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Stabilization of conservation laws for \mathcal{G} -solutions with application to traffic flow.

Thibault Liard ^{*} Swann Marx [†] Vincent Perrollaz [‡]

Abstract

We address a problem of stabilization of a scalar conservation laws with fixed flux pointwise constraints. The PDE models the impact of two toll gates on the traffic flow and the feedback laws is designed to limit the number of cars passing through the two toll gates. The existence of solutions is proved using a Schauder's fixed point theorem, together with a coupled PDE-ODE. To prove stabilization results, we study the structure of the solutions using an extension of backward generalized characteristics.

Keywords: Conservation Laws; non-classical shock; \mathcal{G} -solutions; Stabilization;

AMS classification:

1 Introduction

1.1 Presentation of the Problem

Let $L > 0$. We consider the following scalar conservation laws with local lateral constraints

$$\begin{cases} u_t(t, x) + \partial_x f(u(t, x)) = 0, & t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ f(u(t, 0)) \leq F_1(t, u), & t \in \mathbb{R}_+, \\ f(u(t, L)) \leq F_2(t, u), & t \in \mathbb{R}_+. \end{cases} \quad (1)$$

We assume that $f \in C^2([0, 1]; [0, +\infty))$ is a strictly concave function such that $f(0) = f(1) = 0$. The problem (1) models two toll gates along highway or road lights (see [3]). Since the flux pointwise constraints at $x = 0$ and $x = L$ (the two toll gates) generate shocks that does not verify Oleinik condition, the notion of weak-entropy solutions cannot be used anymore. Thus, we consider here the notion of \mathcal{G} -solution (see [1] or Definition 1.2). We introduce the target function w defined by

$$w(x) = \begin{cases} w_L, & \text{if } 0 \leq x < x_0, \\ w_R, & \text{if } x_0 < x \leq L, \end{cases} \quad (2)$$

where $x_0 \in (0, L)$ and $w_L, w_R \in (0, 1)$ satisfy $w_L < w_R$ and $f(w_L) = f(w_R) < f(\bar{u})$ with $\bar{u} = \operatorname{argmax}_{u \in [0, 1]} f(u)$. Our goal is to find two feedback controllers F_1 and F_2 that stabilizes (1) over $(0, L)$ around the target function w , i.e for any initial data u_0 , the \mathcal{G} -solutions $u(t, \cdot)$ of (1) converges to $w(\cdot)$ in $L^1(0, L)$ as $t \rightarrow \infty$. We consider a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(0) = 0 \text{ and } \forall x \in \mathbb{R}^*, xg(x) > 0 \quad (3)$$

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and the following ODE

$$\begin{cases} \dot{m}(t) = -g(m(t)), \\ m(0) = m_0, \end{cases} \quad (4)$$

admits a unique solution over $[0, \infty)$. In this paper, the feedback controllers F_1 and F_2 have the following forms

$$F_1(t, u) = \max(\min(f(w_L) - g(m(t)), f(\bar{u}) - \varepsilon), 0) \text{ and } F_2(t, u) = f(w_R) \quad (5)$$

where $\varepsilon > 0$ is chosen such that $f(\bar{u}) - f(w_L) - \varepsilon > 0$ and

$$m(t) := \int_0^L (u(t, x) - w(x)) dx. \quad (6)$$

Remark 1 • *Since g is a continuous function, from Peano's Theorem, there exists at least one solution of (4). If there exists $t_0 > 0$ such that $m(t_0) = 0$ then, from (3), any solution $m(\cdot)$ of (4) satisfies for any $t \geq t_0$, $\frac{d}{dt}m^2(t) < 0$. Therefore, for any $t \geq t_0$, $m(t) = 0$.*

- *We have the following maximum and minimum principle*

$$m_0 \geq 0 \implies \forall t \geq 0, m(t) \geq 0 \text{ and } m_0 \leq 0 \implies \forall t \geq 0, m(t) \leq 0.$$

- *The EDO (4) is asymptotically stable around the equilibrium point 0 using (3) and the candidate Lyapunov function $V : x \rightarrow \frac{x^2}{2}$ (see [6]).*
- *The uniqueness of the EDO (4) is used to prove the existence of solutions of (1). More precisely, from a certain time, the unique solution of the coupled PDE-ODE (13) is a solution of (1).*

In [2, 7, 9], stabilization results are obtained in the context of weak-entropy solutions. More precisely, in [2], the authors consider solutions with a finite number of shocks to find the derivative of a Lyapunov function candidate. In [9], they assume that the solutions is only determined by the boundaries and use the Hamilton-Jacobi formulation. In [7], a saturated proportional controller is provided to stabilize asymptotically the weak-entropy solution of LWR around a stationary shocks using the notion of generalized characteristics. To our best knowledge, there are no stabilization results in the context of \mathcal{G} -solutions.

1.2 Definitions and notations

Let $\Pi = [0, \infty) \times \mathbb{R}$. Throughout the paper,

$$\phi : (u, k) \in \mathbb{R} \times \mathbb{R} \mapsto \text{sgn}(u - k)(f(u) - f(k))$$

denotes the entropy flux associated with the Kruzhkov entropy $(u, k) \mapsto |u - k|$ (see [5]).

Definition 1.1 (Admissibility germ $\mathcal{G}(F), [1]$) *Let $F \in [0, f(\bar{u})]$. The admissibility germ $\mathcal{G}(F)$ for the conservation law (1) is the subset of $[0, 1]^2$ defined as the union $\mathcal{G}(F) = \mathcal{G}_1(F) \cup \mathcal{G}_2(F) \cup \mathcal{G}_3(F)$, where*

- $\mathcal{G}_1(F) := \{(c_l, c_r) \in [0, 1]^2; c_l > c_r, f(c_l) = f(c_r) = F\}$,
- $\mathcal{G}_2(F) := \{(c, c) \in [0, 1]^2; f(c) \leq F\}$,

- $\mathcal{G}_3(F) := \{(c_l, c_r) \in [0, 1]^2; c_l < c_r, f(c_l) = f(c_r) \leq F\}$.

Definition 1.2 ([1]) Let $u_0 \in L^\infty(\mathbb{R}, [0, 1])$. We say that $u \in L^\infty(\Pi; [0, 1])$ is a \mathcal{G} -entropy solution of (1) if

1. u is a Kruzhkov entropy solution for $x \in \mathbb{R} \setminus \{0, L\}$, i.e., for all nonnegative test functions $\varphi \in C_c^\infty(\Pi \setminus \{0, L\})$ and all $k \in [0, 1]$,

$$\int_0^{+\infty} \int_{\mathbb{R}} (|u(t, x) - k| \partial_t + \phi(u(t, x), k) \partial_x) \varphi(t, x) dx dt + \int_{\mathbb{R}} |u_0(x) - k| \varphi(0, x) dx \geq 0; \quad (7)$$

2. in addition, for a.e $t > 0$,

$$((\gamma_0^l u)(t), (\gamma_0^r u)(t)) \in \mathcal{G}(F_1) \text{ and } ((\gamma_L^l u)(t), (\gamma_L^r u)(t)) \in \mathcal{G}(F_2). \quad (8)$$

where $\gamma_0^{l,r}$ and $\gamma_L^{l,r}$ denote the operators of left- and right-side strong traces on $\{x = 0\}$ and $\{x = L\}$ and F_1 and F_2 are defined in (5).

2 Main results

Theorem 2.1 (Existence) Let $\varepsilon > 0$ satisfy $f(\bar{u}) - f(w_L) - \varepsilon > 0$. For any $u_0 \in L^\infty(\mathbb{R}; [0, 1])$, there exists at least one \mathcal{G} -entropy solution u of (1) where F_1 and F_2 are defined in (5).

Theorem 2.2 (Asymptotic stability) Let $\varepsilon > 0$ satisfy $f(\bar{u}) - f(w_L) - \varepsilon > 0$. Let $v_0 \in L^\infty([0, L]; [0, 1])$, we consider the initial data u_0 defined by

$$u_0(x; v_0) = \begin{cases} 1, & \text{if } x < 0, \\ v_0(x), & \text{if } 0 < x < L, \\ 0, & \text{if } L < x. \end{cases} \quad (9)$$

For any $v_0 \in L^\infty([0, L]; [0, 1])$, any \mathcal{G} -solution $u(t, \cdot)$ of (1) with initial datum u_0 converges to $w(\cdot)$ in $L^1(0, L)$ as $t \rightarrow \infty$.

Corollary 2.1 (Exponential stability) Let $\varepsilon > 0$ satisfy $f(\bar{u}) - f(w_L) - \varepsilon > 0$ and $g : x \mapsto K_0 x$ with $K_0 > 0$. For any $v_0 \in L^\infty([0, L]; [0, 1])$, any \mathcal{G} -solution $u(t, \cdot)$ of (1) with initial datum u_0 converges exponentially to $w(\cdot)$ in $L^1(0, L)$ as $t \rightarrow \infty$, i.e there exists $C > 0$ such that for any $t \geq 0$,

$$\|u(t, \cdot) - w(\cdot)\|_{L^1(0, L)} \leq C e^{-K_0 t} (\|u_0(\cdot) - w(\cdot)\|_{L^1(0, L)} + C).$$

Corollary 2.2 (Finite time stability) Let $\varepsilon > 0$ satisfy $f(\bar{u}) - f(w_L) - \varepsilon > 0$ and g satisfied (3) and for any $\alpha \in \mathbb{R}^*$, $\int_0^\alpha \frac{dy}{g(y)} > -\infty$. For any $v_0 \in L^\infty([0, L]; [0, 1])$, the \mathcal{G} -solution $u(t, \cdot)$ of (1) with initial datum u_0 converges in finite time to $w(\cdot)$ in $L^1(0, L)$, i.e there exists a time $\bar{T} > 0$ such that for any $t \geq \bar{T}$, for a.e $x \in (0, L)$,

$$u(t, x) = w(x).$$

3 Simulations

To construct an approximate solution of (1), we use a finite volume scheme described in [1]. The code is available at <https://github.com/Thibault-liard/Regulation-Godunov-LWR-fixed>.

In Figure 1, Figure 2, Figure 3, F_1 is a saturated proportional controller, i.e g in (5) is defined by, for any $x \in \mathbb{R}$, $g(x) = K_0x$ where $K_0 > 0$ is the gain of the controller. Any \mathcal{G} -solution of the PDE (1) with the controller F_1 is asymptotically stable around the stationary solution $w(\cdot)$ (see Corollary 2.2). In Figure 4, Figure 5, Figure 6, F_1 is a saturated sliding mode controller, i.e $g : x \rightarrow K_0|x|^{\frac{1}{2}}\text{sign}(x)$ where $K_0 > 0$ is the gain of the controller. Any \mathcal{G} -solution of the PDE (1) with the controller F_1 converges in finite time to the stationary solution $w(\cdot)$ (see Corollary 2.1).

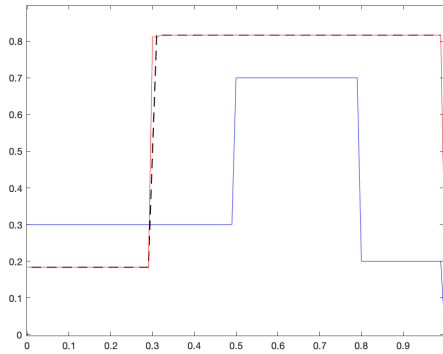


Figure 1: Plottings of the initial data v_0 (—), the target function w (---) and the \mathcal{G} -solution of (1) (—) over $(0, 1)$ at time $t = 20$ with $L = 1$ and for any $x \in \mathbb{R}$, $g(x) = K_0x$ with $K_0 = 0.4$.

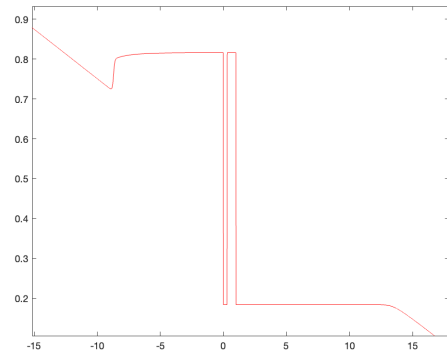


Figure 2: Plotting of the \mathcal{G} -solution of (1) (—) over $(-15, 20)$ at time $t = 20$ with $L = 1$ and for any $x \in \mathbb{R}$, $g(x) = K_0x$ with $K_0 = 0.4$.

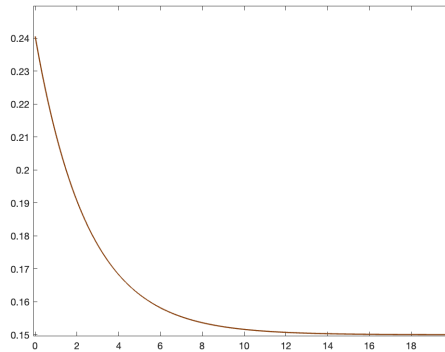


Figure 3: Plotting of the saturated P-controller $F_1(\cdot)$ with respect to time t .

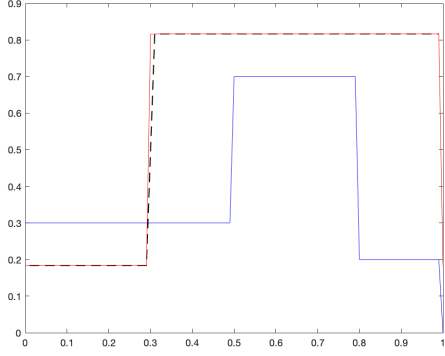


Figure 4: Plottings of the initial data v_0 (—), the target function w (---) and the \mathcal{G} -solution of (1) (—) over $(0, 1)$ at time $t = 20$ with $L = 1$ and for any $x \in \mathbb{R}$, $g(x) = K_0|x|^{\frac{1}{2}}\text{sign}(x)$ with $K_0 = 0.4$.

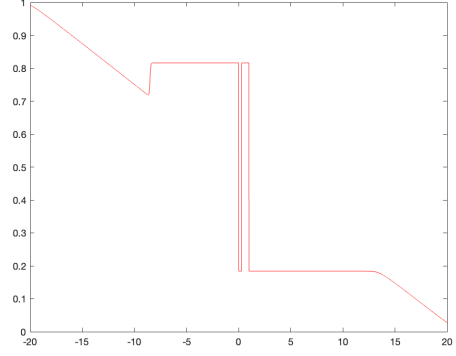


Figure 5: Plotting of the \mathcal{G} -solution of (1) (—) over $(-15, 20)$ at time $t = 20$ with $L = 1$ and for any $x \in \mathbb{R}$, $g(x) = K_0|x|^{\frac{1}{2}}\text{sign}(x)$ with $K_0 = 0.4$.

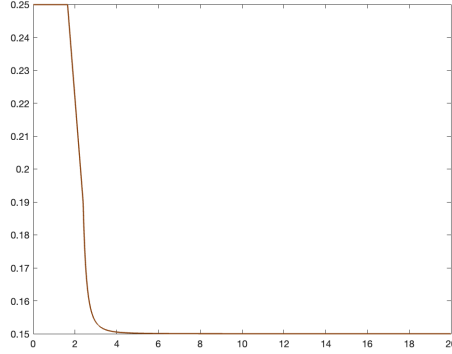


Figure 6: Plotting of the saturated SMC-controller $F_1(\cdot)$ with respect to time t .

