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# Stabilization of conservation laws for $\mathcal{G}$-solutions with application to traffic flow. 

Thibault Liard * Swann Marx ${ }^{\dagger} \quad$ Vincent Perrollaz ${ }^{\ddagger}$


#### Abstract

We address a problem of stabilization of a scalar conservation laws with fixed flux pointwise constraints. The PDE models the impact of two toll gates on the traffic flow and the feedback laws is designed to limit the number of cars passing through the two toll gates. The existence of solutions is proved using a Schauder's fixed point theorem, together with a coupled PDEODE. To prove stabilization results, we study the structure of the solutions using an extension of backward generalized characteristics.


Keywords: Conservation Laws; non-classical shock; $\mathcal{G}$-solutions; Stabilization;

## AMS classification:

## 1 Introduction

### 1.1 Presentation of the Problem

Let $L>0$. We consider the following scalar conservation laws with local lateral constraints

$$
\begin{cases}u_{t}(t, x)+\partial_{x} f(u(t, x))=0, & t \in \mathbb{R}_{+}, x \in \mathbb{R},  \tag{1}\\ u(0, x)=u_{0}(x), & x \in \mathbb{R}, \\ f(u(t, 0)) \leq F_{1}(t, u), & t \in \mathbb{R}_{+}, \\ f(u(t, L)) \leq F_{2}(t, u), & t \in \mathbb{R}_{+}\end{cases}
$$

We assume that $f \in C^{2}([0,1] ;[0,+\infty))$ is a strictly concave function such that $f(0)=f(1)=0$. The problem (1) models two toll gates along highway or road lights (see [3]). Since the flux pointwise constraints at $x=0$ and $x=L$ (the two toll gates) generate shocks that does not verify Oleinik condition, the notion of weak-entropy solutions cannot be used anymore. Thus, we consider here the notion of $\mathcal{G}$-solution (see [1] or Definition 1.2). We introduce the target function $w$ defined by

$$
w(x)= \begin{cases}w_{L}, & \text { if } 0 \leq x<x_{0}  \tag{2}\\ w_{R}, & \text { if } x_{0}<x \leq L\end{cases}
$$

where $x_{0} \in(0, L)$ and $w_{L}, w_{R} \in(0,1)$ satisfy $w_{L}<w_{R}$ and $f\left(w_{L}\right)=f\left(w_{R}\right)<f(\bar{u})$ with $\bar{u}=$ $\operatorname{argmax}_{u \in[0,1]} f(u)$. Our goal is to find two feedback controllers $F_{1}$ and $F_{2}$ that stabilizes (1) over $(0, L)$ around the target function $w$, i.e for any initial data $u_{0}$, the $\mathcal{G}$-solutions $u(t, \cdot)$ of (1) converges to $w(\cdot)$ in $L^{1}(0, L)$ as $t \rightarrow \infty$. We consider a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
g(0)=0 \text { and } \forall x \in \mathbb{R}^{*}, x g(x)>0 \tag{3}
\end{equation*}
$$

[^0]and the following ODE
\[

\left\{$$
\begin{array}{l}
\dot{m}(t)=-g(m(t)),  \tag{4}\\
m(0)=m_{0}
\end{array}
$$\right.
\]

admits a unique solution over $[0, \infty)$. In this paper, the feedback controllers $F_{1}$ and $F_{2}$ have the following forms

$$
\begin{equation*}
F_{1}(t, u)=\max \left(\min \left(f\left(w_{L}\right)-g(m(t)), f(\bar{u})-\varepsilon\right), 0\right) \text { and } F_{2}(t, u)=f\left(w_{R}\right) \tag{5}
\end{equation*}
$$

where $\varepsilon>0$ is chosen such that $f(\bar{u})-f\left(w_{L}\right)-\varepsilon>0$ and

$$
\begin{equation*}
m(t):=\int_{0}^{L}(u(t, x)-w(x)) d x \tag{6}
\end{equation*}
$$

Remark 1 - Since $g$ is a continuous function, from Peano's Theorem, there exists at least one solution of (4). If there exists $t_{0}>0$ such that $m\left(t_{0}\right)=0$ then, from (3), any solution $m(\cdot)$ of (4) satisfies for any $t \geq t_{0}, \frac{d}{d t} m^{2}(t)<0$. Therefore, for any $t \geq t_{0}, m(t)=0$.

- We have the following maximum and minimum principle

$$
m_{0} \geq 0 \Longrightarrow \forall t \geq 0, m(t) \geq 0 \text { and } m_{0} \leq 0 \Longrightarrow \forall t \geq 0, m(t) \leq 0
$$

- The $E D O$ (4) is asymptotically stable around the equilibrium point 0 using (3) and the candidate Lyapunov function $V: x \rightarrow \frac{x^{2}}{2}($ see $[6])$.
- The uniqueness of the EDO (4) is used to prove the existence of solutions of (1). More precisely, from a certain time, the unique solution of the coupled PDE-ODE (13) is a solution of (1).

In $[2,7,9]$, stabilization results are obtained in the context of weak-entropy solutions. More precisely, in [2], the authors consider solutions with a finite number of shocks to find the derivative of a Lyapunov function candidate. In [9], they assume that the solutions is only determined by the boundaries and use the Hamilton-Jacobi formulation. In [7], a saturated proportional controller is provided to stabilize asymptotically the weak-entropy solution of LWR around a stationary shocks using the notion of generalized characteristics. To our best knowledge, there are no stabilization results in the context of $\mathcal{G}$-solutions.

### 1.2 Definitions and notations

Let $\Pi=[0, \infty) \times \mathbb{R}$. Throughout the paper,

$$
\phi:(u, k) \in \mathbb{R} \times \mathbb{R} \mapsto \operatorname{sgn}(u-k)(f(u)-f(k))
$$

denotes the entropy flux associated with the Kruzhkov entropy $(u, k) \mapsto|u-k|$ (see [5]).
Definition 1.1 (Admissibility germ $\mathcal{G}(F),[1])$ Let $F \in[0, f(\bar{u})]$. The admissibility germ $\mathcal{G}(F)$ for the conservation law (1) is the subset of $[0,1]^{2}$ defined as the union $\mathcal{G}(F)=\mathcal{G}_{1}(F) \cup \mathcal{G}_{2}(F) \cup$ $\mathcal{G}_{3}(F)$, where

- $\mathcal{G}_{1}(F):=\left\{\left(c_{l}, c_{r}\right) \in[0,1]^{2} ; c_{l}>c_{r}, f\left(c_{l}\right)=f\left(c_{r}\right)=F\right\}$,
- $\mathcal{G}_{2}(F):=\left\{(c, c) \in[0,1]^{2} ; f(c) \leq F\right\}$,
- $\mathcal{G}_{3}(F):=\left\{\left(c_{l}, c_{r}\right) \in[0,1]^{2} ; c_{l}<c_{r}, f\left(c_{l}\right)=f\left(c_{r}\right) \leq F\right\}$.

Definition $1.2([1])$ Let $u_{0} \in L^{\infty}(\mathbb{R},[0,1])$. We say that $u \in L^{\infty}(\Pi ;[0,1])$ is a $\mathcal{G}$-entropy solution of (1) if

1. $u$ is a Kruzhkov entropy solution for $x \in \mathbb{R} \backslash\{0, L\}$, i.e, for all nonnegative test functions $\varphi \in C_{c}^{\infty}(\Pi \backslash\{0, L\})$ and all $k \in[0,1]$,

$$
\begin{equation*}
\int_{0}^{+\infty} \int_{\mathbb{R}}\left(|u(t, x)-k| \partial_{t}+\phi(u(t, x), k) \partial_{x}\right) \varphi(t, x) d x d t+\int_{\mathbb{R}}\left|u_{0}(x)-k\right| \varphi(0, x) d x \geq 0 ; \tag{7}
\end{equation*}
$$

2. in addition, for a.e $t>0$,

$$
\begin{equation*}
\left(\left(\gamma_{0}^{l} u\right)(t),\left(\gamma_{0}^{r} u\right)(t)\right) \in \mathcal{G}\left(F_{1}\right) \text { and }\left(\left(\gamma_{L}^{l} u\right)(t),\left(\gamma_{L}^{r} u\right)(t)\right) \in \mathcal{G}\left(F_{2}\right) . \tag{8}
\end{equation*}
$$

where $\gamma_{0}^{l, r}$ and $\gamma_{L}^{l, r}$ denote the operators of left- and right-side strong traces on $\{x=0\}$ and $\{x=L\}$ and $F_{1}$ and $F_{2}$ are defined in (5).

## 2 Main results

Theorem 2.1 (Existence) Let $\varepsilon>0$ satisfy $f(\bar{u})-f\left(w_{L}\right)-\varepsilon>0$. For any $u_{0} \in L^{\infty}(\mathbb{R} ;[0,1])$, there exists at least one $\mathcal{G}$-entropy solution $u$ of (1) where $F_{1}$ and $F_{2}$ are defined in (5).

Theorem 2.2 (Asymptotic stability) Let $\varepsilon>0$ satisfy $f(\bar{u})-f\left(w_{L}\right)-\varepsilon>0$. Let $v_{0} \in$ $L^{\infty}([0, L] ;[0,1])$, we consider the initia data $u_{0}$ defined by

$$
u_{0}\left(x ; v_{0}\right)= \begin{cases}1, & \text { if } x<0,  \tag{9}\\ v_{0}(x), & \text { if } 0<0<L, \\ 0, & \text { if } L<x .\end{cases}
$$

For any $v_{0} \in L^{\infty}([0, L] ;[0,1])$, any $\mathcal{G}$-solution $u(t, \cdot)$ of (1) with initial datum $u_{0}$ converges to $w(\cdot)$ in $L^{1}(0, L)$ as $t \rightarrow \infty$.

Corollary 2.1 (Exponential stability) Let $\varepsilon>0$ satisfy $f(\bar{u})-f\left(w_{L}\right)-\varepsilon>0$ and $g: x \mapsto K_{0} x$ with $K_{0}>0$. For any $v_{0} \in L^{\infty}([0, L] ;[0,1])$, any $\mathcal{G}$-solution $u(t, \cdot)$ of (1) with initial datum $u_{0}$ converges exponentially to $w(\cdot)$ in $L^{1}(0, L)$ as $t \rightarrow \infty$, i.e there exists $C>0$ such that for any $t \geq 0$,

$$
\|u(t, \cdot)-w(\cdot)\|_{L^{1}(0, L)} \leq C e^{-K_{0} t}\left(\left\|u_{0}(\cdot)-w(\cdot)\right\|_{L^{1}(0, L)}+C\right) .
$$

Corollary 2.2 (Finite time stability) Let $\varepsilon>0$ satisfy $f(\bar{u})-f\left(w_{L}\right)-\varepsilon>0$ and $g$ satisfied (3) and for any $\alpha \in \mathbb{R}^{*}, \int_{0}^{\alpha} \frac{d y}{g(y)}>-\infty$. For any $v_{0} \in L^{\infty}([0, L] ;[0,1])$, the $\mathcal{G}$-solution $u(t, \cdot)$ of (1) with initial datum $u_{0}$ converges in finite time to $w(\cdot)$ in $L^{1}(0, L)$, i.e there exists a time $\bar{T}>0$ such that for any $t \geq \bar{T}$, for a.e $x \in(0, L)$,

$$
u(t, x)=w(x) .
$$

## 3 Simulations

To construct an approximate solution of (1), we use a finite volume scheme described in [1]. The code is available at https://github.com/Thibault-liard/Regulation-Godunov-LWR-fixed.

In Figure 1, Figure 2, Figure 3, $F_{1}$ is a saturated proportional controller, i.e $g$ in (5) is defined by, for any $x \in \mathbb{R}, g(x)=K_{0} x$ where $K_{0}>0$ is the gain of the controller. Any $\mathcal{G}$-solution of the PDE (1) with the controller $F_{1}$ is asymptotically stable around the stationary solution $w(\cdot)$ (see Corollary 2.2). In Figure 4, Figure 5, Figure 6, $F_{1}$ is a saturated sliding mode controller, i.e $g: x \rightarrow K_{0}|x|^{\frac{1}{2}} \operatorname{sign}(x)$ where $K_{0}>0$ is the gain of the controller. Any $\mathcal{G}$-solution of the PDE (1) with the controller $F_{1}$ converges in finite time to the stationary solution $w(\cdot)$ (see Corollary 2.1).


Figure 1: Plottings of the initial data $v_{0}$ $(-)$, the target function $w(--)$ and the $\mathcal{G}$ solution of $(1)(-)$ over $(0,1)$ at time $t=20$ with $L=1$ and for any $x \in \mathbb{R}, g(x)=K_{0} x$ with $K_{0}=0.4$.


Figure 2: Plotting of the $\mathcal{G}$-solution of (1) $(-)$ over $(-15,20)$ at time $t=20$ with $L=1$ and for any $x \in \mathbb{R}, g(x)=K_{0} x$ with $K_{0}=$ 0.4 .


Figure 3: Plotting of the saturated Pcontroller $F_{1}(\cdot)$ with respect to time $t$.


Figure 4: Plottings of the initial data $v_{0}$ $(-)$, the target function $w(--)$ and the $\mathcal{G}$-solution of $(1)(-)$ over $(0,1)$ at time $t=20$ with $L=1$ and for any $x \in \mathbb{R}$, $g(x)=K_{0}|x|^{\frac{1}{2}} \operatorname{sign}(x)$ with $K_{0}=0.4$.


Figure 5: Plotting of the $\mathcal{G}$-solution of (1) $(-)$ over $(-15,20)$ at time $t=20$ with $L=1$ and for any $x \in \mathbb{R}, g(x)=K_{0}|x|^{\frac{1}{2}} \operatorname{sign}(x)$ with $K_{0}=0.4$.


Figure 6: Plotting of the saturated SMCcontroller $F_{1}(\cdot)$ with respect to time $t$.

## 4 Proofs

### 4.1 Proof of Theorem 2.1

We consider the following scalar conservation laws with local ilateral constraints

$$
\begin{cases}u_{t}(t, x)+\partial_{x} f(u(t, x))=0, & t \in \mathbb{R}_{+}, x \in \mathbb{R},  \tag{10}\\ u(0, x)=u_{0}(x), & x \in \mathbb{R}, \\ f(u(t, 0)) \leq Q_{1}(t), & t \in \mathbb{R}_{+}, \\ f(u(t, L)) \leq Q_{2}(t), & t \in \mathbb{R}_{+}\end{cases}
$$

A $\mathcal{G}$-solution $u$ of (10) is understood in the sense of Definition 1.2 replacing $F_{1}$ and $F_{2}$ by $Q_{1}$ and $Q_{2}$ respectively. Note that, contrary to (1), $Q_{1}$ and $Q_{2}$ in (10) does not depend on the $\mathcal{G}$-solution $u$.

Proposition $4.1([1])$ Assume that $Q_{1}^{1}, Q_{1}^{2}, Q_{2}^{1}, Q_{2}^{1} \in L^{\infty}\left(\mathbb{R}^{+},[0, f(\bar{u})]\right)$ and $u_{0}, v_{0} \in L^{\infty}(\mathbb{R},[0,1])$ such that $\left(u_{0}-v_{0}\right) \in L^{1}(\mathbb{R})$. The $\mathcal{G}$-entropy solutions $u$ and $v$ of

$$
\left\{\begin{array} { l } 
{ u _ { t } ( t , x ) + \partial _ { x } f ( u ( t , x ) ) = 0 , } \\
{ u ( 0 , x ) = u _ { 0 } ( x ) , } \\
{ f ( u ( t , 0 ) ) \leq Q _ { 1 } ^ { 1 } ( t ) , } \\
{ f ( u ( t , L ) ) \leq Q _ { 2 } ^ { 1 } ( t ) , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
v_{t}(t, x)+\partial_{x} f(v(t, x))=0 \\
v(0, x)=v_{0}(x), \\
f(v(t, 0)) \leq Q_{1}^{2}(t) \\
f(v(t, L)) \leq Q_{2}^{2}(t)
\end{array}\right.\right.
$$

satisfy, for a.e $T>0$,

$$
\begin{equation*}
\int_{\mathbb{R}}|u-v|(T, x) d x \leq 2 \int_{0}^{T}\left|Q_{1}^{1}-Q_{1}^{2}\right|(t) d t+2 \int_{0}^{T}\left|Q_{2}^{1}-Q_{2}^{2}\right|(t) d t+\int_{\mathbb{R}}\left|u_{0}-v_{0}\right|(x) d x \tag{11}
\end{equation*}
$$

Definition 4.1 Let us fix $T>0$. We introduce the operator $\mathcal{F}: W^{1, \infty}([0, T]) \rightarrow W^{1, \infty}([0, T])$ defined by

$$
\forall t \in[0, T], \quad \mathcal{F}(z)(t):=\int_{0}^{L}(u(t, x)-w(x)) d x
$$

where $u \in L^{\infty}([0, T] \times \mathbb{R} ;[0,1])$ is a $\mathcal{G}$-entropy solution of (10) with $Q_{1}$ and $Q_{2}$ defined by

$$
Q_{1}(t)=\max \left(\min \left(f\left(w_{L}\right)-g(z(t)), f(\bar{u})-\varepsilon\right), 0\right) \text { and } Q_{2}(t)=f\left(w_{R}\right)
$$

Proposition 4.2 Let us fix $T>0$. $\mathcal{F}$ has a unique fixed point in $W^{1, \infty}([0, T])$.
Proof. We apply Schauder's fixed point theorem [8], i.e suppose $K$ is a compact set in a Fréchet space $X$ and $\mathcal{F}: X \rightarrow K$ is continuous then $\mathcal{F}$ has a fixed point in $K$. Here, $X:=C([0, T] ;[0,1])$ is a Banach space equipped with the norm $\|\cdot\|_{L^{\infty}(0, T)}$ and $K:=\overline{\mathcal{F}(C([0, T] ;[0,1]))} \subset C([0, T] ;[0,1])$.

