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Strong approximations for a class of dependent random variables with semi exponential tails

C. Cuny*, J. Dedecker† and F. Merlevède ‡

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Abstract

We give rates of convergence in the almost sure invariance principle for sums of dependent random variables with semi exponential tails, whose coupling coefficients decrease at a sub-exponential rate. We show that the rates in the strong invariance principle are in powers of $\log n$. We apply our results to iid products of random matrices.

AMS 2020 subject classifications: 60F17 ; 60G10 ; 60J05 ; 60B15

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1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $(\varepsilon_i)_{i \geq 1}$ be independent and identically distributed (iid) random variables defined on Ω , with values in a measurable space G and with common distribution μ . Let W_0 be a random variable defined on Ω with values in a measurable space X , independent of $(\varepsilon_i)_{i \geq 1}$, and let F be a measurable function from $G \times X$ to X . For any $n \geq 1$, define

$$W_n = F(\varepsilon_n, W_{n-1}),$$

and assume that $(W_n, n \geq 1)$ has a stationary distribution ν . Let now h be a measurable function from $G \times X$ to \mathbb{R} and define, for any $n \geq 1$,

$$X_n = h(\varepsilon_n, W_{n-1}). \quad (1)$$

Then $(X_n)_{n \geq 1}$ forms a stationary sequence with stationary distribution, say π . Let $(\mathcal{G}_i)_{i \in \mathbb{Z}}$ be the non-decreasing filtration defined as follows: for any $i < 0$, $\mathcal{G}_i = \{\emptyset, \Omega\}$, $\mathcal{G}_0 = \sigma(W_0)$ and for any $i \geq 1$, $\mathcal{G}_i = \sigma(\varepsilon_i, \dots, \varepsilon_1, W_0)$. It follows that for any $n \geq 1$, X_n is \mathcal{G}_n -measurable.

We can also consider the following model

$$X_n = h(\varepsilon_n, \varepsilon_{n-1}, \dots), \quad (2)$$

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which is in fact included in the preceding situation, by taking $W_0 = (\varepsilon_0, \varepsilon_{-1}, \dots)$.

In what follows we assume that h is such that $\mathbb{E}(X_n) = 0$. For any $n \geq 0$, let us define the sequence $(\delta(n))_{n \geq 0}$ by

$$\delta(0) = \mathbb{E}(|X_1|) \text{ and } \delta(n) = \|X_n - X_n^*\|_1, n \geq 1,$$

where X_n^* is defined as follows : let W_0 and W_0^* be random variables with law ν , and such that W_0^* is independent of $(W_0, (\varepsilon_i)_{i \geq 1})$. For any $n \geq 1$, let

$$X_n^* = h(\varepsilon_n, W_{n-1}^*) \text{ with } W_n^* = F(\varepsilon_n, W_{n-1}^*). \quad (3)$$

Note that the coefficients $\delta(n)$ are well defined if π has a moment of order 1.

In the paper [9], we assumed that the coupling coefficients δ_n decrease at a polynomial rate, and we used a variant of the Berkes-Liu-Wu method (see [3]) to establish strong approximation results for partial sums. Another more restrictive coefficient is also considered in [9] (see also [10] for variants of these coefficients and the resulting conditions). In [9] we applied our results to various classes of (possibly non irreducible) Markov chains; in particular, we obtained optimal rates in the almost sure invariance principle (ASIP) for some functions of the left random walk on $GL_d(\mathbb{R})$ under polynomial moment conditions. In [6] and [7] we adapted the proofs of these results to Birkhoff sums of Hölder observables of non-uniformly expanding dynamical systems, using a representation as functions of the trajectory of a particular Markov Chain due to Korepanov [18]. In the paper [7], we consider dynamical systems with exponential or sub-exponential decay of correlations, and we show that the rates in the ASIP are in powers of $\log n$.

In the present paper, we consider the context of [9], assuming that the coefficients δ_n decrease at an exponential, sub-exponential or super-exponential rate (described by an index γ_1 , see (4)), and that the variables have an exponential, sub-exponential or super-exponential moments (described by an index γ_2 , see (5)). Using arguments from [7], we show that the rates in the ASIP are in powers of $\log n$ (the exponent of the logarithm depending on γ_1 and γ_2). We apply our results to the left random walk on $GL_d(\mathbb{R})$, to iid products of positive matrices, and to a class of non-uniformly contracting auto-regressive processes.

2 Main results

We assume that there exist two positive constants γ_1 and c such that

$$\delta(n) \leq \exp(-cn^{\gamma_1}) \text{ for any positive integer } n, \quad (4)$$

and there are constant $b \in]0, \infty[$ and $\gamma_2 \in]0, \infty]$ such that

$$\mathbb{P}(|X_1| > t) \leq \exp(1 - (t/b)^{\gamma_2}) \text{ for any positive } t, \quad (5)$$

Note that when $\gamma_2 = \infty$ (5) means that $\|X_n\|_\infty \leq b$ a.s. for any integer n . When X_1 satisfies (5) we say either that it has a semi exponential tail of order γ_2 or that it admits a sub-exponential or super-exponential moment of order γ_2 .

Theorem 1 *Let $(X_n)_{n \geq 1}$ be defined by (1) or (2). Let $S_n = \sum_{k=1}^n X_k$. Assume that (4) and (5) are satisfied. Then $n^{-1}\mathbb{E}(S_n^2) \rightarrow \sigma^2$ as $n \rightarrow \infty$ and one can redefine $(X_n)_{n \geq 1}$ without changing*

its distribution on a (richer) probability space on which there exist iid random variables $(N_i)_{i \geq 1}$ with common distribution $\mathcal{N}(0, \sigma^2)$, such that, as $n \rightarrow \infty$,

$$S_n - \sum_{i=1}^n N_i = O((\log n)^{1+\lambda}) \quad \mathbb{P}\text{-a.s. where } \lambda = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}. \quad (6)$$

Proof of Theorem 1. Let $\alpha = 1 + \lambda$. The proof uses similar arguments as those developed in the proof of Theorem 1.6 in [7] with the main difference that the random variables X_k are no longer bounded. we then have to truncate them. With this aim, let us first define for any positive integer k ,

$$M_k = c_1 k^{1/\gamma_2} \text{ where } c_1 = b(2 \log 3)^{1/\gamma_2}, \quad (7)$$

