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# The Solution Path of SLOPE

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## Abstract

The SLOPE estimator has the particularity of having null components (sparsity) and components that are equal in absolute value (clustering). The number of clusters depends on the regularization parameter of the estimator. This parameter can be chosen as a trade-off between interpretability (with a small number of clusters) and accuracy (with a small mean squared error or a small prediction error). Finding such a compromise requires to compute the solution path, that is the function mapping the regularization parameter to the estimator. We provide in this article an algorithm to compute the solution path of SLOPE and show how it can be used to adjust the regularization parameter.

## 1 INTRODUCTION

The SLOPE estimator (Sorted L One Penalized Estimator [Bogdan et al., 2015, Zeng and Figueiredo, 2014]) is defined as a solution to the following convex program:

$$\min_{b \in \mathbb{R}^p} \left\{ \frac{1}{2} \|y - Xb\|_2^2 + \gamma \sum_{i=1}^p \lambda_i |b|_{\downarrow i} \right\} \quad (1)$$

where  $\lambda_1 > 0$ ,  $\lambda_1 \geq \dots \geq \lambda_p \geq 0$  is a given sequence of penalty parameters,  $\gamma > 0$  is the regularization parameter and  $|b|_{\downarrow 1} \geq \dots \geq |b|_{\downarrow p} \geq 0$  are the sorted components of  $b$  in absolute value. The SLOPE estimator generalizes both the LASSO estimator (Least Absolute Shrinkage and Selection Operator [Tibshirani, 1996]) for which  $\lambda_1 = \dots = \lambda_p = 1$ , and the OSCAR

estimator (Octagonal Shrinkage and Clustering Algorithm for Regression [Bondell and Reich, 2008]) for which the sequence  $\lambda_1, \dots, \lambda_p$  is arithmetic. Note that the penalty term of OSCAR satisfies  $\sum_{i=1}^p \lambda_i |b|_{\downarrow i} = \lambda_p \|b\|_1 + \frac{\lambda_1 - \lambda_p}{2} \sum_{1 \leq i < j \leq p} (|b_i + b_j| + |b_i - b_j|)$ , thus OSCAR is a particular generalized LASSO [Tibshirani and Taylor, 2011]; however, in broad generality SLOPE is not a particular generalized LASSO (as proved in supplementary material).

The SLOPE estimator is gaining popularity among statisticians due to its relevant properties such as minmax rates of the estimation and prediction errors [Bellec et al., 2018, Su and Candès, 2016], false discovery rate control [Bogdan et al., 2015] and dimension reduction of the regression model. The latter property comes from the structure of the solutions to the optimization problem (1), which have null components (sparsity) as well as components equal in absolute value (clustering) [Schneider and Tardivel, 2022, Figueiredo and Nowak, 2016, Bondell and Reich, 2008]. In particular, the sparsity and clustering properties of SLOPE are clear when  $X$  is an orthogonal matrix since, in this case, the solution to problem (1) is explicit [Bogdan et al., 2015, Dupuis and Tardivel, 2022, Skalski et al., 2022, Tardivel et al., 2020]. When  $y$  represents the random response of a linear regression model, sparsity has a well-known statistical interpretation: identification of relevant explanatory variables. Clustering also has a statistical interpretation when the design matrix  $X$  is standardized: the explanatory variables having the same regression coefficient have the same impact on the response [Sharma et al., 2013]. On the other hand, without restriction on the design matrix, for a categorical variable having different levels, the equal regression coefficients represent levels that can be grouped together [Stokell et al., 2021, Maj-Kańska et al., 2015]. Therefore, SLOPE estimator can identify relevant explanatory variables, group explanatory variables having the same impact on the response and, more generally, reduce the dimension of the regression model.

The solution path gives the solution of a penalized optimization problem with respect to the regularization parameter  $\gamma > 0$ . For instance, the solution path of the LASSO shows that the number of explanatory variables selected by this estimator tends to decrease when the regularization parameter becomes large (see e.g. [Mairal and Yu, 2012](#), [Rosset and Zhu, 2007](#)) and computing this path is useful to select the regularization parameter. Similarly, the solution path of SLOPE shows that the number of clusters of explanatory variables selected by this estimator tends to decrease when the regularization parameter becomes large. Moreover computing this path is useful to adjust the regularization parameter by minimizing, for instance, the Stein Unbiased Risk Estimate (SURE) formula [Stein, 1981](#) or the sum of residual squares on a validation set.

The generalized LASSO dual path algorithm [Tibshirani and Taylor, 2011](#), implemented in the *genlasso* R package [Arnold and Tibshirani, 2016](#), allows to compute the solution path of the generalized LASSO and therefore of OSCAR but not of SLOPE in broad generality; moreover it requires  $\ker(X) = \{0\}$ . Two articles focus on the solution path of OSCAR: the starting point of their respective algorithm is the ordinary least squared estimator (thus requiring  $\ker(X) = \{0\}$ ) in [Takahashi and Nomura, 2020](#), and a numerical solution of OSCAR in [Gu et al., 2017](#). A recent preprint [Nomura, 2020](#) addresses the solution path of SLOPE, under the assumption  $\ker(X) = \{0\}$  to guarantee the uniqueness of the solution and to use the ordinary least squares estimator as a starting point; it gives no theoretical results on the solution path (such as its continuity, the proof that it is piecewise linear, the characterization of its affine components).

In this article, for sequences of penalty parameters  $\lambda_1 > \dots > \lambda_p > 0$ , we prove that the solution path of SLOPE is continuous and piecewise linear on  $(0, +\infty)$ , we characterize its affine components, and we provide an algorithm to compute the exact solution path of SLOPE. Our algorithm does not require neither  $\ker(X) = \{0\}$  nor to solve SLOPE with an external solver. We dedicate a section to numerical experiments on real data sets to illustrate: the computation of SLOPE solution paths; the exact minimization of SURE for SLOPE (pointing out differences with the LASSO estimator); the performance of our algorithm compared to *genlasso* to compute the OSCAR solution path; the performance of our algorithm compared to the algorithms considered and implemented in [Larsen et al., 2023](#) to compute the SLOPE solution for a single regularization parameter  $\gamma$ .

## 2 BASIC NOTIONS ON SLOPE

Unlike the  $\ell_1$  norm, in broad generality the sorted  $\ell_1$  norm is not separable (the sorted  $\ell_1$  norm cannot be written as a sum of functions of its components). As a result, it is much more challenging to study the SLOPE optimization problem than the LASSO optimization problem. For instance the gradient  $X'(y - X\hat{\beta}^{\text{lasso}})$  (where  $X'$  denotes the transpose matrix of  $X$ ) of the sum of residual squares at the LASSO solution  $\hat{\beta}^{\text{lasso}}$  gives indications on null components of this estimator. Indeed,  $|X'_i(y - X\hat{\beta}^{\text{lasso}}(\gamma))| < \gamma$  implies  $\hat{\beta}_i^{\text{lasso}}(\gamma) = 0$ . Unfortunately, because the sorted  $\ell_1$  norm is not separable, determining null components based on the the gradient  $X'(y - X\hat{\beta})$  of the sum of residual squares at the SLOPE solution  $\hat{\beta}$  is not straightforward (determining non-null clusters is also difficult). The important notions introduced hereafter, already used either in the article of [Schneider and Tardivel, 2022](#) or in the article of [Bogdan et al., 2022](#), allow to overcome this difficulty.

### 2.1 Sorted $\ell_1$ norm and its dual norm

**Definition 1** *The sorted  $\ell_1$  norm associated to  $\lambda \in \mathbb{R}^p$  with  $\lambda_1 \geq \dots \geq \lambda_p \geq 0$  and  $\lambda_1 > 0$  is defined as follows:*

$$J_\lambda(b) = \sum_{i=1}^p \lambda_i |b|_{\downarrow i}, \quad b \in \mathbb{R}^p,$$

where  $|b|_{\downarrow 1} \geq \dots \geq |b|_{\downarrow p}$  are the sorted components of  $b$  with respect to the absolute value.

Given a norm  $\|\cdot\|$  on  $\mathbb{R}^p$ , we recall that its dual norm  $\|\cdot\|^*$  is defined by  $\|v\|^* = \max\{b'v : \|b\| \leq 1\}$ , for  $v \in \mathbb{R}^p$ .

**Remark 1** *The dual sorted  $\ell_1$  norm has an explicit expression given in [Negrinho and Martins, 2014](#) and reminded hereafter:*

$$J_\lambda^*(v) = \max \left\{ \frac{\|v\|_{(1)}}{\lambda_1}, \frac{\|v\|_{(2)}}{\sum_{i=1}^2 \lambda_i}, \dots, \frac{\|v\|_{(p)}}{\sum_{i=1}^p \lambda_i} \right\}, v \in \mathbb{R}^p,$$

where  $\|\cdot\|_{(k)}$  is the  $k$ -norm (the sum of the  $k$  largest components in absolute value).