4 Proofs

4.1 Proof of Theorem 2.1

We consider the following scalar conservation laws with local ilateral constraints

$$\begin{cases} u_t(t, x) + \partial_x f(u(t, x)) = 0, & t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ f(u(t, 0)) \leq Q_1(t), & t \in \mathbb{R}_+, \\ f(u(t, L)) \leq Q_2(t), & t \in \mathbb{R}_+. \end{cases} \quad (10)$$

A \mathcal{G} -solution u of (10) is understood in the sense of Definition 1.2 replacing F_1 and F_2 by Q_1 and Q_2 respectively. Note that, contrary to (1), Q_1 and Q_2 in (10) does not depend on the \mathcal{G} -solution u .

Proposition 4.1 ([1]) *Assume that $Q_1^1, Q_1^2, Q_2^1, Q_2^2 \in L^\infty(\mathbb{R}^+, [0, f(\bar{u})])$ and $u_0, v_0 \in L^\infty(\mathbb{R}, [0, 1])$ such that $(u_0 - v_0) \in L^1(\mathbb{R})$. The \mathcal{G} -entropy solutions u and v of*

$$\begin{cases} u_t(t, x) + \partial_x f(u(t, x)) = 0, \\ u(0, x) = u_0(x), \\ f(u(t, 0)) \leq Q_1^1(t), \\ f(u(t, L)) \leq Q_2^1(t), \end{cases} \quad \text{and} \quad \begin{cases} v_t(t, x) + \partial_x f(v(t, x)) = 0, \\ v(0, x) = v_0(x), \\ f(v(t, 0)) \leq Q_1^2(t), \\ f(v(t, L)) \leq Q_2^2(t), \end{cases}$$

satisfy, for a.e $T > 0$,

$$\int_{\mathbb{R}} |u - v|(T, x) dx \leq 2 \int_0^T |Q_1^1 - Q_1^2|(t) dt + 2 \int_0^T |Q_2^1 - Q_2^2|(t) dt + \int_{\mathbb{R}} |u_0 - v_0|(x) dx \quad (11)$$

Definition 4.1 *Let us fix $T > 0$. We introduce the operator $\mathcal{F} : W^{1,\infty}([0, T]) \rightarrow W^{1,\infty}([0, T])$ defined by*

$$\forall t \in [0, T], \quad \mathcal{F}(z)(t) := \int_0^L (u(t, x) - w(x)) dx,$$

where $u \in L^\infty([0, T] \times \mathbb{R}; [0, 1])$ is a \mathcal{G} -entropy solution of (10) with Q_1 and Q_2 defined by

$$Q_1(t) = \max(\min(f(w_L) - g(z(t)), f(\bar{u}) - \varepsilon), 0) \text{ and } Q_2(t) = f(w_R).$$

Proposition 4.2 *Let us fix $T > 0$. \mathcal{F} has a unique fixed point in $W^{1,\infty}([0, T])$.*

PROOF. We apply Schauder's fixed point theorem [8], i.e suppose K is a compact set in a Fréchet space X and $\mathcal{F} : X \rightarrow K$ is continuous then \mathcal{F} has a fixed point in K . Here, $X := C([0, T]; [0, 1])$ is a Banach space equipped with the norm $\|\cdot\|_{L^\infty(0, T)}$ and $K := \overline{\mathcal{F}(C([0, T]; [0, 1]))} \subset C([0, T]; [0, 1])$.

- Let us fix $y \in C([0, T]; [0, 1])$ and $(y_n)_{n \in \mathbb{N}}$ be a sequence that converges uniformly to y . We denote by u and u_n the \mathcal{G} -entropy solutions involved in the definition of $\mathcal{F}(y)$ and $\mathcal{F}(y_n)$ respectively. Using Proposition 4.1, for any $t \in [0, T]$,

$$\begin{aligned} |\mathcal{F}(y_n)(t) - \mathcal{F}(y)(t)| &\leq 2 \int_0^t |\max(\min(f(w_L) - g(y_n(t)), f(\bar{u}) - \varepsilon), 0) \\ &\quad - \max(\min(f(w_L) - g(y(t)), f(\bar{u}) - \varepsilon)), 0)| dt, \\ &\leq 2 \int_0^t |g(y_n(t)) - g(y(t))| dt, \end{aligned} \quad (12)$$

From Heine's Theorem, $g : [0, 1] \rightarrow \mathbb{R}$ is uniformly continuous and for any $t \in [0, T]$, $y(t) \in [0, 1]$ and $z(t) \in [0, 1]$. Thus, $(g(y_n))_{n \in \mathbb{N}}$ converges uniformly to $g(y)$. Therefore, together with (12), we conclude that \mathcal{F} is continuous on $C([0, T]; [0, 1])$ with respect to norm $\|\cdot\|_{L^\infty(0, T)}$.

- We introduce the set $A = \mathcal{F}(C([0, T]; [0, 1])) \subset C([0, T]; [0, 1])$. Let us fix $z \in C([0, T]; [0, 1])$, we denote by u the \mathcal{G} -entropy solutions involved in the definition of $\mathcal{F}(z)$. Since for any $(t, x) \in \mathbb{R}_+ \times (0, L)$, $0 \leq u(t, x) \leq 1$ then for any $t \in [0, T]$, $|\mathcal{F}(z)(t)| \leq L$ then A is a bounded subset of $C([0, T]; [0, 1])$. Let $0 \leq t < t' \leq T$, from Lemma A.1 with $\gamma_1(\cdot) = 0$, $\gamma_2(\cdot) = L$, $t_1 = t$ et $t_2 = t'$,

$$\int_0^L u(t, x) dx + \int_t^{t'} f(u(t, 0)) dt = \int_0^L u(t', x) dx + \int_t^{t'} f(u(t, L)) dt.$$

Thus, together with (8), we have

$$|\mathcal{F}(z)(t') - \mathcal{F}(z)(t)| \leq f(\bar{u})(t' - t).$$

As a consequence, A is equicontinuous. From Arzelà–Ascoli theorem, $\overline{A} := \overline{\mathcal{F}(C([0, T]; [0, 1]))}$ is a compact set.

From Schauder's fixed point theorem [8], \mathcal{F} has a fixed point in $\overline{\mathcal{F}(C([0, T]; [0, 1]))}$. Since $\overline{\mathcal{F}(C([0, T]; [0, 1]))} \subset W^{1, \infty}([0, T])$, we conclude the proof of Proposition 4.2. \square

Proof of Theorem 2.1. Let $\epsilon > 0$ and $T_5 := T_4 + \epsilon$ where T_4 is defined in Proposition 4.6. From Proposition 4.2, for any $u_0 \in L^\infty(\mathbb{R}; [0, 1])$, there exists at least one \mathcal{G} -entropy solution, denoted by u , of (1) over $[0, T_5]$ where F_1 and F_2 are defined in (5). We consider the following coupled PDE-ODE

$$\begin{cases} v_t(t, x) + \partial_x f(v(t, x)) = 0, & t \in (T_4, \infty), x \in \mathbb{R}, \\ v(T_4, x) = u(T_4, x), & x \in \mathbb{R}, \\ f(v(t, 0)) \leq f(w_L) - g(\gamma(t)), & t \in (T_4, \infty), \\ f(v(t, L)) \leq f(w_R), & t \in (T_4, \infty). \\ \dot{\gamma}(t) = -g(\gamma(t)), & t \in (T_4, \infty). \\ \gamma(T_4) = \int_0^L (u(T_4, x) - w(x)) dx. \end{cases} \quad (13)$$

From Remark 1 and [1, Theorem 2.11], the coupled PDE-ODE admits a unique \mathcal{G} -entropy solution, denoted by v , over $[T_4, \infty)$. From (80), we have $m(T_4) := \int_0^L (u(T_4, x) - w(x)) dx \in [0, f(\bar{u}) - \epsilon]$. From (3) and $\gamma(T_4) = m(T_4)$, we deduce that, for any $t \geq T_4$, $f(w_L) - g(\gamma(t)) \in [0, f(\bar{u}) - \epsilon]$. Thus, since T_4 does not depend on the initial data u_0 , Proposition 4.6 holds replacing $F_1(t, u)$ by $f(w_L) - g(\gamma(t))$. Thus, the function $\sigma : t \rightarrow \int_0^L (v(t, x) - w(x)) dx$ satisfied, for any $t \geq T_4$,

$$\begin{cases} \dot{\sigma}(t) = f(w_L) - g(\gamma(t)) - f(w_R), \\ \sigma(T_4) = m(T_4) \end{cases}$$

From (13), we deduce that $\sigma(t) = \gamma(t)$ for any $t \geq T_4$. Therefore, v is a \mathcal{G} -entropy solution of (1) over $[T_4, \infty)$. By uniqueness of (13) and from Proposition 4.6, for any $t \in [T_4, T_5]$, $u = v$. We conclude that w , defined by $w = u$ over $[0, \frac{T_4 + T_5}{2}] \times \mathbb{R}$ and $w = v$ over $(\frac{T_4 + T_5}{2}, \infty) \times \mathbb{R}$, is a \mathcal{G} -entropy solution of (1).

4.2 Proof of Theorem 2.2

Let $\bar{f} \in [0, f(\bar{u})]$. We denote by $\check{u}_{\bar{f}}$ and $\hat{u}_{\bar{f}}$ the two solutions of $f(u) = \bar{f}$ with $\check{u}_{\bar{f}} < \hat{u}_{\bar{f}}$ (these solutions exist because f is concave). Let u be a \mathcal{G} -entropy solution of (1).

Lemma 4.1 *There exist $T_1 > 0$ (independent of the initial data u_0) and two functions $\beta_1, \beta_2 : [T_1, +\infty) \rightarrow (0, L)$ such that, for any $(\bar{t}, \bar{x}) \in [T_1, +\infty) \times (0, L)$, we have $\beta_1 \leq \beta_2$ and*

$$0 < \bar{x} < \beta_1(\bar{t}) \implies u(\bar{t}, \bar{x}^\pm) \in [0, \check{u}_{f(\bar{u}) - \epsilon}] \quad (14)$$

$$\beta_1(\bar{t}) < \bar{x} < \beta_2(\bar{t}) \implies f(u(\bar{t}, \bar{x}^\pm)) \geq f(\bar{u}) - \frac{\epsilon}{2}, \quad (15)$$

$$\beta_2(\bar{t}) < \bar{x} < L \implies u(\bar{t}, \bar{x}^\pm) \in [w_R, 1] \quad (16)$$

PROOF. Let $(\bar{t}, \bar{x}) \in (0, +\infty) \times (0, L)$. From the point 1. of Definition 1.2 and [4, Theorem 11.1.3], there exists a minimal backward generalized characteristic, denoted by ξ_- , associated to u emanating from (\bar{t}, \bar{x}) defined on an interval $[a, \bar{t}]$ with $a \geq 0$ such that for any $t \in (a, \bar{t})$

$$u(t, \xi_-(t)^-) = u(\bar{t}, \bar{x}^-) = u(t, \xi_-(t)^+). \quad (17)$$

In particular, ξ_- is a genuine characteristic (also called shock-free, see [4, Definition 10.2.4]). We have three different possibilities.

Case 1: $\xi_-(a) = 0$ and $a > 0$. From Proposition B.1 with $\xi = \xi_-$, $a = a$ and $b = \bar{t}$, there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and, for any $t \in (a, \bar{t})$, $u(t, \xi_-(t)) = v$ and $f(v) \leq f(\bar{u}) - \varepsilon$. From (17), we deduce that

$$u(\bar{t}, \bar{x}^-) = v \text{ and } f(v) \leq f(\bar{u}) - \varepsilon.$$

Therefore, using the definition of $\check{u}_{f(\bar{u})-\varepsilon}$ and $\hat{u}_{f(\bar{u})-\varepsilon}$ and the concavity of f , we have $u(\bar{t}, \bar{x}^-) \in [0, \check{u}_{f(\bar{u})-\varepsilon}] \cup (\hat{u}_{f(\bar{u})-\varepsilon}, 1]$. Moreover, since $\xi_-(a) = 0$ and $a > 0$ then $f'(v) > 0$. The concavity of f implies that $v \in [0, \bar{u})$. Thus, $u(\bar{t}, \bar{x}^-) \in [0, \check{u}_{f(\bar{u})-\varepsilon})$.

Case 2: $a = 0$. From Proposition B.1 with $\xi = \xi_-$, $a = 0$ and $b = \bar{t}$, there exists a constant $v \in [0, 1]$ such that, for any $t \in [0, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and, for any $t \in (0, \bar{t})$, $u(t, \xi_-(t)) = v$. From (17), we deduce that, for any $t \in [0, \bar{t}]$,

$$\dot{\xi}_-(t) = f'(u(\bar{t}, \bar{x}^-)).$$

Thus, $\xi_-(\bar{t}) - \xi_-(0) = f'(u(\bar{t}, \bar{x}^-))\bar{t}$. Using that $\xi_-(\bar{t}) = \bar{x}$, we deduce that

$$f'(u(\bar{t}, \bar{x}^-)) = \frac{\bar{x} - \xi_-(0)}{\bar{t}}.$$

Since $0 \leq \xi_-(0) \leq L$, we get

$$\frac{\bar{x} - L}{\bar{t}} \leq f'(u(\bar{t}, \bar{x}^-)) \leq \frac{\bar{x}}{L}$$

which implies that

$$-\frac{L}{\bar{t}} \leq f'(u(\bar{t}, \bar{x}^-)) \leq \frac{L}{\bar{t}}.$$

Using that f is strictly concave and $\bar{u} := \operatorname{argmax}_{u \in [0, R]} f(u)$, we deduce that there exists $T_1 > 0$ such that for any $\bar{t} \geq T_1$, (15) holds.

Case 3: $\xi_-(a) = L$ and $a > 0$. From Proposition B.1 with $\xi = \xi_-$, $a = a$ and $b = \bar{t}$, there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and, for any $t \in (a, \bar{t})$, $u(t, \xi_-(t)) = v$ and $f(v) \leq f(w_R)$. From (17), we deduce that

$$u(\bar{t}, \bar{x}^-) = v \text{ and } f(v) \leq f(w_R)$$

Therefore, using the definition of w and the concavity of f , we deduce that $u(\bar{t}, \bar{x}^-) \in [0, w_L] \cup [w_R, 1]$. Moreover, since $\xi_-(a) = L$ and $a > 0$ then $f'(v) < 0$. The concavity of f implies that $v \in (\bar{u}, 1]$. Thus, $u(\bar{t}, \bar{x}^-) \in [w_R, 1]$.

