S_{N-j,N}$.

Using (6.3) and its analogues, one can easily prove that Ψ is continuous on $\mathbb{R}^N \setminus \mathscr{C}$. On the other hand, the sets $S_{\sigma,\tau}$ form a polyhedral a.e.-partition of Q_1 and, in the interior of $S_{\mathrm{id},\tau}$ and for $k \in \mathbb{Z}^N$ we have (using (6.3) and (6.2))

$$|\nabla\Psi(x^N + 2k)| \lesssim \sum_{m>N-j} \left(\frac{1}{|x_{N-j}|} + \frac{|x_m|}{(x_{N-j})^2} \right) \lesssim \frac{1}{|x_{N-j}|} \leq \frac{1}{d(x^N + 2k, \mathscr{C})}$$

(and similar formulas hold in each $S_{\sigma,\tau}$). Combining (6.3) and its analogues, we obtain (6.1). QED

For the next result, we are in the context of Proposition 2.7, and we use the notation there.

Lemma 6.2. Let $1 \leq p < j+1 \leq N$. Assume that $g \in W^{1,p}(\mathscr{C}_j; \mathbb{R}^\ell)$. Then we have

$$\|g_j\|^p_{L^p(\mathscr{C})} \leq C\|g\|^p_{L^p(\mathscr{C}_j)}, \tag{6.4}$$

$$|g_j|^p_{W^{1,p}(\mathscr{C})} \leq C|g|^p_{W^{1,p}(\mathscr{C}_j)}. \tag{6.5}$$

Equivalently, the map $W^{1,p}(\mathscr{C}_j) \ni g \mapsto g_j \in W^{1,p}(\mathscr{C})$ is continuous.

Proof. By a straightforward induction argument, it suffices to prove that (with the notation at the beginning of the Section 2.2) the map $h := H_{j+1}(g)$ satisfies

$$\|h\|^p_{L^p(\mathscr{C}_{j+1})} \leq C\|g\|^p_{L^p(\mathscr{C}_j)}, \tag{6.6}$$

$$|h|^p_{W^{1,p}(\mathscr{C}_{j+1})} \leq C|g|^p_{W^{1,p}(\mathscr{C}_j)}. \tag{6.7}$$

The validity of (6.6) on each cube $\mathfrak{C} \in \mathscr{C}_{j+1}$, and thus on $\mathscr{C}_{j+1}$, is clear (here, we use $p < j+1$). We next check (6.7) first "cube by cube", next globally. Fix first a cube $\mathfrak{C} \in \mathscr{C}_{j+1}$, of center $0_{\mathfrak{C}}$. When $g_{|\partial\mathfrak{C}}$ is Lipschitz, we find that h is locally Lipschitz in $\mathfrak{C} \setminus \{0_{\mathfrak{C}}\}$, and $|\nabla h(x)| \lesssim 1/d(x, 0_{\mathfrak{C}})$. Moreover, we have $|\nabla h(x)| \sim |\nabla g((x-0_{\mathfrak{C}})/(2|x-0_{\mathfrak{C}}|))$, $\forall\, x \in \mathfrak{C}$. We find that $h \in W^{1,p}(\mathfrak{C})$ and $|h|_{W^{1,p}(\mathfrak{C})} \lesssim |g|_{W^{1,p}(\partial\mathfrak{C})}$. (Here, we use again the fact that $p < j+1$.) The case of a general map $g \in W^{1,p}(\partial\mathfrak{C})$ follows by approximation.

It remains to prove that $h \in W^{1,p}(\mathscr{C}_{j+1})$ (globally). This amounts to $\mathrm{tr}(h_{|\mathfrak{C}}) = g_{|\partial\mathfrak{C}}$, $\forall\, \mathfrak{C} \in \mathscr{C}_{j+1}$. This is clear when $g_{|\partial\mathfrak{C}}$ is Lipschitz; the general case follows by approximation. QED

7 Appendix # 4. Basic topological tools

Lemma 7.1. Let $\pi : \mathscr{E} \to \mathscr{N}$ be a non-trivial covering. Let $z_0 \in \mathscr{N}$, $t_0 \in \pi^{-1}(\{z_0\})$, and $b > 0$. Then there exists some $v \in C^\infty(\mathbb{R}; \mathscr{N})$ with $v(\theta) = z_0$ when $|\theta| \geq b$, with no continuous lifting $\zeta : [-b, b] \to \mathscr{E}$ of u on $[-b, b]$ satisfying $\zeta(b) = \zeta(-b) = t_0$.

Sketch of proof. We prove the existence of a continuous such v. (The existence of a smooth v requires an additional smoothing argument that we omit.) Let $t_1 \in \pi^{-1}(\{z_0\}) \setminus \{t_0\}$. Let $\gamma : [-b, b] \to \mathscr{E}$ be continuous such that $\gamma(-b) = t_0$, $\gamma(b) = t_1$, and set $v(\theta) := \pi \circ \gamma(\theta)$, $\forall\, \theta \in [-b, b]$. We extend v with the value z_0 outside $[-b, b]$. Note that v is continuous. By uniqueness of lifting, γ is the only continuous lifting ζ of v on $[-b, b]$ such that $\zeta(-b) = t_0$, whence the non-existence claimed in the statement. QED

For the next result, see [61, Lemma 3.8].

Lemma 7.2. Let $N \geq 3$. Let A be a finite union of subspaces of dimension $\nu \leq N-3$. Then $((0,1)^N \setminus A$ is simply connected.