- Let us fix $y \in C([0, T] ;[0,1])$ and $\left(y_{n}\right)_{n \in \mathbb{N}}$ be a sequence that converges uniformly to $y$. We denote by $u$ and $u_{n}$ the $\mathcal{G}$-entropy solutions involved in the definition of $\mathcal{F}(y)$ and $\mathcal{F}\left(y_{n}\right)$ respectively. Using Proposition 4.1, for any $t \in[0, T]$,

$$
\begin{align*}
\left|\mathcal{F}\left(y_{n}\right)(t)-\mathcal{F}(y)(t)\right| \leq & \left.\int_{0}^{t} \mid \max \left(\min \left(f\left(w_{L}\right)-g\left(y_{n}(t)\right), f(\bar{u})-\epsilon\right)\right), 0\right) \\
& \left.\left.\quad-\max \left(\min \left(f\left(w_{L}\right)-g(y(t)), f(\bar{u})-\epsilon\right)\right)\right), 0\right) \mid d t  \tag{12}\\
\leq & 2 \int_{0}^{t}\left|g\left(y_{n}(t)\right)-g(y(t))\right| d t
\end{align*}
$$

From Heine's Theorem, $g:[0,1] \rightarrow \mathbb{R}$ is uniformly continuous and for any $t \in[0, T], y(t) \in$ $[0,1]$ and $z(t) \in[0,1]$. Thus, $\left(g\left(y_{n}\right)\right)_{n \in \mathbb{N}}$ converges uniformly to $g(y)$. Therefore, together with (12), we conclude that $\mathcal{F}$ is continuous on $C([0, T] ;[0,1])$ with respect to norm \|. $\|_{L^{\infty}(0, T)}$.

- We introduce the set $A=\mathcal{F}(C([0, T] ;[0,1])) \subset C([0, T] ;[0,1])$. Let us fix $z \in C([0, T] ;[0,1])$, we denote by $u$ the $\mathcal{G}$-entropy solutions involved in the definition of $\mathcal{F}(z)$. Since for any $(t, x) \in \mathbb{R}_{+} \times(0, L), 0 \leq u(t, x) \leq 1$ then for any $t \in[0, T],|\mathcal{F}(z)(t)| \leq L$ then $A$ is a bounded subset of $C([0, T] ;[0,1])$. Let $0 \leq t<t^{\prime} \leq T$, from Lemma A. 1 with $\gamma_{1}(\cdot)=0, \gamma_{2}(\cdot)=L$, $t_{1}=t$ et $t_{2}=t^{\prime}$,

$$
\int_{0}^{L} u(t, x) d x+\int_{t}^{t^{\prime}} f(u(t, 0)) d t=\int_{0}^{L} u\left(t^{\prime}, x\right) d x+\int_{t}^{t^{\prime}} f(u(t, L)) d t
$$

Thus, together with (8), we have

$$
\left|\mathcal{F}(z)\left(t^{\prime}\right)-\mathcal{F}(z)(t)\right| \leq f(\bar{u})\left(t^{\prime}-t\right)
$$

As a consequence, $A$ is equicontinuous. From Arzelà-Ascoli theorem, $\bar{A}:=\overline{\mathcal{F}(C([0, T] ;[0,1]))}$ is a compact set.

From Schauder's fixed point theorem $[8], \mathcal{F}$ has a fixed point in $\overline{\mathcal{F}(C([0, T] ;[0,1]))}$. Since $\overline{\mathcal{F}(C([0, T] ;[0,1]))} \subset$ $W^{1, \infty}([0, T])$, we conclude the proof of Proposition 4.2.

Proof of Theorem 2.1. Let $\epsilon>0$ and $T_{5}:=T_{4}+\epsilon$ where $T_{4}$ is defined in Proposition 4.6. From Proposition 4.2, for any $u_{0} \in L^{\infty}(\mathbb{R} ;[0,1])$, there exists at least one $\mathcal{G}$-entropy solution, denoted by $u$, of (1) over $\left[0, T_{5}\right]$ where $F_{1}$ and $F_{2}$ are defined in (5). We consider the following coupled PDE-ODE

$$
\begin{cases}v_{t}(t, x)+\partial_{x} f(v(t, x))=0, & t \in\left(T_{4}, \infty\right), x \in \mathbb{R},  \tag{13}\\ v\left(T_{4}, x\right)=u\left(T_{4}, x\right), & x \in \mathbb{R}, \\ f(v(t, 0)) \leq f\left(w_{L}\right)-g(\gamma(t)), & t \in\left(T_{4}, \infty\right) \\ f(v(t, L)) \leq f\left(w_{R}\right), & t \in\left(T_{4}, \infty\right) \\ \dot{\gamma}(t)=-g(\gamma(t)), & t \in\left(T_{4}, \infty\right) \\ \gamma\left(T_{4}\right)=\int_{0}^{L}\left(u\left(T_{4}, x\right)-w(x)\right) d x . & \end{cases}
$$

From Remark 1 and [1, Theorem 2.11], the coupled PDE-ODE admits a unique $\mathcal{G}$-entropy solution, denoted by $v$, over $\left[T_{4}, \infty\right)$. From (80), we have $m\left(T_{4}\right):=\int_{0}^{L}\left(u\left(T_{4}, x\right)-w(x)\right) d x \in[0, f(\bar{u})-\epsilon]$. From (3) and $\gamma\left(T_{4}\right)=m\left(T_{4}\right)$, we deduce that, for any $t \geq T_{4}, f\left(w_{L}\right)-g(\gamma(t)) \in[0, f(\bar{u})-\epsilon]$. Thus, since $T_{4}$ does not depend on the initial data $u_{0}$, Proposition 4.6 holds replacing $F_{1}(t, u)$ by $f\left(w_{L}\right)-g(\gamma(t))$. Thus, the function $\sigma: t \rightarrow \int_{0}^{L}(v(t, x)-w(x)) d x$ satisfied, for any $t \geq T_{4}$,

$$
\left\{\begin{array}{l}
\dot{\sigma}(t)=f\left(w_{L}\right)-g(\gamma(t))-f\left(w_{R}\right) \\
\sigma\left(T_{4}\right)=m\left(T_{4}\right)
\end{array}\right.
$$

From (13), we deduce that $\sigma(t)=\gamma(t)$ for any $t \geq T_{4}$. Therefore, $v$ is a $\mathcal{G}$-entropy solution of (1) over $\left[T_{4}, \infty\right)$. By uniqueness of (13) and from Proposition 4.6, for any $t \in\left[T_{4}, T_{5}\right], u=v$. We conclude that $w$, defined by $w=u$ over $\left[0, \frac{T_{4}+T_{5}}{2}\right] \times \mathbb{R}$ and $w=v$ over $\left(\frac{T_{4}+T_{5}}{2}, \infty\right) \times \mathbb{R}$, is a $\mathcal{G}$-entropy solution of (1).

### 4.2 Proof of Theorem 2.2

Let $\bar{f} \in[0, f(\bar{u})]$. We denote by $\check{u}_{\bar{f}}$ and $\hat{u}_{\bar{f}}$ the two solutions of $f(u)=\bar{f}$ with $\check{u}_{\bar{f}}<\hat{u}_{\bar{f}}$ (these solutions exist because $f$ is concave). Let $u$ be a $\mathcal{G}$-entropy solution of (1).

Lemma 4.1 There exist $T_{1}>0$ (independent of the initial data $u_{0}$ ) and two functions $\beta_{1}, \beta_{2}$ : $\left[T_{1},+\infty\right) \rightarrow(0, L)$ such that, for any $(\bar{t}, \bar{x}) \in\left[T_{1},+\infty\right) \times(0, L)$, we have $\beta_{1} \leq \beta_{2}$ and

$$
\begin{align*}
0<\bar{x}<\beta_{1}(\bar{t}) & \Longrightarrow u\left(\bar{t}, \bar{x}^{ \pm}\right) \in\left[0, \check{u}_{f(\bar{u})-\epsilon}\right)  \tag{14}\\
\beta_{1}(\bar{t})<\bar{x}<\beta_{2}(\bar{t}) & \Longrightarrow f\left(u\left(\bar{t}, \bar{x}^{ \pm}\right)\right) \geq f(\bar{u})-\frac{\varepsilon}{2}  \tag{15}\\
\beta_{2}(\bar{t})<\bar{x}<L & \Longrightarrow u\left(\bar{t}, \bar{x}^{ \pm}\right) \in\left[w_{R}, 1\right] \tag{16}
\end{align*}
$$

Proof. Let $(\bar{t}, \bar{x}) \in(0,+\infty) \times(0, L)$. From the point 1. of Definition 1.2 and [4, Theorem 11.1.3], there exists a minimal backward generalized characteristic, denoted by $\xi_{-}$, associated to $u$ emanating from $(\bar{t}, \bar{x})$ defined on an interval $[a, \bar{t}]$ with $a \geq 0$ such that for any $t \in(a, \bar{t})$

$$
\begin{equation*}
u\left(t, \xi_{-}(t)^{-}\right)=u\left(\bar{t}, \bar{x}^{-}\right)=u\left(t, \xi_{-}(t)^{+}\right) \tag{17}
\end{equation*}
$$

In particular, $\xi_{-}$is a genuine characteristic (also called shock-free, see [4, Definition 10.2.4]). We have three different possibilities.

Case 1: $\xi_{-}(a)=0$ and $a>0$. From Proposition B. 1 with $\xi=\xi_{-}, a=a$ and $b=\bar{t}$, there exists a constant $v \in[0,1]$ such that, for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and, for any $t \in(a, \bar{t})$, $u\left(t, \xi_{-}(t)\right)=v$ and $f(v) \leq f(\bar{u})-\varepsilon$. From (17), we deduce that

$$
u\left(\bar{t}, \bar{x}^{-}\right)=v \text { and } f(v) \leq f(\bar{u})-\varepsilon
$$

Therefore, using the definition of $\check{u}_{f(\bar{u})-\varepsilon}$ and $\hat{u}_{f(\bar{u})-\varepsilon}$ and the concavity of $f$, we have $u\left(\bar{t}, \bar{x}^{-}\right) \in\left[0, \check{u}_{f(\bar{u})-\varepsilon}\right) \cup\left(\hat{u}_{f(\bar{u})-\varepsilon}, 1\right]$. Moreover, since $\xi_{-}(a)=0$ and $a>0$ then $f^{\prime}(v)>0$. The concavity of $f$ implies that $v \in[0, \bar{u})$. Thus, $u\left(\bar{t}, \bar{x}^{-}\right) \in\left[0, \check{u}_{f(\bar{u})-\epsilon}\right)$.

Case 2: $a=0$. From Proposition B. 1 with $\xi=\xi_{-}, a=0$ and $b=\bar{t}$, there exists a constant $v \in[0,1]$ such that, for any $t \in[0, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and, for any $t \in(0, \bar{t}), u\left(t, \xi_{-}(t)\right)=v$. From (17), we deduce that, for any $t \in[0, \bar{t}]$,

$$
\dot{\xi}_{-}(t)=f^{\prime}\left(u\left(\bar{t}, \bar{x}^{-}\right)\right)
$$

Thus, $\xi_{-}(\bar{t})-\xi_{-}(0)=f^{\prime}\left(u\left(\bar{t}, \bar{x}^{-}\right)\right) \bar{t}$. Using that $\xi_{-}(\bar{t})=\bar{x}$, we deduce that

$$
f^{\prime}\left(u\left(\bar{t}, \bar{x}^{-}\right)\right)=\frac{\bar{x}-\xi_{-}(0)}{\bar{t}} .
$$

Since $0 \leq \xi_{-}(0) \leq L$, we get

$$
\frac{\bar{x}-L}{\bar{t}} \leq f^{\prime}\left(u\left(\bar{t}, \bar{x}^{-}\right)\right) \leq \frac{\bar{x}}{L}
$$

which implies that

$$
-\frac{L}{\bar{t}} \leq f^{\prime}\left(u\left(\bar{t}, \bar{x}^{-}\right)\right) \leq \frac{L}{\bar{t}}
$$

Using that $f$ is strictly concave and $\bar{u}:=\operatorname{argmax}_{u \in[0, R]} f(u)$, we deduce that there exists $T_{1}>0$ such that for any $\bar{t} \geq T_{1}$, (15) holds.

Case 3: $\xi_{-}(a)=L$ and $a>0$. From Proposition B. 1 with $\xi=\xi_{-}, a=a$ and $b=\bar{t}$, there exists a constant $v \in[0,1]$ such that, for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and, for any $t \in(a, \bar{t})$, $u\left(t, \xi_{-}(t)\right)=v$ and $f(v) \leq f\left(w_{R}\right)$. From (17), we deduce that

$$
u\left(\bar{t}, \bar{x}^{-}\right)=v \text { and } f(v) \leq f\left(w_{R}\right)
$$

Therefore, using the definition of $w$ and the concavity of $f$, we deduce that $u(\bar{t}, \bar{x}-) \in$ $\left[0, w_{L}\right] \cup\left[w_{R}, 1\right]$. Moreover, since $\xi_{-}(a)=L$ and $a>0$ then $f^{\prime}(v)<0$. The concavity of $f$ implies that $v \in(\bar{u}, 1]$. Thus, $u\left(\bar{t}, \bar{x}^{-}\right) \in\left[w_{R}, 1\right]$.
We know that genuine characteristics do not cross over $(0,+\infty) \times(0, L)$. Therefore, given $\bar{t}$, the set of $\bar{x}$ for which we are in first case, second case or third case are connected and they form a partition of $[0, L]$. From a geometrical viewpoint it is obvious that from the left to the right we have points from the first case, points from the second case and points from the last case. At this point we have indeed constructed two functions $\beta_{1}$ and $\beta_{2}$ such that $\beta_{1} \leq \beta_{2}$ and (14), (15) and (16) hold for $x^{-}$. Using that $u\left(\bar{t}, \bar{x}^{+}\right)=\lim _{\epsilon \rightarrow 0^{+}} u\left(\bar{t},(\bar{x}+\epsilon)^{-}\right)$and applying (14), (15) and (16) for $x^{-}$to the right hand of the latter equality, we deduce that (14), (15) and (16) also hold for $x^{+}$. An illustration of the proof is given in Figure 7.

Proposition 4.3 There exists $T_{2}>0$ (independent of the initial data $u_{0}$ ) and a Lipschitz function $\beta:\left[T_{2},+\infty\right] \rightarrow[0, L]$ satisfying, for any $(\bar{t}, \bar{x}) \in\left[T_{2},+\infty\right) \times(0, L)$,

$$
\left\{\begin{array}{l}
0<\bar{x}<\beta(\bar{t}) \Longrightarrow u\left(\bar{t}, \bar{x}^{ \pm}\right) \in\left[0, \check{u}_{f(\bar{u})-\epsilon}\right)  \tag{18}\\
\beta(\bar{t})<\bar{x}<L \Longrightarrow u\left(\bar{t}, \bar{x}^{ \pm}\right) \in\left[w_{R}, 1\right]
\end{array}\right.
$$



Figure 7: Construction of $\beta_{1}(\cdot)$ and $\beta_{2}(\cdot)$; the minimal backward generalized characteristic $\xi_{0}$ emanating from $(\bar{t}, \bar{x}) \in\left[T_{1}, \infty\right) \times\left(0, \beta_{1}(\bar{t})\right.$ touches the left boundary $x=0$ at time $t=a>0$. The minimal backward generalized characteristic $\xi_{1}$ emanating from $(\bar{t}, \bar{x}) \in\left[T_{1}, \infty\right) \times\left(\beta_{1}(\bar{t}), \beta_{2}(\bar{t})\right)$ touches the initial boundary $t=0$. The minimal backward generalized characteristic $\xi_{2}$ emanating from $(\bar{t}, \bar{x}) \in\left[T_{1}, \infty\right) \times\left(\beta_{2}(\bar{t}), L\right)$ touches the right boundary $x=L$ at time $t=b>0$.

