Couplings of Brownian motions on $SU(2, C)$ and $SL(2, R)$

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Abstract

In the subRiemannian manifolds $SU(2,\mathbb{C})$ (resp. $SL(2,\mathbb{R})$), the choice of cylindrical coordinates gives an interesting interpretation of the Brownian motion as a Brownian motion on the sphere (resp. the hyperbolic plane) together with its swept area. After a detailed account of this geometrical interpretation, we use Itô depiction of processes in a moving frame to present models of co-adapted couplings of these Brownian motions. In particular we propose a successful co-adapted coupling on $SU(2,\mathbb{C})$ inspiring ourselves with previous works on the Heisenberg group.

1 Introduction

The notion of coupling has been quite developed these last decades, first for Markov chains, then for Markov processes (see [13] for a general introduction). We begin with reminding the definition of coupling.

Definition 1.1. *Let us consider μ and ν two probability laws on $M \times M$. We call coupling of μ and ν any measure π such that μ is its first marginal distribution and ν its second one.*

In particular, coupling two Brownian motions $(\mathbb{B}_t)_t$ and $(\mathbb{B}'_t)_t$ consists in coupling their laws. The coupling of Brownian motions has been studied on $\mathbb{R}^n$ [11] and also on Riemannian manifolds (see [14]). In the two previous citations, the authors focus in particular on co-adapted couplings, which are couplings where Brownian motions are adapted to the same filtration and which are often quite easy to construct. Couplings can lead to some analysis results involving the heat semi-group just like Poincaré or Sobolev inequalities [12, 5]. In particular, there is a big interest in constructing successful couplings, that is couplings defined such that the first meeting time (or "coupling time") $\tau := inf\{t > 0 | \mathbb{B}_t = \mathbb{B}'_t\}$ of the Brownian motions is almost surely finite. A first motivation in the study of successful couplings is the study of the total variation distance between the laws of the Brownian motions:

$$d_{TV}(\mathcal{L}(\mathbb{B}_t), \mathcal{L}(\mathbb{B}'_t)) := \sup_{A \ measurable} \{\mathbb{P}(\mathbb{B}_t \in A) - \mathbb{P}(\mathbb{B}'_t \in A)\}.$$

We can obtain good estimates of this distance studying the rate of convergence of the coupling $\mathbb{P}(\tau > t)$ and the following inequality:

Proposition 1.1 (Aldous' Inequality)**.** *Let $t > 0$. Considering $(\mathbb{B}_s, \mathbb{B}'_s)_s$ a coupling of Brownian motions, we get:*

$$\mathbb{P}(\tau > t) \geq d_{TV}(\mathcal{L}(\mathbb{B}_t), \mathcal{L}(\mathbb{B}'_t)).$$

An other motivation is the study of harmonic functions. As an example, classical Liouville theorem says that every harmonic bounded function on $\mathbb{R}^2$ is constant. Wang proved in [15], the following result:

Theorem 1.2. *Let M be a Riemannian manifold with Ricci curvature bounded below. There exists a successful coupling if and only if all harmonic bounded function on M are constant.*

Our goal is to focus on the coupling of Brownian motions on subRiemannian manifolds. We first recall quickly the meaning of this structure.

Let M be a smooth connected Riemannian manifold of dimension N and $n \leq N$ an integer. For each $x \in M$, we define a vector subspace $\mathcal{H}_x$ of dimension n of T_xM, the tangent space in x . This way we define a subspace of the tangent space TM denoted by $\mathcal{H}$ and called horizontal space. We can then define the horizontal curves, that is the smooth curves $\gamma : I \subset \mathbb{R} \to M$ such that $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$: the horizontal curves are "moving" only with directions in $\mathcal{H}$. Defining a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}_x}$ on $\mathcal{H}_x$ for all $x \in M$, we obtain the length $L(\gamma)$ of the horizontal curve γ:

$$L(\gamma) := \int_I \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\mathcal{H}_{\gamma(t)}}} dt.$$

The Carnot-Caratheodory distance between x and $y \in M$ is finally defined by:

$$d_{cc}(x, y) = \inf\{L(\gamma) \mid \gamma \text{ horizontal curve between } x \text{ and } y\}.$$