(recall that b is the constant appearing in condition (5)) and

$$\varphi_k(x) = (x \wedge M_k) \vee (-M_k).$$

Then we set

$$X_{k,j} = \varphi_k(X_j) - \mathbb{E}\varphi_k(X_j) \text{ and } W_{k,\ell} = \sum_{i=1+3^{k-1}}^{\ell+3^{k-1}} X_{k,i}.$$

For $n \geq 2$, let $h_n = \lceil (\log n)/(\log 3) \rceil$ (so that h_n is the unique integer such that $3^{h_n-1} < n \leq 3^{h_n}$). Note that

$$S_n = \sum_{k=1}^{h_n-1} \sum_{i=1+3^{k-1}}^{3^k} X_i + \sum_{i=1+3^{h_n-1}}^n X_i,$$

and set

$$S_n^\dagger = \sum_{k=1}^{h_n-1} W_{k,3^k-3^{k-1}} + \sum_{i=1+3^{h_n-1}}^n X_{h_n,i}.$$

Let also $S_1^\dagger = 0$. We first prove that

$$\max_{1 \leq i \leq n} |S_i - S_i^\dagger| = O((\log n)^\alpha) \quad \mathbb{P}\text{-a.s.} \quad (8)$$

With this aim, it suffices to prove via the Kronecker lemma and stationarity that

$$\sum_{k \geq 1} k^{-\alpha} 3^k \mathbb{E}(|X_1| - M_k)_+ < \infty. \quad (9)$$

Obviously this holds when $\gamma_2 = \infty$. Assume now that $\gamma_2 < \infty$ and note that since $3^k = \exp\left(\frac{1}{2}\left(\frac{M_k}{b}\right)^{\gamma_2}\right)$,

$$3^k \mathbb{E}(|X_1| - M_k)_+ = 3^k \int_{M_k}^{\infty} \mathbb{P}(|X_1| > t) dt \leq \int_{M_k}^{\infty} \exp(1 - 2^{-1}(t/b)^{\gamma_2}) dt \leq \frac{b 2^{1/\gamma_2}}{\gamma_2} \Gamma((1-\gamma_2)/\gamma_2) < \infty.$$

Since $\alpha > 1$, (9) is proved and so (8) also.

Now, for any $k \geq 1$, let

$$m_k = \lceil c_2 k^{1/\gamma_1} \rceil + 1 \text{ where } c_2 = (2c^{-1}(\log 3))^{1/\gamma_1}, \quad (10)$$

(recall that c is the constant appearing in condition (4)). We then define

$$\tilde{X}_{k,j} = \mathbb{E}(\varphi_k(X_j) | \varepsilon_j, \varepsilon_{j-1}, \dots, \varepsilon_{j-m_k}) - \mathbb{E}(\varphi_k(X_j)) \text{ for any } j \geq m_k \text{ and } \widetilde{W}_{k,\ell} = \sum_{i=1+3^{k-1}}^{\ell+3^{k-1}} \tilde{X}_{k,i}.$$

Finally, set $\tilde{S}_1 = 0$ and for $n > 1$,

$$\tilde{S}_n = \sum_{k=1}^{h_n-1} \widetilde{W}_{k,3^k-3^{k-1}} + \widetilde{W}_{h_n,n-3^{h_n-1}}.$$

Note that by Lemma 24 in [9], for any $j \geq m_k$,

$$\|X_{k,j} - \tilde{X}_{k,j}\|_1 \leq \iint \mathbb{E}(|X_{m_k+1,x} - X_{m_k+1,y}|) \nu(dx) \nu(dy) \leq \delta(m_k). \quad (11)$$

Hence, for $k \geq k_0$ where k_0 is such that for any $\ell \geq k_0$, $m_\ell \leq 3^{\ell-1}$, the upper bound (11) implies that

$$\left\| \max_{1 \leq \ell \leq 3^k-3^{k-1}} |W_{k,\ell} - \widetilde{W}_{k,\ell}| \right\|_1 \leq \sum_{i=1+3^{k-1}}^{3^k} \|X_{k,i} - \tilde{X}_{k,i}\|_1 \leq 3^k \delta(m_k) \ll 3^k \exp(-c \times c_2^{\gamma_1} k).$$

Hence, since $c \times c_2^{\gamma_1} \geq \log 3$,

$$\sum_{k \geq k_0} k^{-\alpha} \left\| \max_{1 \leq \ell \leq 3^k-3^{k-1}} |W_{k,\ell} - \widetilde{W}_{k,\ell}| \right\|_1 < \infty,$$

which entails by the Kronecker lemma that

$$\max_{1 \leq i \leq n} |S_i^\dagger - \tilde{S}_i| = o((\log n)^\alpha) \quad \mathbb{P}\text{-a.s.} \quad (12)$$

From (8) and (12), the theorem will follow if one can prove that there exists a standard Brownian motion B such that such that, as $n \rightarrow \infty$,

$$\max_{1 \leq i \leq n} |\tilde{S}_i - B(i\sigma^2)| = O((\log n)^\alpha) \quad \mathbb{P}\text{-a.s.} \quad (13)$$

To prove (13), we do exactly as in Steps 3 and 4 of the proof of [7, Theorem 1.6] by replacing the upper bound $\|\psi\|_\infty$ by $2M_k$ (which is a bound of the sup norm of the truncated random variables $X_{k,j}$), and by noticing that $M_k m_k \ll k^{1/\gamma}$. This leads to (3.20) in [7] which combined with (3.9) in [7] gives the following strong approximation: there exists a standard Brownian motion B such that such that, as $n \rightarrow \infty$,

$$\max_{1 \leq i \leq n} |\tilde{S}_i - B(\sigma_i^2)| = O((\log n)^\alpha) \quad \mathbb{P}\text{-a.s.} \quad (14)$$

where σ_i^2 is defined by (3.21) in [7]. It remains to identify the variance of the Brownian motion and to show that

$$\max_{1 \leq i \leq n} |B(\sigma_i^2) - B(i\sigma^2)| = O((\log n)^\alpha) \quad \mathbb{P}\text{-a.s.} \quad (15)$$