### 2.2 SLOPE pattern

The SLOPE pattern introduced in [Schneider and Tardivel, 2022](#), whose definition is reminded below, is a central notion in this article.

**Definition 2** *The SLOPE pattern  $\text{patt}(b) \in \mathbb{Z}^p$  of  $b \in \mathbb{R}^p$  is defined by*

$$\text{patt}(b)_i = \text{sign}(b_i) \text{rank}(|b|)_i, \quad i \in \{1, \dots, p\},$$

where  $\text{rank}(|b|)_i \in \{0, 1, \dots, k\}$ ,  $k$  is the number of nonzero distinct values in  $\{|b_1|, \dots, |b_p|\}$ ,  $\text{rank}(|b|)_i = 0$  if and only if  $b_i = 0$ , and  $\text{rank}(|b|)_i < \text{rank}(|b|)_j$  if  $|b_i| < |b_j|$ .

We denote by  $\mathcal{P}_p^{\text{slope}} = \text{patt}(\mathbb{R}^p)$  the set of SLOPE patterns. Note in the definition above that  $k = \|\text{patt}(b)\|_\infty$  is the number of nonzero clusters of  $b$ .

**Example 1** Let  $b = (4.2, -1.3, 0, 1.3, 4.2)'$ . Then  $\text{patt}(b) = (2, -1, 0, 1, 2)'$ .

**Definition 3** Let  $m \in \mathbb{Z}^p$  be a SLOPE pattern with  $k = \|m\|_\infty \geq 1$ . The associated pattern matrix  $U_m \in \mathbb{R}^{p \times k}$  is defined by

$$(U_m)_{ij} = \text{sign}(m_i) \mathbf{1}_{(|m_i|=k+1-j)}, \\ i \in \{1, \dots, p\}, j \in \{1, \dots, k\}.$$

For  $k \geq 1$  we denote  $\mathbb{R}^{k+} = \{s \in \mathbb{R}^k : s_1 > \dots > s_k > 0\}$ . Definition 3 is such that, for  $b \in \mathbb{R}^p$  and  $m \in \mathbb{Z}^p$  a SLOPE pattern with  $k = \|m\|_\infty \geq 1$ , we have

$$\text{patt}(b) = m \iff \exists s \in \mathbb{R}^{k+} \text{ such that } b = U_m s.$$

Hereafter, the notation  $|m|_\downarrow = (|m|_{\downarrow 1}, \dots, |m|_{\downarrow p})'$  represents the components of  $m$  sorted non-increasingly with respect to the absolute value.

**Example 2** Let  $m = (2, -1, 0, 1, 2)'$ . Then

$$U_m = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}' \\ U_{|m|_\downarrow} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}'.$$

**Definition 4** Let  $m \in \mathbb{Z}^p$  be a SLOPE pattern with  $k = \|m\|_\infty \geq 1$ . The clustered matrix  $\tilde{X}_m \in \mathbb{R}^{n \times k}$  of  $X \in \mathbb{R}^{n \times p}$  is defined by  $\tilde{X}_m = XU_m$ ; the clustered parameter  $\tilde{\lambda}_m \in \mathbb{R}^k$  of  $\lambda \in \mathbb{R}^p$  is defined by  $\tilde{\lambda}_m = U_{|m|_\downarrow}' \lambda$ .

Note that the dimension of the design matrix  $X$  is reduced when it is clustered as  $\tilde{X}_m$  by a pattern  $m$ : a null component  $m_i = 0$  leads to discarding the column  $X_i$  from the design matrix  $X$ , and a cluster  $K \subset \{1, \dots, p\}$  of  $m$  (set of components of  $m$  equal in absolute value) leads to replacement of the columns  $(X_i)_{i \in K}$  by one column equal to the signed sum:  $\sum_{i \in K} \text{sign}(m_i) X_i$ .

**Example 3** Let  $X = (X_1|X_2|X_3|X_4|X_5)$ ,  $m = (2, -1, 0, 1, 2)'$ ,  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)' \in \mathbb{R}^5$ . Then the clustered matrix and the clustered parameter are given by:

$$\tilde{X}_m = (X_1 + X_5 | -X_2 + X_4) \text{ and } \tilde{\lambda}_m = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_3 + \lambda_4 \end{pmatrix}.$$

## 2.3 Subdifferential of the sorted $\ell_1$ norm

The subdifferential of a norm is related to the dual norm  $\|\cdot\|^*$  via the following formula [Hiriart-Urruty and Lemaréchal, 2004, p. 180]:

$$\partial\|\cdot\|(b) = \{v \in \mathbb{R}^p : \|v\|^* \leq 1 \text{ and } b'v = \|b\|\}, \quad b \in \mathbb{R}^p.$$

In particular,  $\partial\|\cdot\|(b)$  is a face of the dual unit ball. For the sorted  $\ell_1$  norm, the above formula can be specified further with the pattern matrix and the clustered parameter associated to  $m = \text{patt}(b)$  for  $b \neq 0$  [Bogdan et al., 2022, Schneider and Tardivel, 2022]:

$$\partial J_\lambda(b) = \left\{ v \in \mathbb{R}^p : J_\lambda^*(v) \leq 1 \text{ and } U_m' v = \tilde{\lambda}_m \right\}. \quad (2)$$

**Remark 2** Given  $\lambda \in \mathbb{R}^{p+}$ , the mapping  $m \mapsto \partial J_\lambda(m)$  is a bijection between the set of SLOPE patterns and the set of faces of the unit ball of  $J_\lambda^*$  (the signed permutahedron) [Schneider and Tardivel, 2022, Theorem 6]. It is no longer true when  $\lambda_1 \geq \dots \geq \lambda_p \geq 0$  is not a decreasing sequence. Therefore we restrict our study to the case where  $\lambda \in \mathbb{R}^{p+}$ , i.e.  $\lambda_1 > \dots > \lambda_p > 0$ .

## 3 SOLUTION, FITTED VALUE AND GRADIENT PATHS

### 3.1 Solution set and fitted value

Given  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}^{p+}$ , and  $\gamma > 0$ , we denote by  $\mathcal{S}_{X,y,\lambda}(\gamma)$  (or simply  $\mathcal{S}(\gamma)$  when there is no ambiguity) the set of solutions to the SLOPE optimization problem (1), namely:

$$\min_{b \in \mathbb{R}^p} \left\{ \frac{1}{2} \|y - Xb\|_2^2 + \gamma J_\lambda(b) \right\}.$$

For any  $\gamma > 0$ , the objective function of the above problem is continuous and coercive thus the solution set  $\mathcal{S}(\gamma)$  is nonempty. Moreover, the fitted value  $\hat{\text{fit}}(\gamma) = X\hat{\beta}$  does not depend on  $\hat{\beta} \in \mathcal{S}(\gamma)$ . When  $\mathcal{S}(\gamma)$  is a singleton, we denote by  $\hat{\beta}(\gamma)$  its unique element. Note that uniqueness is rather a weak assumption, indeed the set

$$\{X \in \mathbb{R}^{n \times p} : \exists y \in \mathbb{R}^n, \exists \gamma > 0 \text{ such that } \mathcal{S}_{X,y,\lambda}(\gamma) \text{ is not a singleton}\}$$

has zero Lebesgue measure [Schneider and Tardivel, 2022, Proposition 3]. Theorem 1 below shows that  $\hat{\text{fit}}(\cdot)$  and  $\hat{\beta}(\cdot)$  are continuous on  $(0, +\infty)$  and affine between two regularization parameters for which SLOPE solutions have the same pattern. Affine expressions of these piecewise linear functions are explicit and intervals are characterized. We denote hereafter by  $A^+$  the Moore-Penrose pseudo-inverse of a matrix  $A$ .

**Theorem 1** Let  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}^{p+}$ , and  $m \in \mathbb{Z}^p$  be a non-null SLOPE pattern with  $k = \|m\|_\infty \geq 1$ .

1. The set  $I_m = \{\gamma > 0 : \exists \hat{\beta} \in \mathcal{S}(\gamma), \text{patt}(\hat{\beta}) = m\}$  is an interval (potentially empty), with the following characterization:

$$\begin{aligned} & \gamma \in I_m \\ & \Updownarrow \\ & \left\{ \begin{array}{l} \exists s \in \mathbb{R}^{k+} \text{ such that } \tilde{X}'_m y - \gamma \tilde{\lambda}_m = \tilde{X}'_m \tilde{X}_m s \\ \text{(positivity condition)} \\ X'(\tilde{X}'_m)^+ \tilde{\lambda}_m + \frac{1}{\gamma} X'(I_n - (\tilde{X}'_m)^+ \tilde{X}'_m) y \in \partial J_\lambda(m) \\ \text{(subdifferential condition)} \end{array} \right. \end{aligned}$$

Moreover,  $U_m s \in \mathcal{S}(\gamma)$  for any  $s \in \mathbb{R}^{k+}$  satisfying the positivity condition at  $\gamma \in I_m$ .