Let us suppose that the horizontal space satisfies Hörmander condition, that is, provided $(X_1, ..., X_n)$ a local basis of vector fields in $\mathcal{H}$, the tangent space TM is spanned by the vectors along with all their commutators (obtained by operation with Lie brackets). Then the Carnot Caratheodory distance is finite and the subRiemannian structure is well defined. In the case of stratified Lie groups, we can choose some globally defined and left-invariant vector fields that we will denote $\bar{X}_1, ..., \bar{X}_n$. Then we can introduce the subLaplacian operator:

$$L := \frac{1}{2} \sum_1^n \bar{X_i}^2.$$

Given this operator, we can define the Brownian motion on these Lie groups as the Markov process $\mathbb{B}_t$ with infinitesimal generator L. This is the case for the Heisenberg group $\mathbb{H}$. In that particular case, the Brownian motion is usually described as a Brownian motion on $\mathbb{R}^2$ together with its Levy area, that is the area swept by the path of the Brownian motion on $\mathbb{R}^2$. Thanks to this description, the problem is reduced to a study of couplings on the Euclidean case. Kendall [9] have constructed some successful co-adapted coupling on $\mathbb{H}$. Banerjee, Gordina and Mariano [1] proposed an other successful coupling on $\mathbb{H}$ but not co-adapted. More generally, Bonnefont and Juillet [5] proved that co-adapted couplings do not stay at bounded distance on $\mathbb{H}$. In this paper we deal with couplings in two other Lie groups with subRiemannian structure: $SU(2, \mathbb{C})$ and $SL(2, \mathbb{R})$.

In the case of $SU(2, \mathbb{C})$, we can define a submersion Π_1 induced by the Hopf fibration from $SU(2, \mathbb{C})$ to the sphere S^2. Similarly, we can define a submersion Π_2 from $SL(2, \mathbb{R})$ to the hyperbolic plane $\mathbf{H}^2$. Using some good coordinate system called cylindrical coordinates, we can bring back the study of Brownian motions on these subRiemannian manifold to the study of processes on Riemannian manifolds:

Theorem 1.3. *The Brownian motion $\mathbb{B}_t$ on $SU(2, \mathbb{C})$ (resp. $SL(2, \mathbb{R})$) can be described by $\Pi_1(\mathbb{B}_t)$ (resp. $\Pi_2(\mathbb{B}_t)$) a Brownian motion on the sphere S^2 (resp. on the hyperbolic plane $\mathbf{H}^2$) together with its swept area up to a sign and modulo 4π.*

While this is explained in [3], we give a different way to prove this result in the second section of this article. We also add a geometric interpretation of the distance between two Brownian motions as it is useful in the study of couplings:

Proposition 1.4. *Let us take $\mathbb{B}_t^x$ and $\mathbb{B}_t^y$ two Brownian motions on $SU(2, \mathbb{C})$ (resp. $SL(2, \mathbb{R})$) starting from x and y respectively. We define A_t the signed swept area, up to a constant, between the two paths $(\Pi_1(\mathbb{B}_s^x))_{s \leq t}$ and $(\Pi_1(\mathbb{B}_s^y))_{s \leq t}$ (resp. $(\Pi_2(\mathbb{B}_s^x))_{s \leq t}$ and $(\Pi_2(\mathbb{B}_s^y))_{s \leq t}$). We denote $\tilde{\mathcal{A}}_t \in]-2\pi, 2\pi]$ such that $\tilde{\mathcal{A}}_t \equiv A_t \mod (4\pi)$. We also denote by R_t the distance between $\Pi_1(\mathbb{B}_t^x)$ and $\Pi_1(\mathbb{B}_t^y)$ (resp. $\Pi_2(\mathbb{B}_t^x)$ and $\Pi_2(\mathbb{B}_t^y)$). Then we have the following equivalence:*

$$d_{cc}^2(\mathbb{B}_t^x, \mathbb{B}_t^y) \sim R_t^2 + |\tilde{\mathcal{A}}_t|$$