With this aim, we shall proceed as in step 3.4 in [3] (see also Step 5 in [7]). We set

$$\nu_k = \sigma^2 + 2 \sum_{i=-m_k}^{m_k} (\tilde{\gamma}_{k,i} - \gamma_i) - 2 \sum_{i \geq m_k+1} \gamma_i, \quad (16)$$

where

$$\tilde{\gamma}_{k,i} = \text{cov}(\tilde{X}_{k,m_k}, \tilde{X}_{k,i+m_k}), \quad \hat{\gamma}_{k,i} = \text{cov}(X_{k,0}, X_{k,i}) \quad \text{and} \quad \gamma_i = \text{cov}(X_0, X_i).$$

To prove (15), we need to prove that

$$(\log n) \max_{k \leq h_n} (m_k \nu_k)^{1/2} = O((\log n)^\alpha), \quad (17)$$

and

$$3^k (\nu_k^{1/2} - \sigma)^2 = O(k^{2\alpha} (\log k)^{-1}). \quad (18)$$

By using the upper bound (11), we have, for any $i \geq 0$,

$$\begin{aligned} |\tilde{\gamma}_{k,i} - \hat{\gamma}_{k,i}| &= |\text{cov}(\tilde{X}_{k,m_k} - X_{k,m_k}, \tilde{X}_{k,i+m_k}) + \text{cov}(X_{k,m_k}, \tilde{X}_{k,i+m_k} - X_{k,i+m_k})| \\ &\leq 2M_k \|\tilde{X}_{k,m_k} - X_{k,m_k}\|_1 + 2M_k \|\tilde{X}_{k,i+m_k} - X_{k,i+m_k}\|_1 \ll k^{1/\gamma_2} \exp(-c \times c_2^{\gamma_1} k). \end{aligned} \quad (19)$$

Next, by using inequality (1.11a) in [21], note that

$$\begin{aligned} |\hat{\gamma}_{k,i} - \gamma_i| &= |\text{cov}(X_{k,0} - X_0, X_{k,i}) + \text{cov}(X_0, X_{k,i} - X_i)| \\ &\leq 2 \int_0^{1/2} Q_{|\varphi_k(X)|}(u) Q_{|g_k(X)|}(u) du + 2 \int_0^{1/2} Q_{|X_1|}(u) Q_{|g_k(X)|}(u) du \leq 4 \int_0^1 Q(u) (Q(u) - M_k)_+ du, \end{aligned}$$

where $g_k(x) = (|x| - M_k)_+ = x - \varphi_k(x)$. But, for any positive t , $\mathbb{P}(|X_1| > t) \leq \exp(1 - (t/b)^{\gamma_2}) := G(t)$. Therefore, for any $u \in]0, 1[$,

$$Q(u) \leq G^{-1}(u) := b(1 - \log u)^{1/\gamma_2}.$$

But

$$G^{-1}(u) > M_k \iff u^{-1} > e^{2k \log 3 - 1}.$$

So, overall,

$$\begin{aligned} |\hat{\gamma}_{k,i} - \gamma_i| &\leq 4b^2 \int_0^1 (\log(e/u))^{2/\gamma_2} \mathbf{1}_{\{u^{-1} > e^{2k \log 3 - 1}\}} du = 4eb^2 \int_{e^{2k \log 3}}^\infty (\log x)^{2/\gamma_2} x^{-2} dx \\ &= 4eb^2 \int_{2k \log 3}^\infty x^{2/\gamma_2} e^{-x} dx \leq \kappa_{\gamma_2} b^2 (1 + k^{2/\gamma_2}) e^{-2k \log 3}, \end{aligned}$$

where κ_{γ_2} is a positive constant depending only on γ_2 .

Next, by using Proposition 1 in [13], we derive

$$\sum_{i \geq m_k} |\text{cov}(X_0, X_i)| \leq 2 \sum_{i \geq m_k} \int_0^{\delta(i)} Q \circ H^{-1}(u) du \leq 2 \sum_{i \geq m_k} \int_0^{H(\delta(i))} Q^2(u) du,$$

where H is the function defined on $[0, 1]$ by $H(v) := \int_0^v Q(u)du$ and H^{-1} is its inverse. Since $Q(u) \leq b(1 - \log u)^{1/\gamma_2}$, we infer that there exists a positive constant κ depending on c, b, γ_1 and γ_2 , such that, for any $i \geq 1$,

$$\int_0^{H(\delta(i))} Q^2(u)du \leq \kappa i^{2\gamma_1/\gamma_2} \exp(-ci^{\gamma_1}).$$

Therefore, since $m_k = \lfloor c_2 k^{1/\gamma_1} \rfloor + 1$,

$$\sum_{i \geq m_k} |\gamma_i| \leq \tilde{\kappa} k^{\frac{2}{\gamma_2} + \frac{1}{\gamma_1} - 1} \exp(-c \times c_2^{\gamma_1} k),$$

where $\tilde{\kappa}$ is a positive constant depending only on c, b, γ_1, γ_2 and c_2 . So, overall, for any positive integer k ,

$$|\nu_k - \sigma^2| \leq K \times k^{\frac{2}{\gamma_2} + \frac{1}{\gamma_1}} \left\{ e^{-2k \log 3} + \exp(-c \times c_2^{\gamma_1} k) \right\}. \quad (20)$$

where K is a positive constant depending only on c, b, γ_1, γ_2 and c_2 . This shows that $\nu_k \rightarrow \sigma^2$, as $k \rightarrow \infty$. Hence (17) is satisfied since $(\log n) \max_{k \leq h_n} (m_k)^{1/2} \ll (\log n)^{1+1/(2\gamma_1)} = O((\log n)^\alpha)$ (indeed $\alpha = 1 + 1/\gamma_1 + 1/\gamma_2$). Note now that since $|\nu_k^{1/2} - \sigma|^2 \leq |\nu_k - \sigma^2|$, (20) implies (18) since we have selected c_2 such that $c \times c_2^{\gamma_1} = 2 \log 3$. The proof is complete. \diamond

3 Applications

3.1 Products of iid invertible matrices

Let $(\varepsilon_n)_{n \geq 1}$ be independent random matrices taking values in $G = GL_d(\mathbb{R})$, $d \geq 2$, with common distribution μ . Let $A_0 = \text{Id}$ and for every $n \geq 1$, $A_n = \varepsilon_n \cdots \varepsilon_1$.