2. The fitted value path  $\gamma \mapsto \widehat{\text{fit}}(\gamma)$  is continuous and piecewise linear on  $(0, +\infty)$ , with the following affine expression on  $I_m$ :

$$\widehat{\text{fit}}(\gamma) = (\tilde{X}'_m)^+ \tilde{X}'_m y - \gamma (\tilde{X}'_m)^+ \tilde{\lambda}_m, \quad \gamma \in I_m.$$

3. If  $\mathcal{S}(\gamma) = \{\hat{\beta}(\gamma)\}$  for all  $\gamma > 0$ , then the solution path  $\gamma \mapsto \hat{\beta}(\gamma)$  is continuous and piecewise linear on  $(0, +\infty)$ , with the following affine expression on  $I_m$ :

$$\hat{\beta}(\gamma) = U_m (\tilde{X}'_m \tilde{X}_m)^{-1} (\tilde{X}'_m y - \gamma \tilde{\lambda}_m), \quad \gamma \in I_m.$$

The characterization of the interval  $I_m$  above is closely related to Theorem 3.1 in [Bogdan et al. \[2022\]](#).

### 3.2 Gradient path and clusters

A solution of the SLOPE optimization problem is characterized by the following two conditions:

$$\hat{\beta} \in \mathcal{S}(\gamma) \Leftrightarrow \begin{cases} J_\lambda^*(X'(y - X\hat{\beta})) \leq \gamma \\ \hat{\beta}' X'(y - X\hat{\beta}) = \gamma J_\lambda(\hat{\beta}) \end{cases}$$

Note that  $X'(y - X\hat{\beta}) = X'(y - \widehat{\text{fit}}(\gamma))$  is the gradient at  $\hat{\beta}$  of the sum of residual squares  $b \mapsto \frac{1}{2} \|y - Xb\|_2^2$ . Subsequently, we call gradient path the expression  $\gamma > 0 \mapsto X'(y - \widehat{\text{fit}}(\gamma))$ . The  $i$ -norm of the gradient satisfies the inequality  $\|X'(y - \widehat{\text{fit}}(\gamma))\|_{(i)} \leq \gamma \sum_{j=1}^i \lambda_j$  for all  $i \in \{1, \dots, p\}$ . Among these inequalities, we denote by  $\mathcal{A}(\gamma)$  the ones which are saturated:

$$\mathcal{A}(\gamma) = \left\{ i \in \{1, \dots, p\} : \frac{\|X'(y - \widehat{\text{fit}}(\gamma))\|_{(i)}}{\sum_{j=1}^i \lambda_j} = \gamma \right\}.$$

According to Theorem [2](#) below, the set  $\mathcal{A}(\gamma)$  provides the number of non-zero clusters, the size of these clusters as well as the number of non-zero components.

**Theorem 2** Let  $\lambda \in \mathbb{R}^{p+}$ ,  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$ ,  $\gamma > 0$  and  $\hat{\beta} \in \mathcal{S}(\gamma)$ .

1. Let  $1 \leq k_1 \leq \dots \leq k_l \leq p$  be a subdivision such that:

$$\begin{aligned} |\text{supp}(\hat{\beta})| &= k_l \text{ and } |\hat{\beta}|_{\downarrow 1} = \dots = |\hat{\beta}|_{\downarrow k_1} \\ &> \dots > |\hat{\beta}|_{\downarrow k_{l-1}+1} = \dots = |\hat{\beta}|_{\downarrow k_l} > 0 \end{aligned}$$

(i.e.  $\hat{\beta}$  has  $l$  non-null clusters, the cluster of the largest value has  $k_1$  elements and so on and  $\hat{\beta}$  has  $k_l$  non-null components). Then,  $\{k_1, \dots, k_l\} \subset \mathcal{A}(\gamma)$ .

2. Conversely, if  $\{k_1, \dots, k_l\} = \mathcal{A}(\gamma)$  then

$$\begin{aligned} |\hat{\beta}|_{\downarrow 1} &= \dots = |\hat{\beta}|_{\downarrow k_1} \geq \dots \geq |\hat{\beta}|_{\downarrow k_{l-1}+1} = \dots \\ &= |\hat{\beta}|_{\downarrow k_l} \geq |\hat{\beta}|_{\downarrow k_l+1} = \dots = |\hat{\beta}|_{\downarrow p} = 0 \end{aligned}$$

(i.e. the number of non-null clusters of  $\hat{\beta}$  is smaller than or equal to  $l$  and the number of non-null components is smaller than or equal to  $k_l$ ).

There are links between Theorem [2](#) and screening rules for SLOPE [\[Elvira and Herzet, 2023\]](#), [\[Larsson et al., 2020\]](#) which identify some null components of this estimator. For instance, running Algorithm 1 in [\[Larsson et al., 2020\]](#) with  $|X'(y - \widehat{\text{fit}}(\gamma))|_{\downarrow}$  returns that a SLOPE solution has at most  $\max\{\mathcal{A}(\gamma)\}$  non-zero components. Otherwise, Theorem 4.1 in [\[Elvira and Herzet, 2023\]](#) is closely related to the following implication:  $|\hat{\beta}|_{\downarrow i} \neq 0 \Rightarrow \exists k \geq i, k \in \mathcal{A}(\gamma)$ .

## 4 ALGORITHMS TO COMPUTE THE SOLUTION PATH

To keep this section simple we assume that  $\mathcal{S}(\gamma) = \{\hat{\beta}(\gamma)\}$  for all  $\gamma > 0$ . Let  $J_\lambda^*(X'y) = \gamma_0 > \gamma_1 > \dots > \gamma_r > \gamma_{r+1} = 0$  be a subdivision such that  $\gamma \mapsto \hat{\beta}(\gamma)$  is affine with pattern  $m^{(i)}$  on the interval  $(\gamma_{i+1}, \gamma_i)$  for  $i = 0, \dots, r$  (i.e the interior of  $I_{m^{(i)}}$  is  $(\gamma_{i+1}, \gamma_i)$ ).

First, let us explain how to compute the SLOPE solution path on  $[\gamma_1, \gamma_0]$ . By construction of  $m^{(0)}$  the following implication holds:

$$\begin{aligned} \forall \gamma \in (\gamma_1, \gamma_0), \quad \text{patt}(\hat{\beta}(\gamma)) &= m^{(0)} \\ &\Rightarrow \frac{1}{\gamma} X'(y - \widehat{\text{fit}}(\gamma)) \in \partial J_\lambda(m^{(0)}). \end{aligned}$$

Moreover, since  $\gamma > 0 \mapsto \widehat{\text{fit}}(\gamma)$  is continuous,  $\widehat{\text{fit}}(\gamma_0) = 0$  and  $\partial J_\lambda(m^{(0)})$  is a closed set, we get

$$\frac{1}{\gamma_0} X'(y - \widehat{\text{fit}}(\gamma_0)) = \frac{1}{\gamma_0} X'y \in \partial J_\lambda(m^{(0)}). \quad (3)$$

Algorithm 1 provides the pattern  $M(\frac{1}{\gamma_0}X'y)$  of the smallest face of the signed permutahedron containing  $\frac{1}{\gamma_0}X'y$ . Therefore, by construction

$$\begin{aligned} \partial J_\lambda \left( M \left( \frac{1}{\gamma_0} X'y \right) \right) &\subset \partial J_\lambda(m^{(0)}) \\ \Rightarrow \left\| M \left( \frac{1}{\gamma_0} X'y \right) \right\|_\infty &\leq \|m^{(0)}\|_\infty \end{aligned} \quad (4)$$

According to (4), if  $\frac{X'y}{\gamma_0}$  lies onto a facet of the signed permutahedron, we have  $m^{(0)} = M\left(\frac{1}{\gamma_0}X'y\right)$ .