In the third section, we describe (X_t, Y_t), a coupling of Brownian motions on S^2 (respectively on $\mathbf{H}^2$), using Itô depiction of diffusion processes in some good frames (e_1^X, e_2^X) and (e_1^Y, e_2^Y) along X and Y respectively:

$$d^\nabla X_t = dU_1(t)e_1^X(t) + dU_2(t)e_2^X(t) \text{ and } d^\nabla Y_t = dV_1(t)e_1^Y(t) + dV_2(t)e_2^Y(t) \tag{1}$$

with $(U_1(t), U_2(t))$ and $(V_1(t), V_2(t))$ some Brownian motions in $\mathbb{R}^2$ adapted to a common filtration. This description can be generalised to every Riemannian manifold M of constant curvature k with dimension 2. With this description and using calculation involving covariant derivatives we obtain general stochastic equations for R_t and A_t:

Proposition 1.5. *Let (X_t, Y_t) be a coupling of Brownian motion defined by (1) on I, a closed interval of $\mathbb{R}^+$. We suppose that $0 < R_t < i(M)$ for all t except maybe at the ends of I, $i(M)$ being the injectivity radius of M. We get*

$$dR_t = dV_1(t) - dU_1(t) + \sqrt{k}\cot(\sqrt{k}R_t)dt - \frac{\sqrt{k}}{\sin(\sqrt{k}R_t)}dU_2(t)\cdot dV_2(t)$$

and

$$dA_t = \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}(dU_2(t) + dV_2(t)) + \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(dU_2 \cdot dV_1(t) - dV_2(t)\cdot dU_1(t))$$

with $dU_i(t)\cdot dV_j(t)$ denoting the derivative of the joint quadratic variation of U_i and V_j. Moreover we have:

$$dR_t \cdot dA_t = \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}R_t}{2}\right)(dV_1(t)\cdot dU_2(t) - dU_1(t)\cdot dV_2(t)).$$

Using these formula, we describe a general way to define co-adapted couplings. We give some examples and, in particular, we describe a reflection coupling as well as a coupling keeping the Brownian motions at constant distance.

In the last section we study successful couplings. In fact, it is evident that there is no such coupling in the case of $SL(2,\mathbb{R})$: because there exists some non constant but bounded harmonic functions on $\mathbf{H}^2$, using Theorem (1.2) we can say that there is no successful coupling of Brownian motions on the hyperbolic plane, and so neither on $SL(2,\mathbb{R})$. In the case of $SU(2,\mathbb{C})$, using the couplings of section 3, we prove that we can generalise to $SU(2,\mathbb{C})$ the co-adapted coupling found on the Heisenberg group by Kendall in [9] to obtain a successful coupling:

Theorem 1.6. *There exists a co-adapted successful coupling in $SU(2)$.*

2 A definition of the Brownian motion in $SU(2,\mathbb{C})$ and $SL(2,\mathbb{R})$

Here, the first subsection deals with the definition and basic properties of the considered Lie groups along with the definitions of the cylindrical coordinates. The second and third subsection contain the calculus of the subLaplacian in the cylindrical coordinates. Finally the last subsections deal with the proofs of Theorem (1.3) and Proposition (1.4).

2.1 Definition of $SU(2,\mathbb{C})$ and $SL(2,\mathbb{R})$

Let us recall the definition of the two Lie groups we are studying and present the cylindrical coordinates.

- The $SU(2,\mathbb{C})$ group, that we will below simply denote $SU(2)$, is the group of the unitary two dimensional matrices with complex coefficients and with determinant 1. Its group law is the one induced by the multiplication of matrices. Considering the manifold structure induced by the usual topology on

the matrices group and, as the application $\left\{ \begin{array}{ccc} SU(2)\times SU(2) & \to & SU(2) \\ (A,B) & \mapsto & A^{-1}\cdot B \end{array} \right.$ is smooth, $SU(2)$ is a Lie group. Note that we have:

$$\begin{aligned} SU(2) &= \left\{ \begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix},\ z_1,\ z_2 \in \mathbb{C},\ |z_1|^2+|z_2|^2=1 \right\} \\ &= \left\{ \begin{pmatrix} \cos(\eta)e^{i\theta_1} & \sin(\eta)e^{i\theta_2} \\ -\sin(\eta)e^{-i\theta_2} & \cos(\eta)e^{-i\theta_1} \end{pmatrix},\ \eta \in \left[0,\frac{\pi}{2}\right],\ \theta_1,\ \theta_2 \in [0,2\pi] \right\}. \end{aligned}$$

