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# The quadratic approximation of Bernoulli relation for shallow water model

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## Abstract

The aim of this paper is to propose a new approach to derive a well-balanced Godunov type scheme to approximate the weak solutions of the shallow water equation. here, we propose an extension of the Godunov-type scheme by adopting a new linearization, which relies on a quadratic formulation, of the Bernoulli relation. This new linearization makes it possible to mimic more precisely the exact behavior of stationary solutions. The choice of this new quadratic formulation, allows us to define, without additional correction, a discretization always well defined whatever the Froude number. We establish that the Godunov type finite volume scheme resulting from our developments verifies the essential properties, namely the preservation of the positivity of the water height, the capture of all stationary solutions for a Froude number different to one and finally an entropy inequality defined relatively to perturbation in  $\mathcal{O}(\Delta x)$ . Several numerical experiments are using to illustrate the relevance of the designed scheme.

**Keywords:** Balance laws, Fully Well-balanced schemes, Hyperbolic conservation laws schemes, Godunov-type schemes, shallow water, stationary solutions.

## 1 Introduction

With climate change, humanity is increasingly faced with natural disasters. In particular, in recent years, we face floods that cause significant loss of life and property damage. Thus, the study of the flows in natural environment is a major stake for the human society. The Saint-Venant equations, introduced in 1871, model free surface flows. It is a system of partial differential equations with a topographic source term and governed by

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\frac{h^2}{2}\right) = -gh\partial_x Z, \end{cases} \quad (1)$$

where  $h(x, t)$  is the height of the water and  $u(x, t)$  is the speed of the water. of the water. It is important to note that the height  $h$  must be positive or zero. The coefficient  $g > 0$  is the gravity constant. The function  $Z(x)$ , assumed to be regular, represents the topography.

Finally, the water flow rate is defined by  $q = hu$ . For the sake of simplicity in the forthcoming notations, we set

$$W = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad F(W) = \begin{pmatrix} hu \\ hu^2 + g\frac{h^2}{2} \end{pmatrix} \quad \text{and} \quad S(W, Z) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix}. \quad (2)$$

In addition, we introduce  $\Omega \subset \mathbb{R}^2$  the set of physical admissible states given by

$$\Omega = \{W \in \mathbb{R}^2; h > 0, u \in \mathbb{R}\}.$$

The purpose of this paper is to make a numerical approximation of the weak solutions of this system (1). However, in order to ensure the relevance of the numerical simulation, the developed numerical scheme must preserve two major properties, namely the preservation of the positivity of the water height and the capture of stationary solutions due to the presence of the topography source term. Indeed, the topography source term implies the existence of particular time-independent solutions governed by the following system:

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x\left(hu^2 + g\frac{h^2}{2}\right) = -gh\partial_x z. \end{cases} \quad (3)$$

The pioneering work of A.Bermudez and M-E.Vazquez [...] and J-M.Greenberg and A-Y.LeRoux [...](see also L.Goss [...] or S.Jin [...]), have shown that these stationary solutions are accurate (see exact) to guarantee the relevance of the designed scheme. During the last two decades, many works have been devoted to the derivation of numerical schemes to approximate the weak solutions of (1) and able to capture with relevance the stationary solutions. According to [...], if the equilibrium solutions are not correctly captured, then errors appear in the solution approximation that increase with time. Since (3) is non-linear, many works consider only the lake at rest, defined by  $u = 0$  and  $h + z$  equal to constant. For example, in [...], E.Audusse and al. propose a so-called hydrostatic reconstruction scheme. It is a scheme that allows in a very simple way to preserve the lake at rest.

Then, more recently, different works have been proposed to develop schemes capturing moving stationary solutions ( $u \neq 0$ ). In [...], the nonlinear system is solved at each interface to obtain a scheme preserving all stationary solutions. This approach is cumbersome because it requires solving a nonlinear system on each interface and at each time iteration. On the other hand, in [...], an extension of the hydrostatic reconstruction is proposed to capture, in addition to the lake at rest, stationary solutions in motion. But this approach poses a problem of preserving the positivity of the water height. Then, in [...], Y.Xing proposes an extension of the discontinuous Galerkin method.

More recently, in [...], a Godunov-type scheme for the system (1) is derived. This scheme is obtained by developing an approximate Riemann solver for equation (1) allowing an accurate consideration of the source term. The obtained scheme then preserves the positivity of the water height and captures exactly all stationary solutions. Unfortunately, here, a strongly nonlinear equation must be solved at each interface. Thus, in the same principle, [...], introduces a linearization technique on each interface. This approach allows to get rid of the highly nonlinear equations to be solved on each interface while preserving the capture of the stationary solutions and preserving the positivity of the water height. Moreover, this approach brings a very interesting calculation cost making this method particularly attractive. However, the discretization of the source term presents inconsistencies that must be solved. Thus, in [...],

we propose a new approximate Riemann solver which allows to remove the inconsistencies present in [...]. However, the scheme proposed in [...], has a shortcoming, because in very specific situations, it is possible to show that the scheme captures stationary solutions which are not entropic. For example, if we consider the initial data of a Riemann problem where the left and right states are characterized by the following relation:

$$\begin{cases} \frac{u_L^2}{2} + gh_L = \frac{u_R^2}{2} + gh_R = B, \\ q_L = q_R = Q. \end{cases} \quad (4)$$

For the function  $Z$  equal to a constant, this initial data coincides with the definition of a local stationary solution. Moreover, the solution associated with this initial data is not a stationary discontinuity but a standard dam break consisting of a 1-rarefaction wave and a 2-shock wave. Moreover, the scheme proposed in [...] coincides with the HLL scheme (see figure) and consequently, the expected solution has been obtained. On the other hand, by introducing a small perturbation in the topography noted  $Z^\delta$  so that  $Z$  is no longer constant, the discretization of the source term in [...] will be active locally around  $Z^\delta$ . Note that this perturbation is taken on a single mesh of size  $\Delta x$ . On this mesh containing the perturbation, we construct a state noted  $W^\delta$  such that:

$$\begin{cases} \mathcal{B}(W^\delta, Z^\delta) = B, \\ q^\delta = Q. \end{cases} \quad (5)$$

This new initial data verifies well the stationarity condition on each mesh. Therefore, the solution will remain stationary at all times by the scheme [...]. Unfortunately, this is not the expected solution since  $Z^\delta$  is a small perturbation of the topography and the entropic solution is in fact made of a 1-rarefaction and a 2-shock modulo the perturbation  $Z^\delta$ . However, it is shown in the numerical applications (see figure), that this stationary approximation seems unstable and the scheme proposed in [...], does not capture the initial stationary data but a weak solution with a small perturbation inside the relaxation and of the same order as the one obtained by other schemes.

In this paper, we will propose an extension of the [...] scheme, in order to correct the defect found on non-entropic stationary discontinuous solutions. to adress such an issue, the papier is organized as follow. First, we will discuss the motivations for this new linearization approach. In the next section, we will make a complete development of the approximated Riemann solver. Then, we highlight the different properties of our scheme. Finally, in the last section, we propose several numerical experiments to show the relevance of our new scheme.

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