Let $\|\cdot\|$ be the euclidean norm on \mathbb{R}^d and for any $g \in G$, let $N(g) := \max(\|g\|, \|g^{-1}\|)$ where $\|g\| = \sup_{\|x\|=1} \|gx\|$. Recall that μ has a moment of order $p \geq 1$ if $\int_G (\log N(g))^p \mu(dg) < \infty$ and a sub-exponential moment of order $\gamma \in]0, 1]$ if there exists $c > 0$ such that

$$\int_G e^{c(\log N(g))^\gamma} \mu(dg) < \infty. \quad (21)$$

Recall also that if μ admits a moment of order 1 then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|A_n\| = \lambda_\mu \quad \mathbb{P}\text{-a.s.}, \quad (22)$$

where $\lambda_\mu := \lim_{n \rightarrow +\infty} n^{-1} \mathbb{E}(\log \|\varepsilon_n \cdots \varepsilon_1\|)$ is the so-called first Lyapunov exponent (see for instance [16]).

Let $X := P_{d-1}(\mathbb{R}^d)$ be the projective space of $\mathbb{R}^d - \{0\}$ and write \bar{x} as the projection of $x \in \mathbb{R}^d - \{0\}$ to X . G is acting on X as follows: $g \cdot \bar{x} = \overline{gx}$ for every $(g, x) \in G \times \mathbb{R}^d - \{0\}$.

We assume that μ is strongly irreducible (i.e. that no proper finite union of subspaces of \mathbb{R}^d are invariant by Γ_μ , the closed semi-group generated by the support of μ) and proximal (i.e. that there exists a matrix in Γ_μ admitting a unique (with multiplicity one) eigenvalue with maximum modulus). Under those assumptions (see e.g. Bougerol-Lacroix [4] or Benoist-Quint

[1]) it is well-known that there exists a unique invariant measure ν on $\mathcal{B}(X)$, meaning that for any continuous and bounded function f from X to \mathbb{R} ,

$$\int_X f(x)\nu(dx) = \int_G \int_X f(g \cdot x)\mu(dg)\nu(dx).$$

The left random walk of law μ on X is the process defined as follows. Let W_0 be an X -valued random variable independent of $(\varepsilon_n)_{n \geq 1}$, and let $W_n = \varepsilon_n \cdot W_{n-1}$ for $n \geq 1$. Note that, if W_0 has distribution ν , then $(W_n)_{n \geq 0}$ is a strictly stationary Markov chain.

Our aim is to study the partial sums associated with the random sequence $(X_n)_{n \geq 1}$ given by

$$X_n := h(\varepsilon_n, W_{n-1}), \quad n \geq 1,$$

where for every $g \in G$ and every $\bar{x} \in X$,

$$h(g, \bar{x}) = \log \left(\frac{\|g\bar{x}\|}{\|\bar{x}\|} \right).$$

As usual, we shall denote by $X_{n,\bar{x}}$ the random variable for which $W_0 = \bar{x}$. We then define $S_{n,\bar{x}} = \sum_{k=1}^n X_{k,\bar{x}}$. Note that, for any $x \in S^{d-1}$,

$$S_{n,\bar{x}} = \sum_{k=1}^n X_{k,\bar{x}} = \log \|A_n x\|.$$

Then, when we keep the notation $(X_n)_{n \geq 1}$, we have in mind that we are in stationary regime, i.e. that W_0 has law ν . In this case we also define

$$S_n = \sum_{k=1}^n X_k = \log \|A_n V_0\|,$$

where V_0 is such that $\|V_0\| = 1$ and $W_0 = \bar{V}_0$.

We denote by $\rho(g)$ the spectral radius of a matrix g . Applying Theorem 1, the following strong approximation with rate holds.

Corollary 2 *Let μ be a proximal and strongly irreducible probability measure on $\mathcal{B}(G)$. Assume that μ has a sub-exponential moment of order $\gamma \in]0, 1]$. Then $n^{-1}\mathbb{E}_\nu((S_n - n\lambda_\mu)^2) \rightarrow \sigma^2$ as $n \rightarrow \infty$ and for every (fixed) $\bar{x} \in X$, one can redefine $(S_{n,\bar{x}})_{n \geq 1}$ without changing its distribution on a (richer) probability space on which there exist iid random variables $(N_i)_{i \geq 1}$ with common distribution $\mathcal{N}(0, \sigma^2)$, such that,*

$$S_{n,\bar{x}} - n\lambda_\mu - \sum_{i=1}^n N_i = O((\log n)^{1+2/\gamma}) \text{ a.s.}$$

Moreover, the result remains true if we replace the sequence $(S_{n,\bar{x}})_{n \geq 1}$ with $(S_n)_{n \geq 1}$, $(\log \|A_n\|)_{n \geq 1}$, $(\log \rho(A_n))_{n \geq 1}$ or $(\log |\langle A_n x, y \rangle|)_{n \geq 1}$, for some $x, y \in S^{d-1}$.

Remark 3 *It follows from item c) of Theorem 4.11 of Benoist-Quint [1] that $\sigma > 0$ if μ is strongly irreducible and the image of Γ_μ in $\mathrm{PGL}_d(\mathbb{R})$ is unbounded.*

Remark 4 If μ has a super-exponential moment meaning that (21) holds with $\gamma > 1$, we infer from the proof of Corollary 2 that for every (fixed) $\bar{x} \in X$, one can redefine $(S_{n,\bar{x}})_{n \geq 1}$ without changing its distribution on a (richer) probability space on which there exist iid random variables $(N_i)_{i \geq 1}$ with common distribution $\mathcal{N}(0, \sigma^2)$, such that,

$$S_{n,\bar{x}} - n\lambda_\mu - \sum_{i=1}^n N_i = O((\log n)^{2+1/\gamma}) \text{ a.s.}$$

In particular if μ has compact support, the rate in the almost sure invariance principle is of order $O((\log n)^2)$. Again, this extension holds with $(S_n)_{n \geq 1}$, $(\log \|A_n\|)_{n \geq 1}$, $(\log \rho(A_n))_{n \geq 1}$ or $(\log |\langle A_n x, y \rangle|)_{n \geq 1}$, for some $x, y \in S^{d-1}$.