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**Algorithm 1** Smallest face pattern containing a vector

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**Input:**  $\lambda \in \mathbb{R}^{p+}$  and  $z \in \mathbb{R}^p$  such that  $J_\lambda^*(z) \leq 1$

Define the set of saturated inequalities as follows:

$$\mathcal{A}(z) = \left\{ i \in \{1, \dots, p\} : \frac{\|z\|_{(i)}}{\sum_{j=1}^i \lambda_j} = 1 \right\}$$

if  $\mathcal{A}(z) = \emptyset$  then

$$\lfloor M(z) = (0, \dots, 0) \in \mathbb{R}^p$$

else

$$\left[ \begin{array}{l} \text{for each } j \in \{1, \dots, p\} \text{ do} \\ \lfloor \text{Set } M_j(z) = \text{sign}(z_j) \sum_{i \in \mathcal{A}(z)} \mathbf{1}(|z_j| \geq \lambda_i) \end{array} \right.$$

return  $M(z)$

---

**Example 4** We illustrate the solution path of OSCAR for  $y = (15, 5)' \in \mathbb{R}^2$ ,  $\lambda = (6, 4, 2)' \in \mathbb{R}^{3+}$  and

$$X = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

**Largest node  $\gamma_0$ :** We have  $X'y = (35, 25, 5)'$ , therefore  $\gamma_0 = J_\lambda^*(X'y) = 6$ .

**Pattern  $m^{(0)}$  in the left neighborhood of  $\gamma_0$ :**

Since  $\frac{1}{\gamma_0}X'y = (35/6, 25/6, 0)'$  lies in the relative interior of  $\partial J_\lambda(1, 1, 0)' = [(6, 2)', (4, 6)'] \times [-2, 2]$  then  $m^{(0)} = M(\frac{1}{\gamma_0}X'y) = (1, 1, 0)'$ .

**Expression of  $\hat{\beta}(\gamma)$  in the left neighborhood of  $\gamma_0$ :**

According to statement 3 in Theorem 1 when  $\gamma < \gamma_0 = 6$  is sufficiently close to  $\gamma_0$  we have  $\hat{\beta}(\gamma) = (\frac{30-5\gamma}{9}, \frac{30-5\gamma}{9}, 0)'$ .

We tried the package `genlasso` to compute this solution path. Since  $\dim(\ker(X)) \neq 0$ , `genlasso` add a small ridge term  $\epsilon\|b\|_2^2$  to the objective function (the default value is  $\epsilon = 10^{-4}$ ); thus `genlasso` solves

$$\begin{aligned} &\min_{b \in \mathbb{R}^3} \left\{ \frac{1}{2} \|y - Xb\|_2^2 + \gamma J_\lambda(b) + \epsilon \|b\|_2^2 \right\}, \\ &= \min_{b \in \mathbb{R}^3} \left\{ \frac{1}{2} \left\| \begin{pmatrix} y \\ 0 \end{pmatrix} - \begin{pmatrix} X \\ \sqrt{2\epsilon} I_3 \end{pmatrix} b \right\|_2^2 + \gamma J_\lambda(b) \right\}. \end{aligned} \quad (5)$$

We computed the solution path of problem (5) when  $\epsilon = 10^{-4}$  with our algorithm and `genlasso`. Surprisingly, the solution path computed with `genlasso` is correct when  $\gamma \geq 5$  but wrong when  $\gamma < 5$ .

Comparatively to the original problem (without adding the ridge term), when  $\epsilon = 10^{-4}$  the solution path have more nodes (especially small nodes). Moreover these paths are extremely different when  $\gamma$  is small since  $\epsilon\|b\|_2^2$  dominates  $\gamma J_\lambda(b)$ .

Algorithm 2 uses the characterisation of  $I_{m^{(0)}}$ , based on the positivity and subdifferential conditions, to provide both the node  $\gamma_1$  as well as the pattern  $m^{(1)}$ .

---

**Algorithm 2** Iterative calculation of nodes and patterns

---

**Input:**  $X \in \mathbb{R}^{n \times p}$ ,  $\lambda \in \mathbb{R}^{p+}$ ,  $\gamma_i > 0$  and  $m^{(i)} \in \mathcal{P}_p^{\text{slope}}$

Set  $k = \|m^{(i)}\|_\infty$

Compute  $s(\gamma) = (\tilde{X}'_{m^{(i)}} \tilde{X}_{m^{(i)}})^{-1} (\tilde{X}'_{m^{(i)}} y - \gamma \tilde{\lambda}_{m^{(i)}})$

if  $s(\gamma) \in \mathbb{R}^{k+}$  for all  $\gamma \in [0, \gamma_i[$  then

$$\lfloor \text{Set } \gamma_{\text{patt}} = 0$$

else

$$\lfloor \text{Set } \gamma_{\text{patt}} = \sup\{\gamma \in [0, \gamma_i[: s(\gamma) \notin \mathbb{R}^{k+}\}$$

if  $X'(y - \tilde{X}_{m^{(i)}} s(\gamma)) \in \gamma \partial J_\lambda(m^{(i)})$  for all  $\gamma \in [\gamma_{\text{patt}}, \gamma_i[$  then

$$\lfloor \text{Set } \gamma_{i+1} = \gamma_{\text{patt}}$$

$$\lfloor \text{Compute } m^{(i+1)} = \text{patt}(U_{m^{(i)}} s(\gamma_{\text{patt}}))$$

$$\lfloor \text{return } \gamma_{i+1}, m^{(i+1)}$$

else

$$\lfloor \text{Set } \gamma_{i+1} = \sup\{\gamma \in [\gamma_{\text{patt}}, \gamma_i[: X'(y - \tilde{X}_{m^{(i)}} s(\gamma)) \notin \gamma \partial J_\lambda(m^{(i)})\}$$

$$\lfloor \text{Compute } m^{(i+1)} = M(\frac{1}{\gamma_{i+1}} X'(y - \tilde{X}_{m^{(i)}} s(\gamma_{i+1})))$$

$$\lfloor \text{using Algorithm 1}$$

$$\lfloor \text{return } \gamma_{i+1}, m^{(i+1)}$$


---

Using iteratively Algorithm 2 allows to compute entirely the SLOPE solution path.

## 5 NUMERICAL EXPERIMENTS

The code of the implementation in Python of our algorithm and of the experiments below is available at <https://github.com/x-dupuis/slope-path>. The computations were carried out on an Apple M1 Pro chip (8-core CPU and 14-core GPU) and 16GB of unified memory.

We use two real data sets:

- the *Wine Quality* data set<sup>1</sup> describes the quality of red “Vinho Verde” wines [Cortez et al.

<sup>1</sup>available at <https://archive.ics.uci.edu/dataset/186/wine+quality>

[2009]. Each column of  $X \in \mathbb{R}^{1599 \times 11}$  represents a physicochemical measurement (density, pH, alcohol, etc.) and  $y \in \mathbb{R}^{1599}$  represents wine quality scores (between 0 and 10);

- the *Riboflavin* data set [2] describes the riboflavin production with *Bacillus subtilis* [Bühlmann et al., 2014]. Each column of  $X \in \mathbb{R}^{71 \times 4088}$  represents a gene expression measurement and  $y \in \mathbb{R}^{71}$  represents production rates.

The matrices  $X$  are mean-centered ( $\forall j, \sum_i X_{ij} = 0$ ) and standardized ( $\forall j, \sum_i X_{ij}^2 = n$ ), the vectors  $y$  are mean-centered ( $\sum_i y_i = 0$ ).

### 5.1 Full paths computation

We illustrate here the computation of SLOPE solution paths on the *Wine Quality* data set. For this numerical experiment we take  $\lambda = (1, \sqrt{2}-1, \sqrt{3}-\sqrt{2}, \dots, \sqrt{11}-\sqrt{10})$ , so that the unit ball of the sorted  $\ell_1$  norm is quasi-spherical [Nomura, 2020]. Figure 1 provides the solution path of SLOPE as well as the solution path of LASSO when  $\lambda_1 = \dots = \lambda_{11} = 1$  and computed via the homotopy algorithm in [Mairal and Yu, 2012] (this algorithm is devoted to solve the solution path of LASSO).

### 5.2 Exact minimization of SURE

The Stein Unbiased Risk Estimate (SURE) formula is an unbiased estimator of the prediction error ( $\mathbb{E}(\|X\hat{\beta} - X\beta\|_2^2)$  where  $\hat{\beta}$  is an estimator of  $\beta$ ). For LASSO and SLOPE, unbiased estimators for the prediction error are reported hereafter [Minami, 2020, Zou et al., 2007]

$$\text{sure}(\gamma) = \begin{cases} \|y - X\hat{\beta}(\gamma)\|_2^2 - n\sigma^2 + 2\sigma^2 \|\text{patt}(\hat{\beta}(\gamma))\|_\infty & \text{when } \hat{\beta}(\gamma) \text{ is a SLOPE estimator} \\ \|y - X\hat{\beta}(\gamma)\|_2^2 - n\sigma^2 + 2\sigma^2 |\text{supp}(\hat{\beta}(\gamma))| & \text{when } \hat{\beta}(\gamma) \text{ is a LASSO estimator} \end{cases}$$

where  $\sigma^2$  is the variance of residuals. A usual way to select the regularization parameter  $\gamma$  is to minimize  $\text{sure}(\gamma)$  [Dossal et al., 2013, Bertrand et al., 2022]. For both SLOPE and LASSO the solution path is piecewise linear, therefore the SURE formula is quadratic between two adjacent nodes (*i.e* the SURE formula restricted to the interval  $(\gamma_{i+1}, \gamma_i)$  is quadratic). As a result, solving exactly the solution path allows to minimize exactly the SURE formula [3]. For this numerical experiment we substitute  $\sigma^2$  in the expression

<sup>2</sup>available at <https://www.annualreviews.org/doi/suppl/10.1146/annurev-statistics-022513-115545>

<sup>3</sup>One may similarly minimize exactly the sum of residual squares on a validation set  $\gamma > 0 \mapsto \|y^{\text{val}} - X^{\text{val}}\hat{\beta}(\gamma)\|_2^2$ .