We know that genuine characteristics do not cross over $(0, +\infty) \times (0, L)$. Therefore, given \bar{t} , the set of \bar{x} for which we are in first case, second case or third case are connected and they form a partition of $[0, L]$. From a geometrical viewpoint it is obvious that from the left to the right we have points from the first case, points from the second case and points from the last case. At this point we have indeed constructed two functions β_1 and β_2 such that $\beta_1 \leq \beta_2$ and (14), (15) and (16) hold for x^- . Using that $u(\bar{t}, \bar{x}^+) = \lim_{\epsilon \rightarrow 0^+} u(\bar{t}, (\bar{x} + \epsilon)^-)$ and applying (14), (15) and (16) for x^- to the right hand of the latter equality, we deduce that (14), (15) and (16) also hold for x^+ . An illustration of the proof is given in Figure 7. \square

Proposition 4.3 *There exists $T_2 > 0$ (independent of the initial data u_0) and a Lipschitz function $\beta : [T_2, +\infty) \rightarrow [0, L]$ satisfying, for any $(\bar{t}, \bar{x}) \in [T_2, +\infty) \times (0, L)$,*

$$\begin{cases} 0 < \bar{x} < \beta(\bar{t}) \implies u(\bar{t}, \bar{x}^\pm) \in [0, \check{u}_{f(\bar{u})-\varepsilon}] \\ \beta(\bar{t}) < \bar{x} < L \implies u(\bar{t}, \bar{x}^\pm) \in [w_R, 1] \end{cases} \quad (18)$$

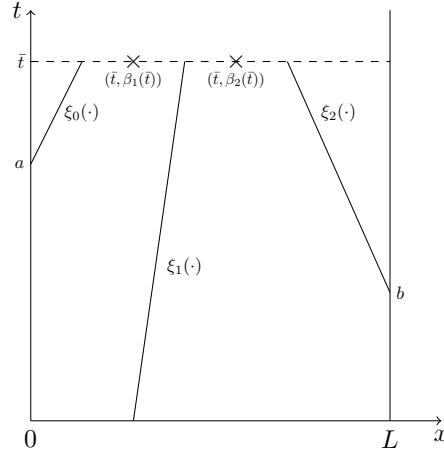


Figure 7: Construction of $\beta_1(\cdot)$ and $\beta_2(\cdot)$; the minimal backward generalized characteristic ξ_0 emanating from $(\bar{t}, \bar{x}) \in [T_1, \infty) \times (0, \beta_1(\bar{t}))$ touches the left boundary $x = 0$ at time $t = a > 0$. The minimal backward generalized characteristic ξ_1 emanating from $(\bar{t}, \bar{x}) \in [T_1, \infty) \times (\beta_1(\bar{t}), \beta_2(\bar{t}))$ touches the initial boundary $t = 0$. The minimal backward generalized characteristic ξ_2 emanating from $(\bar{t}, \bar{x}) \in [T_1, \infty) \times (\beta_2(\bar{t}), L)$ touches the right boundary $x = L$ at time $t = b > 0$.

PROOF. We assume that $\beta_1(T_1) < \beta_2(T_1)$ and (8) holds at time $t = T_1$. We now prove that $\beta_1(T_1) > 0$ and $\beta_2(T_1) < L$ by contradiction. We assume that $\beta_1(T_1) = 0$. From (15) and using that $u(T_1, \bar{0}^+) = \lim_{x \rightarrow 0^+} u(T_1, x^-)$, we have

$$f(u(T_1, \bar{0}^+)) \geq f(\bar{u}) - \frac{\varepsilon}{2}.$$

Since (8) holds at time $t = T_1$, we get

$$f(u(T_1, \bar{0}^+)) \leq f(\bar{u}) - \varepsilon.$$

This leads to a contradiction. Thus, we have $\beta_1(T_1) > 0$. The same argument works for β_2 . From [4, 11.1.4 Theorem] and $(T_1, \beta_1(T_1)) \in (0, +\infty) \times (0, L)$, there exists a unique forward characteristics $\gamma_1(\cdot)$, issues from $(T_1, \beta_1(T_1))$, defined on a certain interval $[T_1, c_1)$ where $c_1 > T_1$ is chosen such that for any $t \in [T_1, c_1)$, $\gamma_1(t) \in (0, L)$, γ_1 is a Lipschitz function over $[T_1, c_1)$. We will now prove that $\beta_1 = \gamma_1$. To that end, let us fix $t \in (T_1, c_1)$.

- Let $x \in (0, \gamma_1(t))$. We denote by ξ_- , the minimal backward characteristic emanating from (t, x) , defined maximally on an interval $[b, t]$. By uniqueness of forward characteristics, we have

$$\forall s \in [\max(T_1, b), t], \quad \xi_-(s) < \gamma_1(s).$$

We have two alternatives.

- If $b > T_1$, we have $\xi_-(b) = 0$ and $b > 0$. Thus, by definition of β_1 , $x < \beta_1(t)$.
- If $b \leq T_1$, then $\xi_-(T_1) < \gamma_1(T_1) = \beta_1(T_1)$. Therefore, by definition of β_1 , ξ_- is the minimal backward characteristic emanating from $(T_1, \xi_-(T_1))$, defined maximally on an interval $[b, T_1]$ and $\xi_-(b) = 0$ with $b > 0$.

We conclude that, for any $x \in (0, \gamma_1(t))$, $x < \beta_1(t)$. Thus, we have

$$\gamma_1(t) \leq \beta_1(t). \tag{19}$$

- Let $x \in (\gamma_1(t), L)$. We denote by ξ_- , the minimal backward characteristic emanating from (t, x) , defined maximally on an interval $[b, t]$. By uniqueness of forward characteristics, we have

$$\forall s \in [\max(T_1, b), t], \quad \gamma_1(s) < \xi_-(s).$$

We have two alternatives.

- If $b > T_1$, we have $\xi_-(b) = L$ and $b > 0$. Thus, by definition of β_1 and β_2 , $\beta_1(t) \leq \beta_2(t) < x$.
- If $b \leq T_1$, then $\gamma_1(T_1) = \beta_1(T_1) < \xi_-(T_1)$. Then, by definition of β_1 and β_2 , ξ_- is the minimal backward characteristic emanating from $(T_1, \xi_-(T_1))$, defined maximally on an interval $[b, T_1]$ and either $b = 0$ or $\xi_-(b) = L$ with $b > 0$. For both cases, $\beta_1(t) < x$.

We conclude that, for any $x \in (\gamma_1(t), L)$, $\beta_1(t) < x$. Thus, we have

$$\beta_1(t) \leq \gamma_1(t). \quad (20)$$

From (19) and (20), we deduce that for any $t \in [T_1, c_1)$, $\beta_1(t) = \gamma_1(t)$. The same argument works for β_2 . More precisely, β_2 coincides with the unique forward characteristic γ_2 , issues from $(T_1, \beta_2(T_1))$, defined on a certain interval $[T_1, c_2)$ where $c_2 > T_1$ is chosen such that for any $t \in [T_1, c_2)$, $\gamma_2(t) \in (0, L)$. As a consequence, β_1 and β_2 are two Lipschitz functions over $(T_1, \min(c_1, c_2))$ and for a.e $t \in (T_1, \min(c_1, c_2))$,

$$\dot{\beta}_i(t) = \frac{f(u(t, \beta_i(t)^-)) - f(u(t, \beta_i(t)^+))}{u(t, \beta_i(t)^-) - u(t, \beta_i(t)^+)}, \quad i \in \{1, 2\}. \quad (21)$$

From Lemma 4.1, $\beta_1(T_1) < \beta_2(T_1)$, (21) and the concavity of f , we deduce that, for a.e $t \in (T_1, \min(c_1, c_2))$

$$\dot{\beta}_1(t) \geq \frac{(f(\bar{u}) - \epsilon) - (f(\bar{u}) - \frac{\epsilon}{2})}{\check{u}_{f(\bar{u})-\epsilon} - \hat{u}_{f(\bar{u})-\frac{\epsilon}{2}}} := d_1 > 0, \quad (22)$$

$$\dot{\beta}_2(t) \leq \frac{(f(\bar{u}) - \epsilon) - (f(\bar{u}) - \frac{\epsilon}{2})}{\hat{u}_{f(\bar{u})-\epsilon} - \check{u}_{f(\bar{u})-\frac{\epsilon}{2}}} := d_2 < 0, \quad (23)$$

and there exists $T \in (T_1, \min(c_1, c_2))$ such that for any $t \in [T_1, T)$, $0 < \beta_1(t) < \beta_2(t) < L$ and $\beta_1(T) = \beta_2(T)$. Moreover, we have

$$d_1(T - T_1) \leq \beta_1(T) = \beta_2(T) \leq L + d_2(T - T_1).$$

This implies that $T \leq T_2 := L + \frac{T_1}{d_1 + d_2}$ and for any $t \geq [T, \min(c_1, c_2))$,

$$\beta(t) := \beta_1(t) = \beta_2(t).$$

By definition of β_1 and β_2 , (18) holds over $[T, \min(c_1, c_2))$. As a consequence, we have for a.e $t \in [T, \min(c_1, c_2))$

$$\frac{-f(\bar{u}) + \epsilon}{1 - \check{u}_{f(\bar{u})-\epsilon}} \leq \dot{\beta}(t) = \frac{f(u(t, \beta(t)^-)) - f(u(t, \beta(t)^+))}{u(t, \beta(t)^-) - u(t, \beta(t)^+)} \leq \frac{f(w_R)}{w_R}. \quad (24)$$

Thus, $c := c_1 = c_2$ and β is a Lipschitz function over $[T, c)$. By definition of c_1 and c_2 , c is the first time when $\beta(c) = 0$ or $\beta(c) = L$ (by convention, $c = +\infty$ if $\beta(t) \in (0, L)$ for any $t \geq T_2$).

- If $c = +\infty$ then β is a Lipschitz function over $[T_2, +\infty)$ and by definition of β_1 and β_2 , (18) holds over $[T_2, \infty)$.

- If $c < +\infty$ then β is a Lipschitz function over $[T_2, c)$ and there exists a final time $\bar{c} \geq c$ such that for any $t \in [c, \bar{c}]$ either $\beta(t) = L$ or $\beta(t) = 0$ (by convention $\bar{c} = +\infty$ if for any $t \geq c$, either $\beta(t) = L$ or $\beta(t) = 0$). If $c \neq +\infty$, there exists $\bar{c}_1 > 0$ such that for any $t \in (\bar{c}, \bar{c}_1)$, $\beta(t) \in (0, L)$ using the continuity of β . To simplify the proof, we only consider the case, for any $t \in [c, \bar{c}]$, $\beta(t) = 0$ (the same arguments work with the case $\beta(t) = L$).
- If $\bar{c} = +\infty$, then β is a Lipschitz function over $[T_2, c)$ with Lipschitz constant denoted by K and $\beta(t) = 0$ for any $t \geq c$. Thus, β is continuous over $[T_2, +\infty)$. Moreover, for any $t_1 \in [T_2, c)$ and $t_2 \in [c, \infty)$ we have

$$\begin{aligned} |\beta(t_1) - \beta(t_2)| &\leq |\beta(t_1) - \beta(t_n)| + |\beta(t_n) - \beta(c)| + |\beta(c) - \beta(t_2)| \\ &\leq K|t_1 - t_n| + |\beta(t_n) - \beta(c)| \end{aligned} \quad (25)$$

where $(t_n)_{n \in \mathbb{N}}$ a sequence such that for any $n \in \mathbb{N}$, $t_n < c$ and $\lim_{n \rightarrow \infty} t_n = c$. Passing to the limit as n tends to $+\infty$ in (25), we deduce that β is a Lipschitz function over $[T_2, +\infty)$. By definition of β_1 and β_2 , (18) holds over $[T_2, +\infty)$.

- If $\bar{c} < +\infty$, then as in the case $\bar{c} = +\infty$, β is a Lipschitz function over $[T_2, \bar{c}]$ and by definition of β_1 and β_2 , (18) holds over $[T_2, \bar{c}]$. Moreover, for any $\bar{t} \in (\bar{c}, \bar{c}_1)$, β coincides with the unique forward characteristic γ , issues from $(\bar{t}, \beta(\bar{t}))$, defined on $[\bar{t}, \bar{c}_1)$. Thus, for any $\bar{t} \in (\bar{c}, \bar{c}_1)$, β is a Lipschitz function over $[\bar{t}, \bar{c}_1)$ with Lipschitz constant denoted by K . By definition of β_1 and β_2 , (18) holds over $[\bar{t}, \bar{c}_1)$. As a consequence, β is continuous over $[T_2, \bar{c}_1)$. Let the sequence $(t_n)_{n \in \mathbb{N}}$ be defined by $t_n = \bar{t} + \frac{1}{n}$ for any $n \in \mathbb{N}$. For any $t_1 \in [c, \bar{c})$ and $t_2 \in (\bar{c}, \bar{c}_1)$, we have

$$\begin{aligned} |\beta(t_1) - \beta(t_2)| &\leq |\beta(t_1) - \beta(t_n)| + |\beta(t_n) - \beta(\bar{c})| + |\beta(\bar{c}) - \beta(t_2)| \\ &\leq K|t_1 - t_n| + |\beta(t_n) - \beta(\bar{c})| \end{aligned} \quad (26)$$

Passing to the limit as n tends to $+\infty$ in (26), we deduce that β is a Lipschitz function over $[T_2, \bar{c}_1)$. By iteration, we deduce that β is a Lipschitz function over $[T_2, +\infty)$ and (18) holds over $[T_2, +\infty)$.

Note that (8) holds at time $t = T_1$ can be removed since β_1 and β_2 are two continuous functions and (8) holds almost everywhere. Moreover, if $\beta_1(T_1) = \beta_2(T_1)$ then for any $t \in [T_1, \infty)$, $\beta(t) := \beta_1(t) = \beta_2(t)$ and (18) holds. Moreover, if $\beta(t) > 0$ with $t \in [T_1, \infty)$, we prove as above that $\beta(\cdot)$ coincides with the unique forward characteristic $\gamma_1(\cdot)$ emanating from $(t, \beta(t))$, over $[t, c_1)$ where $c_1 > t$ is chosen such that for any $t \in [t, c_1)$, $\beta(t) = \gamma_1(t) \in (0, L)$. So β is a Lipschitz function over $[T_1, +\infty)$ \square

Proposition 4.4 *Let $\bar{t} \geq T_2$ such that (8) holds at time $t = \bar{t}$*

$$\beta(\bar{t}) > 0 \implies f(u(\bar{t}, 0^+)) = F_1(\bar{t}, u) \text{ and } u(\bar{t}, 0^+) \leq \bar{u} \quad (27)$$

$$\beta(\bar{t}) < L \implies f(u(\bar{t}, L^-)) = F_2(\bar{t}, u) \text{ and } u(\bar{t}, L^-) \geq \bar{u} \quad (28)$$

PROOF. Let $\bar{t} \geq T_2$ such that (8) holds at time $t = \bar{t}$ and $\beta(\bar{t}) > 0$. From (18), $u(\bar{t}, 0^+) = \lim_{x \rightarrow 0^+} u(\bar{t}, x^+) \in [0, \check{u}_{f(\bar{u})-e}]$. Let us fix $\bar{x} \in (-\infty, 0)$. From the point 1. of Definition 1.2 and [4, Theorem 11.1.3] and (17), there exists a minimal generalized characteristic ξ_- emanating from (\bar{t}, \bar{x}) , defined maximally on an interval $[a, \bar{t}]$. Moreover, for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and, for any $t \in (a, \bar{t}]$, we have $u(t, \xi_-(t)^-) = v$. We have two alternatives.

Case 1: $a = 0$ and $\xi_-(0) < 0$. From [4, Theorem 11.1.3], we have $u_0(\xi_-(0)^+) \leq u(\bar{t}, \bar{x}^-) \leq u_0(\xi_-(0)^-)$. By definition of u_0 in (9), we have $u_0(\xi_-(0)^+) = u_0(\xi_-(0)^-) = 1$. Then, we deduce that $u(\bar{t}, \bar{x}^-) = 1$.