Proof of Corollary 2. It follows from Theorem 2 (ii) in [5] that $n^{-1}\mathbb{E}_\nu((S_n - n\lambda_\mu)^2) \rightarrow \sigma^2$ as $n \rightarrow \infty$. Then, the strong invariance principle in stationary regime, i.e. for $(S_n)_{n \geq 1}$, is a direct application of Theorem 1 by taking into account the estimate given in Proposition 5 below. The proof of Proposition 5 is given in appendix.

Proposition 5 Assume that μ satisfies (21) with $\gamma > 0$. Then, there exists $\beta > 0$ such that, for any positive integer k ,

$$\sup_{\bar{x}, \bar{y} \in X} \mathbb{E}(|X_{k,\bar{x}} - X_{k,\bar{y}}|) \ll e^{-\beta k^{\min(\gamma, 1)}}. \quad (23)$$

As in [5] we obtain the result for the matrix norms $(\log(\|A_n\|))_{n \geq 1}$ from the stationary regime, using an argument from [4]. Indeed, it suffices to apply the estimate given just after equation (66) of [5] to get the ASIP in that case. Similarly, we get the ASIP for $(S_{n,\bar{x}})_{n \geq 1}$, for any $x \in S^{d-1}$.

It remains to handle the case of the matrix coefficients and of the spectral radius. The proof follows a well-known scheme that has been used in [2] under exponential moments and in [1] and [12] under polynomial moments. Those proofs rely on large deviations inequalities. In the case of sub-exponential moments, the needed large deviations estimates were obtained in [8].

We start with an auxiliary result of independent interest, see Theorem 2.6 of [17], for a related result.

Proposition 6 Under the condition of Corollary 2, there exists $\eta > 0$ such that

$$\sup_{\bar{x} \in X} \int_X e^{\eta |\log \delta(\bar{x}, \bar{y})|^\gamma} d\nu(\bar{y}) < \infty,$$

where $\delta(\bar{x}, \bar{y}) = |\langle x, y \rangle| / (\|x\| \|y\|)$.

We can now prove the result for the matrix coefficients.

Let η be as in Proposition 6. Let $n \in \mathbb{N}$ and $\bar{x} \in X$. Using that ν is μ -invariant and Markov's inequality, we have

$$\mathbb{P}\left(-\log \delta(W_n, \bar{x}) \geq (2 \log n / \eta)^{1/\gamma}\right) = \nu\left(\left\{\bar{y} \in X : e^{\eta |\log \delta(\bar{y}, \bar{x})|^\gamma} \geq n^2\right\}\right) \leq K/n^2.$$

Hence, by the Borel-Cantelli lemma we see that, \mathbb{P} -a.s.

$$-\log \delta(W_n, \bar{x}) \leq (2 \log n / \eta)^{1/\gamma},$$

for all but finitely many $n \in \mathbb{N}$. Combining this with the computations after (67) (and page 1864) in [5], we infer that for every $\bar{x}, \bar{y} \in X$,

$$|\log \delta(A_n \cdot \bar{x}, \bar{y})| = O((\log n)^{1/\gamma}),$$

and the desired result follows since

$$\log |\langle A_n x, y \rangle| = \log \|A_n x\| + \log \|y\| + \log \delta(A_n \cdot \bar{x}, \bar{y}).$$

It remains to prove the ASIP for the spectral radius. The result follows from the following lemma, with $\ell \geq (2 \log n / c)^{1/\gamma}$, combined with the Borel-Cantelli lemma.

Lemma 7 *Under the assumption of Corollary 2, for every $\varepsilon > 0$, there exists $C, c > 0$ and $\ell_0 \in \mathbb{N}$ such that for every $\ell_0 \leq \ell \leq n$,*

$$\mathbb{P}(\log \rho(A_n) - \|A_n\| \geq -\varepsilon \ell) \geq 1 - Ce^{-c\ell^\gamma}.$$

The lemma may be proved as Lemma 8 in [12] using our Lemma 12 from the Appendix instead of Lemma 6 in [12].

3.2 Products of iid positive matrices

Let us assume now that G is the semi-group of d -dimensional positive allowable matrices: by positive, we mean that all entries are greater than or equal to 0, by allowable, we mean that any row and any column admits a strictly positive element.

In this case, G is acting on $X = S^{d-1} \cap (\mathbb{R}^+)^d$ as follows: $g \cdot x = gx / \|gx\|$, and we define $N(g)$ by $N(g) = \max(\|g\|, 1/v(g))$, where $v(g) := \inf_{x \in X} \|gx\|$. Here, $\|\cdot\|$ stands for the ℓ^1 norm on \mathbb{R}^d which is more convenient for the problem, see [11]. This change does not affect the final results since all norms on \mathbb{R}^d are equivalent.

Then, similarly to the previous section, we define polynomial moments of order $p \geq 1$ for μ as well as sub-exponential moments of order $\gamma \in]0, 1]$.

We shall also say that μ is strictly contracting if its support contains a matrix whose all entries are *strictly* positive. When μ is strictly contracting there exists a unique μ -invariant probability ν on $\mathcal{B}(X)$, see Section 3 of [11].

We have the following analogue to Corollary 2 with $S_{n,x} = \log \|A_n x\|$, $x \in X$, where $A_n = \varepsilon_n \cdots \varepsilon_1$ with $(\varepsilon_n)_{n \geq 1}$ a sequence of independent random matrices taking values in G , $d \geq 2$, with common distribution μ . Let also W_0 be an X -valued random variable with law ν and independent of $(\varepsilon_n)_{n \geq 1}$, and $S_n = \log \|A_n W_0\|$. The following corollary was announced in [11].