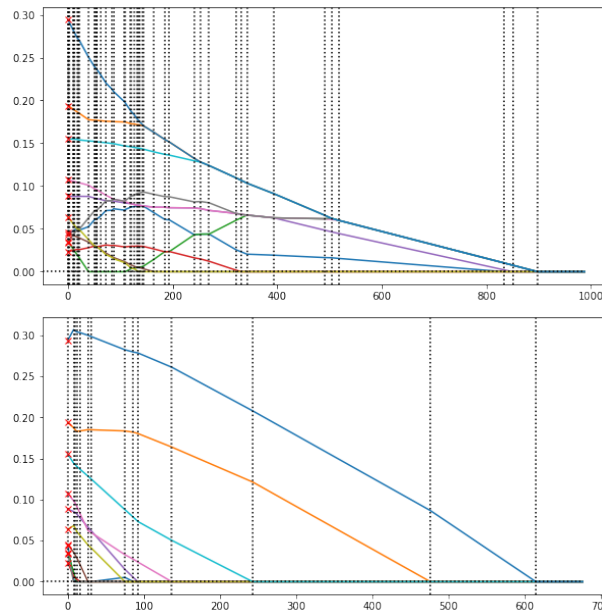


Figure 1: Solution paths in absolute value of SLOPE (top) and LASSO (bottom) as functions of  $\gamma > 0$ . On top some curves partially superimpose or partially coincide with the x-axis, illustrating the clustering and sparsity properties of SLOPE. At the bottom some curves just partially coincide with the x-axis, illustrating the sparsity property of LASSO.

of  $\text{sure}(\gamma)$  by  $\hat{\sigma}^2 = \|(I_n - X(X'X)^{-1}X')y\|/1588 = 0.4197$ . Note that when  $\gamma$  is very large the SURE formula satisfies  $\text{sure}(\gamma) = \|y\|_2^2 - 1599\hat{\sigma}^2 = 371.1382$ . Moreover when  $\gamma$  tends to 0, both SLOPE and LASSO converge to the ordinary least squares estimator therefore  $\lim_{\gamma \rightarrow 0} \text{sure}(\gamma) = 11\hat{\sigma}^2 = 4.6162$ . We report the regularization parameter minimizing the SURE formula in Table 5.2

Table 1: Minimizer and minimum of the SURE formula for both SLOPE and LASSO.

	$\gamma_{\text{sure}}$	$\text{sure}(\gamma_{\text{sure}})$
SLOPE	18.6292	3.4641
LASSO	11.7602	4.1297

The explanatory variables “fixed acidity” (corresponding to  $X_1$ ) and “pH” (corresponding to  $X_9$ ) are the most correlated ones (the largest off-diagonal components of  $X'X$ , in absolute value, is  $|X_1'X_9| = 1092.0821$ ). The explanatory variables “fixed acidity” and “density” (corresponding to  $X_8$ ) are also strongly correlated ( $|X_1'X_8| = 1068.2076$ ). These three variables are clustered by the SLOPE estimator  $\hat{\beta}(\gamma_{\text{sure}})$  (corresponding to the cluster “4” in  $\text{patt}(\hat{\beta}(\gamma_{\text{sure}})) = (4, -8, -1, 2, -5, 3, -6, -4, -4, 7, 9)$ )

whereas the LASSO estimator  $\hat{\beta}^{\text{lasso}}(\gamma_{\text{sure}})$  only selects one: the ‘‘pH’’ (actually  $\hat{\beta}_1^{\text{lasso}}(\gamma_{\text{sure}}) = \hat{\beta}_8^{\text{lasso}}(\gamma_{\text{sure}}) = 0$ ). Clustering property of SLOPE for highly correlated variables had been discussed in Figueiredo and Nowak [2016] and intuitively we believe that this property is beneficial for the prediction error.

### 5.3 Full path solvers benchmark

For this benchmark we focus on the solution path of OSCAR as, in the literature, no algorithm for solving the solution path of SLOPE is available online (the code for solving the solution path of SLOPE in the preprint Nomura, 2020 is not available). A natural competitor to our algorithm is *genlasso*. Hereafter  $X$  and  $y$  are provided by the *Wine Quality* data set and  $\lambda$  is an arithmetic progression where  $\lambda_1 = 4$  and  $\lambda_{11} = 1$ . In table 2 we compare the time needed to compute the solution path as well as the value of the objective function of OSCAR at  $\gamma \in \left\{ \frac{J_\lambda^*(X'y)}{2}, \frac{J_\lambda^*(X'y)}{10} \right\}$ .

Table 2: Time in seconds to compute the solution path and value of the objective function. Our algorithm is much faster than *genlasso*. Moreover, the value obtained with our algorithm is lower than the one obtained with *genlasso* at  $\gamma = \frac{J_\lambda^*(X'y)}{10}$ , illustrating that the solution provided by *genlasso* is not accurate.

	genlasso	SLOPE path (our)
Time	4.96e-01	<b>1.31e-02</b>
Value at $\frac{J_\lambda^*(X'y)}{2}$	483.4367	483.4367
Value at $\frac{J_\lambda^*(X'y)}{10}$	379.8561	<b>378.5511</b>

Comparison between *genlasso* and our algorithm on the Riboflavin data set is not tractable; indeed the  $D$  matrix such that  $J_\lambda(b) = \|Db\|_1$  belongs to  $\mathbb{R}^{4088^2 \times 4088}$  and even if it is sparse, the package *genlasso* cannot handle such a big matrix.

### 5.4 SLOPE solvers benchmark

Computing the full solution path of SLOPE on  $(0, +\infty)$  is a more ambitious task than solving the SLOPE optimization problem for a single regularization parameter  $\gamma$ . Therefore, given such a  $\gamma$ , we can compute the solution path on  $[\gamma, +\infty)$  and thus define a SLOPE solver (called *SLOPE path* hereafter). We compare it to the following algorithms implemented in the extensive benchmark of SLOPE solvers Larsson et al., 2023:

- Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) Beck and Teboulle, 2009;

- Anderson acceleration for Proximal Gradient Descent (Anderson PGD) Zhang et al., 2020;
- Alternating Direction Method of Multipliers (ADMM) Boyd et al., 2011 with the augmented Lagrangian parameter  $\rho = 100$ ;
- Coordinate Descent for SLOPE (hybrid CD) Larsson et al., 2023.

We used their code<sup>4</sup> and set as stopping criterion a primal-dual gap smaller than  $1e-12$  (which is satisfied by our algorithm all along the path). When  $\lambda$  is an arithmetic progression where  $\lambda_1 = 4$  and  $\lambda_p = 1$  the benchmarks on the two real data sets are reported in tables 3 and 4.

Table 3: Time in seconds to compute the solution for the *Wine Quality* data set. In this case, where  $p = 11$  is small, our algorithm is the fastest one.

$\gamma$	$\frac{J_\lambda^*(X'y)}{2}$	$\frac{J_\lambda^*(X'y)}{10}$
FISTA	1.36e-02	4.47e-02
Anderson PGD	5.26e-03	7.02e-02
ADMM ( $\rho = 100$ )	2.38e-02	7.18e-03
hybrid CD	2.39e-03	7.91e-03
SLOPE path (our)	<b>6.58e-04</b>	<b>4.51e-03</b>

Table 4: Time in seconds to compute the solution for the *Riboflavin* data set. In this case where  $p = 4088$  is large, our algorithm is still the fastest one when  $\gamma$  is large ( $\gamma = \frac{J_\lambda^*(X'y)}{2}$ ) but is over-performed by hybrid CD when  $\gamma$  is small ( $\gamma = \frac{J_\lambda^*(X'y)}{10}$ ). The missing values correspond to algorithms not reaching the required primal-dual gap ( $1e-12$ ).