Case 2: $a > 0$ or $(a = 0 \text{ and } \xi_-(0) = 0)$. Since ξ_- is a straight line, $\xi_-(\bar{t}) = \bar{x} < 0$ and $\xi_-(a) = 0$, we have for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v) < 0$. By concavity of f , we deduce that $u(\bar{t}, \bar{x}^-) \in (\bar{u}, 1]$.

Thus, for any $\bar{x} \in (-\infty, 0)$, $u(\bar{t}, \bar{x}^-) \in (\bar{u}, 1]$. As a consequence, $u(\bar{t}, 0^-) = \lim_{\bar{x} \rightarrow 0^-} u(\bar{t}, \bar{x}^-) \in [\bar{u}, 1]$ and $u(\bar{t}, 0^+) \in [0, \tilde{u}_{f(\bar{u})-\epsilon}]$ with $\tilde{u}_{f(\bar{u})-\epsilon} < \bar{u}$. Since (8) holds at time $t = \bar{t}$, we conclude that

$$((\gamma_0^l u)(\bar{t}), (\gamma_0^r u)(\bar{t})) \in \mathcal{G}_1(F_1),$$

with $(\gamma_0^l u)(\bar{t}) = u(\bar{t}, 0^-)$ and $(\gamma_0^r u)(\bar{t}) = u(\bar{t}, 0^+)$ and \mathcal{G}_1 defined in Definition 1.1. Thus, (27) holds. The same argument works for $\beta(t) < L$. \square

We will state some preliminary lemmas. Let $(t, x) \in (0, +\infty) \times [0, L]$, we introduce $\tau_-(t, x) \in (0, +\infty)$ and $\tau_+(t, x) \in (0, +\infty)$ defined by

$$\tau_+(t, x) = \frac{x}{f'(u(t, x^-))} \text{ and } \tau_-(t, x) = \frac{x - L}{f'(u(t, x^-))} \quad (29)$$

with u the \mathcal{G} -entropy solution of (1).

Proposition 4.5 *Let $\bar{t} \geq \tilde{T}_2 := T_2 + \tau_{\max}$ with $\tau_{\max} = \max\left(-\frac{L}{f'(w_R)}, \frac{L}{f'(\tilde{u}_{f(\bar{u})-\epsilon})}\right)$ such that (8) holds at time $t = \bar{t}$.*

$$\beta(\bar{t}) > 0 \implies \forall x \in (0, \beta(\bar{t})), f(u(\bar{t}, x^\pm)) = F_1(\bar{t} - \tau_+(\bar{t}, x), u) \text{ and } u(\bar{t}, x^\pm) \in [0, \bar{u}), \quad (30)$$

$$\beta(\bar{t}) < L \implies \forall x \in (\beta(\bar{t}), L), u(\bar{t}, x^\pm) = w_R. \quad (31)$$

PROOF. Let $\bar{t} \geq \tilde{T}_2 := T_2 + \tau_{\max}$ with $\tau_{\max} = \max\left(-\frac{L}{f'(w_R)}, \frac{L}{f'(\tilde{u}_{f(\bar{u})-\epsilon})}\right)$ such that (8) holds at time $t = \bar{t}$ and $x \in (0, \beta(\bar{t}))$. From the point 1. of Definition 1.2, [4, Theorem 11.1.3] and the definition of β , there exists a minimal backward generalized characteristic, denoted by ξ_- , associated to u emanating from (\bar{t}, x) defined on an interval $[a, \bar{t}]$ with $a > 0$ and $\xi_-(a) = 0$ such that for any $t \in (a, \bar{t})$

$$u(t, \xi_-(t)^-) = u(\bar{t}, x^-) = u(t, \xi_-(t)^+). \quad (32)$$

In particular, ξ_- is a genuine characteristic (also called shock-free, see [4, Definition 10.2.4]). From Proposition B.1 with $\xi = \xi_-$, $a = a$ and $b = \bar{t}$, there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v) > 0$ and, for any $t \in (a, \bar{t})$, $u(t, \xi_-(t)^-) = v$ and $f(v) \leq f(\bar{u}) - \epsilon$. In particular, we have $a = \bar{t} - \tau_+(\bar{t}, x) = \bar{t} - \frac{x}{f'(u(\bar{t}, x^-))} \geq \tilde{T}_2 - \frac{L}{f'(\tilde{u}_{f(\bar{u})-\epsilon})} \geq T_2$. From (32), we deduce that for any $t \in (a, \bar{t}]$

$$u(t, \xi_-(t)^-) = v \text{ and } f(v) \leq f(\bar{u}) - \epsilon.$$

- We assume that there exists a time $t_0 \in (a, \bar{t})$ such that $\xi_-(t_0) = \beta(t_0) \in (0, L)$. Then ξ_- interacts with any maximal backward generalized characteristic emanating from $(t, \beta(t))$ with $t \in (t_0, \bar{t})$. This leads to a contradiction since from [4, 1.1.2 Corollary], two genuine characteristics cannot interact. Thus, we have for any $t \in (a, \bar{t}]$,

$$\xi_-(t) < \beta(t). \quad (33)$$

- We assume that $\beta(a) = \xi_-(a)$ (see Figure 8). Since $\beta(a) = 0$, from Proposition 4.3, for any $y \in (0, L)$,

$$u(a, y^\pm) \in [w_R, 1]. \quad (34)$$

Let $\delta > 0$ such that $x + \delta \in (0, \beta(\bar{t}))$. From Lemma A.1 with $\gamma_1(\cdot) = \xi_-(\cdot)$, $\gamma_2(\cdot) = \xi_-(\cdot) + \delta$, $t_1 = a$ et $t_2 = \bar{t}$, we have

$$\begin{aligned} \int_0^\delta u(a, z) dz + \int_a^{\bar{t}} (f(v) - f'(v)v) dt \\ = \int_x^{x+\delta} u(\bar{t}, z) dz + \int_a^{\bar{t}} (f(u(t, (\xi_-(t) + \delta)^-)) - \dot{\xi}_-(t)u(t, (\xi(t) + \delta)^-)) dt \end{aligned} \quad (35)$$

Using that $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and the concavity of f , we deduce that

$$f(u(t, (\xi_-(t) + \delta)^-)) - f(v) - f'(v)(u(t, (\xi(t) + \delta)^-) - v) \leq 0 \quad (36)$$

From (35) and (36), we have

$$\int_0^\delta u(a, z) dz \leq \int_x^{x+\delta} u(\bar{t}, z) dz. \quad (37)$$

Since for any $z \in (x, x + \delta)$, $0 < z < x + \delta < \beta(\bar{t})$, from Proposition 4.3, $u(\bar{t}, z^\pm) \in [0, \check{u}_{f(\bar{u})-\epsilon}]$. Therefore, together with (34), we have

$$\int_x^{x+\delta} u(\bar{t}, z) dz < \delta \check{u}_{f(\bar{u})-\epsilon} \leq \delta w_R \leq \int_0^\delta u(a, z) dz$$

This leads to a contradiction using (37). Thus, we have

$$\xi_-(a) < \beta(a). \quad (38)$$

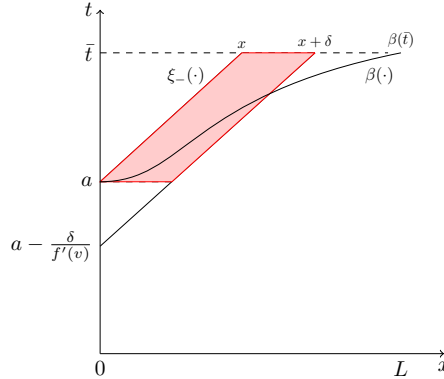


Figure 8: Case $\beta(a) = \xi_-(a)$ where ξ_- is a minimal backward generalized characteristic associated to u emanating from (\bar{t}, x) defined on an interval $[a, \bar{t}]$ with $a > 0$ and $\xi_-(a) = 0$. Moreover, for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and, for any $t \in (a, \bar{t}]$, $u(t, \xi_-(t)^-) = v$ and $f(v) \leq f(\bar{u}) - \epsilon$ with $v \in [0, 1]$.

Using that for any $t \in [a, \bar{t}]$, $\xi_-(t) < \beta(t)$, $\xi_-(a) = 0$ and β is a Lipschitz function (see Proposition 4.3), there exists $\bar{\delta} > 0$ such that for any $t \in [a - \bar{\delta}, a + \bar{\delta}]$, $\beta(t) > 0$. Therefore, from Proposition 4.4, we have for a.e $t \in [a - \bar{\delta}, a + \bar{\delta}]$,

$$f(u(t, 0^+)) = F_1(t, u) \text{ and } u(t, 0^+) \leq \bar{u}$$

From Proposition B.1 with $\xi = \xi_-$, $a = a$, $b = \bar{t}$, $c = a - \bar{\delta}$, $d = a + \bar{\delta}$, $g = \min_{t \in [a - \bar{\delta}, a + \bar{\delta}]} F_1(t, u)$ and $G = \max_{t \in [a - \bar{\delta}, a + \bar{\delta}]} F_1(t, u)$, we deduce that for any $t \in (b, \bar{t})$,

$$\min_{t \in [a - \bar{\delta}, a + \bar{\delta}]} F_1(t, u) \leq f(u(\bar{t}, x^-)) \leq \max_{t \in [a - \bar{\delta}, a + \bar{\delta}]} F_1(t, u). \quad (39)$$

Note that the minimum and the maximum of $F_1(t, u)$ makes sense since using (5), $t \mapsto F_1(t, u)$ is a continuous function and $[a - \bar{\delta}, a + \bar{\delta}]$ is a compact set. Passing to the limit in (39) as $\bar{\delta} \rightarrow 0$, we conclude that

$$f(u(\bar{t}, x^-)) = F_1(a, u) \text{ with } a = \bar{t} - \tau_+(\bar{t}, x). \quad (40)$$

Using that $u(\bar{t}, x^+) = \lim_{\epsilon \rightarrow 0^+} u(\bar{t}, (x + \epsilon)^-)$, we deduce that (40) also hold for x^+ .

Let $\bar{t} \geq \tilde{T}_2 := T_2 + \tau_{\max}$ with $\tau_{\max} = \max\left(-\frac{L}{f'(w_R)}, \frac{L}{f'(\tilde{u}_{f(\bar{u})-\epsilon})}\right)$ such that (8) holds at time $t = \bar{t}$ and $x \in (\beta(\bar{t}), L)$. From the point 1. of Definition 1.2, [4, Theorem 11.1.3] and the definition of β , there exists a minimal backward generalized characteristic, denoted by ξ_- , associated to u emanating from (\bar{t}, x) defined on an interval $[a, \bar{t}]$ with $a > 0$ and $\xi_-(a) = L$ such that for any $t \in (a, \bar{t})$, (32) holds. In particular, ξ_- is a genuine characteristic (also called shock-free, see [4, Definition 10.2.4]). From Proposition B.1 with $\xi = \xi_-$, $a = a$ and $b = \bar{t}$, there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v) < 0$ and, for any $t \in (a, \bar{t})$, $u(t, \xi_-(t)^-) = v$ and $f(v) \leq f(w_R)$. In particular, we have $a = \bar{t} - \tau_-(\bar{t}, x) = \bar{t} - \frac{x-L}{f'(u(\bar{t}, x^-))} \geq \tilde{T}_2 - \frac{L}{f'(w_R)} \geq T_2$. From (32), we deduce that for any $t \in (a, \bar{t}]$

$$u(t, \xi_-(t)^-) = v \text{ and } f(v) \leq f(w_R).$$

- We assume that there exists a time $t_0 \in (a, \bar{t})$ such that $\xi_-(t_0) = \beta(t_0) \in (0, L)$. Then ξ_- interacts with any maximal backward generalized characteristic emanating from $(t, \beta(t))$ with $t \in (t_0, \bar{t})$. This leads to a contradiction since from [4, 1.1.2 Corollary], two genuine characteristics cannot interact. Thus, we have for any $t \in (a, \bar{t}]$,

$$\beta(t) < \xi_-(t). \quad (41)$$

- We assume that $\beta(a) = \xi_-(a)$. Since $\beta(a) = L$, from Proposition 4.3, for any $y \in (0, L)$,

$$u(a, y^\pm) \in [0, \tilde{u}_{f(\bar{u})-\epsilon}). \quad (42)$$

Let $\delta > 0$ such that $x - \delta \in (\beta(\bar{t}), L)$. From Lemma A.1 with $\gamma_1(\cdot) = \xi_-(\cdot) - \delta$, $\gamma_2(\cdot) = \xi_-(\cdot)$, $t_1 = \bar{t}$ et $t_2 = a$, we have

$$\begin{aligned} \int_{L-\delta}^L u(a, x) dx + \int_a^{\bar{t}} (f(u(t, (\xi_-(t) - \delta)^+)) - \dot{\xi}_-(t) u(t, (\xi(t) - \delta)^-)) dt \\ = \int_{\bar{x}-\delta}^{\bar{x}} u(\bar{t}, x) dx + \int_a^{\bar{t}} (f(v) - f'(v)v) dt \end{aligned} \quad (43)$$

Using that $t \in [a, \bar{t}]$, $\dot{\xi}_-(t) = f'(v)$ and the concavity of f , we deduce that

$$f(u(t, (\xi_-(t) - \delta)^+)) - f(v) - f'(v)(u(t, (\xi(t) - \delta)^-) - v) \leq 0 \quad (44)$$

From (43) and (44), we have

$$\int_{\bar{x}-\delta}^{\bar{x}} u(\bar{t}, x) dx \leq \int_{L-\delta}^L u(a, x) dx. \quad (45)$$

Since, for any $z \in (x - \delta, x)$, $\beta(\bar{t}) < x - \delta < z < L$, from Proposition 4.3, $u(\bar{t}, z^\pm) \in [w_R, 1]$. Therefore, together with (42), we have

$$\int_{L-\delta}^L u(a, x) dx \leq \delta \check{u}_{f(\bar{u})-\epsilon} < \delta w_R \leq \int_{\bar{x}-\delta}^{\bar{x}} u(\bar{t}, x) dx$$

This leads to a contradiction using (45). Thus, we have

$$\beta(a) < \xi_-(a). \quad (46)$$

Using that for any $t \in [a, \bar{t}]$, $\beta(t) < \xi_-(t)$, $\xi_-(a) = L$ and β is a Lipschitz function (see Proposition 4.3), there exists $\delta > 0$ such that for any $t \in [a - \delta, a + \delta]$, $\beta(t) < L$. Therefore, from Proposition 4.4, we have for a.e $t \in [a - \delta, a + \delta]$,

$$f(u(t, L^-)) = f(w_R) \text{ and } \bar{u} < u(t, L^-)$$

From Proposition B.1 with $\xi = \xi_-$, $a = a$, $b = \bar{t}$, $c = a - \delta$, $d = a + \delta$, $g = f(w_R) = G$, we deduce that for any $t \in (b, \bar{t})$,

$$u(\bar{t}, x^-) = w_R \text{ with } a = \tau_-(\bar{t}, x) \quad (47)$$

Using that $u(\bar{t}, x^+) = \lim_{\epsilon \rightarrow 0^+} u(\bar{t}, (x + \epsilon)^-)$, we deduce that (47) also hold for x^+ . \square

Lemma 4.2 For a.e $t \geq \tilde{T}_2 := T_2 + 2\tau_{\max}$ with $\tau_{\max} = \max\left(-\frac{L}{f'(w_R)}, \frac{L}{f'(\check{u}_{f(\bar{u})-\epsilon})}\right)$.

$$\begin{cases} \dot{m}(t) = 0 & \text{if } \beta(t) = 0, \\ \dot{m}(t) = F_1(t, u) - F_2(t, u) & \text{if } 0 < \beta(t) < L, \\ \dot{m}(t) \geq F_1(t, u) - F_2(t, u) & \text{if } \beta(t) = L, \end{cases} \quad (48)$$

where $F_1(t, u)$ and $F_2(t, u)$ are defined in (5).