Corollary 8 *Let μ be a strictly contracting probability measure on $\mathcal{B}(G)$. Assume that μ has a sub-exponential moment of order $\gamma \in]0, 1]$. Then $n^{-1} \mathbb{E}_\nu((S_n - n\lambda_\mu)^2) \rightarrow \sigma^2$ as $n \rightarrow \infty$ and for every (fixed) $x \in X$, one can redefine $(S_{n,x})_{n \geq 1}$ without changing its distribution on a (richer)*

probability space on which there exist iid random variables $(N_i)_{i \geq 1}$ with common distribution $\mathcal{N}(0, \sigma^2)$, such that,

$$S_{n,x} - n\lambda_\mu - \sum_{i=1}^n N_i = O((\log n)^{2+1/\gamma}) \text{ a.s.}$$

Moreover, the result remains true if we replace the sequence $(S_{n,x})_{n \geq 1}$ with $(S_n)_{n \geq 1}$, $(\log \|A_n\|)_{n \geq 1}$, $(\log \rho(A_n))_{n \geq 1}$, $(\log v(A_n))_{n \geq 1}$ or $(\log |\langle A_n x, y \rangle|)_{n \geq 1}$, for some $x, y \in X$.

Remark 9 When μ is aperiodic, see Definition 5.1 of [11], then, by Proposition 5.2 of [11], $\sigma^2 > 0$. When μ admits super-exponential moments, it is possible to improve the rate as in the remark after Corollary 2.

Proof of Corollary 8. Again, we start with the stationary regime. The existence of the asymptotic variance follows from Proposition 5.2 of [11].

By Proposition 3.2 of [11], when μ is strictly contracting and admits a moment of order greater than one, (23) holds. Then, the result (for $(S_n)_{n \geq 1}$) follows from Theorem 1.

The ASIP for $(\log \|A_n\|)_{n \geq 1}$ and $(\log v(A_n))_{n \geq 1}$ follows from (4.3) in [11] and the fact that for every $x \in X$, $v(A_n) \leq \|A_n x\| \leq \|A_n\|$. Similarly, we obtain the ASIP for $(S_{n,x})_{n \geq 1}$ for any $x \in X$.

Then, the ASIP for $(\log \rho(A_n))_{n \geq 1}$ follows from (4.3) in [11], combined with the inequalities, for every $n \geq 1$, $\log v(A_n) \leq \log \rho(A_n) \leq \log \|A_n\|$.

The ASIP for the matrix coefficients is proved in [11], see Theorem 6.4.

3.3 Lipschitz autoregressive models

We consider the autoregressive Lipschitz model introduced by Dedecker and Rio [14]. Let $\tau \in [0, 1)$, $C \in (0, 1]$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ a 1-Lipschitz function such that

$$f(0) = 0 \quad \text{and} \quad |f'(t)| \leq 1 - \frac{C}{(1 + |t|)^\tau} \quad \text{for almost every } t.$$

Let $(\varepsilon_i)_{i \geq 1}$ be iid real-valued random valued with common law μ and define for any $n \geq 1$

$$W_n = f(W_{n-1}) + \varepsilon_n, \quad \text{with } W_0 \text{ independent of } (\varepsilon_i)_{i \geq 1}. \quad (24)$$

Let $S_n(g) = \sum_{k=1}^n g(W_k)$ for any measurable function g . Assume that μ has a sub-exponential moment of order $\eta \in]0, 1]$ (hence $\mathbb{E}(e^{c|\varepsilon_0|^\eta}) < \infty$, for some $c > 0$). For this model, there exists an unique invariant probability measure ν (see Proposition 2 of [14]) and the following result holds:

Corollary 10 Assume that μ has a sub-exponential moment of order $\eta \in]0, 1]$. Let $(W_n)_{n \geq 0}$ be defined by (24) with $\mathcal{L}(W_0) = \nu$. Assume furthermore that g is Lipschitz and that, for some $\zeta \in [0, 1]$ and some positive constant κ , $|g(x)| \leq \kappa(1 + |x|^\zeta)$. Suppose that $\tau + \zeta > 0$. Then, $n^{-1} \text{Var}(S_n(g)) \rightarrow \sigma^2(g)$ as $n \rightarrow \infty$ and one can redefine $(W_n)_{n \geq 0}$ without changing its

distribution on a (richer) probability space on which there exist iid random variables $(N_i)_{i \geq 1}$ with common distribution $\mathcal{N}(0, \sigma^2(g))$, such that,

$$S_n(g) - n\nu(g) - \sum_{i=1}^n N_i = O((\log n)^{1+1/\gamma_1+1/\gamma_2}) \quad \mathbb{P}\text{-a.s.}$$

where $\gamma_1 = \eta(1 - \tau)(\eta(1 - \tau) + \tau)^{-1}$ and $\gamma_2 = \eta(1 - \tau)/\zeta$.

Proof. The result comes from an application of Theorem 1 by taking into account Proposition 3 of [20] and its proof. More precisely, [20, (4.3)] and [20, Remark 5] ensure that $g(W_1) - \nu(g)$ satisfies (5) with index $\gamma_2 = \eta(1 - \tau)/\zeta$. In addition, as it appears clearly in the proof of [20, (4.4)], the control of the τ -coefficients is done via a control of the coupling coefficients $\delta(n)$ (see [20, (4.6)]). \square

4 Appendix

4.1 Proof of Proposition 5

The proof of the proposition is based on the following lemma whose statement needs the following notation: For $\bar{x}, \bar{y} \in X$, define

$$d(\bar{x}, \bar{y}) := \frac{\|x \wedge y\|}{\|x\| \|y\|}$$

where \wedge stands for the exterior product, see e.g. [4, page 61] for the definition and some properties. Then d is a metric on X .

Lemma 11 *Assume that μ satisfies (21) with $\gamma > 0$. Then, for any positive integer k and any $\kappa > 0$, there exists $\ell > 0$, not depending on k , such that*

$$\max_{k \leq j \leq 2k} \sup_{\bar{x}, \bar{y} \in X, \bar{x} \neq \bar{y}} \mathbb{P}(\log(d(A_{j-1} \cdot \bar{x}, A_{j-1} \cdot \bar{y})) \geq -\ell k) \ll e^{-\kappa k^{\min(\gamma, 1)}}.$$

Proof of Lemma 11. Without loss of generality, let us assume that $c > 4$ in condition (21). The general case can be deduced by normalizing suitably the quantities of interest. We proceed with similar arguments as in the proof of Lemma 6 in [5]; namely, we use a martingale decomposition.