The associated Lie algebra is $\mathfrak{su}(2) = \{M \in M_2(\mathbb{C}),\ \exp(tM) \in SU(2)\ \forall t>0\}$ constituted by the skew-adjoint two dimensional matrices with complex coefficients and trace 0. It is also the tangent space of $SU(2)$ at point I_2. A basis of this algebra can be constituted by the Pauli matrices. We will use these Pauli matrices up to the multiplicative coefficient $\frac{1}{2}$, we denote:

$$X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \text{ and } Z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then (X,Y,Z) is a basis of $\mathfrak{su}(2)$ and, thanks to the multiplicative coefficient, it also satisfies:

$$[X,Y] = Z\ ,\ [Y,Z] = X \text{ and } [Z,X] = Y. \tag{2}$$

This last property will be useful in the following sections.
It is important to notice that all matrices in $SU(2)$ can be written on the form:

$$\exp(\varphi(\cos(\theta)X+\sin(\theta)Y))\exp(zZ) = \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right)e^{i\frac{z}{2}} & e^{i(\theta-\frac{z}{2})}\sin\left(\frac{\varphi}{2}\right) \\ -e^{-i(\theta-\frac{z}{2})}\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)e^{-i\frac{z}{2}} \end{pmatrix}$$

(this result is trivial taking $\varphi = 2\eta$, $z = 2\theta_1$ and $\theta \equiv \theta_2 - \theta_1 \mod (2\pi)$). Thus, we have described a system of coordinates (φ,θ,z) of $SU(2)$ with $\varphi \in [0,\pi]$, $z \in]-2\pi,2\pi]$ and $\theta \in [0,2\pi[$ called cylindrical coordinates. We can also consider the coordinate system induced by $\exp(xX+yY)\exp(zZ)$ with $(x,y) \in \mathbb{R}^2$, $z \in]-2\pi,2\pi]$.
Let us remark that the cylindrical coordinates are a good way to observe the link between the sphere S^2 and $SU(2)$. Indeed, as there is a trivial diffeormophism between $SU(2)$ and S^3, using the Hopf fibration, we can define a submersion from $SU(2)$ to S^2. For example, we can define it using quaternions. For $\begin{pmatrix} z_1 & z_2 \\ -\bar{z_2} & \bar{z_1} \end{pmatrix} \in SU(2)$, with $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$, we define a unique quarternion $q = x_1 + x_2 i + y_2 j + y_1 k$. Denoting $p = k$ the north pole in S^3, we define:

$$\begin{array}{ccccl} \Pi_1 : SU(2) & \to & S^2 & & \\ q & \mapsto & qpq^* & = 2(x_1y_2+x_2y_1)i + 2(y_1y_2-x_1x_2)j + (x_1^2-x_2^2-y_2^2+y_1^2)k & . \end{array}$$

We can show ([3]) that Π_1 define a submersion. Using the cylindrical coordinates (φ,θ,z) for q, we obtain:

$$\Pi_1(q) = \sin(\varphi)\sin(\theta)i - \sin(\varphi)\cos(\theta)j + \cos(\varphi)k.$$

Thus Π_1 sends every element of $SU(2)$ described by the cylindrical coordinates (φ,θ,z) on the point of S^2 described by the spherical coordinates (φ,θ). Moreover the fiber over (φ,θ) of this projection is described by $\{(\varphi,\theta,z),\ z \in]-2\pi,2\pi]\}$.