PROOF. From Lemma A.1 with $\gamma_1 = 0$, $\gamma_2 = L$, $t_1 = 0$ et $t_2 = t$ and the definition of m in (6), we have for a.e $t > 0$,

$$\dot{m}(t) = f(u(t, 0^+)) - f(u(t, L^-)), \quad (49)$$

where (49) is understood in the sense of Carathéodory solutions. More precisely, Caratheodory solutions are absolutely continuous curves that satisfy the integral version of the differential equation (49), that is,

$$m(t) = m(0) + \int_0^t (f(u(s, 0^+)) - f(u(s, L^-))) ds.$$

Let us fix $\bar{t} \geq \tilde{T}_2$ such that (8), (27), (28) and (49) hold. We have three alternatives.

Case 1: $\beta(\bar{t}) = 0$. From Proposition 4.5, for any $x \in (0, L)$, $u(\bar{t}, x^\pm) = w_R$. Using that $u(\bar{t}, 0^+) = \lim_{x \rightarrow 0} u(\bar{t}, x^+)$ and $u(\bar{t}, L^-) = \lim_{x \rightarrow L} u(\bar{t}, x^-)$, we conclude that $u(\bar{t}, 0^+) = u(\bar{t}, L^-) = w_R$. Thus, from (49), for a.e $t \geq \tilde{T}_2$ such that $\beta(t) = 0$, we have $\dot{m}(t) = 0$.

Case 2: $0 < \beta(\bar{t}) < L$. From (27), (28) and (49), we have

$$\dot{m}(\bar{t}) = F_1(\bar{t}, u) - F_2(\bar{t}, u).$$

Case 3 : $\beta(\bar{t}) = L$. From (8) and (27), $f(u(\bar{t}, 0^+)) = F_1(\bar{t}, u)$ and $f(u(\bar{t}, L^-)) \leq f(w_R) = F_2(\bar{t}, u)$. Therefore, using (49),

$$\dot{m}(t) = f(u(t, 0^+)) - f(u(t, L^-)) \geq F_1(t, u) - F_2(t, u).$$

□

Lemma 4.3 *We assume that there exist $t_1 > 0$ and $t_2 > 0$ such that $T_2 + \frac{L}{f'(\bar{u}_{f(\bar{a})-\epsilon})} < t_1 < t_2$ and for any $t \in [t_1, t_2]$, $\beta(t) = L$. Then, we have*

$$t_2 - t_1 < \frac{L}{f'(\bar{u}_{f(\bar{a})-\epsilon})}.$$

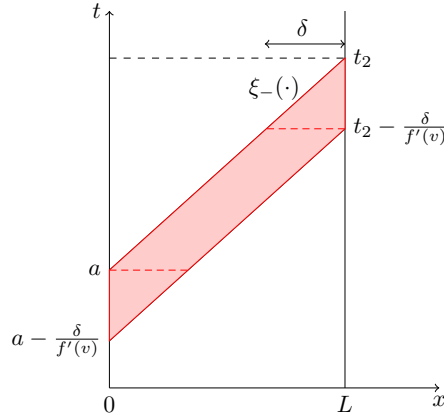


Figure 9: Plotting of the minimal backward generalized characteristic ξ_- , emanating from (t_2, L) , defined maximally over $[a, t_2]$ with $a > 0$ and $\xi_-(a) = 0$.

PROOF. We assume that $t_2 \geq \frac{L}{f'(\bar{u}_{f(\bar{a})-\epsilon})} + t_1$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence such that for any $n \in \mathbb{N}$, $x_n \in (0, L)$ and $x_n \rightarrow L$ as $n \rightarrow \infty$. Since $f \in C^2([0, 1])$, we have

$$\lim_{n \rightarrow \infty} t_2 - \tau_+(t_2, x_n) = \lim_{n \rightarrow \infty} t_2 - \frac{x_n}{f'(u(t_2, x_n^-))} = t_2 - \tau_+(t_2, L). \quad (50)$$

From the point 1. of Definition 1.2, [4, Theorem 11.1.3] and the definition of β , there exists a minimal backward generalized characteristic, denoted by ξ_-^n , associated to u emanating from (t_2, x_n) defined on an interval $[a_n, t_2]$ with $a_n := t_2 - \tau_+(t_2, x_n) > 0$ and $\xi_-^n(a_n) = 0$ such that for any $t \in (a_n, t_2)$

$$u(t, \xi_-^n(t)^-) = u(t_2, x_n^-) = u(t, \xi_-^n(t)^+). \quad (51)$$

From [4, Section 10.2], (50) and (51), $(\xi_-^n)_{n \in \mathbb{N}}$ converges to the minimal generalized characteristic $\xi(\cdot)$ emanating from (t_2, L) defined on an interval $[a, t_2]$ with $a := t_2 - \tau_+(t_2, L) > 0$ and $\xi_-(a) = 0$ such that, for any $t \in (a, t_2]$,

$$u(t, \xi_-(t)^-) = u(t_2, L^-) = u(t, \xi_-(t)^+). \quad (52)$$

In particular, ξ_- is a genuine characteristic (see [4, Definition 10.2.4]). From Proposition B.1 with $\xi = \xi_-$, $a = a$ and $b = t_2$, there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, t_2]$,

$\dot{\xi}_-(t) = f'(v) > 0$ and, for any $t \in (a, t_2)$, $u(t, \xi_-(t)^-) = v$ and $f(v) \leq f(\bar{u}) - \varepsilon$. In particular, we have

$$a = t_2 - \tau_+(t, L) = t_2 - \frac{L}{f'(u(t_2, L^-))} > t_2 - \frac{L}{f'(\check{u}_{f(\bar{u})-\varepsilon})} \geq t_1. \quad (53)$$

From (52), we deduce that for any $t \in (a, t_2]$

$$u(t, \xi_-(t)^-) = v \text{ and } f(v) \leq f(\bar{u}) - \varepsilon. \quad (54)$$

Let us fix $\delta > 0$ such that $a - \frac{\delta}{f'(v)} \geq t_1$. From Lemma A.1 with $\gamma_1(\cdot) = 0$, $\gamma_2(\cdot) = \xi_-(\cdot) + \delta$, $t_1 = a - \frac{\delta}{f'(v)}$ and $t_2 = a$ (see Figure 9), we have

$$\int_{a - \frac{\delta}{f'(v)}}^a f(u(t, 0^+)) dt + 0 = \int_0^\delta u(a, x) dx + \int_{a - \frac{\delta}{f'(v)}}^a (f(u(t, (\xi_-(t) + \delta)^-)) - \dot{\xi}_-(t) u(t, (\xi_-(t) + \delta)^-)) dt$$

Using that f is concave and for any $t \in [a, t_2]$, $\dot{\xi}_-(t) = f'(v)$, we have for any $t \in [a - \frac{\delta}{f'(v)}, a]$

$$f(u(t, (\xi_-(t) + \delta)^-)) - f(v) - f'(v)(u(t, (\xi_-(t) + \delta)^-) - v) \leq 0.$$

Therefore,

$$\int_{a - \frac{\delta}{f'(v)}}^a f(u(t, 0^+)) dt + 0 \leq \int_0^\delta u(a, x) dx + \frac{\delta}{f'(v)} (f(v) - f'(v)v) \quad (55)$$

From Lemma A.1 with $\gamma_1(\cdot) = \xi_-(\cdot)$, $\gamma_2(\cdot) = \xi_-(\cdot) + \delta$, $t_1 = a$ and $t_2 = t_2 - \frac{\delta}{f'(v)}$ (see Figure 9), we have that

$$\begin{aligned} \int_0^\delta u(a, x) dx + \int_a^{t_2 - \frac{\delta}{f'(v)}} (f(v) - f'(v)v) dt \\ = \int_{L-\delta}^L u(t_2 - \frac{\delta}{f'(v)}, x) dx + \int_a^{t_2 - \frac{\delta}{f'(v)}} (f(u(t, (\xi_-(t) + \delta)^-)) - \dot{\xi}_-(t) u(t, (\xi_-(t) + \delta)^-)) dt \end{aligned}$$

Using that f is concave and for any $t \in [a, t_2]$, $\dot{\xi}_-(t) = f'(v)$, we have for any $t \in [a, t_2 - \frac{\delta}{f'(v)}]$

$$f(u(t, (\xi_-(t) + \delta)^-)) - f(v) - f'(v)(u(t, (\xi_-(t) + \delta)^-) - v) \leq 0.$$

Thus, we have

$$\int_0^\delta u(a, x) dx \leq \int_{L-\delta}^L u\left(t_2 - \frac{\delta}{f'(v)}, x\right) dx \quad (56)$$

From Lemma A.1 with $\gamma_1(\cdot) = \xi_-(\cdot)$, $\gamma_2(\cdot) = L$, $t_1 = t_2 - \frac{\delta}{f'(v)}$ and $t_2 = t_2$ (see Figure 9), we have

$$\int_{L-\delta}^L u\left(t_2 - \frac{\delta}{f'(v)}, x\right) dx + \frac{\delta}{f'(v)} (f(v) - f'(v)v) = \int_{t_2 - \frac{\delta}{f'(v)}}^{t_2} f(u(t, L^-)) dt \quad (57)$$

From (55), (56) and (57), we have

$$\int_{a - \frac{\delta}{f'(v)}}^a f(u(t, 0^+)) dt \leq \int_{t_2 - \frac{\delta}{f'(v)}}^{t_2} f(u(t, L^-)) dt. \quad (58)$$

Since (8) holds almost everywhere, we have for a.e $t \in [t_1, t_2]$

$$f(u(t, L^-)) \leq f(w_R). \quad (59)$$

From Lemma A.1 with $\gamma_1(\cdot) = \xi_-(\cdot)$, $\gamma_2(\cdot) = L$, $t_1 = a$ and $t_2 = t_2$, we have

$$\int_0^L u(a, x) dx + \int_a^{t_2} (f(v) - f'(v)v) dt = 0 + \int_a^{t_2} f(u(t, L^-)) dt \quad (60)$$

From (59) and $f(w_R) = f(w_L)$, we have

$$\int_0^L u(a, x) dx \leq (t_2 - a)(f(w_L) - f(v) + f'(v)v) \quad (61)$$

Since f is a concave function, we have

$$f(w_L) - f(v) - f'(v)(v - w_L) \leq 0. \quad (62)$$

From (29), $a := t_2 - \tau_+(t_2, L) \geq t_1 > 0$ and (54),

$$t_2 - a = \frac{L}{f'(u(t_2, L^-))} = \frac{L}{f'(v)}. \quad (63)$$

Combining (61) with (62) and (63), we deduce that

$$\int_0^L u(a, x) dx \leq \frac{L}{f'(v)} f'(v)w_L = Lw_L.$$

Therefore, together with (2) and (6), we have

$$m(a) = \int_0^L (u(a, x) - w(x))dx \leq Lw_L - \int_0^L w(x)dx < 0.$$

Using that m is a continuous function and (53), we deduce that there exists $\bar{\delta} > 0$ such that for any $t \in [a - \bar{\delta}, a] \subset [t_1, t_2]$, $m(t) < 0$. Therefore, together with Proposition 4.4, (5) and for any $t \in [t_1, t_2]$, $\beta(t) = L$, we have for a.e $t \in [a - \bar{\delta}, a]$,

$$f(u(t, 0^+)) = F_1(t, u) > f(w_L) \quad (64)$$

From (58) with $\delta = f'(v)\bar{\delta}$, (59) and (64), we deduce that

$$\bar{\delta}f(w_L) < \int_{a-\bar{\delta}}^a f(u(t, 0^+))dt \leq \int_{t_2-\bar{\delta}}^{t_2} f(u(t, L^-))dt \leq \bar{\delta}f(w_R)$$

This leads to a contradiction since $f(w_L) = f(w_R)$. \square

Proposition 4.6 *There exists $T_4 > 0$ (independent of the initial data u_0) such that for any $t \geq T_4$, $\beta(t) \in (0, L)$ and*

$$\dot{m}(t) = -g(m(t)) \quad (65)$$

PROOF. Let us fix $\tau_{\max} = \max\left(-\frac{L}{f'(w_R)}, \frac{L}{f'(\bar{u}_{f(\bar{a})-\epsilon})}\right)$. We will prove that for a.e $t > \tilde{T}_2 + 2\tau_{\max}$, $\beta(t) \in (0, L)$.

Case 1 : there exists $t_0 \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$ such that $m(t_0) > 0$. Using (5), Remark 1 and Lemma 4.2, for any $t \geq t_0$, $m(t) \geq 0$. Since $m(\cdot)$ is a continuous function, without loss of generality, we can additionally assume that (8) holds at time $t = t_0$. Moreover, $t - \tau_+(t, \beta(t)) = t - \frac{\beta(t)}{f'(u(t, \beta(t)^-))} > t_0 - \tau_{\max} \geq \tilde{T}_2$. We assume that $\beta(t_0) = 0$. From Proposition 4.5, for any $x \in (0, L)$, $u(t_0, x^\pm) = w_R$. In particular, we have $f(u(t_0, 0^+)) = f(w_R)$. Moreover, using (8), (5) and $m(t_0) > 0$, we have $f(u(t_0, 0^+)) \leq F_1(t_0, u) < f(w_L)$. This leads to a contradiction since $f(w_R) = f(w_L)$. Thus, we have

$$\beta(t_0) > 0.$$

Since β is a continuous function and $\beta(t_0) > 0$, there exists $t_b \in \mathbb{R} \cup \{+\infty\}$ such that for any $t \in [t_0, t_b)$, $\beta(t) \in (0, L)$ and $\beta(t_b) \in \{0, L\}$. From Proposition 4.5, for a.e $t \in [t_0, t_b)$, for any $x \in (0, \beta(t))$, $f(u(t, x)) = F_1(t - \tau_+(t, x), u)$ and for any $x \in (\beta(t), L)$, $u(t, x^\pm) = w_R$. In particular, we have for a.e $t \in [t_0, t_b)$,

$$f(u(t, \beta(t)^-)) = F_1(t - \tau_+(t, \beta(t)), u) \text{ and } u(t, \beta(t)^+) = w_R$$

Therefore, together with (5) and for any $t \in [t_0, t_b)$, $m(t) \geq 0$, we have

$$u(t, \beta(t)^-) \in [0, w_L] \text{ and } u(t, \beta(t)^+) = w_R \quad (66)$$

From (24), (66), $f(w_L) = f(w_R)$ and the concavity of f , we deduce that, for a.e $t \in [t_0, t_b)$,

$$\dot{\beta}(t) = \frac{f(u(t, \beta(t)^-)) - f(u(t, \beta(t)^+))}{u(t, \beta(t)^-) - u(t, \beta(t)^+)} \geq 0. \quad (67)$$