Since $\int_G \log N(g) \mu(dg) < \infty$, we can define the following bounded function F_1 :

$$F_1(\bar{x}, \bar{y}) = \int_G \log(d(g \cdot \bar{x}, g \cdot \bar{y}) / d(\bar{x}, \bar{y})) \mu(dg), \quad \forall \bar{x}, \bar{y} \in X, \bar{x} \neq \bar{y}.$$

Then we define the following centered cocycle:

$$\sigma_1(g, (\bar{x}, \bar{y})) = \log(d(g \cdot \bar{x}, d(g \cdot \bar{y}) / d(\bar{x}, \bar{y})) - F_1(\bar{x}, \bar{y}).$$

Setting

$$M_n = M_n(\bar{x}, \bar{y}) := \sum_{k=1}^n \sigma_1(\varepsilon_k, (A_{k-1} \cdot \bar{x}, A_{k-1} \cdot \bar{y})) \quad \text{and} \quad R_n = R_n(\bar{x}, \bar{y}) := \sum_{k=1}^n F_1(A_{k-1} \cdot \bar{x}, A_{k-1} \cdot \bar{y}),$$

we have

$$\log(d(A_n \cdot \bar{x}, A_n \cdot \bar{y})/d(\bar{x}, \bar{y})) = M_n + R_n.$$

Hence, since $d(\bar{x}, \bar{y}) \leq 1$, the lemma will be proved if one can show that for any $\kappa > 0$, there exists a $\ell > 0$ such that

$$\max_{k \leq j \leq 2k} \sup_{\bar{x}, \bar{y} \in X, \bar{x} \neq \bar{y}} \mathbb{P}(R_j(\bar{x}, \bar{y}) \geq -2\ell k) \ll e^{-\kappa k^{\min(\gamma, 1)}} \quad (25)$$

and

$$\max_{k \leq j \leq 2k} \sup_{\bar{x}, \bar{y} \in X, \bar{x} \neq \bar{y}} \mathbb{P}(|M_j(\bar{x}, \bar{y})| \geq \ell k) \ll e^{-\kappa k^{\min(\gamma, 1)}}. \quad (26)$$

The estimate (25) comes from the proof of (53) in [5]. Indeed from the last inequality in the proof of [5, (53), page 1857], there exist $\rho \in]0, 1[$, $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that

$$\max_{k \leq j \leq 2k} \sup_{\bar{x}, \bar{y} \in X, \bar{x} \neq \bar{y}} \mathbb{P}(R_j(\bar{x}, \bar{y}) \geq -\alpha k) \leq C\rho^{k/(2n_0)}.$$

Hence (25) follows by taking $\ell \geq \alpha n_0/(|\log \rho|)$.

We turn to the proof of (26). Let $D_k = \sigma_1(\varepsilon_k, (A_{k-1} \cdot \bar{x}, A_{k-1} \cdot \bar{y}))$ and $\mathcal{F}_k = \sigma(W_0, \varepsilon_1, \dots, \varepsilon_k)$. Note that $(D_k, \mathcal{F}_k)_{k \geq 0}$ is a sequence of martingale differences. Moreover setting $X = 4 \log(N(g))$, Lemma 4.2 page 56 in [4] (see also their Proposition 5.3 page 62) ensures that

$$\|\mathbb{P}(D_k \geq x | \mathcal{F}_{k-1})\|_\infty \leq \mathbb{P}(X + \mathbb{E}(X) \geq x),$$

implying that

$$\left\| \sum_{k=1}^n \mathbb{E}(D_k^2 \exp\{|D_k|^\gamma\} | \mathcal{F}_{k-1}) \right\|_\infty \leq n \mathbb{E}((X + \mathbb{E}(X))^2 \exp\{(X + \mathbb{E}(X))^\gamma\}) := nK.$$

Note that since μ satisfies (21) with $c > 4$, we have $K < \infty$. It follows from [15, Theorem 2.1] (in case $\gamma \in]0, 1[$) or from [19, Theorem 1.1] (when $\gamma \geq 1$) that there exist c_1 and c_2 such that, for any $k \geq 1$ and any $\ell > 0$,

$$\max_{k \leq j \leq 2k} \sup_{\bar{x}, \bar{y} \in X, \bar{x} \neq \bar{y}} \mathbb{P}(|M_j(\bar{x}, \bar{y})| \geq \ell k) \ll \exp(-c_1 \ell^2 k / K) + \exp(-c_2 (\ell k)^{\min(\gamma, 1)}),$$

which proves (26) and ends the proof of the lemma. \square

We turn now to the proof of Proposition 5.

Proof of Proposition 5. Let $\bar{x}, \bar{y} \in X$ and $\kappa > 0$. Let $\ell > 0$ be as in Lemma 11. We start from inequality (18) in [5], namely: setting

$$A = \{\log N(\varepsilon_k) \geq \ell k / 2\} \quad \text{and} \quad B = \{\log(d(A_{k-1} \cdot \bar{x}, d(A_{k-1} \cdot \bar{y})) \geq -\ell k\},$$

we write

$$\begin{aligned} |X_{k, \bar{x}} - X_{k, \bar{y}}| &\leq |\sigma(\varepsilon_k, A_{k-1} \cdot \bar{x}) - \sigma(\varepsilon_k, A_{k-1} \cdot \bar{y})| \mathbf{1}_A + |\sigma(\varepsilon_k, A_{k-1} \cdot \bar{x}) - \sigma(\varepsilon_k, A_{k-1} \cdot \bar{y})| \mathbf{1}_B \\ &\quad + |\sigma(\varepsilon_k, A_{k-1} \cdot \bar{x}) - \sigma(\varepsilon_k, A_{k-1} \cdot \bar{y})| \mathbf{1}_{\{A^c \cap B^c\}}. \end{aligned}$$