- We now deal with the $SL(2,\mathbb{R})$ group, simply denoted $SL(2)$ afterward. That is the group of two dimensional matrices with real coefficients and with determinant 1, for the group law induced by the multiplication. As for $SU(2)$ this is a Lie group with the topology on the matrices groups. The associated Lie algebra, denoted $\mathfrak{sl}(2)$, is constituted by the two dimensional matrices with real coefficients and trace 0. It is also the tangent space of $SL(2)$ at point I_2. The following matrices, using the same

notation as for $SU(2)$, form a basis of $\mathfrak{sl}(2)$: $X = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $Y = \frac{1}{2}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ and $Z = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. This time, the relation induced by the Lie brackets are:

$$[X,Y] = Z \ , \ [Y,Z] = -X \text{ and } [Z,X] = -Y. \tag{3}$$

As before, we can proove that every element of $SL(2)$ can be written on the form :

$$\exp(\varphi(\cos(\theta)X + \sin(\theta)Y))\exp(zZ) = \begin{pmatrix}\cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right) + \sinh\left(\frac{\varphi}{2}\right)\cos\left(\theta+\frac{z}{2}\right) & -\cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right) - \sinh\left(\frac{\varphi}{2}\right)\sin\left(\theta+\frac{z}{2}\right)\\ \cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right) - \sinh\left(\frac{\varphi}{2}\right)\sin\left(\theta+\frac{z}{2}\right) & \cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right) - \sinh\left(\frac{\varphi}{2}\right)\cos\left(\theta+\frac{z}{2}\right)\end{pmatrix}$$

with $(x,y) \in \mathbb{R}^2$, $z \in]-2\pi, 2\pi]$, $\varphi > 0$ and $\theta \in [0, 2\pi[$ as seen in [6] for example. We can thus define as well a system of cylindrical coordinates on $SL(2)$.
Similarly to $SU(2)$, we can define a submersion from $SL(2)$ to $\mathbf{H}^2$, the Poincaré upper half-plane, defining:

$$\begin{aligned}\Pi_2 : SL(2) &\rightarrow \mathbf{H}^2\\ M = \begin{pmatrix}a & b\\ c & d\end{pmatrix} &\mapsto \frac{ai+b}{ci+d}\ .\end{aligned}$$

As before Π_2 define a submersion and, using the cylindrical coordinates (φ, θ, z) for M, we get:

$$\Pi_2(M) = \frac{i - \sinh(\varphi)\sin(\theta)}{\cosh(\varphi) - \sinh(\varphi)cos(\theta)}.$$

With the help of the Cartesian formula of the hyperbolic metric and trigonometric relations, we obtain that $\Pi_1(M)$ is described by the polar coordinates (φ, θ) relative to the pole i. Thus the fiber over (φ, θ) of this projection is given by $\{(\varphi, \theta, z),\ z \in]-2\pi, 2\pi]\}$.

For the two cases we denote $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the left-invariant vector fields associated to X, Y, Z, that is the vector fields induced by:

$$\frac{\partial}{\partial\epsilon}_{|\epsilon=0}\left(\exp\left(\varphi(\cos(\theta)X + \sin(\theta)Y)\right)\exp(zZ)\exp(\epsilon M)\right) \text{ for } M = X, Y, Z.$$

We can then provide a subRiemannian structure to $SU(2)$ (respectively $SL(2)$) considering $\mathcal{H} = Vect\langle\bar{X}, \bar{Y}\rangle$ as the horizontal plane and the associated Carnot-Caratheodory distance d_{cc}. The subLaplacian operator is then given by $L = \bar{X}^2 + \bar{Y}^2$. Note that, as the considered vector fields are chosen left invariant, the subRiemannian structure is in fact only determined by the horizontal plane at point I_2, $\mathcal{H}_{I_2} = Vect\langle X, Y\rangle$. It has been proven by Baudoin and Bonnefont in [2, 4, 3], that $d^2_{cc}(0, (\varphi, \theta, z))$ is equivalent to $\varphi^2 + |z|$ noting (φ, θ, z) an element of $SU(2)$ (respectively $SL(2)$) in cylindrical coordinates.

2.2 Calculation of the subLaplacian on SU(2)

To obtain the left invariant vectors $\bar{X}$ and $\bar{Y}$, we have to express in cylindrical coordinates:

$$\exp\left(\varphi(\cos(\theta)X + \sin(\theta)Y)\right)\exp(zZ)\exp(\epsilon X) \text{ and } \exp\left(\varphi(\cos(\theta)X + \sin(\theta)Y)\right)\exp(zZ)\exp(\epsilon Y).$$

We will then study the derivation of these coordinates in $\epsilon = 0$. We first deal with $SU(2)$. In [3], calculations are made directly using matrices. Here we propose another way only using the Lie brackets relations (2) and easy to adapt to $SL(2)$.