As a consequence, for any $t \in [t_0, t_b)$, $\beta(t) \geq \beta(t_0) > 0$ and $\beta(t_b) \neq 0$. We now assume that $\beta(t_b) = L$. Since for a.e $t \geq t_0$ (67) holds as soon as $\beta(t) \in (0, L)$, we have for any $t \geq t_b$, $\beta(t) = L$. This leads to a contradiction using Lemma 4.3. We conclude that for any $t \geq \tilde{T}_2 + 2\tau_{\max} \geq t_0$,

$$\beta(t) \in (0, L).$$

Case 2 : for any $t \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$, $m(t) \leq 0$. For a.e $t \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$ such that $\beta(t) \in (0, L)$, we have

$$f(u(t, \beta(t)^-)) = F_1(t - \tau_+(t, \beta(t)), u) \text{ and } u(t, \beta(t)^+) = w_R.$$

Therefore, together with (5) and for any $t \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$, $m(t) \leq 0$, we have

$$u(t, \beta(t)^-) \in [w_L, \tilde{u}_{f(\bar{u})-\epsilon}] \text{ and } u(t, \beta(t)^+) = w_R. \quad (68)$$

From (24), (68), $f(w_L) = f(w_R)$ and the concavity of f , we deduce that, for a.e $t \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$ such that $\beta(t) \in (0, L)$,

$$\dot{\beta}(t) = \frac{f(u(t, \beta(t)^-)) - f(u(t, \beta(t)^+))}{u(t, \beta(t)^-) - u(t, \beta(t)^+)} \leq 0. \quad (69)$$

If $\beta(\tilde{T}_2 + \tau_{\max}) \in [0, L)$ then from (69), for any $t \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$, $\beta(t) < L$ and so $\beta(\tilde{T}_2 + 2\tau_{\max}) < L$. If $\beta(\tilde{T}_2 + \tau_{\max}) = L$ then from Lemma 4.3 and (69), $\beta(\tilde{T}_2 + 2\tau_{\max}) < L$. We conclude that

$$\beta(\tilde{T}_2 + 2\tau_{\max}) < L. \quad (70)$$

Let $\delta = \frac{x_0(w_R - w_L)}{2w_R}$, we assume that there exists $\bar{t} \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$ such that (8) holds at time $t = \bar{t}$ and $\beta(\bar{t}) \in [0, \delta]$. From Proposition 4.5, for any $x \in (\beta(\bar{t}), L)$, $u(t, x^\pm) = w_R$. Then, from (2), (6) and (68),

$$\begin{aligned} m(\bar{t}) &= \int_0^L u(\bar{t}, x) dx - (L - x_0)w_R - x_0w_L \\ &\geq (L - \beta(\bar{t}))w_R - (L - x_0)w_R - x_0w_L \\ &\geq (L - \delta)w_R - (L - x_0)w_R - x_0w_L \\ &> 0 \end{aligned} \quad (71)$$

This leads to a contradiction since $m(\bar{t}) \leq 0$ and so $\beta(\bar{t}) > \delta$. Thus, using that β is a continuous function, for any $t \in [\tilde{T}_2 + \tau_{\max}, \tilde{T}_2 + 2\tau_{\max}]$, we have $\beta(t) \in [\delta, L]$. In particular,

$$\beta(\tilde{T}_2 + 2\tau_{\max}) \geq \delta. \quad (72)$$

From (70) and (72), we have

$$\delta \leq \beta(\tilde{T}_2 + 2\tau_{\max}) < L. \quad (73)$$

From (73) and using that β is a continuous function, there exists $t_b > \tilde{T}_2 + 2\tau_{\max}$ such that $\beta(t_b) = 0$ or $\beta(L) = 0$ and for any $t \in [\tilde{T}_2 + 2\tau_{\max}, t_b)$, $\beta(t) \in (0, L)$ (by convention $t_b = +\infty$ if $\forall t \geq \tilde{T}_2 + 2\tau_{\max}$, $\beta(t) \in (0, L)$). From Lemma 4.2, for a.e $t \in (\tilde{T}_2 + 2\tau_{\max}, t_b)$,

$$\dot{m}(t) = F_1(t, u) - f(w_R).$$

Therefore, together with (5), Remark 1 and the continuity of m , for any $t \in (\tilde{T}_2 + 2\tau_{\max}, t_b)$,

$$m(t) \leq 0. \quad (74)$$

From Proposition 4.5, for a.e $t \in [\tilde{T}_2 + 2\tau_{\max}, t_b)$, for any $x \in (0, \beta(t))$, $f(u(t, x)) = F_1(t - \tau_+(t, x), u)$ and for any $x \in (\beta(t), L)$, $u(t, x^\pm) = w_R$. In particular, we have for a.e $t \in [\tilde{T}_2 + 2\tau_{\max}, t_b)$,

$$f(u(t, \beta(t)^-)) = F_1(t - \tau_+(t, \beta(t)), u) \text{ and } u(t, \beta(t)^+) = w_R$$

Therefore, together with (5) and (74), we have

$$u(t, \beta(t)^-) \in [w_L, \check{u}_{f(\bar{u})-\epsilon}] \text{ and } u(t, \beta(t)^+) = w_R \quad (75)$$

From (24), (75), $f(w_L) = f(w_R)$ and the concavity of f , we deduce that, for a.e $t \in [\tilde{T}_2 + 2\tau_{\max}, t_b)$,

$$\dot{\beta}(t) = \frac{f(u(t, \beta(t)^-)) - f(u(t, \beta(t)^+))}{u(t, \beta(t)^-) - u(t, \beta(t)^+)} \leq 0. \quad (76)$$

As a consequence, together with (73), for any $t \in [\tilde{T}_2 + 2\tau_{\max}, t_b)$, $\beta(t) \leq \beta(\tilde{T}_2 + 2\tau_{\max}) < L$. We assume that there exists $\bar{t} \in [\tilde{T}_2 + 2\tau_{\max}, t_b]$ such that (8) holds at time $t = \bar{t}$ and $\beta(\bar{t}) \in [0, \delta]$ with $\delta = \frac{x_0(w_R - w_L)}{2w_R}$. From Proposition 4.5, for any $x \in (\beta(\bar{t}), L)$, $u(t, x^\pm) = w_R$. Then, from (2), (6) and (75),

$$\begin{aligned} m(\bar{t}) &= \int_0^L u(\bar{t}, x) dx - (L - x_0)w_R - x_0w_L \\ &\geq (L - \beta(\bar{t}))w_R - (L - x_0)w_R - x_0w_L \\ &\geq (L - \delta)w_R - (L - x_0)w_R - x_0w_L \\ &> 0 \end{aligned} \quad (77)$$

From (74), this leads to a contradiction and so $\beta(\bar{t}) > \delta$. Since β is continuous, for any $t \in [\tilde{T}_2 + 2\tau_{\max}, t_b)$, $0 < \delta \leq \beta(t)$. We deduce that for any $t \geq \tilde{T}_2 + 2\tau_{\max}$,

$$\beta(t) \in (0, L).$$

We conclude that for a.e $t \geq T_3 := \tilde{T}_2 + 2\tau_{\max}$, $\beta(t) \in (0, L)$. From Proposition 4.2, for a.e $t \geq T_3 := \tilde{T}_2 + 2\tau_{\max}$,

$$\dot{m}(t) = F_1(t, u) - F_2(t, u). \quad (78)$$

Using that for any $(t, x) \in \mathbb{R}_+ \times (0, L)$, $u(t, x) \in [0, 1]$, (2) and (6), we have for any $t \geq 0$,

$$-x_0 w_L - (L - x_0) w_R \leq m(t) \leq L - x_0 w_L - (L - x_0) w_R \quad (79)$$

From (5), Remark 4, g is a continuous function, (78) and (79), there exists a finite time $T_4 > 0$ such that for any $t \geq T_4$,

$$f(w_L) - g(m(t)) \in [0, f(\bar{u}) - \epsilon]. \quad (80)$$

Thus, from (5) and (78), we deduce that for a.e $t \geq T_4$,

$$\dot{m}(t) = -g(m(t)).$$

□

Proof of Theorem 2.2. From Proposition 4.6, for a.e $t \geq T_4$,

$$\beta(t) \in (0, L) \text{ and } \dot{m}(t) = -g(m(t)), \quad (81)$$

with $m(t) := \int_0^L (u(t, x) - w(x)) dx$. Therefore, from Remark 4,

$$\lim_{t \rightarrow \infty} m(t) = 0. \quad (82)$$

From (81) and Proposition 4.5, for a.e $t \geq T_4$,

$$\forall x \in (0, \beta(t)), \quad f(u(t, x^\pm)) = F_1(t - \tau_+(t, x), u) \text{ and } u(t, x^\pm) \in [0, \bar{u}] \quad (83)$$

and

$$\forall x \in (\beta(t), L), \quad u(t, x^\pm) = w_R. \quad (84)$$

Using that $f \in C^2([0, 1])$ is strictly concave and $\bar{u} = \operatorname{argmax}_{u \in [0, 1]} f(u)$, then $f : [0, \bar{u}] \rightarrow [0, f(\bar{u})]$ is bijective and we denote by f_+^{-1} its inverse function. Since for any $x \in (0, \bar{u})$, $f'(x) > 0$ then, using the Inverse function theorem, $f_+^{-1} \in C^2([0, f(\bar{u})])$. From (80), (83) and (84), we have, for a.e $t \geq T_4$,

$$\begin{aligned} m(t) + \int_0^T w(x) dx &= \int_0^T u(t, x) dx \\ &= \int_0^{\beta(t)} f_+^{-1}(f(w_L) - g(m(t - \tau_+(t, x)))) dx + \int_{\beta(t)}^L w_R dx \end{aligned} \quad (85)$$

From (29) and (83), for any $x \in [0, L]$,

$$t - \frac{L}{f'(\tilde{u}_{f(\bar{u}) - \epsilon})} \leq t - \tau_+(t, x) \leq t. \quad (86)$$

We introduce two functions $a(\cdot)$ and $b(\cdot)$ defined by, for any $t \geq T_4$

$$a(t) := \min_{s \in [t - \frac{L}{f'(\tilde{u}_{f(\bar{u}) - \epsilon})}, t]} g(m(s)) \text{ and } b(t) := \max_{s \in [t - \frac{L}{f'(\tilde{u}_{f(\bar{u}) - \epsilon})}, t]} g(m(s)). \quad (87)$$

From (85) and using that f_+^{-1} is an increasing function,

$$\frac{m(t) + \int_0^T w(x)dx - Lw_R}{f_+^{-1}(f(w_L) - b(t)) - w_R} \leq \beta(t) \leq \frac{m(t) + \int_0^T w(x)dx - Lw_R}{f_+^{-1}(f(w_L) - a(t)) - w_R} \quad (88)$$

We introduce two functions α_a and α_b defined by, for any $t \geq T_4 + \tau_{\max}$,

$$\alpha_a(t) = f_+^{-1}(f(w_L) - K_0 a(t)) - w_L \text{ and } \alpha_b(t) = f_+^{-1}(f(w_L) - K_0 b(t)) - w_L. \quad (89)$$

From (2), (86), (88) and (89), for any $t \geq T_4$,

$$\frac{m(t) - \alpha_b(t)}{w_L - w_R + \alpha_b(t)} \leq \beta(t) - x_0 \leq \frac{m(t) - \alpha_a(t)}{w_L - w_R + \alpha_a(t)} \quad (90)$$

From (82), (87) and (88), we have,

$$\lim_{t \rightarrow \infty} \max(\alpha_a(t), \alpha_b(t)) = 0 \quad (91)$$

From (82), (90) and (91), we deduce that

$$\lim_{t \rightarrow \infty} \beta(t) = x_0. \quad (92)$$

For any $t \geq T_4 + \tau_{\max}$,

$$\begin{aligned} \|u(t, \cdot) - w(\cdot)\|_{L^1(0,L)} &= \int_0^{\min(\beta(t), x_0)} |u(t, x) - w(x)| dx + \int_{\min(\beta(t), x_0)}^{\max(\beta(t), x_0)} |u(t, x) - w(x)| dx \\ &\quad + \int_{\max(\beta(t), x_0)}^L |u(t, x) - w(x)| dx \end{aligned}$$

From (5), (86), (87), (91), we have for any $t \geq T_4 + \tau_{\max}$,

$$\begin{aligned} \int_0^{\min(\beta(t), x_0)} |u(t, x) - w(x)| dx &= \int_0^{\min(\beta(t), x_0)} |F_1(t - \tau_+(t, x)) - w_L| dx, \\ &= \int_0^{\min(\beta(t), x_0)} |f_+^{-1}(f(w_L) - g(m(t - \tau_+(t, x)))) - w_L|, \\ &\leq \min(\beta(t), x_0) \max(|\alpha_a(t)|, |\alpha_b(t)|). \end{aligned}$$

Thus, from (91) and (92),

$$\lim_{t \rightarrow \infty} \int_0^{\min(\beta(t), x_0)} |u(t, x) - w(x)| dx = 0.$$

Moreover,

$$\int_{\max(\beta(t), x_0)}^L |u(t, x) - w(x)| dx = \int_{\max(\beta(t), x_0)}^L |w_R - w_L| dx = 0,$$

We have

$$\begin{aligned} \int_{\min(\beta(t), x_0)}^{\max(\beta(t), x_0)} |u(t, x) - w(x)| dx &\leq (\max(\beta(t), x_0) - \min(\beta(t), x_0)), \\ &\leq |\beta(t) - x_0|. \end{aligned}$$

Thus, from (92),

$$\lim_{t \rightarrow \infty} \int_{\min(\beta(t), x_0)}^{\max(\beta(t), x_0)} |u(t, x) - w(x)| dx = 0.$$

We conclude that

$$\lim_{t \rightarrow \infty} \|u(t, \cdot) - w(\cdot)\|_{L^1(0,L)} = 0 \quad (93)$$

4.3 Proofs of Corollary 2.1 and Corollary 2.2

Proof of Corollary 2.1. From Proposition 4.6, for a.e $t \geq T_4$,

$$\beta(t) \in (0, L) \text{ and } f(w_L) - g(m(t)) \in [0, f(\bar{u}) - \epsilon] \text{ and } \dot{m}(t) = -K_0 m(t), \quad (94)$$

with $m(t) := \int_0^L (u(t, x) - w(x)) dx$ and

$$T_4 := \max \left(\frac{L - x_0 w_L - (L - x_0) w_R - \frac{f(w_L)}{K_0}}{f(w_L)}, \frac{-x_0 w_L - (L - x_0) w_R + \frac{f(w_L) - f(\bar{u}) + \epsilon}{K_0}}{f(\bar{u}) - \epsilon - f(w_L)} \right) + T_3.$$