Proof. We assume that $\beta_{1}\left(T_{1}\right)<\beta_{2}\left(T_{1}\right)$ and (8) holds at time $t=T_{1}$. We now prove that $\beta_{1}\left(T_{1}\right)>0$ and $\beta_{2}\left(T_{1}\right)<L$ by contradiction. We assume that $\beta_{1}\left(T_{1}\right)=0$. From (15) and using that $u\left(T_{1}, \overline{0}^{+}\right)=\lim _{\substack{x \rightarrow 0 \\ x>0}} u\left(T_{1}, x^{-}\right)$, we have

$$
f\left(u\left(T_{1}, \overline{0}^{+}\right)\right) \geq f(\bar{u})-\frac{\varepsilon}{2}
$$

Since (8) holds at time $t=T_{1}$, we get

$$
f\left(u\left(T_{1}, \overline{0}^{+}\right)\right) \leq f(\bar{u})-\epsilon
$$

This leads to a contradiction. Thus, we have $\beta_{1}\left(T_{1}\right)>0$. The same argument works for $\beta_{2}$. From [4, 11.1.4 Theorem] and $\left(T_{1}, \beta_{1}\left(T_{1}\right)\right) \in(0,+\infty) \times(0, L)$, there exists a unique forward characteristics $\gamma_{1}(\cdot)$, issues from $\left(T_{1}, \beta_{1}\left(T_{1}\right)\right)$, defined on a certain interval $\left[T_{1}, c_{1}\right)$ where $c_{1}>T_{1}$ is chosen such that for any $t \in\left[T_{1}, c_{1}\right), \gamma_{1}(t) \in(0, L), \gamma_{1}$ is a Lipschitz function over $\left[T_{1}, c_{1}\right)$. We will now prove that $\beta_{1}=\gamma_{1}$. To that end, let us fix $t \in\left(T_{1}, c_{1}\right)$.

- Let $x \in\left(0, \gamma_{1}(t)\right)$. We denote by $\xi_{-}$, the minimal backward characteristic emanating from $(t, x)$, defined maximally on an interval $[b, t]$. By uniqueness of forward characteristics, we have

$$
\forall s \in\left[\max \left(T_{1}, b\right), t\right], \quad \xi_{-}(s)<\gamma_{1}(s)
$$

We have two alternatives.

- If $b>T_{1}$, we have $\xi_{-}(b)=0$ and $b>0$. Thus, by definition of $\beta_{1}, x<\beta_{1}(t)$.
- If $b \leq T_{1}$, then $\xi_{-}\left(T_{1}\right)<\gamma_{1}\left(T_{1}\right)=\beta_{1}\left(T_{1}\right)$. Therefore, by definition of $\beta_{1}, \xi_{-}$is the minimal backward characteristic emanating from $\left(T_{1}, \xi_{-}\left(T_{1}\right)\right)$, defined maximally on an interval $\left[b, T_{1}\right]$ and $\xi_{-}(b)=0$ with $b>0$.

We conclude that, for any $x \in\left(0, \gamma_{1}(t)\right), x<\beta_{1}(t)$. Thus, we have

$$
\begin{equation*}
\gamma_{1}(t) \leq \beta_{1}(t) \tag{19}
\end{equation*}
$$

- Let $x \in\left(\gamma_{1}(t), L\right)$. We denote by $\xi_{-}$, the minimal backward characteristic emanating from $(t, x)$, defined maximally on an interval $[b, t]$. By uniqueness of forward characteristics, we have

$$
\forall s \in\left[\max \left(T_{1}, b\right), t\right], \quad \gamma_{1}(s)<\xi_{-}(s)
$$

We have two alternatives.

- If $b>T_{1}$, we have $\xi_{-}(b)=L$ and $b>0$. Thus, by definition of $\beta_{1}$ and $\beta_{2}, \beta_{1}(t) \leq$ $\beta_{2}(t)<x$.
- If $b \leq T_{1}$, then $\gamma_{1}\left(T_{1}\right)=\beta_{1}\left(T_{1}\right)<\xi_{-}\left(T_{1}\right)$. Then, by definition of $\beta_{1}$ and $\beta_{2}, \xi_{-}$is the minimal backward characteristic emanating from $\left(T_{1}, \xi_{-}\left(T_{1}\right)\right)$, defined maximally on an interval $\left[b, T_{1}\right]$ and either $b=0$ or $\xi_{-}(b)=L$ with $b>0$. For both cases, $\beta_{1}(t)<x$.

We conclude that, for any $x \in\left(\gamma_{1}(t), L\right), \beta_{1}(t)<x$. Thus, we have

$$
\begin{equation*}
\beta_{1}(t) \leq \gamma_{1}(t) \tag{20}
\end{equation*}
$$

From (19) and (20), we deduce that for any $t \in\left[T_{1}, c_{1}\right), \beta_{1}(t)=\gamma_{1}(t)$. The same argument works for $\beta_{2}$. More precisely, $\beta_{2}$ coincides with the unique forward characteristic $\gamma_{2}$, issues from $\left(T_{1}, \beta_{2}\left(T_{1}\right)\right)$, defined on a certain interval $\left[T_{1}, c_{2}\right)$ where $c_{2}>T_{1}$ is chosen such that for any $t \in\left[T_{1}, c_{2}\right)$, $\gamma_{2}(t) \in(0, L)$. As a consequence, $\beta_{1}$ and $\beta_{2}$ are two Lipschitz functions over $\left(T_{1}, \min \left(c_{1}, c_{2}\right)\right)$ and for a.e $t \in\left(T_{1}, \min \left(c_{1}, c_{2}\right)\right)$,

$$
\begin{equation*}
\dot{\beta}_{i}(t)=\frac{f\left(u\left(t, \beta_{i}(t)^{-}\right)\right)-f\left(u\left(t, \beta_{i}(t)^{+}\right)\right)}{u\left(t, \beta_{i}(t)^{-}\right)-u\left(t, \beta_{i}(t)^{+}\right)}, \quad i \in\{1,2\} . \tag{21}
\end{equation*}
$$

From Lemma 4.1, $\beta_{1}\left(T_{1}\right)<\beta_{2}\left(T_{1}\right)$, (21) and the concavity of $f$, we deduce that, for a.e $t \in$ $\left(T_{1}, \min \left(c_{1}, c_{2}\right)\right)$

$$
\begin{align*}
& \dot{\beta}_{1}(t) \geq \frac{(f(\bar{u})-\epsilon)-\left(f(\bar{u})-\frac{\epsilon}{2}\right)}{\check{u}_{f(\bar{u})-\epsilon}-\hat{u}_{f(\bar{u})-\frac{\epsilon}{2}}}:=d_{1}>0,  \tag{22}\\
& \dot{\beta}_{2}(t) \leq \frac{(f(\bar{u})-\epsilon)-\left(f(\bar{u})-\frac{\epsilon}{2}\right)}{\hat{u}_{f(\bar{u})-\epsilon}-\breve{u}_{f(\bar{u})-\frac{\epsilon}{2}}}:=d_{2}<0, \tag{23}
\end{align*}
$$

and there exists $T \in\left(T_{1}, \min \left(c_{1}, c_{2}\right)\right)$ such that for any $t \in\left[T_{1}, T\right), 0<\beta_{1}(t)<\beta_{2}(t)<L$ and $\beta_{1}(T)=\beta_{2}(T)$. Moreover, we have

$$
d_{1}\left(T-T_{1}\right) \leq \beta_{1}(T)=\beta_{2}(T) \leq L+d_{2}\left(T-T_{1}\right)
$$

This implies that $T \leq T_{2}:=L+\frac{T_{1}}{d_{1}+d_{2}}$ and for any $t \geq\left[T, \min \left(c_{1}, c_{2}\right)\right)$,

$$
\beta(t):=\beta_{1}(t)=\beta_{2}(t)
$$

By definition of $\beta_{1}$ and $\beta_{2}$, (18) holds over $\left[T, \min \left(c_{1}, c_{2}\right)\right)$. As a consequence, we have for a.e $t \in\left[T, \min \left(c_{1}, c_{2}\right)\right)$

$$
\begin{equation*}
\frac{-f(\bar{u})+\epsilon}{1-\check{u}_{f(\bar{u})-\epsilon}} \leq \dot{\beta}(t)=\frac{f\left(u\left(t, \beta(t)^{-}\right)\right)-f\left(u\left(t, \beta(t)^{+}\right)\right)}{u\left(t, \beta(t)^{-}\right)-u\left(t, \beta(t)^{+}\right)} \leq \frac{f\left(w_{R}\right)}{w_{R}} \tag{24}
\end{equation*}
$$

Thus, $c:=c_{1}=c_{2}$ and $\beta$ is a Lipschitz function over $[T, c)$. By definition of $c_{1}$ and $c_{2}, c$ is the first time when $\beta(c)=0$ or $\beta(c)=L$ (by convention, $c=+\infty$ if $\beta(t) \in(0, L)$ for any $t \geq T_{2}$ ).

- If $c=+\infty$ then $\beta$ is a Lipschitz function over $\left[T_{2},+\infty\right)$ and by definition of $\beta_{1}$ and $\beta_{2}$, (18) holds over $\left[T_{2}, \infty\right)$.
- If $c<+\infty$ then $\beta$ is a Lipschitz function over $\left[T_{2}, c\right)$ and there exists a final time $\bar{c} \geq c$ such that for any $t \in[c, \bar{c}]$ either $\beta(t)=L$ or $\beta(t)=0$ (by convention $\bar{c}=+\infty$ if for any $t \geq c$, either $\beta(t)=L$ or $\beta(t)=0)$. If $c \neq+\infty$, there exists $\bar{c}_{1}>0$ such that for any $t \in\left(\bar{c}, \bar{c}_{1}\right)$, $\beta(t) \in(0, L)$ using the continuity of $\beta$. To simplify the proof, we only consider the case, for any $t \in[c, \bar{c}], \beta(t)=0$ (the same arguments work with the case $\beta(t)=L$ ).
- If $\bar{c}=+\infty$, then $\beta$ is a Lipschitz function over $\left[T_{2}, c\right.$ ) with Lipschitz constant denoted by $K$ and $\beta(t)=0$ for any $t \geq c$. Thus, $\beta$ is continuous over $\left[T_{2},+\infty\right)$. Moreover, for any $t_{1} \in\left[T_{2}, c\right)$ and $t_{2} \in[c, \infty)$ we have

$$
\begin{align*}
\left|\beta\left(t_{1}\right)-\beta\left(t_{2}\right)\right| & \leq\left|\beta\left(t_{1}\right)-\beta\left(t_{n}\right)\right|+\left|\beta\left(t_{n}\right)-\beta(c)\right|+\left|\beta(c)-\beta\left(t_{2}\right)\right| \\
& \leq K\left|t_{1}-t_{n}\right|+\left|\beta\left(t_{n}\right)-\beta(c)\right| \tag{25}
\end{align*}
$$

where $\left(t_{n}\right)_{n \in \mathbb{N}}$ a sequence such that for any $n \in \mathbb{N}, t_{n}<c$ and $\lim _{n \rightarrow \infty} t_{n}=c$. Passing to the limit as $n$ tends to $+\infty$ in (25), we deduce that $\beta$ is a Lipschitz function over $\left[T_{2},+\infty\right)$. By definition of $\beta_{1}$ and $\beta_{2},(18)$ holds over $\left[T_{2},+\infty\right)$.

- If $\bar{c}<+\infty$, then as in the case $\bar{c}=+\infty, \beta$ is a Lipschitz function over $\left[T_{2}, \bar{c}\right]$ and by definition of $\beta_{1}$ and $\beta_{2}$, (18) holds over $\left[T_{2}, \bar{c}\right]$. Moreover, for any $\bar{t} \in\left(\bar{c}, \bar{c}_{1}\right), \beta$ coincides with the unique forward characteristic $\gamma$, issues from $\left(\bar{t}, \beta(\bar{t})\right.$ ), defined on $\left[\bar{t}, \bar{c}_{1}\right)$. Thus, for any $\bar{t} \in\left(\bar{c}, \bar{c}_{1}\right), \beta$ is a Lipschitz function over $\left[\bar{t}, \bar{c}_{1}\right]$ with Lipschitz constant denoted by $K$. By definition of $\beta_{1}$ and $\beta_{2}$, (18) holds over $\left[t, \bar{c}_{1}\right)$. As a consequence, $\beta$ is continuous over $\left[T_{2}, \bar{c}_{1}\right)$. Let the sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ be defined by $t_{n}=\bar{t}+\frac{1}{n}$ for any $n \in \mathbb{N}$. For any $t_{1} \in[c, \bar{c})$ and $t_{2} \in\left(\bar{c}, \bar{c}_{1}\right)$, we have

$$
\begin{align*}
\left|\beta\left(t_{1}\right)-\beta\left(t_{2}\right)\right| & \leq\left|\beta\left(t_{1}\right)-\beta\left(t_{n}\right)\right|+\left|\beta\left(t_{n}\right)-\beta(\bar{c})\right|+\left|\beta(\bar{c})-\beta\left(t_{2}\right)\right|  \tag{26}\\
& \leq K\left|t_{1}-t_{n}\right|+\left|\beta\left(t_{n}\right)-\beta(\bar{c})\right|
\end{align*}
$$

Passing to the limit as $n$ tends to $+\infty$ in (26), we deduce that $\beta$ is a Lipschitz function over $\left[T_{2}, \bar{c}_{1}\right)$. By iteration, we deduce that $\beta$ is a Lipschitz function over $\left[T_{2},+\infty\right)$ and (18) holds over $\left[T_{2},+\infty\right)$.

Note that (8) holds at time $t=T_{1}$ can be removed since $\beta_{1}$ and $\beta_{2}$ are two continuous functions and (8) holds almost everywhere. Moreover, if $\beta_{1}\left(T_{1}\right)=\beta_{2}\left(T_{1}\right)$ then for any $t \in\left[T_{1}, \infty\right)$, $\beta(t):=\beta_{1}(t)=\beta_{2}(t)$ and (18) holds. Moreover, if $\beta(t)>0$ with $t \in\left[T_{1}, \infty\right)$, we prove as above that $\beta(\cdot)$ coincides with the unique forward characteristic $\gamma_{1}(\cdot)$ emanating from $(t, \beta(t))$, over $\left[t, c_{1}\right)$ where $c_{1}>t$ is chosen such that for any $t \in\left[t, c_{1}\right), \beta(t)=\gamma_{1}(t) \in(0, L)$. So $\beta$ is a Lipschitz function over $\left[T_{1},+\infty\right)$

Proposition 4.4 Let $\bar{t} \geq T_{2}$ such that (8) holds at time $t=\bar{t}$

$$
\begin{align*}
& \beta(\bar{t})>0 \Longrightarrow f\left(u\left(\bar{t}, 0^{+}\right)\right)=F_{1}(\bar{t}, u) \text { and } u\left(\bar{t}, 0^{+}\right) \leq \bar{u}  \tag{27}\\
& \beta(\bar{t})<L \Longrightarrow f\left(u\left(\bar{t}, L^{-}\right)\right)=F_{2}(\bar{t}, u) \text { and } u\left(\bar{t}, L^{-}\right) \geq \bar{u} \tag{28}
\end{align*}
$$

Proof. Let $\bar{t} \geq T_{2}$ such that (8) holds at time $t=\bar{t}$ and $\beta(\bar{t})>0$. From (18), $u\left(\bar{t}, 0^{+}\right)=$ $\lim _{\substack{x \rightarrow 0 \\ x>0}} u\left(\bar{t}, x^{+}\right) \in\left[0, \breve{u}_{f(\bar{u})-\epsilon)}\right]$. Let us fix $\bar{x} \in(-\infty, 0)$. From the point 1 . of Definition 1.2 and [4, Theorem 11.1.3] and (17), there exists a minimal generalized characteristic $\xi_{\text {- emanating from }}$ $(\bar{t}, \bar{x})$, defined maximally on an interval $[a, \bar{t}]$. Moreover, for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and, for any $t \in(a, \not t]$, we have $u\left(t, \xi_{-}(t)^{-}\right)=v$. We have two alternatives.

Case 1: $a=0$ and $\xi_{-}(0)<0$. From [4, Theorem 11.1.3], we have $u_{0}\left(\xi_{-}(0)^{+}\right) \leq u\left(\bar{t}, \bar{x}^{-}\right) \leq$ $u_{0}\left(\xi_{-}(0)^{-}\right)$. By definition of $u_{0}$ in (9), we have $u_{0}\left(\xi_{-}(0)^{+}\right)=u_{0}\left(\xi_{-}(0)^{-}\right)=1$. Then, we deduce that $u\left(\bar{t}, \bar{x}^{-}\right)=1$.