The triplet (X, Y, Z) is a basis of the Lie Algebra $\mathfrak{su}(2)$ satisfying (2). Thus it can be seen as a sort of direct orthonormal basis for the action of rotation $[\cdot,\cdot]$ which we will denote $\cdot \wedge \cdot$ as the tensor operator, in

order to simplify the calculations. In the following lemmas we will deal with some basis (u, v, w) of the Lie algebra satisfying (2) that is, such that $u \wedge v = w$, $v \wedge w = u$ and $w \wedge u = v$. We will focus on expressions of the type

$$\exp(\alpha u + \beta v + \gamma w) \exp(\epsilon(\alpha' u + \beta' v + \gamma' w))$$

for $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ real numbers and ϵ real near 0.

Lemma 2.1. *Let (u, v, w) be a basis satisfying (2). Then we get:*

$$\exp(\alpha u) \exp(\epsilon \beta v) = \exp\left(\alpha u + \epsilon \frac{\beta \alpha}{2}\left(\cot\left(\frac{\alpha}{2}\right) v + w\right)\right) + \mathcal{O}(\epsilon^2). \tag{4}$$

More generally we have:

$$\exp(\alpha u) \exp(\beta v) = \exp(\beta(\cos(\alpha) v + \sin(\alpha) w)) \exp(\alpha u). \tag{5}$$

Proof. Let us deal with (4). Using the Campbell Hausdorff formula we have

$$\exp(\alpha u) \exp(\epsilon \beta v) = \exp(\alpha u + \epsilon \beta \psi(-\mathrm{ad}_{\alpha u})(v)) + \mathcal{O}(\epsilon^2)$$

with $\psi(z) = \frac{z}{e^z - 1} = \sum_{n \geq 0} \frac{B_n}{n!} z^n$, B_n denoting the Bernoulli coefficients.

We have: $\mathrm{ad}_{(\alpha u)}(v) = \alpha u \wedge v = \alpha w$ and $\mathrm{ad}^{(2)}_{(\alpha u)}(v) = \alpha^2 u \wedge w = -\alpha^2 v$.
So we obtain:

$$\mathrm{ad}^{(k)}_{(\alpha u)}(v) = \begin{cases} (-1)^n \alpha^{2n} v & \text{if } k = 2n,\ n \geq 0; \\ (-1)^n \alpha^{2n+1} w & \text{if } k = 2n+1,\ n \geq 0. \end{cases}$$

Thus:

$$\psi\left(-\mathrm{ad}_{(\alpha u)}\right) = \sum_{n \geq 0} \frac{B_{2n}}{(2n)!} (-1)^n \alpha^{2n} v - \sum_{n \geq 0} \frac{B_{2n+1}}{(2n+1)!} (-1)^n \alpha^{2n+1} w = Re(\psi(i\alpha))\, v - Im(\psi(i\alpha))\, w.$$

To obtain the announced equality, we just need to note that:

$$\psi(i\alpha) = \frac{i\alpha}{e^{i\alpha} - 1} = \frac{i\alpha e^{-i\frac{\alpha}{2}}}{e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}}} = \frac{\alpha e^{-i\frac{\alpha}{2}}}{2\sin(\frac{\alpha}{2})} = \frac{\alpha}{2}\left(\cot\left(\frac{\alpha}{2}\right) - i\right).$$

Notice that $\exp(A)\exp(B)\exp(-A) = \exp(\mathrm{Ad}(\exp(A))(B)) = \exp\left(e^{\mathrm{ad}_A}(B)\right)$ for any matrix A, B with real or complex coefficients. Thus:

$$\begin{aligned} \exp(\alpha u) \exp(\beta v) &= \exp(\alpha u) \exp(\beta v) \exp(-\alpha u) \exp(\alpha u) \\ &= \exp\left(e^{\mathrm{ad}_{\alpha u}}(\beta v)\right) \exp(\alpha u). \end{aligned}$$