$\gamma$	$\frac{J_\lambda^*(X'y)}{2}$	$\frac{J_\lambda^*(X'y)}{10}$
FISTA	9.01e+01	-
Anderson PGD	1.45e+01	-
ADMM ( $\rho = 100$ )	-	4.93e+00
hybrid CD	4.19e-02	<b>8.84e-01</b>
SLOPE path (our)	<b>3.83e-02</b>	3.72e+00

## 6 CONCLUSION AND FUTURE WORKS

One of the main result in this article is Theorem 1 proving that the SLOPE solution path is piecewise linear and providing the characterization of the intervals where the path is affine. Moreover algorithms 1 and 2 allow to solve exactly this path. The computational

<sup>4</sup>available at <https://github.com/jolars/slopecd>



time of our numerical scheme depends mainly on the number of nodes. In our illustration on real data sets, the number of intervals is not too large. However the number of intervals where the path is affine is bounded by the number of SLOPE patterns in  $\mathbb{R}^p$  and potentially, similarly as for LASSO [Mairal and Yu, 2012], this huge upper bound might be reached. Therefore solving the solution path of SLOPE on  $(0, +\infty)$  might be intractable for some pathological examples and, in such a situation, our algorithm can only compute partially the solution path. A first algorithmic perspective would be to generalize this method to a wide class of penalized estimators. Indeed, the crucial notion of SLOPE pattern might be generalized to a polyhedral gauge penalty [Graczyk et al., 2023] (the SLOPE pattern is just the pattern associated to the sorted  $\ell_1$  (polyhedral) norm). Another methodological perspective is to derive, based on Theorem 2, screening rules identifying null components and clusters for SLOPE.

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## The Solution Path of SLOPE: Supplementary Materials

### Proof of Theorem 1

**1:  $I_m$  is an interval**) Hereafter we suppose that  $I_m \neq \emptyset$ . Let  $\gamma_0, \gamma_1 \in I_m$  and pick  $\widehat{\beta}(\gamma_0) \in \mathcal{S}(\gamma_0)$ ,  $\widehat{\beta}(\gamma_1) \in \mathcal{S}(\gamma_1)$  such that  $\text{patt}(\widehat{\beta}(\gamma_0)) = \text{patt}(\widehat{\beta}(\gamma_1)) = m$ . Let  $\alpha \in [0, 1]$ ,  $\bar{\gamma} = \alpha\gamma_0 + (1 - \alpha)\gamma_1$  and  $\bar{\beta} = \alpha\widehat{\beta}(\gamma_0) + (1 - \alpha)\widehat{\beta}(\gamma_1)$  then  $\text{patt}(\bar{\beta}) = m$ . Indeed, if  $m = 0$  then clearly  $\text{patt}(\bar{\beta}) = 0$ . Otherwise, let  $k = \|m\|_\infty \geq 1$  then  $\widehat{\beta}(\gamma_0) = U_m s_0$  for some  $s_0 \in \mathbb{R}^{k+}$ ,  $\widehat{\beta}(\gamma_1) = U_m s_1$  for some  $s_1 \in \mathbb{R}^{k+}$  therefore  $\bar{\beta} = U_m \bar{s}$  where  $\bar{s} = \alpha s_0 + (1 - \alpha)s_1 \in \mathbb{R}^{k+}$ . To prove that  $I_m$  is an interval it remains to show that  $\bar{\beta} \in \mathcal{S}(\bar{\gamma})$ . Because both  $\widehat{\beta}(\gamma_0)$  and  $\widehat{\beta}(\gamma_1)$  are SLOPE minimizers, we have

$$X'(y - X\widehat{\beta}(\gamma_0)) \in \gamma_0 \partial J_\lambda(m) \text{ and } X'(y - X\widehat{\beta}(\gamma_1)) \in \gamma_1 \partial J_\lambda(m).$$

By construction of  $\bar{\beta}$  the following equality occurs:

$$\alpha X'(y - X\widehat{\beta}(\gamma_0)) + (1 - \alpha)X'(y - X\widehat{\beta}(\gamma_1)) = X'(y - X\bar{\beta}).$$

Moreover, since  $\partial J_\lambda(m)$  is a convex set, we have  $\alpha\gamma_0 \partial J_\lambda(m) + (1 - \alpha)\gamma_1 \partial J_\lambda(m) \subset \bar{\gamma} \partial J_\lambda(m)$ . Consequently,  $X'(y - X\bar{\beta}) \in \bar{\gamma} \partial J_\lambda(m) = \bar{\gamma} \partial J_\lambda(\bar{\beta})$  thus  $\bar{\beta} \in \mathcal{S}(\bar{\gamma})$ .

**1: characterization of  $I_m$ )** The proof of this characterization is closely related to the proof of Theorem 3.1 in [Bogdan et al. \[2022\]](#).

*Necessity.* If  $\gamma \in I_m$ , then there exists  $\widehat{\beta} \in \mathcal{S}(\gamma)$  such that  $\text{patt}(\widehat{\beta}) = m$ . Consequently,  $\widehat{\beta} = U_m s$  for some  $s \in \mathbb{R}^{k+}$ . Because  $\widehat{\beta}$  is a element of  $\mathcal{S}(\gamma)$  whose pattern is  $m$  then  $X'(y - \widehat{\text{fit}}(\gamma)) \in \gamma \partial J_\lambda(\widehat{\beta}) = \gamma \partial J_\lambda(m)$ . Multiplying this inclusion by  $U'_m$ , we get  $\tilde{X}'_m(y - \widehat{\text{fit}}(\gamma)) = \gamma \tilde{\lambda}_m$  and so

$$\tilde{X}'_m y - \gamma \tilde{\lambda}_m = \tilde{X}'_m \widehat{\text{fit}}(\gamma) = \tilde{X}'_m \tilde{X}_m s. \quad (6)$$

The positivity condition is proven.

We apply  $(\tilde{X}'_m)^+$  from the left to [\(6\)](#) and use the fact that  $(\tilde{X}'_m)^+ \tilde{X}'_m$  is the projection onto  $\text{col}(\tilde{X}_m)$ . Since  $\widehat{\text{fit}}(\gamma) \in \text{col}(\tilde{X}_m)$ , we have  $(\tilde{X}'_m)^+ \tilde{X}'_m \widehat{\text{fit}}(\gamma) = \widehat{\text{fit}}(\gamma)$ . Thus,

$$(\tilde{X}'_m)^+ \tilde{X}'_m y - \gamma (\tilde{X}'_m)^+ \tilde{\lambda}_m = \widehat{\text{fit}}(\gamma).$$

The above equality gives the subdifferential condition:

$$\begin{aligned} \partial J_\lambda(m) \ni \frac{1}{\gamma} X'(y - \widehat{\text{fit}}(\gamma)) &= \frac{1}{\gamma} X'(y - ((\tilde{X}'_m)^+ \tilde{X}'_m y - \gamma (\tilde{X}'_m)^+ \tilde{\lambda}_m)) \\ &= X'(\tilde{X}'_m)^+ \tilde{\lambda}_m + \frac{1}{\gamma} X'(I_n - (\tilde{X}'_m)^+ \tilde{X}'_m) y. \end{aligned} \quad (7)$$

*Sufficiency.* Assume that the positivity condition and the subdifferential conditions hold true. Then, by the positivity condition, one may pick  $s \in \mathbb{R}^{k+}$  for which

$$\gamma \tilde{\lambda}_m = \tilde{X}'_m y - \tilde{X}'_m \tilde{X}_m s. \quad (8)$$

Let us show that  $U_m s \in \mathcal{S}(\gamma)$ . By definition of  $U_m$ , we have  $\text{patt}(U_m s) = m$  thus  $\partial J_\lambda(U_m s(\gamma)) = \partial J_\lambda(m)$ . Moreover, using (7) and (8) one may deduce

$$\begin{aligned} \partial J_\lambda(U_m s) &\ni \frac{1}{\gamma} X'(y - (\tilde{X}'_m)^+ \tilde{X}'_m y + \gamma (\tilde{X}'_m)^+ \tilde{\lambda}_m) \\ &= \frac{1}{\gamma} X'(y - (\tilde{X}'_m)^+ \tilde{X}'_m y + (\tilde{X}'_m)^+ (\tilde{X}_m y - \tilde{X}'_m \tilde{X}_m s)) \\ &= \frac{1}{\gamma} X'(y - X U_m s). \end{aligned}$$

Consequently  $U_m s \in \mathcal{S}(\gamma)$ .