Case 2: $a>0$ or $\left(a=0\right.$ and $\left.\xi_{-}(0)=0\right)$. Since $\xi_{-}$is a straight line, $\xi_{-}(\bar{t})=\bar{x}<0$ and $\xi_{-}(a)=0$, we have for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)<0$. By concavity of $f$, we deduce that $u\left(\bar{t}, \bar{x}^{-}\right) \in(\bar{u}, 1]$.

Thus, for any $\bar{x} \in(-\infty, 0), u\left(\bar{t}, \bar{x}^{-}\right) \in(\bar{u}, 1]$. As a consequence, $u\left(\bar{t}, 0^{-}\right)=\lim _{\substack{\bar{x}<0 \\ \bar{x} \rightarrow 0}} u\left(\bar{t}, \bar{x}^{-}\right) \in[\bar{u}, 1]$ and $u\left(\bar{t}, 0^{+}\right) \in\left[0, \check{u}_{f(\bar{u})-\epsilon}\right)$ with $\check{u}_{f(\bar{u})-\epsilon}<\bar{u}$. Since (8) holds at time $t=\bar{t}$, we conclude that

$$
\left(\left(\gamma_{0}^{l} u\right)(\bar{t}),\left(\gamma_{0}^{r} u\right)(\bar{t})\right) \in \mathcal{G}_{1}\left(F_{1}\right)
$$

with $\left(\gamma_{0}^{l} u\right)(\bar{t})=u\left(\bar{t}, 0^{-}\right)$and $\left(\gamma_{0}^{r} u\right)(\bar{t})=u\left(\bar{t}, 0^{+}\right)$and $\mathcal{G}_{1}$ defined in Definition 1.1. Thus, (27) holds. The same argument works for $\beta(t)<L$.

We will state some preliminary lemmas. Let $(t, x) \in(0,+\infty) \times[0, L]$, we introduce $\tau_{-}(t, x) \in$ $(0,+\infty)$ and $\tau_{+}(t, x) \in(0,+\infty)$ defined by

$$
\begin{equation*}
\tau_{+}(t, x)=\frac{x}{f^{\prime}\left(u\left(t, x^{-}\right)\right)} \text {and } \tau_{-}(t, x)=\frac{x-L}{f^{\prime}\left(u\left(t, x^{-}\right)\right)} \tag{29}
\end{equation*}
$$

with $u$ the $\mathcal{G}$-entropy solution of (1).
Proposition 4.5 Let $\bar{t} \geq \tilde{T}_{2}:=T_{2}+\tau_{\max }$ with $\tau_{\max }=\max \left(-\frac{L}{f^{\prime}\left(w_{R}\right)}, \frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right.}\right)$ such that (8) holds at time $t=\bar{t}$.

$$
\begin{gather*}
\beta(\bar{t})>0 \Longrightarrow \forall x \in(0, \beta(\bar{t})), f\left(u\left(\bar{t}, x^{ \pm}\right)\right)=F_{1}\left(\bar{t}-\tau_{+}(\bar{t}, x), u\right) \text { and } u\left(\bar{t}, x^{ \pm}\right) \in[0, \bar{u})  \tag{30}\\
\beta(\bar{t})<L \Longrightarrow \forall x \in(\beta(\bar{t}), L), u\left(\bar{t}, x^{ \pm}\right)=w_{R} \tag{31}
\end{gather*}
$$

Proof. Let $\bar{t} \geq \tilde{T}_{2}:=T_{2}+\tau_{\max }$ with $\tau_{\max }=\max \left(-\frac{L}{f^{\prime}\left(w_{R}\right)}, \frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon}\right)}\right)$ such that (8) holds at time $t=\bar{t}$ and $x \in(0, \beta(\bar{t}))$. From the point 1. of Definition 1.2, [4, Theorem 11.1.3] and the definition of $\beta$, there exists a minimal backward generalized characteristic, denoted by $\xi_{-}$, associated to $u$ emanating from $(\bar{t}, x)$ defined on an interval $[a, \bar{t}]$ with $a>0$ and $\xi_{-}(a)=0$ such that for any $t \in(a, \bar{t})$

$$
\begin{equation*}
u\left(t, \xi_{-}(t)^{-}\right)=u\left(\bar{t}, x^{-}\right)=u\left(t, \xi_{-}(t)^{+}\right) \tag{32}
\end{equation*}
$$

In particular, $\xi_{-}$is a genuine characteristic (also called shock-free, see [4, Definition 10.2.4]). From Proposition B. 1 with $\xi=\xi_{-}, a=a$ and $b=\bar{t}$, there exists a constant $v \in[0,1]$ such that, for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)>0$ and, for any $t \in(a, \bar{t}), u\left(t, \xi_{-}(t)^{-}\right)=v$ and $f(v) \leq f(\bar{u})-\varepsilon$. In particular, we have $a=\bar{t}-\tau_{+}(\bar{t}, x)=\bar{t}-\frac{x}{f^{\prime}\left(u\left(t, x^{-}\right)\right)} \geq \tilde{T}_{2}-\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right.} \geq T_{2}$. From (32), we deduce that for any $t \in(a, \bar{t}]$

$$
u\left(t, \xi_{-}(t)^{-}\right)=v \text { and } f(v) \leq f(\bar{u})-\varepsilon
$$

- We assume that there exists a time $t_{0} \in(a, \bar{t})$ such that $\xi_{-}\left(t_{0}\right)=\beta\left(t_{0}\right) \in(0, L)$. Then $\xi_{-}$interacts with any maximal backward generalized characteristic emanating from $(t, \beta(t))$ with $t \in\left(t_{0}, \bar{t}\right)$. This leads to a contradiction since from [4, 1.1.2 Corollary], two genuine characteristics cannot interact. Thus, we have for any $t \in(a, \bar{t}]$,

$$
\begin{equation*}
\xi_{-}(t)<\beta(t) \tag{33}
\end{equation*}
$$

- We assume that $\beta(a)=\xi_{-}(a)$ (see Figure 8). Since $\beta(a)=0$, from Proposition 4.3, for any $y \in(0, L)$,

$$
\begin{equation*}
u\left(a, y^{ \pm}\right) \in\left[w_{R}, 1\right] \tag{34}
\end{equation*}
$$

Let $\delta>0$ such that $x+\delta \in(0, \beta(\bar{t}))$. From Lemma A. 1 with $\gamma_{1}(\cdot)=\xi_{-}(\cdot), \gamma_{2}(\cdot)=\xi_{-}(\cdot)+\delta$, $t_{1}=a$ et $t_{2}=\bar{t}$, we have

$$
\begin{align*}
& \int_{0}^{\delta} u(a, z) d z+\int_{a}^{\bar{t}}\left(f(v)-f^{\prime}(v) v\right) d t \\
& \quad=\int_{x}^{x+\delta} u(\bar{t}, z) d z+\int_{a}^{\bar{t}}\left(f\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right)-\dot{\xi}_{-}(t) u\left(t,(\xi(t)+\delta)^{-}\right)\right) d t \tag{35}
\end{align*}
$$

Using that $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and the concavity of $f$, we deduce that

$$
\begin{equation*}
f\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,\left(\xi(t)_{+} \delta\right)^{-}\right)-v\right) \leq 0 \tag{36}
\end{equation*}
$$

From (35) and (36), we have

$$
\begin{equation*}
\int_{0}^{\delta} u(a, z) d z \leq \int_{x}^{x+\delta} u(\bar{t}, z) d z \tag{37}
\end{equation*}
$$

Since for any $z \in(x, x+\delta), 0<z<x+\delta<\beta(\bar{t})$, from Proposition $4.3, u\left(\bar{t}, z^{ \pm}\right) \in\left[0, \breve{u}_{f(\bar{u})-\epsilon}\right)$. Therefore, together with (34), we have

$$
\int_{x}^{x+\delta} u(\bar{t}, z) d z<\delta \check{u}_{f(\bar{u})-\epsilon} \leq \delta w_{R} \leq \int_{0}^{\delta} u(a, z) d z
$$

This leads to a contradiction using (37). Thus, we have

$$
\begin{equation*}
\xi_{-}(a)<\beta(a) \tag{38}
\end{equation*}
$$



Figure 8: Case $\beta(a)=\xi_{-}(a)$ where $\xi_{-}$is a minimal backward generalized characteristic associated to $u$ emanating from $(\bar{t}, x)$ defined on an interval $[a, \bar{t}]$ with $a>0$ and $\xi_{-}(a)=0$. Moreover, for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and, for any $t \in(a, \bar{t}], u\left(t, \xi_{-}(t)^{-}\right)=v$ and $f(v) \leq f(\bar{u})-\varepsilon$ with $v \in[0,1]$.

Using that for any $t \in[a, \bar{t}], \xi_{-}(t)<\beta(t), \xi_{-}(a)=0$ and $\beta$ is a Lipschitz function (see Proposition 4.3), there exists $\bar{\delta}>0$ such that for any $t \in[a-\bar{\delta}, a+\bar{\delta}], \beta(t)>0$. Therefore, from Proposition 4.4, we have for a.e $t \in[a-\bar{\delta}, a+\bar{\delta}]$,

$$
f\left(u\left(t, 0^{+}\right)\right)=F_{1}(t, u) \text { and } u\left(t, 0^{+}\right) \leq \bar{u}
$$

From Proposition B. 1 with $\xi=\xi_{-}, a=a, b=\bar{t}, c=a-\bar{\delta}, d=a+\bar{\delta}, g=\min _{t \in[a-\bar{\delta}, a+\bar{\delta}]} F_{1}(t, u)$ and $G=\max _{t \in[a-\bar{\delta}, a+\bar{\delta}]} F_{1}(t, u)$, we deduce that for any $t \in(b, \bar{t})$,

$$
\begin{equation*}
\min _{t \in[a-\bar{\delta}, a+\bar{\delta}]} F_{1}(t, u) \leq f\left(u\left(\bar{t}, x^{-}\right)\right) \leq \max _{t \in[a-\bar{\delta}, a+\bar{\delta}]} F_{1}(t, u) . \tag{39}
\end{equation*}
$$

Note that the minimum and the maximum of $F_{1}(t, u)$ makes sense since using (5), $t \mapsto F_{1}(t, u)$ is a continuous function and $[a-\bar{\delta}, a+\bar{\delta}]$ is a compact set. Passing to the limit in (39) as $\bar{\delta} \rightarrow 0$, we conclude that

$$
\begin{equation*}
f\left(u\left(\bar{t}, x^{-}\right)\right)=F_{1}(a, u) \text { with } a=\bar{t}-\tau_{+}(\bar{t}, x) \tag{40}
\end{equation*}
$$

Using that $u\left(\bar{t}, x^{+}\right)=\lim _{\epsilon \rightarrow 0^{+}} u\left(\bar{t},(x+\epsilon)^{-}\right)$, we deduce that (40) also hold for $x^{+}$.
Let $\bar{t} \geq \tilde{T}_{2}:=T_{2}+\tau_{\max }$ with $\tau_{\max }=\max \left(-\frac{L}{f^{\prime}\left(w_{R}\right)}, \frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right)}\right)$ such that (8) holds at time $t=\bar{t}$ and $x \in(\beta(\bar{t}), L)$. From the point 1. of Definition 1.2, [4, Theorem 11.1.3] and the definition of $\beta$, there exists a minimal backward generalized characteristic, denoted by $\xi_{-}$, associated to $u$ emanating from $(\bar{t}, x)$ defined on an interval $[a, \bar{t}]$ with $a>0$ and $\xi_{-}(a)=L$ such that for any $t \in(a, \bar{t})$, (32) holds. In particular, $\xi_{-}$is a genuine characteristic (also called shock-free, see [4, Definition 10.2.4]). From Proposition B. 1 with $\xi=\xi_{-}, a=a$ and $b=\bar{t}$, there exists a constant $v \in[0,1]$ such that, for any $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)<0$ and, for any $t \in(a, \bar{t}), u\left(t, \xi_{-}(t)^{-}\right)=v$ and $f(v) \leq f\left(w_{R}\right)$. In particular, we have $a=\bar{t}-\tau_{-}(\bar{t}, x)=\bar{t}-\frac{x-L}{f^{\prime}\left(u\left(t, x^{-}\right)\right)} \geq \tilde{T}_{2}-\frac{L}{f^{\prime}\left(w_{R}\right)} \geq T_{2}$. From (32), we deduce that for any $t \in(a, \bar{t}]$

$$
u\left(t, \xi_{-}(t)^{-}\right)=v \text { and } f(v) \leq f\left(w_{R}\right)
$$

- We assume that there exists a time $t_{0} \in(a, \bar{t})$ such that $\xi_{-}\left(t_{0}\right)=\beta\left(t_{0}\right) \in(0, L)$. Then $\xi_{-}$interacts with any maximal backward generalized characteristic emanating from $(t, \beta(t))$ with $t \in\left(t_{0}, \bar{t}\right)$. This leads to a contradiction since from [4, 1.1.2 Corollary], two genuine characteristics cannot interact. Thus, we have for any $t \in(a, \bar{t}]$,

$$
\begin{equation*}
\beta(t)<\xi_{-}(t) \tag{41}
\end{equation*}
$$

- We assume that $\beta(a)=\xi_{-}(a)$. Since $\beta(a)=L$, from Proposition 4.3, for any $y \in(0, L)$,

$$
\begin{equation*}
u\left(a, y^{ \pm}\right) \in\left[0, \check{u}_{f(\bar{u})-\epsilon}\right) . \tag{42}
\end{equation*}
$$

Let $\delta>0$ such that $x-\delta \in(\beta(\bar{t}), L)$. From Lemma A. 1 with $\gamma_{1}(\cdot)=\xi_{-}(\cdot)-\delta, \gamma_{2}(\cdot)=\xi_{-}(\cdot)$, $t_{1}=\bar{t}$ et $t_{2}=a$, we have

$$
\begin{align*}
\int_{L-\delta}^{L} u(a, x) d x+\int_{a}^{\bar{t}}\left(f\left(u\left(t,\left(\xi_{-}(t)-\delta\right)^{+}\right)\right)\right. & \left.-\dot{\xi}_{-}(t) u\left(t,(\xi(t)-\delta)^{-}\right)\right) d t \\
& =\int_{\bar{x}-\delta}^{\bar{x}} u(\bar{t}, x) d x+\int_{a}^{\bar{t}}\left(f(v)-f^{\prime}(v) v\right) d t \tag{43}
\end{align*}
$$

Using that $t \in[a, \bar{t}], \dot{\xi}_{-}(t)=f^{\prime}(v)$ and the concavity of $f$, we deduce that

$$
\begin{equation*}
f\left(u\left(t,\left(\xi_{-}(t)-\delta\right)^{+}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)-\delta)^{-}\right)-v\right) \leq 0 \tag{44}
\end{equation*}
$$

From (43) and (44), we have

$$
\begin{equation*}
\int_{\bar{x}-\delta}^{\bar{x}} u(\bar{t}, x) d x \leq \int_{L-\delta}^{L} u(a, x) d x \tag{45}
\end{equation*}
$$

Since, for any $z \in(x-\delta, x), \beta(\bar{t})<x-\delta<z<L$, from Proposition 4.3, $u\left(\bar{t}, z^{ \pm}\right) \in\left[w_{R}, 1\right]$. Therefore, together with (42), we have

$$
\int_{L-\delta}^{L} u(a, x) d x \leq \delta \check{u}_{f(\bar{u})-\epsilon}<\delta w_{R} \leq \int_{\bar{x}-\delta}^{\bar{x}} u(\bar{t}, x) d x
$$

This leads to a contradiction using (45). Thus, we have

$$
\begin{equation*}
\beta(a)<\xi_{-}(a) \tag{46}
\end{equation*}
$$

Using that for any $t \in[a, \bar{t}], \beta(t)<\xi_{-}(t), \xi_{-}(a)=L$ and $\beta$ is a Lipschitz function (see Proposition 4.3), there exists $\delta>0$ such that for any $t \in[a-\delta, a+\delta], \beta(t)<L$. Therefore, from Proposition 4.4, we have for a.e $t \in[a-\delta, a+\delta]$,

$$
f\left(u\left(t, L^{-}\right)\right)=f\left(w_{R}\right) \text { and } \bar{u}<u\left(t, L^{-}\right)
$$

From Proposition B. 1 with $\xi=\xi_{-}, a=a, b=\bar{t}, c=a-\delta, d=a+\delta, g=f\left(w_{R}\right)=G$, we deduce that for any $t \in(b, \bar{t})$,

$$
\begin{equation*}
u\left(\bar{t}, x^{-}\right)=w_{R} \text { with } a=\tau_{-}(\bar{t}, x) \tag{47}
\end{equation*}
$$

Using that $u\left(\bar{t}, x^{+}\right)=\lim _{\epsilon \rightarrow 0^{+}} u\left(\bar{t},(x+\epsilon)^{-}\right)$, we deduce that (47) also hold for $x^{+}$.