Therefore, for any $T \geq T_4$

$$m(t) = m(T_4) e^{-K_0(t-T_4)}. \quad (95)$$

From (94) and Proposition 4.5, for a.e $t \geq T_4$,

$$\forall x \in (0, \beta(t)), \quad f(u(t, x^\pm)) = F_1(t - \tau_+(t, x), u) \text{ and } u(t, x^\pm) \in [0, \bar{u}] \quad (96)$$

and

$$\forall x \in (\beta(t), L), \quad u(t, x^\pm) = w_R. \quad (97)$$

Using that $f \in C^2([0, 1])$ is strictly concave and $\bar{u} = \operatorname{argmax}_{u \in [0, 1]} f(u)$, then $f : [0, \bar{u}] \rightarrow [0, f(\bar{u})]$ is bijective and we denote by f_+^{-1} its inverse function. Since for any $x \in (0, \bar{u})$, $f'(x) > 0$ then, using the Inverse function theorem, $f_+^{-1} \in C^2([0, f(\bar{u})])$. From (96) and (97), we have, for a.e $t \geq T_4$,

$$\begin{aligned} m(t) + \int_0^T w(x) dx &= \int_0^T u(t, x) dx \\ &= \int_0^{\beta(t)} f_+^{-1}(f(w_L) - K_0 m(t - \tau_+(t, x))) dx + \int_{\beta(t)}^L w_R dx \end{aligned} \quad (98)$$

From (29) and (96), for any $x \in [0, L]$,

$$t - \frac{L}{f'(\bar{u}_{f(\bar{u}) - \epsilon})} \leq t - \tau_+(t, x) \leq t. \quad (99)$$

We introduce two functions $a(\cdot)$ and $b(\cdot)$ defined by, for any $t \geq T_4$

$$a(t) = \min_{s \in [t - \frac{L}{f'(\bar{u}_{f(\bar{u}) - \epsilon})}, t]} m(s) \text{ and } b(t) = \max_{s \in [t - \frac{L}{f'(\bar{u}_{f(\bar{u}) - \epsilon})}, t]} m(s). \quad (100)$$

From (98) and using that f_+^{-1} is an increasing function,

$$\frac{m(t) + \int_0^T w(x) dx - L w_R}{f_+^{-1}(f(w_L) - K_0 b(t)) - w_R} \leq \beta(t) \leq \frac{m(t) + \int_0^T w(x) dx - L w_R}{f_+^{-1}(f(w_L) - K_0 a(t)) - w_R} \quad (101)$$

We introduce two functions α_a and α_b defined by, for any $t \geq T_4 + \tau_{\max}$,

$$\alpha_a(t) = f_+^{-1}(f(w_L) - K_0 a(t)) - w_L \text{ and } \alpha_b(t) = f_+^{-1}(f(w_L) - K_0 b(t)) - w_L. \quad (102)$$

From (2), (99), (101) and (102), for any $t \geq T_4$,

$$\frac{m(t) - \alpha_b(t)}{w_L - w_R + \alpha_b(t)} \leq \beta(t) - x_0 \leq \frac{m(t) - \alpha_a(t)}{w_L - w_R + \alpha_a(t)} \quad (103)$$

From (95), (100) and (101), we have, for any $t \geq T_4 + \tau_{\max}$,

$$\max(|\alpha_a(t)|, |\alpha_b(t)|) \leq \max_{x \in [0, L]} [(f^{-1})'(x)] K_0 |m(T_4)| e^{-K_0 \left(t - \frac{L}{f'(\bar{u}_{f(\bar{u}) - \epsilon})} - T_4 \right)} \quad (104)$$

From (103) and (104), we deduce that there exists $C > 0$ such that, for any $t \geq T_4 + \tau_{\max}$,

$$|\beta(t) - x_0| \leq Ce^{-K_0(t - \tau_{\max} - T_4)} |m(T_4)|. \quad (105)$$

For any $t \geq T_4 + \tau_{\max}$,

$$\begin{aligned} \|u(t, \cdot) - w(\cdot)\|_{L^1(0,L)} &= \int_0^{\min(\beta(t), x_0)} |u(t, x) - w(x)| dx + \int_{\min(\beta(t), x_0)}^{\max(\beta(t), x_0)} |u(t, x) - w(x)| dx \\ &\quad + \int_{\max(\beta(t), x_0)}^L |u(t, x) - w(x)| dx \end{aligned}$$

From (5), (99), (100), (104), we have for any $t \geq T_4 + \tau_{\max}$,

$$\begin{aligned} \int_0^{\min(\beta(t), x_0)} |u(t, x) - w(x)| dx &= \int_0^{\min(\beta(t), x_0)} |F_1(t - \tau_+(t, x)) - w_L| dx, \\ &= \int_0^{\min(\beta(t), x_0)} |f_+^{-1}(f(w_L) - K_0 m(t - \tau_+(t, x))) - w_L|, \\ &\leq \min(\beta(t), x_0) \max(|\alpha_a(t)|, |\alpha_b(t)|), \\ &\leq x_0 \max_{x \in [0, L]} [(f^{-1})'(x)] K_0 |m(T_4)| e^{-K_0 \left(t - \frac{L}{f'(\bar{u} - \epsilon)} - T_4 \right)}, \end{aligned}$$

and

$$\int_{\max(\beta(t), x_0)}^L |u(t, x) - w(x)| dx = \int_{\max(\beta(t), x_0)}^L |w_R - w_R| dx = 0,$$

and from (105), there exists $C > 0$ such that

$$\begin{aligned} \int_{\min(\beta(t), x_0)}^{\max(\beta(t), x_0)} |u(t, x) - w(x)| dx &\leq (\max(\beta(t), x_0) - \min(\beta(t), x_0)), \\ &\leq |\beta(t) - x_0|, \\ &\leq Ce^{-K_0(t - \tau_{\max} - T_4)} |m(T_4)|. \end{aligned}$$

Therefore, together with (95) at time $t = T_5 := T_4 + \tau_{\max}$, there exists $C > 0$ such that for any $t \geq T_5$,

$$\|u(t, \cdot) - w(\cdot)\|_{L^1(0,L)} \leq Ce^{-K_0(t - T_5)} \|u(T_5, \cdot) - w(\cdot)\|_{L^1(0,L)}. \quad (106)$$

Adapting the proof of [1, Proposition 2.10] and using (5),

$$\int_0^L |u(T_5, x) - w(x)| dx \leq 2 \int_0^{T_5} |f(\bar{u}) - \epsilon - f(w_L)| dt + \int_{\mathbb{R}} |u_0(x) - w(x)| dx \quad (107)$$

From (106) and (107), there exists $C > 0$ such that for any $t \geq 0$,

$$\|u(t, \cdot) - w(\cdot)\|_{L^1(0,L)} \leq Ce^{-K_0 t} (\|u_0(\cdot) - w(\cdot)\|_{L^1(0,L)} + C).$$

Proof of Corollary 2.2. From Proposition 4.6, for a.e $t \geq T_4$,

$$\beta(t) \in (0, L) \text{ and } f(w_L) - g(m(t)) \in [0, f(\bar{u}) - \epsilon] \text{ and } \dot{m}(t) = -g(m(t)), \quad (108)$$

with $m(t) := \int_0^L (u(t, x) - w(x)) dx$. From Proposition [6, Proposition 1], there exists a time $T_5 := T_4 + \int_0^1 \frac{dy}{g(y)}$ such that for any $t \geq T_5$,

$$m(t) = 0. \quad (109)$$

From (108) and Proposition 4.5, for a.e $t \geq T_4$,

$$\forall x \in (0, \beta(t)), \quad f(u(t, x^\pm)) = F_1(t - \tau_+(t, x), u) \text{ and } u(t, x^\pm) \in [0, \bar{u}] \quad (110)$$

and

$$\forall x \in (\beta(t), L), \quad u(t, x^\pm) = w_R. \quad (111)$$

From (29) and (110), for any $x \in [0, L]$,

$$t - \frac{L}{f'(\tilde{u}_{f(\bar{u})-\epsilon})} \leq t - \tau_+(t, x) \leq t. \quad (112)$$

From (5), (109), (110), (111) and (112), for any $t \geq T_4 + \tau_{\max}$,

$$\forall x \in (0, \beta(t)), u(t, x^\pm) = w_L \text{ and } \forall x \in (\beta(t), L), u(t, x^\pm) = w_R \text{ and } m(t) = 0.$$

Therefore, together with (2) and (6), for a.e $x \in (0, L)$, we have $u(t, x) = w(x)$.

A Conservation of mass

Lemma A.1 *Let $t_1 < t_2$ and let $(\gamma_1, \gamma_2) \in C^{0,1}([t_1, t_2]; [0, L])^2$ satisfy $\gamma_1(t) < \gamma_2(t)$ for any $t \in (t_1, t_2)$. We have*

$$\begin{aligned} \int_{\gamma_1(t_1)}^{\gamma_2(t_1)} u(t_1, x) dx + \int_{t_1}^{t_2} F(u(t, \gamma_1(t)^+)) dt \\ = \int_{\gamma_1(t_2)}^{\gamma_2(t_2)} u(t_2, x) dx + \int_{t_1}^{t_2} F(u(t, \gamma_2(t)^-)) dt \end{aligned} \quad (113)$$

with u the \mathcal{G} -entropy solution of (1) and for every $t \in \mathbb{R}^+$, for every $\gamma \in C^{0,1}([0, T]; \mathbb{R})$,

$$F(u(t, \gamma(t)^\pm)) = f(u(t, \gamma(t)^\pm) - \dot{\gamma}(t)u(t, \gamma(t)^\pm)).$$

PROOF. It directly follows from the point 1. of Definition 1.2 and [7, Lemma 1]. \square

B Notion of generalized backward characteristics with \mathcal{G} -entropy boundary conditions

Proposition B.1 *Let u be the unique \mathcal{G} -entropy solution of (1) and we assume that there exist $c, d > 0$ such that $c < d$ and for a.e $t \in [c, d]$*

$$g \leq f(u(t, 0^+)) \leq G \text{ and } l \leq f(u(t, L^-)) \leq L \quad (114)$$

where g, G, l, L are given constants. We consider ξ a genuine characteristic on an interval $[a, b]$ such that

$$\forall t \in (a, b], \quad \xi(t) \in (0, L).$$

Then there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, b]$, $\dot{\xi}(t) = f'(v)$ and, for any $t \in (a, b)$, $u(t, \xi(t)) = v$. Moreover,

$$\xi(a) = 0 \text{ and } a > 0 \text{ implies } f'(v) > 0 \text{ and } \begin{cases} g \leq f(v) \leq G & \text{if } a \in (c, d), \\ 0 \leq f(v) \leq f(\bar{u}) - \epsilon & \text{otherwise,} \end{cases} \quad (115)$$

$$a = 0 \text{ implies } u(a, \xi(a)^+) \leq v \leq u(a, \xi(a)^-), \quad (116)$$

$$\xi(a) = L \text{ and } a > 0 \text{ implies } f'(v) < 0 \text{ and } \begin{cases} l \leq f(v) \leq L & \text{if } a \in (c, d), \\ 0 \leq f(v) \leq f(w_R) & \text{otherwise.} \end{cases} \quad (117)$$

Remark 2 Note that if the inequality in (114) is strict then the inequalities in (115), (116) and (117) are also strict. For instance, if $g \leq f(u(t, 0^+)) < G$ then (115) is replaced by $g \leq f(v) < G$.

PROOF. From Definition 1.2, the unique \mathcal{G} -entropy is a Kruzhkov entropy solution over $\mathbb{R}_+ \times (0, L)$. Thus, From [4, Theorem 10.2.3 and Theorem 11.1.1], there exists a constant $v \in [0, 1]$ such that, for any $t \in [a, b]$, $\xi(t) = f'(v)$ and, for any $t \in (a, b)$, $u(t, \xi(t)) = v$. We have three different possibilities :

Case 1: $\xi(a) = 0$ and $a > 0$). We immediately have $f'(v) > 0$. Let us fix $\delta > 0$, from Lemma A.1 with $t_1 = a$, $t_2 = a + \frac{\delta}{f'(v)}$, $\gamma_1(\cdot) = 0$ and $\gamma_2(\cdot) = \xi(\cdot)$ (see Figure 10a), we have

$$\int_a^{a + \frac{\delta}{f'(v)}} f(u(t, 0^+)) dt + 0 = \int_0^\delta u \left(a + \frac{\delta}{f'(v)}, x \right) dx + \int_a^{a + \frac{\delta}{f'(v)}} (f(v) - f'(v)v) dt. \quad (118)$$

From (8), for a.e $t > 0$, $f(u(t, 0)) \leq F_1(t, u) \leq f(\bar{u}) - \epsilon$. Thus, we have

$$\int_0^\delta u \left(a + \frac{\delta}{f'(v)}, x \right) dx + \int_a^{a + \frac{\delta}{f'(v)}} (f(v) - f'(v)v) dt \leq \frac{\delta}{f'(v)} (f(\bar{u}) - \epsilon). \quad (119)$$

Let $\bar{c} \in (a, b)$. From Lemma A.1 with $t_1 = a + \frac{\delta}{f'(v)}$, $t_2 = \bar{c}$, $\gamma_1(\cdot) = \xi(\cdot) - \delta$ and $\gamma_2(\cdot) = \xi(\cdot)$ (see Figure 10a), we have

$$\begin{aligned} \int_0^\delta u \left(a + \frac{\delta}{f'(v)}, x \right) dx + \int_{a + \frac{\delta}{f'(v)}}^{\bar{c}} (f(u(t, (\xi(t) - \delta)^+)) - \dot{\xi}(t)u(t, (\xi(t) - \delta)^+)) dt \\ = \int_{\xi(\bar{c}) - \delta}^{\xi(\bar{c})} u(\bar{c}, x) dx + \int_{a + \frac{\delta}{f'(v)}}^{\bar{c}} (f(v) - f'(v)v) dt \end{aligned}$$

Using that f is concave and for any $t \in [a, b]$, $\dot{\xi}(t) = f'(v)$, we have for any $t \in [a + \frac{\delta}{f'(v)}, \bar{c}]$

$$f(u(t, (\xi(t) - \delta)^+)) - f(v) - f'(v)(u(t, (\xi(t) - \delta)^+) - v) \leq 0.$$

Thus, we have

$$\int_0^\delta u \left(a + \frac{\delta}{f'(v)}, x \right) dx \geq \int_{\xi(\bar{c}) - \delta}^{\xi(\bar{c})} u(\bar{c}, x) dx. \quad (120)$$

From (119) and (120),

$$\int_{\xi(\bar{c}) - \delta}^{\xi(\bar{c})} u(\bar{c}, x) dx + \frac{\delta}{f'(v)} (f(v) - f'(v)v) \leq \frac{\delta}{f'(v)} (f(\bar{u}) - \epsilon).$$

Dividing by δ and taking $\delta \rightarrow 0^+$ and using that $u(\bar{c}, \xi(\bar{c})) = v$, we deduce that

$$f(v) \leq f(\bar{u}) - \epsilon.$$

If $a \in (c, d)$ then we choose $\delta > 0$ such that $[a - \frac{\delta}{f'(v)}, a + \frac{\delta}{f'(v)}] \in (c, d)$. From (114), we have for a.e $t \in [a - \frac{\delta}{f'(v)}, a + \frac{\delta}{f'(v)}]$, $g \leq f(u(t, 0^+)) \leq G$. Thus, using (118), we get

$$\int_0^\delta u \left(a + \frac{\delta}{f'(v)}, x \right) dx + \int_a^{a + \frac{\delta}{f'(v)}} (f(v) - f'(v)v) dt \leq \frac{\delta}{f'(v)} G.$$

Moreover, (120) holds. Thus, we deduce that

$$\int_{\xi(\bar{c}) - \delta}^{\xi(\bar{c})} u(\bar{c}, x) dx + \frac{\delta}{f'(v)} (f(v) - f'(v)v) \leq \frac{\delta}{f'(v)} G.$$