Recall the estimates (13) and (14) in [5]: there exists $C > 0$ such that for every $\bar{x}, \bar{y} \in X$,

$$|\sigma(g, \bar{x}) - \sigma(g, \bar{y})| \leq CN(g)d(\bar{x}, \bar{y}),$$

and

$$|\sigma(g, \bar{x})| \leq \log N(g).$$

We then infer that

$$\begin{aligned} \mathbb{E}(|X_{k, \bar{x}} - X_{k, \bar{y}}|) &\leq 2\mathbb{E}((\log N(\varepsilon_k))\mathbf{1}_A) + 2\mathbb{E}((\log N(\varepsilon_k))\mathbf{1}_B) \\ &\quad + C\mathbb{E}(N(\varepsilon_k)d(A_{k-1} \cdot \bar{x}, A_{k-1} \cdot \bar{y})|\mathbf{1}_{\{A^c \cap B^c\}}). \end{aligned} \quad (27)$$

Now

$$\mathbb{E}((\log N(\varepsilon_k))\mathbf{1}_A) \leq e^{-(\ell k/2)^\gamma} \mathbb{E}(\log N(g)e^{(\log N(g))^\gamma}). \quad (28)$$

On another hand, by independence between ε_k and B , and Lemma 11,

$$\mathbb{E}((\log N(\varepsilon_k))\mathbf{1}_B) = \mathbb{P}(B)\mathbb{E}(\log N(g)) \leq e^{-\kappa k^\gamma} \mathbb{E}(\log N(g)). \quad (29)$$

Moreover,

$$\mathbb{E}(N(\varepsilon_k)d(A_{k-1} \cdot \bar{x}, A_{k-1} \cdot \bar{y})|\mathbf{1}_{\{A^c \cap B^c\}}) \leq e^{-\ell k} \mathbb{E}(N(g)\mathbf{1}_{\{\log N(g) \leq \ell k/2\}}) \leq e^{-\ell k/2}. \quad (30)$$

Starting from (27) and taking into account the upper bounds (28)-(30), the lemma follows. \square

4.2 Proof of Proposition 6

It suffices to find $\eta > 0$ such that (notice that $\delta(\cdot, \cdot) \leq 1$)

$$\sum_{n \geq 1} e^{n^\gamma} \sup_{\bar{x} \in X} \nu(\{\bar{y} \in X : -\log \delta(\bar{x}, \bar{y}) \geq \eta n\}) < \infty.$$

Now, using that ν is μ -invariant, for every $n \in \mathbb{N}$ and $\bar{x} \in X$,

$$\begin{aligned} \nu(\{\bar{y} \in X : -\log \delta(\bar{x}, \bar{y}) \geq \eta n\}) &= \nu \otimes \mu^{*n}(\{(\bar{y}, g) \in X \times G : -\log \delta(\bar{x}, g \cdot \bar{y}) \geq \eta n\}) \\ &\leq \sup_{\bar{x}, \bar{y} \in X} \mathbb{P}(\log \delta(\bar{x}, A_n \cdot \bar{y}) \leq -\eta n). \end{aligned}$$

The fact that one may find such a $\eta > 0$ may be proved as Lemma 4.8 (and more particularly (4.13)) of [1], using the next lemma.

Lemma 12 *Under the assumption of Corollary 2, for every $\varepsilon > 0$, there exist $C, c > 0$, such that*

$$\sup_{x \in S^{d-1}} \mathbb{P}\left(\max_{1 \leq k \leq n} |\log \|A_k x\| - k\lambda_\mu| > \varepsilon n\right) \leq Ce^{-cn^\gamma}, \quad (31)$$

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |\log \|A_k\| - n\lambda_\mu| > \varepsilon n\right) \leq Ce^{-cn^\gamma}; \quad (32)$$

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |\log \|\Lambda^2(A_k)\| - k(\lambda_\mu + \gamma_\mu)| > \varepsilon n\right) \leq Ce^{-cn^\gamma}. \quad (33)$$

Remark 13 Let us recall that, for any $A \in GL_d(\mathbb{R})$, $\Lambda^2(A)$ is the matrix on $\Lambda^2(\mathbb{R}^d)$ defined by $\Lambda^2(A)(x \wedge y) = Ax \wedge Ay$, where \wedge is the exterior product. In addition, in (33), γ_μ is the second Lyapunov exponent of μ . With the notations of [2, Section 14], λ_μ is denoted either $\lambda_{1,\mu}$ or λ_1 , while γ_μ is either denoted $\lambda_{2,\mu}$ or λ_2 .

Proof of Lemma 12. (31) follows directly from Theorem 2.2 of [8]. Then (32) follows from (31) and the fact that there exists $C > 0$ such that for every $g \in G$ and every $x \in S^{d-1}$,

$$\|gx\| \leq \|g\| \leq C \max_{1 \leq i \leq d} \|ge_i\|,$$

with $\{e_i : 1 \leq i \leq d\}$ the canonical basis of \mathbb{R}^d .

The proof of (33) may be done similarly as above. The group G acts on $P(\Lambda^2(\mathbb{R}^d))$, the projective space of $\Lambda^2(\mathbb{R}^d)$, by $g \cdot \bar{\xi} := \Lambda^2(g)\xi/\|\xi\|$, for every $g \in G$ and $\xi \in \Lambda^2(\mathbb{R}^d)$, where $\bar{\xi}$ is the projection of ξ on $P(\Lambda^2(\mathbb{R}^d))$. Then, $(g, \bar{\xi}) \mapsto \|\Lambda^2(g)(\xi)\|/\|\xi\|$ defines a cocycle and one may prove, similarly to (31) that (see Section 6 of [8] where the situation of general cocycles is considered)

$$\sup_{\xi \in \Lambda^2(\mathbb{R}^d) : \|\xi\|=1} \mathbb{P} \left(\max_{1 \leq k \leq n} |\log \|\Lambda^2(A_k)(\xi) - k(\lambda_\mu + \gamma_\mu)\| > \varepsilon n \right) \leq Ce^{-cn^\gamma}.$$

Then, we deduce (33) as we deduced (32) from (31).

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