Lemma 4.2 For a.e $t \geq \tilde{T}_{2}:=T_{2}+2 \tau_{\max }$ with $\tau_{\max }=\max \left(-\frac{L}{f^{\prime}\left(w_{R}\right)}, \frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right)}\right)$.

$$
\begin{cases}\dot{m}(t)=0 & \text { if } \beta(t)=0  \tag{48}\\ \dot{m}(t)=F_{1}(t, u)-F_{2}(t, u) & \text { if } 0<\beta(t)<L \\ \dot{m}(t) \geq F_{1}(t, u)-F_{2}(t, u) & \text { if } \beta(t)=L\end{cases}
$$

where $F_{1}(t, u)$ and $F_{2}(t, u)$ are defined in (5).
Proof. From Lemma A. 1 with $\gamma_{1}=0, \gamma_{2}=L, t_{1}=0$ et $t_{2}=t$ and the definition of $m$ in (6), we have for a.e $t>0$,

$$
\begin{equation*}
\dot{m}(t)=f\left(u\left(t, 0^{+}\right)\right)-f\left(u\left(t, L^{-}\right)\right) \tag{49}
\end{equation*}
$$

where (49) in understood in the sense of Carathéodory solutions. More precisely, Caratheodory solutions are absolutely continuous curves that satisfy the integral version of the differential equation (49), that is,

$$
m(t)=m(0)+\int_{0}^{t}\left(f\left(u\left(s, 0^{+}\right)\right)-f\left(u\left(s, L^{-}\right)\right)\right) d s
$$

Let us fix $\bar{t} \geq \tilde{T}_{2}$ such that (8), (27), (28) and (49) hold. We have three alternatives.
Case 1: $\beta(\bar{t})=0$. From Proposition 4.5, for any $x \in(0, L), u\left(\bar{t}, x^{ \pm}\right)=w_{R}$. Using that $u\left(\bar{t}, 0^{+}\right)=\lim _{\substack{x \rightarrow 0 \\ x>0}} u\left(\bar{t}, x^{+}\right)$and $u\left(\bar{t}, L^{-}\right)=\lim _{\substack{x \rightarrow 0 \\ x<L}} u\left(\bar{t}, x^{-}\right)$, we conclude that $u\left(\bar{t}, 0^{+}\right)=$ $u\left(\bar{t}, L^{-}\right)=w_{R}$. Thus, from (49), for a.e $t \geq \tilde{T}_{2}$ such that $\beta(t)=0$, we have $\dot{m}(t)=0$.
Case 2: $0<\beta(\bar{t})<L$. From (27), (28) and (49), we have

$$
\dot{m}(\bar{t})=F_{1}(\bar{t}, u)-F_{2}(\bar{t}, u)
$$

$\underline{\text { Case } 3}: \beta(\bar{t})=L$. From (8) and (27), $f\left(u\left(\bar{t}, 0^{+}\right)\right)=F_{1}(\bar{t}, u)$ and $f\left(u\left(\bar{t}, L^{-}\right)\right) \leq f\left(w_{R}\right)=$ $F_{2}(\bar{t}, u)$. Therefore, using (49),

$$
\dot{m}(t)=f\left(u\left(t, 0^{+}\right)\right)-f\left(u\left(t, L^{-}\right)\right) \geq F_{1}(t, u)-F_{2}(t, u)
$$

Lemma 4.3 We assume that there exist $t_{1}>0$ and $t_{2}>0$ such that $T_{2}+\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right.}<t_{1}<t_{2}$ and for any $t \in\left[t_{1}, t_{2}\right], \beta(t)=L$. Then, we have

$$
t_{2}-t_{1}<\frac{L}{f^{\prime}\left(\breve{u}_{f(\bar{u})-\epsilon}\right)}
$$



Figure 9: Plotting of the minimal backward generalized characteristic $\xi_{-}$, emanating from $\left(t_{2}, L\right)$, defined maximally over $\left[a, t_{2}\right]$ with $a>0$ and $\xi_{-}(a)=0$.

Proof. We assume that $t_{2} \geq \frac{L}{f^{\prime}\left(\check{u}_{f(\bar{u})-\epsilon)}\right.}+t_{1}$. Let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a sequence such that for any $n \in \mathbb{N}$, $x_{n} \in(0, L)$ and $x_{n} \rightarrow L$ as $n \rightarrow \infty$. Since $f \in C^{2}([0,1])$, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} t_{2}-\tau_{+}\left(t_{2}, x_{n}\right)=\lim _{n \rightarrow \infty} t_{2}-\frac{x_{n}}{f^{\prime}\left(u\left(t_{2}, x_{n}^{-}\right)\right)}=t_{2}-\tau_{+}\left(t_{2}, L\right) \tag{50}
\end{equation*}
$$

From the point 1. of Definition 1.2, [4, Theorem 11.1.3] and the definition of $\beta$, there exists a minimal backward generalized characteristic, denoted by $\xi_{-}^{n}$, associated to $u$ emanating from $\left(t_{2}, x_{n}\right)$ defined on an interval $\left[a_{n}, t_{2}\right]$ with $a_{n}:=t_{2}-\tau_{+}\left(t_{2}, x_{n}\right)>0$ and $\xi_{-}^{n}\left(a_{n}\right)=0$ such that for any $t \in\left(a_{n}, t_{2}\right)$

$$
\begin{equation*}
u\left(t, \xi_{-}^{n}(t)^{-}\right)=u\left(t_{2}, x_{n}^{-}\right)=u\left(t, \xi_{-}^{n}(t)^{+}\right) \tag{51}
\end{equation*}
$$

From [4, Section 10.2], (50) and (51), ( $\left.\xi_{-}^{n}\right)_{n \in \mathbb{N}}$ converges to the minimal generalized characteristic $\xi(\cdot)$ emanating from $\left(t_{2}, L\right)$ defined on an interval $\left[a, t_{2}\right]$ with $a:=t_{2}-\tau_{+}\left(t_{2}, L\right)>0$ and $\xi_{-}(a)=0$ such that, for any $t \in\left(a, t_{2}\right]$,

$$
\begin{equation*}
u\left(t, \xi_{-}(t)^{-}\right)=u\left(t_{2}, L^{-}\right)=u\left(t, \xi_{-}(t)^{+}\right) \tag{52}
\end{equation*}
$$

In particular, $\xi_{-}$is a genuine characteristic (see [4, Definition 10.2.4]). From Proposition B. 1 with $\xi=\xi_{-}, a=a$ and $b=t_{2}$, there exists a constant $v \in[0,1]$ such that, for any $t \in\left[a, t_{2}\right]$,
$\dot{\xi}_{-}(t)=f^{\prime}(v)>0$ and, for any $t \in\left(a, t_{2}\right), u\left(t, \xi_{-}(t)^{-}\right)=v$ and $f(v) \leq f(\bar{u})-\varepsilon$. In particular, we have

$$
\begin{equation*}
a=t_{2}-\tau_{+}(\bar{t}, L)=t_{2}-\frac{L}{f^{\prime}\left(u\left(t_{2}, L^{-}\right)\right)}>t_{2}-\frac{L}{f^{\prime}\left(\check{u}_{f(\bar{u})-\epsilon}\right)} \geq t_{1} \tag{53}
\end{equation*}
$$

From (52), we deduce that for any $t \in\left(a, t_{2}\right]$

$$
\begin{equation*}
u\left(t, \xi_{-}(t)^{-}\right)=v \text { and } f(v) \leq f(\bar{u})-\varepsilon \tag{54}
\end{equation*}
$$

Let us fix $\delta>0$ such that $a-\frac{\delta}{f^{\prime}(v)} \geq t_{1}$. From Lemma A. 1 with $\gamma_{1}(\cdot)=0, \gamma_{2}(\cdot)=\xi_{-}(\cdot)+\delta$, $t_{1}=a-\frac{\delta}{f^{\prime}(v)}$ and $t_{2}=a$ (see Figure 9), we have
$\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, 0^{+}\right)\right) d t+0=\int_{0}^{\delta} u(a, x) d x+\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a}\left(f\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right)-\dot{\xi}_{-}(t) u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right) d t$
Using that $f$ is concave and for any $t \in\left[a, t_{2}\right], \dot{\xi}_{-}(t)=f^{\prime}(v)$, we have for any $t \in\left[a-\frac{\delta}{f^{\prime}(v)}, a\right]$

$$
f\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)-v\right) \leq 0
$$

Therefore,

$$
\begin{equation*}
\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, 0^{+}\right)\right) d t+0 \leq \int_{0}^{\delta} u(a, x) d x+\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right) \tag{55}
\end{equation*}
$$

From Lemma A. 1 with $\gamma_{1}(\cdot)=\xi_{-}(\cdot), \gamma_{2}(\cdot)=\xi_{-}(\cdot)+\delta, t_{1}=a$ and $t_{2}=t_{2}-\frac{\delta}{f^{\prime}(v)}$ (see Figure 9), we have that

$$
\begin{aligned}
\int_{0}^{\delta} u(a, x) d x & +\int_{a}^{t_{2}-\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t \\
& =\int_{L-\delta}^{L} u\left(t_{2}-\frac{\delta}{f^{\prime}(v)}, x\right) d x+\int_{a}^{t_{2}-\frac{\delta}{f^{\prime}(v)}}\left(f\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right)-\dot{\xi}_{-}(t) u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right) d t
\end{aligned}
$$

Using that $f$ is concave and for any $t \in\left[a, t_{2}\right], \dot{\xi}_{-}(t)=f^{\prime}(v)$, we have for any $t \in\left[a, t_{2}-\frac{\delta}{f^{\prime}(v)}\right]$

$$
f\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,\left(\xi_{-}(t)+\delta\right)^{-}\right)-v\right) \leq 0
$$

Thus, we have

$$
\begin{equation*}
\int_{0}^{\delta} u(a, x) d x \leq \int_{L-\delta}^{L} u\left(t_{2}-\frac{\delta}{f^{\prime}(v)}, x\right) d x \tag{56}
\end{equation*}
$$

From Lemma A. 1 with $\gamma_{1}(\cdot)=\xi_{-}(\cdot), \gamma_{2}(\cdot)=L, t_{1}=t_{2}-\frac{\delta}{f^{\prime}(v)}$ and $t_{2}=t_{2}$ (see Figure 9), we have

$$
\begin{equation*}
\int_{L-\delta}^{L} u\left(t_{2}-\frac{\delta}{f^{\prime}(v)}, x\right) d x+\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)=\int_{t_{2}-\frac{\delta}{f^{\prime}(v)}}^{t_{2}} f\left(u\left(t, L^{-}\right)\right) d t \tag{57}
\end{equation*}
$$

From (55), (56) and (57), we have

$$
\begin{equation*}
\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, 0^{+}\right)\right) d t \leq \int_{t_{2}-\frac{\delta}{f^{\prime}(v)}}^{t_{2}} f\left(u\left(t, L^{-}\right)\right) d t \tag{58}
\end{equation*}
$$

Since (8) holds almost everywhere, we have for a.e $t \in\left[t_{1}, t_{2}\right]$

$$
\begin{equation*}
f\left(u\left(t, L^{-}\right)\right) \leq f\left(w_{R}\right) \tag{59}
\end{equation*}
$$

From Lemma A. 1 with $\gamma_{1}(\cdot)=\xi_{-}(\cdot), \gamma_{2}(\cdot)=L, t_{1}=a$ and $t_{2}=t_{2}$, we have

$$
\begin{equation*}
\int_{0}^{L} u(a, x) d x+\int_{a}^{t_{2}}\left(f(v)-f^{\prime}(v) v\right) d t=0+\int_{a}^{t_{2}} f\left(u\left(t, L^{-}\right)\right) d t \tag{60}
\end{equation*}
$$

From (59) and $f\left(w_{R}\right)=f\left(w_{L}\right)$, we have

$$
\begin{equation*}
\int_{0}^{L} u(a, x) d x \leq\left(t_{2}-a\right)\left(f\left(w_{L}\right)-f(v)+f^{\prime}(v) v\right) \tag{61}
\end{equation*}
$$

Since $f$ is a concave function, we have

$$
\begin{equation*}
f\left(w_{L}\right)-f(v)-f^{\prime}(v)\left(v-w_{L}\right) \leq 0 \tag{62}
\end{equation*}
$$

From (29), $a:=t_{2}-\tau_{+}\left(t_{2}, L\right) \geq t_{1}>0$ and (54),

$$
\begin{equation*}
t_{2}-a=\frac{L}{f^{\prime}\left(u\left(t_{2}, L^{-}\right)\right)}=\frac{L}{f^{\prime}(v)} \tag{63}
\end{equation*}
$$

Combining (61) with (62) and (63), we deduce that

$$
\int_{0}^{L} u(a, x) d x \leq \frac{L}{f^{\prime}(v)} f^{\prime}(v) w_{L}=L w_{L}
$$

Therefore, together with (2) and (6), we have

$$
m(a)=\int_{0}^{L}(u(a, x)-w(x)) d x \leq L w_{L}-\int_{0}^{L} w(x) d x<0
$$

Using that $m$ is a continuous function and (53), we deduce that there exists $\bar{\delta}>0$ such that for any $t \in[a-\bar{\delta}, a] \subset\left[t_{1}, t_{2}\right], m(t)<0$. Therefore, together with Proposition 4.4, (5) and for any $t \in\left[t_{1}, t_{2}\right], \beta(t)=L$, we have for a.e $t \in[a-\bar{\delta}, a]$,

$$
\begin{equation*}
f\left(u\left(t, 0^{+}\right)\right)=F_{1}(t, u)>f\left(w_{L}\right) \tag{64}
\end{equation*}
$$

From (58) with $\delta=f^{\prime}(v) \bar{\delta}$, (59) and (64), we deduce that

$$
\bar{\delta} f\left(w_{L}\right)<\int_{a-\bar{\delta}}^{a} f(u(t, 0+)) d t \leq \int_{t_{2}-\bar{\delta}}^{t_{2}} f\left(u\left(t, L^{-}\right)\right) d t \leq \bar{\delta} f\left(w_{R}\right)
$$

This leads to a contradiction since $f\left(w_{L}\right)=f\left(w_{R}\right)$.

Proposition 4.6 There exists $T_{4}>0$ (independent of the initial data $u_{0}$ ) such that for any $t \geq T_{4}$, $\beta(t) \in(0, L)$ and

$$
\begin{equation*}
\dot{m}(t)=-g(m(t)) \tag{65}
\end{equation*}
$$

Proof. Let us fix $\tau_{\max }=\max \left(-\frac{L}{f^{\prime}\left(w_{R}\right)}, \frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right)}\right)$. We will prove that for a.e $t>\tilde{T}_{2}+2 \tau_{\max }$, $\beta(t) \in(0, L)$.