**2 and 3: continuity** Let  $\gamma \in (0, +\infty)$ ,  $(\gamma_n)_{n \in \mathbb{N}}$  be a sequence converging to  $\gamma$  and  $\hat{\beta}(\gamma_n) \in S_{X, \gamma_n, J_\lambda}(y)$ . Both sequences  $(\hat{\beta}(\gamma_n))_{n \in \mathbb{N}}$  and  $(\hat{\text{fit}}(\gamma_n))_{n \in \mathbb{N}}$  are bounded therefore, up to extract a subsequence, one may assume that both  $(\hat{\beta}(\gamma_n))_{n \in \mathbb{N}}$  and  $(\hat{\text{fit}}(\gamma_n))_{n \in \mathbb{N}}$  converge respectively to a limit point  $l \in \mathbb{R}^p$  and  $Xl \in \mathbb{R}^n$ . Let  $\hat{\beta}(\gamma) \in \mathcal{S}(\gamma)$ . Because  $\hat{\beta}(\gamma_n)$  is a minimizer, the following inequality occurs.

$$\frac{1}{2} \|y - \hat{\text{fit}}(\gamma_n)\|_2^2 + \gamma_n J_\lambda(\hat{\beta}(\gamma_n)) \leq \frac{1}{2} \|y - \hat{\text{fit}}(\gamma)\|_2^2 + \gamma_n J_\lambda(\hat{\beta}(\gamma)).$$

Taking the limit in the above expression gives

$$\frac{1}{2} \|y - Xl\|_2^2 + \gamma J_\lambda(l) \leq \frac{1}{2} \|y - \hat{\text{fit}}(\gamma)\|_2^2 + \gamma J_\lambda(\hat{\beta}(\gamma)).$$

Because  $\hat{\beta}(\gamma) \in \mathcal{S}(\gamma)$ , one may deduce that  $l \in \mathcal{S}(\gamma)$  and thus  $Xl = \hat{\text{fit}}(\gamma)$ . Therefore, the unique limit point of the bounded sequence  $(\hat{\text{fit}}(\gamma_n))_{n \in \mathbb{N}}$  is  $\hat{\text{fit}}(\gamma)$ . Consequently,  $\lim_{n \rightarrow +\infty} \hat{\text{fit}}(\gamma_n) = \hat{\text{fit}}(\gamma)$  and thus the function  $\gamma \in (0, +\infty) \mapsto \hat{\text{fit}}(\gamma)$  is continuous. Similarly, if  $\mathcal{S}(\gamma)$  is a singleton then  $l = \hat{\beta}(\gamma)$ , the unique limit point of the bounded sequence  $(\hat{\beta}(\gamma_n))_{n \in \mathbb{N}}$  is  $\hat{\beta}(\gamma)$  and thus  $\lim_{n \rightarrow +\infty} \hat{\beta}(\gamma_n) = \hat{\beta}(\gamma)$ . Therefore the function  $\gamma \in (0, +\infty) \mapsto \hat{\beta}(\gamma)$  is continuous.

2) When  $\gamma \in I_m$  then multiplying both side of the positivity condition by  $(\tilde{X}'_m)^+$  and using the fact that  $(\tilde{X}'_m)^+ \tilde{X}'_m$  is the projection onto  $\text{col}(\tilde{X}_m)$  gives

$$(\tilde{X}'_m)^+ \tilde{X}'_m y - \gamma (\tilde{X}'_m)^+ \tilde{\lambda}_m = (\tilde{X}'_m)^+ \tilde{X}'_m \tilde{X}_m s = \tilde{X}_m s = \hat{\text{fit}}(\gamma).$$

3) The proof of statement 3) relies on Lemma 1 (proved further).

**Lemma 1** *Let  $X \in \mathbb{R}^{n \times p}$ ,  $y \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}^{p+}$ . There exists  $\hat{\beta} \in \mathcal{S}(\gamma)$  for which the pattern  $m = \text{patt}(\hat{\beta})$  satisfies  $\ker(\tilde{X}_m) = \{0\}$  (or equivalently  $\|m\|_\infty = \text{rk}(\tilde{X}_m)$ ).*

Lemma 1 provides a statement more precise than both [Kremer et al., 2022, Theorem 2.1] and [Schneider and Tardivel, 2022, Corollary 9], proving that, under the assumption of uniqueness, the unique element  $\hat{\beta}$  of  $\mathcal{S}(\gamma)$  has a number of non-null clusters smaller or equal to  $\text{rk}(X)$ : *i.e.*  $\|\text{patt}(\hat{\beta})\|_\infty \leq \text{rk}(X)$  (note that  $\text{rk}(\tilde{X}_m) \leq \text{rk}(X)$ ).

Consequently, when  $\gamma \in I_m$  and  $\mathcal{S}(\gamma)$  is a singleton then  $\ker(\tilde{X}_m) = \{0\}$ , where  $m = \text{patt}(\hat{\beta}(\gamma))$ . Since  $\tilde{X}'_m \tilde{X}_m$  is invertible, the positivity condition gives

$$\hat{\beta}(\gamma) = U_m s = U_m (\tilde{X}'_m \tilde{X}_m)^{-1} (\tilde{X}'_m y - \gamma \tilde{\lambda}_m).$$

## Basic notions on subdifferential, permutahedron and signed permutahedron

The results of this section will be useful to establish the proof of Proposition 1. We denote by  $S_p$  the set of permutations on the set  $\{1, \dots, p\}$ . Given  $\lambda \in \mathbb{R}^{p+}$ , the subdifferential calculus of the sorted  $\ell_1$  norm satisfies the following properties [Dupuis and Tardivel, 2022, Schneider and Tardivel, 2022, Tardivel et al., 2020]:

**Subdifferential at 0: signed permutahedron** The following equality holds:

$$\partial J_\lambda(0) = \text{conv}\{(\sigma_1 \lambda_{\pi(1)}, \dots, \sigma_p \lambda_{\pi(p)}), \sigma_1, \dots, \sigma_p \in \{-1, 1\}, \pi \in S_p\}.$$

The V-polytope  $P^\pm(\lambda_1, \dots, \lambda_p) = \text{conv}\{(\sigma_1 \lambda_{\pi(1)}, \dots, \sigma_p \lambda_{\pi(p)}), \sigma_1, \dots, \sigma_p \in \{-1, 1\}, \pi \in S_p\}$  is called the *signed permutahedron* and can be described as a H-polytope as follows [Godland and Kabluchko 2023](#):

$$P^\pm(\lambda_1, \dots, \lambda_p) = \left\{ x \in \mathbb{R}^p : \forall j \in \{1, \dots, p\}, \sum_{i=1}^j |x|_{\downarrow i} \leq \sum_{i=1}^j \lambda_i \right\}.$$

This polytope is actually the unit ball of the dual sorted  $\ell_1$  norm [Negrinho and Martins 2014](#).

**Subdifferential at a constant vector: permutahedron** Let  $c > 0$ . Then the following equality holds:

$$\partial J_\lambda(c, \dots, c) = \text{conv}\{(\lambda_{\pi(1)}, \dots, \lambda_{\pi(p)}), \pi \in S_p\}.$$

The V-polytope  $P(\lambda_1, \dots, \lambda_p) = \text{conv}\{(\lambda_{\pi(1)}, \dots, \lambda_{\pi(p)}), \pi \in S_p\}$  is called the *permutahedron* and can be described as an H-polytope as follows [Godland and Kabluchko 2023](#), [Negrinho and Martins 2014](#):

$$P(\lambda_1, \dots, \lambda_p) = \left\{ b \in \mathbb{R}^p : \sum_{i=1}^p b_i = \sum_{i=1}^p \lambda_i \text{ and } \sum_{i=1}^j b_{\downarrow i} \leq \sum_{i=1}^j \lambda_i \forall j \in \{1, \dots, p-1\} \right\}. \quad (9)$$

**Subdifferential computation rule** Let  $b \in \mathbb{R}^p$  be such that  $b_1 \geq \dots \geq b_k > b_{k+1} \geq \dots \geq b_p \geq 0$ . Then

$$\partial J_\lambda(b) = \partial J_{\lambda_1, \dots, \lambda_k}(b_1, \dots, b_k) \times \partial J_{\lambda_{k+1}, \dots, \lambda_p}(b_{k+1}, \dots, b_p). \quad (10)$$

## Proof of Theorem 2

Let  $\pi \in S_p$  and  $\epsilon \in \{-1, 1\}^p$  be such that

$$|\hat{\beta}|_{\downarrow} = (\epsilon_1 \hat{\beta}_{\pi(1)}, \dots, \epsilon_p \hat{\beta}_{\pi(p)}),$$

and let  $\phi$  be the orthogonal transformation defined as follows:

$$\forall x \in \mathbb{R}^p \quad \phi(x) = (\epsilon_1 x_{\pi(1)}, \dots, \epsilon_p x_{\pi(p)}).$$

**Proof of 1)** Because  $\hat{\beta} \in \mathcal{S}(\gamma)$  is a SLOPE minimizer, the following equivalence holds:

$$\frac{1}{\gamma} X'(y - \hat{\text{fit}}(\gamma)) \in \partial J_\lambda(\hat{\beta}) \Leftrightarrow \phi \left( \frac{1}{\gamma} X'(y - \hat{\text{fit}}(\gamma)) \right) \in \phi(\partial J_\lambda(\hat{\beta})) = \partial J_\lambda(|\hat{\beta}|_{\downarrow}).$$