For the next result, see, e. g., Bethuel and Chiron [9, Proof of Proposition 2].

Lemma 7.3. Assume that $\mathscr{E}$ is non-compact. Then, for each $t \in \mathscr{E}$, there exists some smooth map $\gamma_t : [0, \infty) \to \mathscr{E}$ such that $\gamma_t(0) = t$ and $d_{\mathscr{E}}(\gamma_t(\theta_1), \gamma_t(\theta_2)) = |\theta_1 - \theta_2|$, $\forall\, \theta_1, \theta_2 \geq 0$.

For the next result, we are in the context of Propositions 2.7 and 2.8, and we use the notation there.

Lemma 7.4. Let $0 \leq j \leq N-1$. Then there exists a Lipschitz homotopy $G = G(x, \theta) : \mathscr{C}_N \times [0, 1] \to \mathscr{C}_N$ such that:

a) $G(x, 0) = x$, $\forall\, x \in \mathscr{C}_N$.

b) $G(x, \theta) = a$, for some (fixed) point $a \in \mathscr{C}_{j+1}$, $\forall\, x \in \mathscr{C}_N$, $\forall\, \theta \geq 1/2$.

c) $G(x,\theta) \in \mathscr{C}_{j+1}, \forall\, x \in \mathscr{C}_j, \forall\, \theta$.

Proof. The proof is by induction on $N-j$ and "concatenation" of successive homotopies. When $N-j=1$, we may let $a=(1/2,\dots,1/2)$, $G(x,\theta)=\begin{cases} 2\theta a+(1-2\theta)x, & \text{if } 0\le\theta\le 1/2 \\ a, & \text{if } \theta\ge 1/2 \end{cases}$.

Assume next that $N-j=k\ge 2$ and the the lemma has been established for $N-1$ and j. Let us note that

$$\mathscr{C}_j=\{x\in\mathscr{C}_N;\ x \text{ has at least } (N-j) \text{ coordinates in } (1/2+\mathbb{Z})^N\}.$$

Using this, we see that $G_1(x,\theta)=G_1(x',x_N,\theta)=\begin{cases} (x',\theta+(1-2\theta)x_N), & \text{if } 0\le\theta\le 1/2 \\ (x',1/2), & \text{if } \theta\ge 1/2 \end{cases}$, is a homotopy satisfying conclusions a) and c) above. Moreover, if we set $\widetilde{\mathscr{C}}_m:=\mathscr{C}_m\cap(\mathbb{R}^{N-1}\times\{1/2\})$, then $G(\mathscr{C}_N,1)\subset\widetilde{\mathscr{C}}_{N-1}$ and $G_1(x,\theta)=x, \forall\, x\in\widetilde{\mathscr{C}}_{N-1}, \forall\,\theta$. It then suffices to concatenate G_1 with an appropriate map (given by the induction hypothesis) $G_2:\widetilde{\mathscr{C}}_N\times[0,1]\to\widetilde{\mathscr{C}}_N$. QED

8 Appendix # 5. Uniqueness

In what follows, we let $\Omega=(0,1)^N$.

For the next result, see [12, Theorem B.1].

Lemma 8.1. Let $F\subset\mathbb{R}^k$ be a discrete closed set. Let $f\in W^{s,p}(\Omega;F)$. If $sp\ge 1$, then f is constant.

Lemma 8.2. Assume that $sp\ge 1$. Let $\varphi\in W^{s,p}(\Omega;\mathscr{E})$ be such that $u:=\pi\circ\varphi$ is continuous. Then φ is continuous. Moreover, if u is smooth, then φ is smooth.

Proof. We may assume that $s\le 1$. Given the local nature of the problem, we may also assume that $u(\Omega)\subset B$ for some geodesic ball or radius r smaller than the injectivity radius of $\mathscr{N}$. Write $\pi^{-1}(B)$ as an at most countable union of balls B_j, each one diffeomorphic with B. Pick j such that the set $\varphi^{-1}(B_j)$ has positive measure. Let ζ be the continuous lifting of u with values into B_j. By Theorem 4.13, we have $\zeta\in W^{s,p}$ and, by choice of B_j, the set $[\varphi=\zeta]$ has positive measure. We claim that $\varphi=\zeta$ a.e. (and this completes the proof).

For this purpose, we consider some smooth map $g:[0,\infty)\to[0,1]$ such that $g(0)=0$ and $g(\theta)=1$ when $\theta\ge r$. Let

$$f(x):=g(d_{\mathscr{E}}(\varphi(x),\zeta(x)),\ \forall\, x\in\Omega,$$

where $d_{\mathscr{E}}$ denotes the geodesic distance on $\mathscr{E}$. By triangle's inequality, we have

$$|d_{\mathscr{E}}(\varphi(x),\zeta(x))-d_{\mathscr{E}}(\varphi(y),\zeta(y))|\le d_{\mathscr{E}}(\varphi(x),\varphi(y))+d_{\mathscr{E}}(\zeta(x),\zeta(y)). \tag{8.1}$$

Since $\varphi,\zeta\in W^{s,p}$ and g is Lipschitz, we obtain from (8.1) that $f\in W^{s,p}$ (recall that $s\le 1$).

On the other hand, we clearly have, by the formula of g and the property $\pi\circ\varphi=\pi\circ\zeta$, that $f:\Omega\to\{0,1\}$. By Lemma 8.1 and the choice of B_j, we find that $f=0$ a.e., and thus φ is continuous. QED

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