Case 1 : there exists $t_{0} \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right]$ such that $m\left(t_{0}\right)>0$. Using (5), Remark 1 and Lemma 4.2, for any $t \geq t_{0}, m(t) \geq 0$. Since $m(\cdot)$ is a continuous function, without loss of generality, we can additionally assume that (8) holds at time $t=t_{0}$. Moreover, $t-\tau_{+}(t, \beta(t))=t-\frac{\beta(t)}{f^{\prime}\left(u\left(t, \beta(t)^{-}\right)\right)}>t_{0}-\tau_{\max } \geq \tilde{T}_{2}$. We assume that $\beta\left(t_{0}\right)=0$. From Proposition 4.5, for any $x \in(0, L), u\left(t_{0}, x^{ \pm}\right)=w_{R}$. In particular, we have $f\left(u\left(t_{0}, 0^{+}\right)\right)=$ $f\left(w_{R}\right)$. Moreover, using (8), (5) and $m\left(t_{0}\right)>0$, we have $f\left(u\left(t_{0}, 0^{+}\right)\right) \leq F_{1}\left(t_{0}, u\right)<f\left(w_{L}\right)$. This leads to a contradiction since $f\left(w_{R}\right)=f\left(w_{L}\right)$. Thus, we have

$$
\beta\left(t_{0}\right)>0
$$

Since $\beta$ is a continuous function and $\beta\left(t_{0}\right)>0$, there exists $t_{b} \in \mathbb{R} \cup\{+\infty\}$ such that for any $t \in\left[t_{0}, t_{b}\right), \beta(t) \in(0, L)$ and $\beta\left(t_{b}\right) \in\{0, L\}$. From Proposition 4.5, for a.e $t \in\left[t_{0}, t_{b}\right)$, for any $x \in(0, \beta(t)), f(u(t, x))=F_{1}\left(t-\tau_{+}(t, x), u\right)$ and for any $x \in(\beta(t), L), u\left(t, x^{ \pm}\right)=w_{R}$. In particular, we have for a.e $t \in\left[t_{0}, t_{b}\right)$,

$$
f\left(u\left(t, \beta(t)^{-}\right)\right)=F_{1}\left(t-\tau_{+}(t, \beta(t)), u\right) \text { and } u\left(t, \beta(t)^{+}\right)=w_{R}
$$

Therefore, together with (5) and for any $t \in\left[t_{0}, t_{b}\right), m(t) \geq 0$, we have

$$
\begin{equation*}
u\left(t, \beta(t)^{-}\right) \in\left[0, w_{L}\right] \text { and } u\left(t, \beta(t)^{+}\right)=w_{R} \tag{66}
\end{equation*}
$$

From (24), (66), $f\left(w_{L}\right)=f\left(w_{R}\right)$ and the concavity of $f$, we deduce that, for a.e $t \in\left[t_{0}, t_{b}\right)$,

$$
\begin{equation*}
\dot{\beta}(t)=\frac{f\left(u\left(t, \beta(t)^{-}\right)\right)-f\left(u\left(t, \beta(t)^{+}\right)\right)}{u\left(t, \beta(t)^{-}\right)-u\left(t, \beta(t)^{+}\right)} \geq 0 \tag{67}
\end{equation*}
$$

As a consequence, for any $t \in\left[t_{0}, t_{b}\right), \beta(t) \geq \beta\left(t_{0}\right)>0$ and $\beta\left(t_{b}\right) \neq 0$. We now assume that $\beta\left(t_{b}\right)=L$. Since for a.e $t \geq t_{0}(67)$ holds as soon as $\beta(t) \in(0, L)$, we have for any $t \geq t_{b}, \beta(t)=L$. This leads to a contradiction using Lemma 4.3. We conclude that for any $t \geq \tilde{T}_{2}+2 \tau_{\max } \geq t_{0}$,

$$
\beta(t) \in(0, L)
$$

Case 2 : for any $t \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right], m(t) \leq 0$. For a.e $t \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right]$ such that $\beta(t) \in(0, L)$, we have

$$
f\left(u\left(t, \beta(t)^{-}\right)\right)=F_{1}\left(t-\tau_{+}(t, \beta(t)), u\right) \text { and } u\left(t, \beta(t)^{+}\right)=w_{R}
$$

Therefore, together with (5) and for any $t \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right], m(t) \leq 0$, we have

$$
\begin{equation*}
u\left(t, \beta(t)^{-}\right) \in\left[w_{L}, \check{u}_{f(\bar{u})-\epsilon}\right] \text { and } u\left(t, \beta(t)^{+}\right)=w_{R} \tag{68}
\end{equation*}
$$

From (24), (68), $f\left(w_{L}\right)=f\left(w_{R}\right)$ and the concavity of $f$, we deduce that, for a.e $t \in\left[\tilde{T}_{2}+\right.$ $\left.\tau_{\text {max }}, \tilde{T}_{2}+2 \tau_{\text {max }}\right]$ such that $\beta(t) \in(0, L)$,

$$
\begin{equation*}
\dot{\beta}(t)=\frac{f\left(u\left(t, \beta(t)^{-}\right)\right)-f\left(u\left(t, \beta(t)^{+}\right)\right)}{u\left(t, \beta(t)^{-}\right)-u\left(t, \beta(t)^{+}\right)} \leq 0 . \tag{69}
\end{equation*}
$$

If $\beta\left(\tilde{T}_{2}+\tau_{\max }\right) \in[0, L)$ then from (69), for any $t \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right], \beta(t)<L$ and so $\beta\left(\tilde{T}_{2}+2 \tau_{\max }\right)<L$. If $\beta\left(\tilde{T}_{2}+\tau_{\max }\right)=L$ then from Lemma 4.3 and $(69), \beta\left(\tilde{T}_{2}+2 \tau_{\max }\right)<L$. We conclude that

$$
\begin{equation*}
\beta\left(\tilde{T}_{2}+2 \tau_{\max }\right)<L \tag{70}
\end{equation*}
$$

Let $\delta=\frac{x_{0}\left(w_{R}-w_{L}\right)}{2 w_{R}}$, we assume that there exists $\bar{t} \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right]$ such that (8) holds at time $t=\bar{t}$ and $\beta(\bar{t}) \in[0, \delta]$. From Proposition 4.5, for any $x \in(\beta(\bar{t}), L), u\left(t, x^{ \pm}\right)=w_{R}$. Then, from (2), (6) and (68),

$$
\begin{align*}
m(\bar{t}) & =\int_{0}^{L} u(\bar{t}, x) d x-\left(L-x_{0}\right) w_{R}-x_{0} w_{L} \\
& \geq(L-\beta(\bar{t})) w_{R}-\left(L-x_{0}\right) w_{R}-x_{0} w_{L} \\
& \geq(L-\delta) w_{R}-\left(L-x_{0}\right) w_{R}-x_{0} w_{L}  \tag{71}\\
& >0
\end{align*}
$$

This leads to a contradiction since $m(\bar{t}) \leq 0$ and so $\beta(\bar{t})>\delta$. Thus, using that $\beta$ is a continuous function, for any $t \in\left[\tilde{T}_{2}+\tau_{\max }, \tilde{T}_{2}+2 \tau_{\max }\right]$, we have $\beta(t) \in[\delta, L]$. In particular,

$$
\begin{equation*}
\beta\left(\tilde{T}_{2}+2 \tau_{\max }\right) \geq \delta \tag{72}
\end{equation*}
$$

From (70) and (72), we have

$$
\begin{equation*}
\delta \leq \beta\left(\tilde{T}_{2}+2 \tau_{\max }\right)<L \tag{73}
\end{equation*}
$$

From (73) and using that $\beta$ is a continuous function, there exists $t_{b}>\tilde{T}_{2}+2 \tau_{\max }$ such that $\beta\left(t_{b}\right)=0$ or $\beta(L)=0$ and for any $t \in\left[\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right), \beta(t) \in(0, L)$ (by convention $t_{b}=+\infty$ if $\left.\forall t \geq \tilde{T}_{2}+2 \tau_{\max }, \beta(t) \in(0, L)\right)$. From Lemma 4.2, for a.e $t \in\left(\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right)$,

$$
\dot{m}(t)=F_{1}(t, u)-f\left(w_{R}\right)
$$

Therefore, together with (5), Remark 1 and the continuity of $m$, for any $t \in\left(\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right)$,

$$
\begin{equation*}
m(t) \leq 0 \tag{74}
\end{equation*}
$$

From Proposition 4.5, for a.e $t \in\left[\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right)$, for any $x \in(0, \beta(t)), f(u(t, x))=F_{1}(t-$ $\left.\tau_{ \pm}(t, x), u\right)$ and for any $x \in(\beta(t), L), u\left(t, x^{ \pm}\right)=w_{R}$. In particular, we have for a.e $t \in$ $\left[\tilde{T}_{2}+2 \tau_{\text {max }}, t_{b}\right)$,

$$
f\left(u\left(t, \beta(t)^{-}\right)\right)=F_{1}\left(t-\tau_{+}(t, \beta(t)), u\right) \text { and } u\left(t, \beta(t)^{+}\right)=w_{R}
$$

Therefore, together with (5) and (74), we have

$$
\begin{equation*}
u\left(t, \beta(t)^{-}\right) \in\left[w_{L}, \check{u}_{f(\bar{u})-\epsilon}\right) \text { and } u\left(t, \beta(t)^{+}\right)=w_{R} \tag{75}
\end{equation*}
$$

From (24), (75), f( $\left.w_{L}\right)=f\left(w_{R}\right)$ and the concavity of $f$, we deduce that, for a.e $t \in\left[\tilde{T}_{2}+\right.$ $\left.2 \tau_{\text {max }}, t_{b}\right)$,

$$
\begin{equation*}
\dot{\beta}(t)=\frac{f\left(u\left(t, \beta(t)^{-}\right)\right)-f\left(u\left(t, \beta(t)^{+}\right)\right)}{u\left(t, \beta(t)^{-}\right)-u\left(t, \beta(t)^{+}\right)} \leq 0 \tag{76}
\end{equation*}
$$

As a consequence, together with (73), for any $t \in\left[\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right), \beta(t) \leq \beta\left(\tilde{T}_{2}+2 \tau_{\max }\right)<L$. We assume that there exists $\bar{t} \in\left[\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right]$ such that (8) holds at time $t=\bar{t}$ and $\beta(\bar{t}) \in[0, \delta]$ with $\delta=\frac{x_{0}\left(w_{R}-w_{L}\right)}{2 w_{R}}$. From Proposition 4.5, for any $x \in(\beta(\bar{t}), L), u\left(t, x^{ \pm}\right)=w_{R}$. Then, from (2), (6) and (75),

$$
\begin{align*}
m(\bar{t}) & =\int_{0}^{L} u(\bar{t}, x) d x-\left(L-x_{0}\right) w_{R}-x_{0} w_{L} \\
& \geq(L-\beta(\bar{t})) w_{R}-\left(L-x_{0}\right) w_{R}-x_{0} w_{L}  \tag{77}\\
& \geq(L-\delta) w_{R}-\left(L-x_{0}\right) w_{R}-x_{0} w_{L}
\end{align*}
$$

From (74), this leads to a contradiction and so $\beta(\bar{t})>\delta$. Since $\beta$ is continuous, for any $t \in\left[\tilde{T}_{2}+2 \tau_{\max }, t_{b}\right), 0<\delta \leq \beta(t)$. We deduce that for any $t \geq \tilde{T}_{2}+2 \tau_{\max }$,

$$
\beta(t) \in(0, L)
$$

We conclude that for a.e $t \geq T_{3}:=\tilde{T}_{2}+2 \tau_{\max }, \beta(t) \in(0, L)$. From Proposition 4.2, for a.e $t \geq T_{3}:=\tilde{T}_{2}+2 \tau_{\max }$,

$$
\begin{equation*}
\dot{m}(t)=F_{1}(t, u)-F_{2}(t, u) . \tag{78}
\end{equation*}
$$

Using that for any $(t, x) \in \mathbb{R}_{+} \times(0, L), u(t, x) \in[0,1],(2)$ and (6), we have for any $t \geq 0$,

$$
\begin{equation*}
-x_{0} w_{L}-\left(L-x_{0}\right) w_{R} \leq m(t) \leq L-x_{0} w_{L}-\left(L-x_{0}\right) w_{R} \tag{79}
\end{equation*}
$$

From (5), Remark 4, $g$ is a continuous function, (78) and (79), there exists a finite time $T_{4}>0$ such that for any $t \geq T_{4}$,

$$
\begin{equation*}
f\left(w_{L}\right)-g(m(t)) \in[0, f(\bar{u})-\epsilon] . \tag{80}
\end{equation*}
$$

Thus, from (5) and (78), we deduce that for a.e $t \geq T_{4}$,

$$
\dot{m}(t)=-g(m(t))
$$

Proof of Theorem 2.2. From Proposition 4.6, for a.e $t \geq T_{4}$,

$$
\begin{equation*}
\beta(t) \in(0, L) \text { and } \dot{m}(t)=-g(m(t)) \tag{81}
\end{equation*}
$$

with $m(t):=\int_{0}^{L}(u(t, x)-w(x)) d x$. Therefore, from Remark 4,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} m(t)=0 \tag{82}
\end{equation*}
$$

From (81) and Proposition 4.5, for a.e $t \geq T_{4}$,

$$
\begin{equation*}
\forall x \in(0, \beta(t)), \quad f\left(u\left(t, x^{ \pm}\right)\right)=F_{1}\left(t-\tau_{+}(t, x), u\right) \text { and } u\left(t, x^{ \pm}\right) \in[0, \bar{u}) \tag{83}
\end{equation*}
$$

and

$$
\begin{equation*}
\forall x \in(\beta(t), L), \quad u\left(t, x^{ \pm}\right)=w_{R} \tag{84}
\end{equation*}
$$

Using that $f \in C^{2}([0,1])$ is strictly concave and $\bar{u}=\operatorname{argmax}_{u \in[0,1]} f(u)$, then $f:[0, \bar{u}] \rightarrow[0, f(\bar{u})]$ is bijective and we denote by $f_{+}^{-1}$ its inverse function. Since for any $x \in(0, \bar{u}), f^{\prime}(x)>0$ then, using the Inverse function theorem, $f_{+}^{-1} \in C^{2}([0, f(\bar{u})])$. From (80), (83) and (84), we have, for a.e $t \geq T_{4}$,

$$
\begin{align*}
m(t)+\int_{0}^{T} w(x) d x & =\int_{0}^{T} u(t, x) d x \\
& =\int_{0}^{\beta(t)} f_{+}^{-1}\left(f\left(w_{L}\right)-g\left(m\left(t-\tau_{+}(t, x)\right)\right)\right) d x+\int_{\beta(t)}^{L} w_{R} d x \tag{85}
\end{align*}
$$

From (29) and (83), for any $x \in[0, L]$,

$$
\begin{equation*}
t-\frac{L}{f^{\prime}\left(\check{u}_{f(\bar{u})-\epsilon}\right)} \leq t-\tau_{+}(t, x) \leq t \tag{86}
\end{equation*}
$$

We introduce two functions $a(\cdot)$ and $b(\cdot)$ defined by, for any $t \geq T_{4}$

$$
\begin{equation*}
a(t):=\min _{s \in\left[t-\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right.}, t\right]} g(m(s)) \text { and } b(t):=\max _{s \in\left[t-\frac{\tilde{L}^{\prime}}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right)} t\right]} g(m(s)) . \tag{87}
\end{equation*}
$$

From (85) and using that $f_{+}^{-1}$ is an increasing function,

$$
\begin{equation*}
\frac{m(t)+\int_{0}^{T} w(x) d x-L w_{R}}{\left.f_{+}^{-1}\left(f\left(w_{L}\right)-b(t)\right)\right)-w_{R}} \leq \beta(t) \leq \frac{m(t)+\int_{0}^{T} w(x) d x-L w_{R}}{\left.f_{+}^{-1}\left(f\left(w_{L}\right)-a(t)\right)\right)-w_{R}} \tag{88}
\end{equation*}
$$

We introduce two functions $\alpha_{a}$ and $\alpha_{b}$ defined by, for any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{equation*}
\left.\left.\alpha_{a}(t)=f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} a(t)\right)\right)-w_{L} \text { and } \alpha_{b}(t)=f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} b(t)\right)\right)-w_{L} \tag{89}
\end{equation*}
$$

From (2), (86), (88) and (89), for any $t \geq T_{4}$,

$$
\begin{equation*}
\frac{m(t)-\alpha_{b}(t)}{w_{L}-w_{R}+\alpha_{b}(t)} \leq \beta(t)-x_{0} \leq \frac{m(t)-\alpha_{a}(t)}{w_{L}-w_{R}+\alpha_{a}(t)} \tag{90}
\end{equation*}
$$

From (82), (87) and (88), we have,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \max \left(\alpha_{a}(t), \alpha_{b}(t)\right)=0 \tag{91}
\end{equation*}
$$

From (82), (90) and (91), we deduce that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \beta(t)=x_{0} \tag{92}
\end{equation*}
$$

For any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{aligned}
\|u(t, \cdot)-w(\cdot)\|_{L^{1}(0, L)} & =\int_{0}^{\min \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x+\int_{\min \left(\beta(t), x_{0}\right)}^{\max \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x \\
& +\int_{\max \left(\beta(t), x_{0}\right)}^{L}|u(t, x)-w(x)| d x
\end{aligned}
$$

From (5), (86), (87), (91), we have for any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{aligned}
\int_{0}^{\min \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x & =\int_{0}^{\min \left(\beta(t), x_{0}\right)}\left|F_{1}\left(t-\tau_{+}(t, x)\right)-w_{L}\right| d x \\
& =\int_{0}^{\min \left(\beta(t), x_{0}\right)}\left|f_{+}^{-1}\left(f\left(w_{L}\right)-g\left(m\left(t-\tau_{+}(t, x)\right)\right)\right)-w_{L}\right| \\
& \leq \min \left(\beta(t), x_{0}\right) \max \left(\left|\alpha_{a}(t)\right|,\left|\alpha_{b}(t)\right|\right)
\end{aligned}
$$

Thus, from (91) and (92),

$$
\lim _{t \rightarrow \infty} \int_{0}^{\min \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x=0
$$

Moreover,

$$
\int_{\max \left(\beta(t), x_{0}\right)}^{L}|u(t, x)-w(x)| d x=\int_{\max \left(\beta(t), x_{0}\right)}^{L}\left|w_{R}-w_{R}\right| d x=0
$$

We have

$$
\begin{aligned}
\int_{\min \left(\beta(t), x_{0}\right)}^{\max \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x & \leq\left(\max \left(\beta(t), x_{0}\right)-\min \left(\beta(t), x_{0}\right)\right) \\
& \leq\left|\beta(t)-x_{0}\right|
\end{aligned}
$$