Since the components of  $|\hat{\beta}|_{\downarrow}$  are decreasing,  $\partial J_\lambda(|\hat{\beta}|_{\downarrow})$  is a Cartesian product of permutahedra with potentially a signed permutahedron (if  $\hat{\beta}$  has a null component) [Dupuis and Tardivel 2022](#), [Schneider and Tardivel 2022](#). Specifically, we have

$$\partial J_\lambda(|\hat{\beta}|_{\downarrow}) = \begin{cases} P(\lambda_1, \dots, \lambda_{k_1}) \times \dots \times P(\lambda_{k_{l-1}+1}, \dots, \lambda_{k_l}) & \text{if } k_l = p, \\ P(\lambda_1, \dots, \lambda_{k_1}) \times \dots \times P(\lambda_{k_{l-1}+1}, \dots, \lambda_{k_l}) \times P^\pm(\lambda_{k_l+1}, \dots, \lambda_p) & \text{if } k_l < p. \end{cases}$$

According to [9](#), if  $b \in P(\lambda_1, \dots, \lambda_{k_1}) \times \dots \times P(\lambda_{k_{l-1}+1}, \dots, \lambda_{k_l})$ , then the following equalities hold:

$$\forall i \in \{k_1, \dots, k_l\}, \sum_{j=1}^i b_j = \|b\|_{(i)} = \sum_{j=1}^i \lambda_j.$$

Finally, since the  $i$ -norm  $\|\cdot\|_{(i)}$  is invariant by the transformation  $\phi$ , one may deduce the following equalities:

$$\forall i \in \{k_1, \dots, k_l\}, \frac{\left\| \phi \left( \frac{1}{\gamma} X'(y - \hat{\text{fit}}(\gamma)) \right) \right\|_{(i)}}{\sum_{j=1}^i \lambda_j} = \frac{\left\| \frac{1}{\gamma} X'(y - \hat{\text{fit}}(\gamma)) \right\|_{(i)}}{\sum_{j=1}^i \lambda_j} = 1.$$

**Proof of 2)** First, let us establish for  $b \in \mathbb{R}^p$  such that  $b_1 \geq \dots \geq b_p > 0$  the following inclusion:

$$\partial J_\lambda(b) \subset \text{conv} \{(\lambda_{\pi(1)}, \dots, \lambda_{\pi(p)}), \pi \in \mathcal{S}_p\} = P(\lambda_1, \dots, \lambda_p). \quad (11)$$

Since the sorted  $\ell_1$  norm is polyhedral, namely

$$J_\lambda(b) = \max \left\{ \sum_{i=1}^p \epsilon_i \lambda_{\pi(i)} b_i : \epsilon_1, \dots, \epsilon_p \in \{-1, 1\}, \pi \in \mathcal{S}_p \right\},$$

its subdifferential is given by

$$\partial J_\lambda(b) = \text{conv} \left\{ (\epsilon_1 \lambda_{\pi(1)}, \dots, \epsilon_p \lambda_{\pi(p)}), \epsilon_1, \dots, \epsilon_p \in \{-1, 1\}, \pi \in \mathcal{S}_p : \sum_{i=1}^p \epsilon_i \lambda_{\pi(i)} b_i = J_\lambda(b) \right\}.$$

Moreover, if  $\epsilon_{i_0} = -1$  for some  $i_0 \in \{1, \dots, p\}$ , then

$$\sum_{i=1}^p \epsilon_i \lambda_{\pi(i)} b_i < \lambda_{i_0} b_{i_0} + \sum_{i \neq i_0} \epsilon_i \lambda_{\pi(i)} b_i \leq J_\lambda(b).$$

Therefore  $(\epsilon_1 \lambda_{\pi(1)}, \dots, \epsilon_p \lambda_{\pi(p)}) \notin \partial J_\lambda(x)$ , which proves inclusion (11).

Now, let us assume that there exists  $i \notin \mathcal{A}(\gamma)$  such that

$$\begin{cases} |\widehat{\beta}|_{\downarrow i} > |\widehat{\beta}|_{\downarrow i+1} & \text{if } i \leq p-1, \\ |\widehat{\beta}|_{\downarrow i} > 0 & \text{if } i = p. \end{cases}$$

Then, according to (10) and (11), we have  $\partial J_{\lambda_1, \dots, \lambda_i}(|\widehat{\beta}|_{\downarrow 1}, \dots, |\widehat{\beta}|_{\downarrow i}) \subset P(\lambda_1, \dots, \lambda_i)$ . Consequently

$$\frac{\left\| \phi \left( \frac{1}{\gamma} X'(y - \widehat{\text{fit}}(\gamma)) \right) \right\|_{(i)}}{\sum_{j=1}^i \lambda_j} = \frac{\left\| \frac{1}{\gamma} X'(y - \widehat{\text{fit}}(\gamma)) \right\|_{(i)}}{\sum_{j=1}^i \lambda_j} = 1.$$

Therefore  $i \in \mathcal{A}(\gamma)$ , which leads to a contradiction.

## Proof of Lemma 1

If  $0 \in \mathcal{S}(\gamma)$  and since every elements in  $\mathcal{S}(\gamma)$  have the same sorted  $\ell_1$  norm, one may deduce that  $\mathcal{S}(\gamma) = \{0\}$ . Now, let us assume that  $0 \notin \mathcal{S}(\gamma)$ . Let  $\widehat{\beta} \in \mathcal{S}(\gamma)$  be such that the number of non-null clusters  $k = \|\text{patt}(\widehat{\beta})\|_\infty = \|m\|_\infty \geq 1$  is minimal. Let us prove that  $\ker(\tilde{X}_m) = \{0\}$ . If  $\dim(\ker(\tilde{X}_m)) \geq 1$ , then pick  $h \in \ker(\tilde{X}_m)$ ,  $h \neq 0$ . Then set  $\widehat{\beta} = U_m s$  where  $s \in \mathbb{R}^{k+}$  and  $c(t) = \widehat{\beta} + t U_m h = U_m(s + th)$ . Since  $\tilde{X}_m h = X U_m h = 0$ , then  $X'(y - Xc(t)) = X'(y - X\widehat{\beta})$ . Let  $t_{\min} = \inf\{t : s + th \notin \mathbb{R}^{k+}\} > 0$ ; by construction, for  $t \in (-t_{\min}, t_{\min})$ ,  $s + th \in \mathbb{R}^{k+}$  and thus  $\text{patt}(c(t)) = m$ . Consequently,

$$\begin{aligned} \forall t \in (-t_{\min}, t_{\min}) \quad X'(y - Xc(t)) &\in \partial J_\lambda(m) = \partial J_\lambda(c(t)), \\ \Rightarrow \forall t \in (-t_{\min}, t_{\min}) \quad c(t) &\in \mathcal{S}(\gamma). \end{aligned}$$

Since  $\mathcal{S}(\gamma)$  is a closed set, one may deduce that  $c(\pm t_{\min}) \in \mathcal{S}(\gamma)$ . Finally, by construction of  $t_{\min}$ , one of the vectors  $s + t_{\min} h$  or  $s - t_{\min} h$  does not have  $k$  distinct components, therefore  $\|\text{patt}(c(t_{\min}))\|_\infty < k$  or  $\|\text{patt}(c(-t_{\min}))\|_\infty < k$  which contradicts the fact that  $\widehat{\beta} \in \mathcal{S}(\gamma)$  has a minimal number of non-null clusters.

## SLOPE is a generalized LASSO if and only if $\lambda$ is an arithmetic progression

Let  $D \in \mathbb{R}^{m \times p}$ . The subdifferential at 0 of the function  $b \in \mathbb{R}^p \mapsto \|Db\|_1$  is  $D'[-1, 1]^m$ . The polytope  $D'[-1, 1]^m$  is a zonotope (the image of a cube under an affine transformation). On the other hand the signed permutahedron (the subdifferential at 0 of  $J_\lambda$ ) is a zonotope if and only if  $\lambda$  is an arithmetic progression [Godland and Kabluchko, 2023, Theorem 4.13]. Consequently, when  $\lambda$  is not an arithmetic progression one cannot pick a matrix  $D \in \mathbb{R}^{m \times p}$  such that  $J_\lambda(\cdot) = \|D\cdot\|_1$  thus SLOPE is not a generalized LASSO. Finally OSCAR (*i.e.* SLOPE when  $\lambda_1 \geq \dots \geq \lambda_p \geq 0$  is an arithmetic progression) is clearly a particular generalized LASSO.