Dividing by δ and taking $\delta \rightarrow 0^+$ and using that $u(\bar{c}, \xi(\bar{c})) = v$, we deduce that

$$f(v) \leq G.$$

From Lemma A.1 with $t_1 = a - \frac{\delta}{f'(v)}$, $t_2 = a$, $\gamma_1(\cdot) = 0$ and $\gamma_2(\cdot) = \xi(\cdot) + \delta$ (see Figure 10b), we have

$$\int_{a - \frac{\delta}{f'(v)}}^a f(u(t, 0^+)) dt + 0 = \int_0^\delta u(a, x) dx + \int_{a - \frac{\delta}{f'(v)}}^a (f(u(t, (\xi(t) + \delta)^-)) - \dot{\xi}(t)u(t, (\xi(t) + \delta)^-)) dt \quad (121)$$

Using that f is concave and for any $t \in [a, b]$, $\dot{\xi}(t) = f'(v)$, we have for any $t \in [a - \frac{\delta}{f'(v)}, a]$

$$f(u(t, (\xi(t) + \delta)^-)) - f(v) - f'(v)(u(t, (\xi(t) + \delta)^-) - v) \leq 0.$$

Therefore,

$$\int_{a - \frac{\delta}{f'(v)}}^a f(u(t, 0^+)) dt + 0 \leq \int_0^\delta u(a, x) dx + \frac{\delta}{f'(v)}(f(v) - f'(v)v).$$

From (114) and $[a - \frac{\delta}{f'(v)}, a + \frac{\delta}{f'(v)}] \in (c, d)$, we deduce that

$$\frac{\delta}{f'(v)}g \leq \int_0^\delta u(a, x) dx + \frac{\delta}{f'(v)}(f(v) - f'(v)v). \quad (122)$$

Let $\bar{c} \in (a, b)$. From Lemma A.1 with $t_1 = a$, $t_2 = \bar{c}$, $\gamma_1(\cdot) = \xi(\cdot)$ and $\gamma_2(\cdot) = \xi(\cdot) + \delta$, we have

$$\begin{aligned} \int_0^\delta u(a, x) dx + \int_a^{\bar{c}} (f(v) - f'(v)v) dt \\ = \int_{\xi(\bar{c})}^{\xi(\bar{c}) + \delta} u(\bar{c}, x) dx + \int_a^{\bar{c}} (f(u(t, (\xi(t) + \delta)^-)) - \dot{\xi}(t)u(t, (\xi(t) + \delta)^-)) dt \end{aligned}$$

Using that f is concave and for any $t \in [a, b]$ $\dot{\xi}(t) = f'(v)$, we have for any $t \in [a, \bar{c}]$

$$f(u(t, (\xi(t) + \delta)^-)) - f(v) - f'(v)(u(t, (\xi(t) + \delta)^-) - v) \leq 0.$$

Thus, we have

$$\int_0^\delta u(a, x) dx \leq \int_{\xi(\bar{c})}^{\xi(\bar{c}) + \delta} u(\bar{c}, x) dx \quad (123)$$

From (122) and (123),

$$\frac{\delta}{f'(v)}g - \frac{\delta}{f'(v)}(f(v) - f'(v)v) \leq \int_{\xi(\bar{c})}^{\xi(\bar{c}) + \delta} u(\bar{c}, x) dx$$

Dividing by δ and taking $\delta \rightarrow 0^+$ and using that $u(\bar{c}, \xi(\bar{c})) = v$, we deduce that

$$g \leq f(v).$$

Case 2: $a = 0$. From the point 1. of Definition 1.2 and from [4, Theorem 11.1.1], (116) holds.

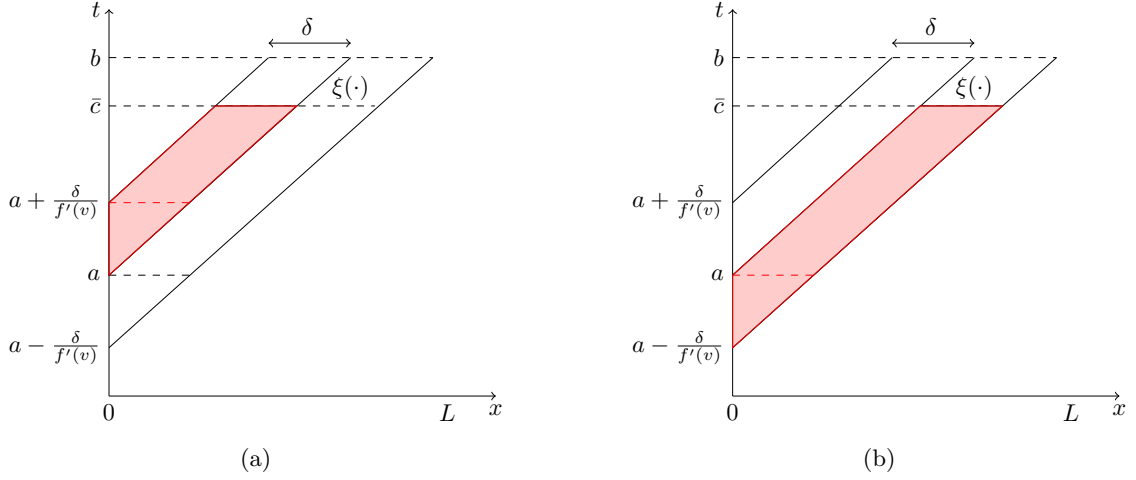


Figure 10: Plotting of a genuine characteristic ξ on an interval $[a, b]$ with $a < b$ such that $\xi(a) = 0$ with $a > 0$ and $\dot{\xi}(t) = f'(v) > 0$ with $v \in [0, \check{u}_{f(\bar{u})-\epsilon}]$.

Case 3: $\xi(a) = L$ and $a > 0$. We immediately have $f'(v) < 0$. Let us fix $\delta > 0$, from Lemma A.1 with $t_1 = a$, $t_2 = a - \frac{\delta}{f'(v)}$, $\gamma_1(\cdot) = \xi(\cdot)$ and $\gamma_2(\cdot) = L$ (see Figure 11a), we have

$$\int_a^{a - \frac{\delta}{f'(v)}} (f(v) - f'(v)v) dt + 0 = \int_a^{a - \frac{\delta}{f'(v)}} f(u(t, L^-)) dt + \int_{L-\delta}^L u\left(a - \frac{\delta}{f'(v)}, x\right) dx. \quad (124)$$

From (8), for a.e $t > 0$, $f(u(t, L)) \leq F_2(t, u) = f(w_R)$. Thus, we have

$$\int_a^{a - \frac{\delta}{f'(v)}} (f(v) - f'(v)v) dt - \int_{L-\delta}^L u\left(a - \frac{\delta}{f'(v)}, x\right) dx \leq \frac{-\delta}{f'(v)} f(w_R). \quad (125)$$

Let $\bar{c} \in (a, b)$. Using Lemma A.1 with $t_1 = a - \frac{\delta}{f'(v)}$, $t_2 = \bar{c}$, $\gamma_1(\cdot) = \xi(\cdot)$ and $\gamma_2(\cdot) = \xi(\cdot) + \delta$ (see Figure 11a), we deduce that

$$\begin{aligned} & \int_{a - \frac{\delta}{f'(v)}}^{\bar{c}} (f(v) - f'(v)v) dt + \int_{L-\delta}^L u\left(a - \frac{\delta}{f'(v)}, x\right) dx \\ &= \int_{\xi(\bar{c})}^{\xi(\bar{c}) + \delta} u(\bar{c}, x) dx + \int_{a - \frac{\delta}{f'(v)}}^{\bar{c}} (f(u(t, (\xi(t) + \delta)^-)) - \dot{\xi}(t)u(t, (\xi(t) + \delta)^-)) dt. \end{aligned}$$

Using that f is concave and for any $t \in [a - \frac{\delta}{f'(v)}, \bar{c}]$, $\dot{\xi}(t) = f'(v)$, we have for any $t \in [a - \frac{\delta}{f'(v)}, \bar{c}]$

$$f(u(t, (\xi(t) + \delta)^-)) - f(v) - f'(v)(u(t, (\xi(t) + \delta)^-) - v) \leq 0.$$

Thus, we have

$$\int_{L-\delta}^L u\left(a - \frac{\delta}{f'(v)}, x\right) dx \leq \int_{\xi(\bar{c})}^{\xi(\bar{c}) + \delta} u(\bar{c}, x) dx. \quad (126)$$

From (125) and (126),

$$\frac{-\delta}{f'(v)} (f(v) - f'(v)v) - \int_{\xi(\bar{c})}^{\xi(\bar{c}) + \delta} u(\bar{c}, x) dx \leq \frac{-\delta}{f'(v)} f(w_R).$$

Dividing by δ and taking $\delta \rightarrow 0^+$, using that $u(\bar{c}, \xi(\bar{c})) = v$, we deduce that

$$f(v) \leq f(w_R).$$

If $a \in (c, d)$ then we choose $\delta > 0$ such that $[a + \frac{\delta}{f'(v)}, a - \frac{\delta}{f'(v)}] \in (c, d)$. From (114), we have for a.e $t \in [a + \frac{\delta}{f'(v)}, a - \frac{\delta}{f'(v)}]$, $l \leq f(u(t, L^-)) \leq L$. Thus, using (124), we get

$$\int_a^{a - \frac{\delta}{f'(v)}} (f(v) - f'(v)v)dt - \int_{L-\delta}^L u\left(a - \frac{\delta}{f'(v)}, x\right) dx \leq \frac{-\delta}{f'(v)}L.$$

Moreover, (126) holds. Thus, we deduce that

$$\frac{-\delta}{f'(v)}(f(v) - f'(v)v) - \int_{\xi(\bar{c})}^{\xi(\bar{c})+\delta} u(\bar{c}, x)dx \leq \frac{-\delta}{f'(v)}L.$$

Dividing by δ and taking $\delta \rightarrow 0^+$ and using that $u(\bar{c}, \xi(\bar{c})) = v$, we deduce that

$$f(v) \leq L.$$

From Lemma A.1 with $t_1 = a + \frac{\delta}{f'(v)}$, $t_2 = a$, $\gamma_1(\cdot) = \xi(\cdot) - \delta$ and $\gamma_2(\cdot) = L$ (see Figure 11b), we have

$$\int_{a + \frac{\delta}{f'(v)}}^a (f(u(t, (\xi(t) - \delta)^+)) - \dot{\xi}(t)u(t, (\xi(t) - \delta)^+))dt = \int_{L-\delta}^L u(a, x) dx + \int_{a + \frac{\delta}{f'(v)}}^a f(u(t, L^-))dt. \quad (127)$$

Using that f is concave and for any $t \in [a, b]$, $\dot{\xi}(t) = f'(v)$, we have for any $t \in [a + \frac{\delta}{f'(v)}, a]$

$$f(u(t, (\xi(t) - \delta)^+)) - f(v) - f'(v)(u(t, (\xi(t) - \delta)^+) - v) \leq 0.$$

Therefore,

$$\int_{a + \frac{\delta}{f'(v)}}^a f(u(t, L^-))dt \leq -\frac{\delta}{f'(v)}(f(v) - f'(v)v) - \int_{L-\delta}^L u(a, x) dx.$$

From (114) and $[a + \frac{\delta}{f'(v)}, a - \frac{\delta}{f'(v)}] \in (c, d)$, we deduce that

$$-\frac{\delta}{f'(v)}l \leq -\frac{\delta}{f'(v)}(f(v) - f'(v)v) - \int_{L-\delta}^L u(a, x) dx. \quad (128)$$

Let $\bar{c} \in (a, b)$. From Lemma A.1 with $t_1 = a$, $t_2 = \bar{c}$, $\gamma_1(\cdot) = \xi(\cdot) - \delta$ and $\gamma_2(\cdot) = \xi(\cdot)$, we have

$$\begin{aligned} \int_{L-\delta}^L u(a, x)dx + \int_a^{\bar{c}} (f(u(t, (\xi(t) - \delta)^+)) - \dot{\xi}(t)u(t, (\xi(t) - \delta)^+))dt \\ = \int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x)dx + \int_a^{\bar{c}} (f(v) - f'(v)v)dt. \end{aligned}$$

Using that f is concave and for any $t \in [a, b]$ $\dot{\xi}(t) = f'(v)$, we have for any $t \in [a, \bar{c}]$

$$f(u(t, (\xi(t) - \delta)^+)) - f(v) - f'(v)(u(t, (\xi(t) - \delta)^+) - v) \leq 0.$$

Thus, we have

$$\int_{\xi(\bar{c})}^{\xi(\bar{c})-\delta} u(\bar{c}, x)dx \leq \int_{L-\delta}^L u(a, x)dx. \quad (129)$$

From (128) and (129),

$$-\frac{\delta}{f'(v)}l \leq -\frac{\delta}{f'(v)}(f(v) - f'(v)v) - \int_{\xi(\bar{c})-\delta}^{\xi(\bar{c})} u(\bar{c}, x)dx.$$

Dividing by δ and taking $\delta \rightarrow 0^+$ and using that $u(\bar{c}, \xi(\bar{c})) = v$, we deduce that

$$l \leq f(v).$$

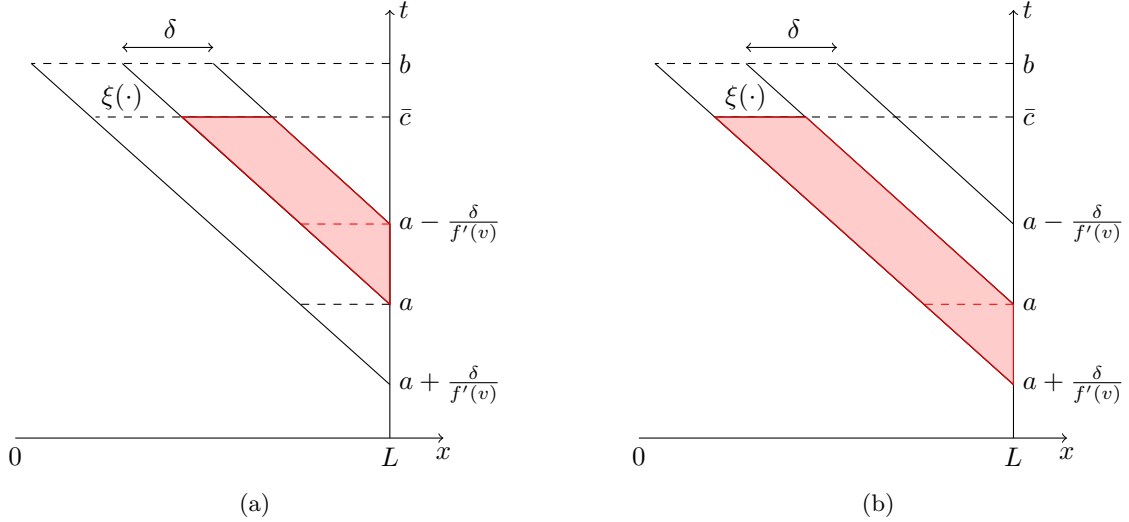


Figure 11: Plotting of a genuine characteristic ξ on an interval $[a, b]$ with $a < b$ such that $\xi(a) = L$ with $a > 0$ and $\dot{\xi}(t) = f'(v) < 0$ with $v \in [w_R, 1]$.

□

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