Thus, from (92),

$$
\lim _{t \rightarrow \infty} \int_{\min \left(\beta(t), x_{0}\right)}^{\max \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x=0
$$

We conclude that

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\|u(t, \cdot)-w(\cdot)\|_{L^{1}(0, L)}=0 \tag{93}
\end{equation*}
$$

### 4.3 Proofs of Corollary 2.1 and Corollary 2.2

Proof of Corollary 2.1. From Proposition 4.6, for a.e $t \geq T_{4}$,

$$
\begin{equation*}
\beta(t) \in(0, L) \text { and } f\left(w_{L}\right)-g(m(t)) \in[0, f(\bar{u})-\epsilon] \text { and } \dot{m}(t)=-K_{0} m(t) \tag{94}
\end{equation*}
$$

with $m(t):=\int_{0}^{L}(u(t, x)-w(x)) d x$ and

$$
T_{4}:=\max \left(\frac{L-x_{0} w_{L}-\left(L-x_{0}\right) w_{R}-\frac{f\left(w_{L}\right)}{K_{0}}}{f\left(w_{L}\right)}, \frac{-x_{0} w_{L}-\left(L-x_{0}\right) w_{R}+\frac{f\left(w_{L}\right)-f(\bar{u})+\epsilon}{K_{0}}}{f(\bar{u})-\epsilon-f\left(w_{L}\right)}\right)+T_{3}
$$

Therefore, for any $T \geq T_{4}$

$$
\begin{equation*}
m(t)=m\left(T_{4}\right) e^{-K_{0}\left(t-T_{4}\right)} \tag{95}
\end{equation*}
$$

From (94) and Proposition 4.5, for a.e $t \geq T_{4}$,

$$
\begin{equation*}
\forall x \in(0, \beta(t)), \quad f\left(u\left(t, x^{ \pm}\right)\right)=F_{1}\left(t-\tau_{+}(t, x), u\right) \text { and } u\left(t, x^{ \pm}\right) \in[0, \bar{u}) \tag{96}
\end{equation*}
$$

and

$$
\begin{equation*}
\forall x \in(\beta(t), L), \quad u\left(t, x^{ \pm}\right)=w_{R} \tag{97}
\end{equation*}
$$

Using that $f \in C^{2}([0,1])$ is strictly concave and $\bar{u}=\operatorname{argmax}_{u \in[0,1]} f(u)$, then $f:[0, \bar{u}] \rightarrow[0, f(\bar{u})]$ is bijective and we denote by $f_{+}^{-1}$ its inverse function. Since for any $x \in(0, \bar{u}), f^{\prime}(x)>0$ then, using the Inverse function theorem, $f_{+}^{-1} \in C^{2}([0, f(\bar{u})])$. From (96) and (97), we have, for a.e $t \geq T_{4}$,

$$
\begin{align*}
m(t)+\int_{0}^{T} w(x) d x & =\int_{0}^{T} u(t, x) d x  \tag{98}\\
& =\int_{0}^{\beta(t)} f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} m\left(t-\tau_{+}(t, x)\right)\right) d x+\int_{\beta(t)}^{L} w_{R} d x
\end{align*}
$$

From (29) and (96), for any $x \in[0, L]$,

$$
\begin{equation*}
t-\frac{L}{f^{\prime}\left(\check{u}_{f(\bar{u})-\epsilon}\right)} \leq t-\tau_{+}(t, x) \leq t \tag{99}
\end{equation*}
$$

We introduce two functions $a(\cdot)$ and $b(\cdot)$ defined by, for any $t \geq T_{4}$

$$
\begin{equation*}
a(t)=\min _{s \in\left[t-\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon}\right)}, t\right]} m(s) \text { and } b(t)=\max _{s \in\left[t-\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}, t\right]}\right.} m(s) . \tag{100}
\end{equation*}
$$

From (98) and using that $f_{+}^{-1}$ is an increasing function,

$$
\begin{equation*}
\frac{m(t)+\int_{0}^{T} w(x) d x-L w_{R}}{\left.f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} b(t)\right)\right)-w_{R}} \leq \beta(t) \leq \frac{m(t)+\int_{0}^{T} w(x) d x-L w_{R}}{\left.f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} a(t)\right)\right)-w_{R}} \tag{101}
\end{equation*}
$$

We introduce two functions $\alpha_{a}$ and $\alpha_{b}$ defined by, for any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{equation*}
\left.\left.\alpha_{a}(t)=f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} a(t)\right)\right)-w_{L} \text { and } \alpha_{b}(t)=f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} b(t)\right)\right)-w_{L} \tag{102}
\end{equation*}
$$

From (2), (99), (101) and (102), for any $t \geq T_{4}$,

$$
\begin{equation*}
\frac{m(t)-\alpha_{b}(t)}{w_{L}-w_{R}+\alpha_{b}(t)} \leq \beta(t)-x_{0} \leq \frac{m(t)-\alpha_{a}(t)}{w_{L}-w_{R}+\alpha_{a}(t)} \tag{103}
\end{equation*}
$$

From (95), (100) and (101), we have, for any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{equation*}
\max \left(\left|\alpha_{a}(t)\right|,\left|\alpha_{b}(t)\right|\right)\left|\leq \max _{x \in[0, L]}\left[\left(f^{-1}\right)^{\prime}(x)\right] K_{0}\right| m\left(T_{4}\right) \left\lvert\, e^{-K_{0}\left(t-\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon)}\right.}-T_{4}\right)}\right. \tag{104}
\end{equation*}
$$

From (103) and (104), we deduce that there exists $C>0$ such that, for any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{equation*}
\left|\beta(t)-x_{0}\right| \leq C e^{-K_{0}\left(t-\tau_{\max }-T_{4}\right)}\left|m\left(T_{4}\right)\right| . \tag{105}
\end{equation*}
$$

For any $t \geq T_{4}+\tau_{\text {max }}$,

$$
\begin{aligned}
\|u(t, \cdot)-w(\cdot)\|_{L^{1}(0, L)} & =\int_{0}^{\min \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x+\int_{\min \left(\beta(t), x_{0}\right)}^{\max \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x \\
& +\int_{\max \left(\beta(t), x_{0}\right)}^{L}|u(t, x)-w(x)| d x
\end{aligned}
$$

From (5), (99), (100), (104), we have for any $t \geq T_{4}+\tau_{\max }$,

$$
\begin{aligned}
\int_{0}^{\min \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x & =\int_{0}^{\min \left(\beta(t), x_{0}\right)}\left|F_{1}\left(t-\tau_{+}(t, x)\right)-w_{L}\right| d x, \\
& =\int_{0}^{\min \left(\beta(t), x_{0}\right)}\left|f_{+}^{-1}\left(f\left(w_{L}\right)-K_{0} m\left(t-\tau_{+}(t, x)\right)\right)-w_{L}\right|, \\
& \leq \min \left(\beta(t), x_{0}\right) \max \left(\left|\alpha_{a}(t)\right|,\left|\alpha_{b}(t)\right|\right), \\
& \left.\leq x_{0} \max _{x \in[0, L]}\left[\left(f^{-1}\right)^{\prime}(x)\right] K_{0}\left|m\left(T_{4}\right)\right| e^{-K_{0}\left(t-\frac{L}{f^{\prime}\left(\tilde{u}_{f(\bar{u})-\epsilon}\right)}-T_{4}\right.}\right),
\end{aligned}
$$

and

$$
\int_{\max \left(\beta(t), x_{0}\right)}^{L}|u(t, x)-w(x)| d x=\int_{\max \left(\beta(t), x_{0}\right)}^{L}\left|w_{R}-w_{R}\right| d x=0
$$

and from (105), there exists $C>0$ such that

$$
\begin{aligned}
\int_{\min \left(\beta(t), x_{0}\right)}^{\max \left(\beta(t), x_{0}\right)}|u(t, x)-w(x)| d x & \leq\left(\max \left(\beta(t), x_{0}\right)-\min \left(\beta(t), x_{0}\right)\right), \\
& \leq\left|\beta(t)-x_{0}\right|, \\
& \leq C e^{-K_{0}\left(t-\tau_{\max }-T_{4}\right)}\left|m\left(T_{4}\right)\right|
\end{aligned}
$$

Therefore, together with (95) at time $t=T_{5}:=T_{4}+\tau_{\max }$, there exists $C>0$ such that for any $t \geq T_{5}$,

$$
\begin{equation*}
\|u(t, \cdot)-w(\cdot)\|_{L^{1}(0, L)} \leq C e^{-K_{0}\left(t-T_{5}\right)}\left\|u\left(T_{5}, \cdot\right)-w(\cdot)\right\|_{L^{1}(0, L)} \tag{106}
\end{equation*}
$$

Adapting the proof of [1, Proposition 2.10] and using (5),

$$
\begin{equation*}
\int_{0}^{L}\left|u\left(T_{5}, x\right)-w(x)\right| d x \leq 2 \int_{0}^{T_{5}}\left|f(\bar{u})-\epsilon-f\left(w_{L}\right)\right| d t+\int_{\mathbb{R}}\left|u_{0}(x)-w(x)\right| d x \tag{107}
\end{equation*}
$$

From (106) and (107), there exists $C>0$ such that for any $t \geq 0$,

$$
\|u(t, \cdot)-w(\cdot)\|_{L^{1}(0, L)} \leq C e^{-K_{0} t}\left(\left\|u_{0}(\cdot)-w(\cdot)\right\|_{L^{1}(0, L)}+C\right)
$$

Proof of Corollary 2.2. From Proposition 4.6, for a.e $t \geq T_{4}$,

$$
\begin{equation*}
\beta(t) \in(0, L) \text { and } f\left(w_{L}\right)-g(m(t)) \in[0, f(\bar{u})-\epsilon] \text { and } \dot{m}(t)=-g(m(t)), \tag{108}
\end{equation*}
$$

with $m(t):=\int_{0}^{L}(u(t, x)-w(x)) d x$. From Proposition [6, Proposition 1], there exists a time $T_{5}:=T_{4}+\int_{0}^{1} \frac{d y}{g(y)}$ such that for any $t \geq T_{5}$,

$$
\begin{equation*}
m(t)=0 . \tag{109}
\end{equation*}
$$

From (108) and Proposition 4.5, for a.e $t \geq T_{4}$,

$$
\begin{equation*}
\forall x \in(0, \beta(t)), \quad f\left(u\left(t, x^{ \pm}\right)\right)=F_{1}\left(t-\tau_{+}(t, x), u\right) \text { and } u\left(t, x^{ \pm}\right) \in[0, \bar{u}) \tag{110}
\end{equation*}
$$

and

$$
\begin{equation*}
\forall x \in(\beta(t), L), \quad u\left(t, x^{ \pm}\right)=w_{R} \tag{111}
\end{equation*}
$$

From (29) and (110), for any $x \in[0, L]$,

$$
\begin{equation*}
t-\frac{L}{f^{\prime}\left(\check{u}_{f(\bar{u})-\epsilon}\right)} \leq t-\tau_{+}(t, x) \leq t \tag{112}
\end{equation*}
$$

From (5), (109), (110), (111) and (112), for any $t \geq T_{4}+\tau_{\max }$,

$$
\forall x \in(0, \beta(t)), u\left(t, x^{ \pm}\right)=w_{L} \text { and } \forall x \in(\beta(t), L), u\left(t, x^{ \pm}\right)=w_{R} \text { and } m(t)=0
$$

Therefore, together with (2) and (6), for a.e $x \in(0, L)$, we have $u(t, x)=w(x)$.

## A Conservation of mass

Lemma A. 1 Let $t_{1}<t_{2}$ and let $\left(\gamma_{1}, \gamma_{2}\right) \in C^{0,1}\left(\left[t_{1}, t_{2}\right] ;[0, L]\right)^{2}$ satisfy $\gamma_{1}(t)<\gamma_{2}(t)$ for any $t \in\left(t_{1}, t_{2}\right)$. We have

$$
\begin{align*}
& \int_{\gamma_{1}\left(t_{1}\right)}^{\gamma_{2}\left(t_{1}\right)} u\left(t_{1}, x\right) d x+\int_{t_{1}}^{t_{2}} F\left(u\left(t, \gamma_{1}(t)^{+}\right)\right) d t \\
& \quad=\int_{\gamma_{1}\left(t_{2}\right)}^{\gamma_{2}\left(t_{2}\right)} u\left(t_{2}, x\right) d x+\int_{t_{1}}^{t_{2}} F\left(u\left(t, \gamma_{2}(t)^{-}\right)\right) d t \tag{113}
\end{align*}
$$

with $u$ the $\mathcal{G}$-entropy solution of (1) and for every $t \in \mathbb{R}^{+}$, for every $\gamma \in C^{0,1}([0, T] ; \mathbb{R})$,

$$
F\left(u\left(t, \gamma(t)^{ \pm}\right)\right)=f\left(u(t, \gamma(t) \pm)-\dot{\gamma}(t) u\left(t, \gamma(t)^{ \pm}\right)\right.
$$

Proof. It directly follows from the point 1. of Definition 1.2 and [7, Lemma 1].

## B Notion of generalized backward characteristics with $\mathcal{G}$ entropy boundary conditions

Proposition B. 1 Let $u$ be the unique $\mathcal{G}$-entropy solution of (1) and we assume that there exist $c, d>0$ such that $c<d$ and for a.e $t \in[c, d]$

$$
\begin{equation*}
g \leq f\left(u\left(t, 0^{+}\right)\right) \leq G \text { and } l \leq f\left(u\left(t, L^{-}\right)\right) \leq L \tag{114}
\end{equation*}
$$

where $g, G, l, L$ are given constants. We consider $\xi$ a genuine characteristic on an interval $[a, b]$ such that

$$
\forall t \in(a, b], \quad \xi(t) \in(0, L)
$$

Then there exists a constant $v \in[0,1]$ such that, for any $t \in[a, b], \dot{\xi}(t)=f^{\prime}(v)$ and, for any $t \in(a, b), u(t, \xi(t))=v$. Moreover,

$$
\begin{align*}
\xi(a) & =0 \text { and } a>0 \text { implies } f^{\prime}(v)>0 \text { and } \begin{cases}g \leq f(v) \leq G & \text { if } a \in(c, d), \\
0 \leq f(v) \leq f(\bar{u})-\varepsilon & \text { otherwise, }\end{cases}  \tag{115}\\
a & =0 \text { implies } u(a, \xi(a)+) \leq v \leq u(a, \xi(a)-),  \tag{116}\\
\xi(a) & =L \text { and } a>0 \text { implies } f^{\prime}(v)<0 \text { and } \begin{cases}l \leq f(v) \leq L & \text { if } a \in(c, d), \\
0 \leq f(v) \leq f\left(w_{R}\right) & \text { otherwise. }\end{cases} \tag{117}
\end{align*}
$$

Remark 2 Note that if the inequality in (114) is strict then the inequalities in (115), (116) and (117) are also strict. For instance, if $g \leq f\left(u\left(t, 0^{+}\right)\right)<G$ then (115) is replaced by $g \leq f(v)<G$.

Proof. From Definition 1.2 , the unique $\mathcal{G}$-entropy is a Kruzhkov entropy solution over $\mathbb{R}_{+} \times(0, L)$. Thus, From [4, Theorem 10.2.3 and Theorem 11.1.1], there exists a constant $v \in[0,1]$ such that, for any $t \in[a, b], \dot{\xi}(t)=f^{\prime}(v)$ and, for any $t \in(a, b), u(t, \xi(t))=v$. We have three different possibilities :

Case 1: $\xi(a)=0$ and $a>0)$. We immediately have $f^{\prime}(v)>0$. Let us fix $\delta>0$, from Lemma A. 1 with $t_{1}=a, t_{2}=a+\frac{\delta}{f^{\prime}(v)}, \gamma_{1}(\cdot)=0$ and $\gamma_{2}(\cdot)=\xi(\cdot)$ (see Figure 10a), we have

$$
\begin{equation*}
\int_{a}^{a+\frac{\delta}{f^{\prime}(v)}} f\left(u\left(t, 0^{+}\right)\right) d t+0=\int_{0}^{\delta} u\left(a+\frac{\delta}{f^{\prime}(v)}, x\right) d x+\int_{a}^{a+\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t \tag{118}
\end{equation*}
$$

From (8), for a.e $t>0, f(u(t, 0)) \leq F_{1}(t, u) \leq f(\bar{u})-\varepsilon$. Thus, we have

$$
\begin{equation*}
\int_{0}^{\delta} u\left(a+\frac{\delta}{f^{\prime}(v)}, x\right) d x+\int_{a}^{a+\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t \leq \frac{\delta}{f^{\prime}(v)}(f(\bar{u})-\epsilon) \tag{119}
\end{equation*}
$$

Let $\bar{c} \in(a, b)$. From Lemma A. 1 with $t_{1}=a+\frac{\delta}{f^{\prime}(v)}, t_{2}=\bar{c}, \gamma_{1}(\cdot)=\xi(\cdot)-\delta$ and $\gamma_{2}(\cdot)=\xi(\cdot)$ (see Figure 10a), we have

$$
\begin{array}{r}
\int_{0}^{\delta} u\left(a+\frac{\delta}{f^{\prime}(v)}, x\right) d x+\int_{a+\frac{\delta}{f^{\prime}(v)}}^{\bar{c}}\left(f\left(u\left(t,(\xi(t)-\delta)^{+}\right)\right)-\dot{\xi}(t) u\left(t,(\xi(t)-\delta)^{+}\right)\right) d t \\
=\int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x) d x+\int_{a+\frac{\delta}{f^{\prime}(v)}}^{\bar{c}}\left(f(v)-f^{\prime}(v) v\right) d t
\end{array}
$$

Using that $f$ is concave and for any $t \in[a, b], \dot{\xi}(t)=f^{\prime}(v)$, we have for any $t \in\left[a+\frac{\delta}{f^{\prime}(v)}, \bar{c}\right]$

$$
f\left(u\left(t,(\xi(t)-\delta)^{+}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)-\delta)^{+}\right)-v\right) \leq 0
$$

Thus, we have

$$
\begin{equation*}
\int_{0}^{\delta} u\left(a+\frac{\delta}{f^{\prime}(v)}, x\right) d x \geq \int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x) d x \tag{120}
\end{equation*}
$$

From (119) and (120),

$$
\int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x) d x+\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right) \leq \frac{\delta}{f^{\prime}(v)}(f(\bar{u})-\epsilon)
$$

Dividing by $\delta$ and taking $\delta \rightarrow 0^{+}$and using that $u(\bar{c}, \xi(\bar{c}))=v$, we deduce that

$$
f(v) \leq f(\bar{u})-\epsilon
$$

If $a \in(c, d)$ then we choose $\delta>0$ such that $\left[a-\frac{\delta}{f^{\prime}(v)}, a+\frac{\delta}{f^{\prime}(v)}\right] \in(c, d)$. From (114), we have for a.e $t \in\left[a-\frac{\delta}{f^{\prime}(v)}, a+\frac{\delta}{f^{\prime}(v)}\right], g \leq f\left(u\left(t, 0^{+}\right)\right) \leq G$. Thus, using (118), we get

$$
\int_{0}^{\delta} u\left(a+\frac{\delta}{f^{\prime}(v)}, x\right) d x+\int_{a}^{a+\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t \leq \frac{\delta}{f^{\prime}(v)} G
$$

Moreover, (120) holds. Thus, we deduce that

$$
\int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x) d x+\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right) \leq \frac{\delta}{f^{\prime}(v)} G
$$

Dividing by $\delta$ and taking $\delta \rightarrow 0^{+}$and using that $u(\bar{c}, \xi(\bar{c}))=v$, we deduce that

$$
f(v) \leq G
$$

From Lemma A. 1 with $t_{1}=a-\frac{\delta}{f^{\prime}(v)}, t_{2}=a, \gamma_{1}(\cdot)=0$ and $\gamma_{2}(\cdot)=\xi(\cdot)+\delta$ (see Figure 10b), we have

$$
\begin{equation*}
\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, 0^{+}\right)\right) d t+0=\int_{0}^{\delta} u(a, x) d x+\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a}\left(f\left(u\left(t,(\xi(t)+\delta)^{-}\right)\right)-\dot{\xi}(t) u\left(t,(\xi(t)+\delta)^{-}\right)\right) d t \tag{121}
\end{equation*}
$$

Using that $f$ is concave and for any $t \in[a, b], \dot{\xi}(t)=f^{\prime}(v)$, we have for any $t \in\left[a-\frac{\delta}{f^{\prime}(v)}, a\right]$

$$
f\left(u\left(t,(\xi(t)+\delta)^{-}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)+\delta)^{-}\right)-v\right) \leq 0
$$

Therefore,

$$
\int_{a-\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, 0^{+}\right)\right) d t+0 \leq \int_{0}^{\delta} u(a, x) d x+\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)
$$

From (114) and $\left[a-\frac{\delta}{f^{\prime}(v)}, a+\frac{\delta}{f^{\prime}(v)}\right] \in(c, d)$, we deduce that

$$
\begin{equation*}
\frac{\delta}{f^{\prime}(v)} g \leq \int_{0}^{\delta} u(a, x) d x+\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right) \tag{122}
\end{equation*}
$$

Let $\bar{c} \in(a, b)$. From Lemma A. 1 with $t_{1}=a, t_{2}=\bar{c}, \gamma_{1}(\cdot)=\xi(\cdot)$ and $\gamma_{2}(\cdot)=\xi(\cdot)+\delta$, we have

$$
\begin{aligned}
\int_{0}^{\delta} u(a, x) d x & +\int_{a}^{\bar{c}}\left(f(v)-f^{\prime}(v) v\right) d t \\
& =\int_{\xi(\bar{c})}^{\xi(\bar{c})+\delta} u(\bar{c}, x) d x+\int_{a}^{\bar{c}}\left(f\left(u\left(t,(\xi(t)+\delta)^{-}\right)\right)-\dot{\xi}(t) u\left(t,(\xi(t)+\delta)^{-}\right)\right) d t
\end{aligned}
$$

Using that $f$ is concave and for any $t \in[a, b] \dot{\xi}(t)=f^{\prime}(v)$, we have for any $t \in[a, \bar{c}]$

$$
f\left(u\left(t,(\xi(t)+\delta)^{-}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)+\delta)^{-}\right)-v\right) \leq 0
$$

Thus, we have

$$
\begin{equation*}
\int_{0}^{\delta} u(a, x) d x \leq \int_{\xi(\bar{c})}^{\xi(\bar{c})-\delta} u(\bar{c}, x) d x \tag{123}
\end{equation*}
$$

From (122) and (123),

$$
\frac{\delta}{f^{\prime}(v)} g-\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right) \leq \int_{\xi(\bar{c})}^{\xi(\bar{c})-\delta} u(c, x) d x
$$

Dividing by $\delta$ and taking $\delta \rightarrow 0^{+}$and using that $u(\bar{c}, \xi(\bar{c}))=v$, we deduce that

$$
g \leq f(v)
$$

Case 2: $a=0$. From the point 1. of Definition 1.2 and from [4, Theorem 11.1.1], (116) holds.


Figure 10: Plotting of a genuine characteristic $\xi$ on an interval $[a, b]$ with $a<b$ such that $\xi(a)=0$ with $a>0$ and $\dot{\xi}(t)=f^{\prime}(v)>0$ with $v \in\left[0, \check{u}_{f(\bar{u})-\epsilon}\right]$.

Case 3: $\xi(a)=L$ and $a>0$. We immediately have $f^{\prime}(v)<0$. Let us fix $\delta>0$, from Lemma A. 1 with $t_{1}=a, t_{2}=a-\frac{\delta}{f^{\prime}(v)}, \gamma_{1}(\cdot)=\xi(\cdot)$ and $\gamma_{2}(\cdot)=L$ (see Figure 11a), we have

$$
\begin{equation*}
\int_{a}^{a-\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t+0=\int_{a}^{a-\frac{\delta}{f^{\prime}(v)}} f\left(u\left(t, L^{-}\right)\right) d t+\int_{L-\delta}^{L} u\left(a-\frac{\delta}{f^{\prime}(v)}, x\right) d x \tag{124}
\end{equation*}
$$

From (8), for a.e $t>0, f(u(t, L)) \leq F_{2}(t, u)=f\left(w_{R}\right)$. Thus, we have

$$
\begin{equation*}
\int_{a}^{a-\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t-\int_{L-\delta}^{L} u\left(a-\frac{\delta}{f^{\prime}(v)}, x\right) d x \leq \frac{-\delta}{f^{\prime}(v)} f\left(w_{R}\right) \tag{125}
\end{equation*}
$$

Let $\bar{c} \in(a, b)$. Using Lemma A. 1 with $t_{1}=a-\frac{\delta}{f^{\prime}(v)}, t_{2}=\bar{c}, \gamma_{1}(\cdot)=\xi(\cdot)$ and $\gamma_{2}(\cdot)=\xi(\cdot)+\delta$ (see Figure 11a), we deduce that

$$
\begin{aligned}
\int_{a-\frac{\delta}{f^{\prime}(v)}}^{\bar{c}}(f(v) & \left.-f^{\prime}(v) v\right) d t+\int_{L-\delta}^{L} u\left(a-\frac{\delta}{f^{\prime}(v)}, x\right) d x \\
& =\int_{\xi(\bar{c})}^{\xi \xi(\bar{c})+\delta} u(\bar{c}, x) d x+\int_{a-\frac{\delta}{f^{\prime}(v)}}^{\bar{c}}\left(f\left(u\left(t,(\xi(t)+\delta)^{-}\right)\right)-\dot{\xi}(t) u\left(t,(\xi(t)+\delta)^{-}\right)\right) d t .
\end{aligned}
$$

Using that $f$ is concave and for any $t \in\left[a-\frac{\delta}{f^{\prime}(v)}, \bar{c}\right], \dot{\xi}(t)=f^{\prime}(v)$, we have for any $t \in$ $\left[a-\frac{\delta}{f^{\prime}(v)}, \bar{c}\right]$

$$
f\left(u\left(t,(\xi(t)+\delta)^{-}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)+\delta)^{-}\right)-v\right) \leq 0
$$

Thus, we have

$$
\begin{equation*}
\int_{L-\delta}^{L} u\left(a-\frac{\delta}{f^{\prime}(v)}, x\right) d x \leq \int_{\xi(\bar{c})}^{\xi(\bar{c})+\delta} u(\bar{c}, x) d x \tag{126}
\end{equation*}
$$

From (125) and (126),

$$
\frac{-\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)-\int_{\xi(\bar{c})}^{\xi(\bar{c})+\delta} u(\bar{c}, x) d x \leq \frac{-\delta}{f^{\prime}(v)} f\left(w_{R}\right)
$$

Dividing by $\delta$ and taking $\delta \rightarrow 0^{+}$, using that $u(\bar{c}, \xi(\bar{c}))=v$, we deduce that

$$
f(v) \leq f\left(w_{R}\right)
$$

If $a \in(c, d)$ then we choose $\delta>0$ such that $\left[a+\frac{\delta}{f^{\prime}(v)}, a-\frac{\delta}{f^{\prime}(v)}\right] \in(c, d)$. From (114), we have for a.e $t \in\left[a+\frac{\delta}{f^{\prime}(v)}, a-\frac{\delta}{f^{\prime}(v)}\right], l \leq f\left(u\left(t, L^{-}\right)\right) \leq L$. Thus, using (124), we get

$$
\int_{a}^{a-\frac{\delta}{f^{\prime}(v)}}\left(f(v)-f^{\prime}(v) v\right) d t-\int_{L-\delta}^{L} u\left(a-\frac{\delta}{f^{\prime}(v)}, x\right) d x \leq \frac{-\delta}{f^{\prime}(v)} L
$$

Moreover, (126) holds. Thus, we deduce that

$$
\frac{-\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)-\int_{\xi(\bar{c})}^{\xi(\bar{c})+\delta} u(\bar{c}, x) d x \leq \frac{-\delta}{f^{\prime}(v)} L
$$

Dividing by $\delta$ and taking $\delta \rightarrow 0^{+}$and using that $u(\bar{c}, \xi(\bar{c}))=v$, we deduce that

$$
f(v) \leq L
$$

From Lemma A. 1 with $t_{1}=a+\frac{\delta}{f^{\prime}(v)}, t_{2}=a, \gamma_{1}(\cdot)=\xi(\cdot)-\delta$ and $\gamma_{2}(\cdot)=L$ (see Figure 11b), we have

$$
\begin{equation*}
\int_{a+\frac{\delta}{f^{\prime}(v)}}^{a}\left(f\left(u\left(t,(\xi(t)-\delta)^{+}\right)\right)-\dot{\xi}(t) u\left(t,(\xi(t)-\delta)^{+}\right)\right) d t=\int_{L-\delta}^{L} u(a, x) d x+\int_{a+\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, L^{-}\right)\right) d t \tag{127}
\end{equation*}
$$

Using that $f$ is concave and for any $t \in[a, b], \dot{\xi}(t)=f^{\prime}(v)$, we have for any $t \in\left[a+\frac{\delta}{f^{\prime}(v)}, a\right]$

$$
f\left(u\left(t,(\xi(t)-\delta)^{+}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)-\delta)^{+}\right)-v\right) \leq 0
$$

Therefore,

$$
\int_{a+\frac{\delta}{f^{\prime}(v)}}^{a} f\left(u\left(t, L^{-}\right)\right) d t \leq-\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)-\int_{L-\delta}^{L} u(a, x) d x
$$

From (114) and $\left[a+\frac{\delta}{f^{\prime}(v)}, a-\frac{\delta}{f^{\prime}(v)}\right] \in(c, d)$, we deduce that

$$
\begin{equation*}
-\frac{\delta}{f^{\prime}(v)} l \leq-\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)-\int_{L-\delta}^{L} u(a, x) d x \tag{128}
\end{equation*}
$$

Let $\bar{c} \in(a, b)$. From Lemma A. 1 with $t_{1}=a, t_{2}=\bar{c}, \gamma_{1}(\cdot)=\xi(\cdot)-\delta$ and $\gamma_{2}(\cdot)=\xi(\cdot)$, we have

$$
\begin{gathered}
\int_{L-\delta}^{L} u(a, x) d x+\int_{\bar{a}}^{\bar{c}}\left(f\left(u\left(t,(\xi(t)-\delta)^{+}\right)\right)-\dot{\xi}(t) u\left(t,(\xi(t)-\delta)^{+}\right)\right) d t \\
=\int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x) d x+\int_{a}^{\bar{c}}\left(f(v)-f^{\prime}(v) v\right) d t .
\end{gathered}
$$

Using that $f$ is concave and for any $t \in[a, b] \dot{\xi}(t)=f^{\prime}(v)$, we have for any $t \in[a, \bar{c}]$

$$
f\left(u\left(t,(\xi(t)-\delta)^{+}\right)\right)-f(v)-f^{\prime}(v)\left(u\left(t,(\xi(t)-\delta)^{+}\right)-v\right) \leq 0
$$

Thus, we have

$$
\begin{equation*}
\int_{\xi(\bar{c})}^{\xi(\bar{c})-\delta} u(\bar{c}, x) d x \leq \int_{L-\delta}^{L} u(a, x) d x \tag{129}
\end{equation*}
$$

From (128) and (129),

$$
-\frac{\delta}{f^{\prime}(v)} l \leq-\frac{\delta}{f^{\prime}(v)}\left(f(v)-f^{\prime}(v) v\right)-\int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x) d x
$$

Dividing by $\delta$ and taking $\delta \rightarrow 0^{+}$and using that $u(\bar{c}, \xi(\bar{c}))=v$, we deduce that

$$
l \leq f(v)
$$



Figure 11: Plotting of a genuine characteristic $\xi$ on an interval $[a, b]$ with $a<b$ such that $\xi(a)=L$ with $a>0$ and $\dot{\xi}(t)=f^{\prime}(v)<0$ with $v \in\left[w_{R}, 1\right]$.

## References

[1] Boris Andreianov, Paola Goatin, and Nicolas Seguin. Finite volume schemes for locally constrained conservation laws. Numerische Mathematik, 115(4):609-645, 2010.
[2] Sébastien Blandin, Xavier Litrico, Maria Laura Delle Monache, Benedetto Piccoli, and Alexandre Bayen. Regularity and lyapunov stabilization of weak entropy solutions to scalar conservation laws. IEEE Transactions on Automatic Control, 62(4):1620-1635, 2017.
[3] Rinaldo M. Colombo and Paola Goatin. A well posed conservation law with a variable unilateral constraint. Journal of Differential Equations, 234(2):654-675, 2007.
[4] Constantine M Dafermos. Hyperbolic conservation laws in continuum physics, volume 325 of grundlehren der mathematischen wissenschaften [fundamental principles of mathematical sciences], 2010.
[5] Stanislav N Kružkov. First order quasilinear equations in several independent variables. Mathematics of the USSR-Sbornik, 10(2):217, 1970.
[6] E. Moulay and W. Perruquetti. Finite time stability of nonlinear systems. In 42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475), volume 4, pages 3641-3646 vol.4, 2003.
[7] Vincent Perrollaz. Asymptotic stabilization of stationnary shock waves using a boundary feedback law. arXiv preprint arXiv:1801.06335, 2018.
[8] Walter Rudin. Functional analysis, 2nd edition. McGraw-Hill, 1991.
[9] Liudmila Tumash, Carlos Canudas-de Wit, and Maria Laura Delle Monache. Boundary control design for traffic with nonlinear dynamics. IEEE Transactions on Automatic Control, 67(3):1301-1313, 2022.


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