We finally obtain (5) as:

$$\begin{aligned} e^{\mathrm{ad}_{\alpha u}}(\beta v) &= \sum_{k \geq 0} \frac{1}{k!} \mathrm{ad}^{(k)}_{\alpha u}(\beta v) = \beta \left(\sum_{n \geq 0} \frac{\alpha^{2n}}{(2n)!} (-1)^n v + \sum_{n \geq 0} \frac{\alpha^{2n+1}}{(2n+1)!} (-1)^n w \right) \\ &= \beta(\cos(\alpha) v + \sin(\alpha) w). \end{aligned}$$

□

Lemma 2.2. *Let (u, v, w) be a basis of our Lie algebra satisfying (2) and t such that $u \wedge t = \rho w$; $\rho \neq 0$. Then we get:*

$$\exp(\alpha u) \exp(\epsilon t) = \exp\left(\alpha u + \epsilon\left(t + \rho\left(-v + \frac{\alpha}{2}\cot\left(\frac{\alpha}{2}\right) v + \frac{\alpha}{2} w\right)\right)\right) + \mathcal{O}(\epsilon^2). \tag{6}$$

Proof. We use exactly the same strategy as for the previous case using the Campbell Hausdorff formula. This time we have: $\mathrm{ad}_{(\alpha u)}(t) = \alpha u \wedge t = \alpha\rho w$; $\mathrm{ad}^{(2)}_{(\alpha u)}(t) = -\alpha^2\rho v$; $\mathrm{ad}^{(3)}_{(\alpha u)}(v) = -\alpha^3\rho w$.
So we obtain:

$$\mathrm{ad}^{(k)}_{(\alpha u)}(t) = \begin{cases} t & \text{if } k = 0 \\ (-1)^n\alpha^{2n}\rho v & \text{if } k = 2n, n \geq 1 \\ (-1)^n\alpha^{2n+1}\rho w & \text{if } k = 2n+1,\ n \geq 0 \end{cases}$$

and

$$\begin{aligned}\psi\left(-\mathrm{ad}_{\alpha u}\right)(t) &= t + \rho\left(\mathrm{Re}\left(\psi(i\alpha)\right) - 1\right)v - \rho\mathrm{Im}\left(\psi(i\alpha)\right)w \\ &= t + \rho\left(\frac{\alpha}{2}\cot\left(\frac{\alpha}{2}\right) - 1\right)v + \rho\frac{\alpha}{2}w.\end{aligned}$$

□

Remark. *Note that the two lemmas stay true if the coefficients α and β are complex.*

Now we can find the expressions of the invariant vectors on $SU(2)$ in cylindrical coordinates. We take (φ, θ, z) some cylindrical coordinates of an element of $SU(2)$. We will denote $x = \varphi\cos(\theta)$ and $y = \varphi\sin(\theta)$. We look for the cylindrical coordinates of $\exp(xX + yY)\exp(zZ)\exp(\epsilon(aX + bY))$, with a,$b \in \mathbb{R}$ and ϵ small.

- First, we use Lemma 2.1 with $u = Z$, $v = \frac{aX+bY}{r}$, $\alpha = z$, $\beta = r$ and $r = \sqrt{a^2+b^2}$. Note that $w = \frac{-bX+aY}{r}$. we get:

$$\begin{aligned}\exp(zZ)\exp(\epsilon(aX + bY)) &= \exp(\epsilon r(\cos(z)v + \sin(z)w))\exp(zZ) \\ &= \exp(\epsilon(\cos(z)(aX + bY) + \sin(z)(-bX + aY)))\exp(zZ) \\ &= \exp(\epsilon(A(z)X + B(z)Y))\exp(zZ) \\ &\text{with } A(z) = a\cos(z) - b\sin(z) \text{ and } B(z) = a\sin(z) + b\cos(z).\end{aligned}$$

  If $x = y = 0$, we obtain the wanted expression in cylindrical coordinates. For what will follow, we will suppose that $(x, y) \neq (0, 0)$. We recall that $\varphi = \sqrt{x^2+y^2}$.

- To obtain the expression of $\exp(xX + yY)\exp(\epsilon(A(z)X + B(z)Y))$ in cylindrical coordinates, we use Lemma 2.2 with $u = \frac{xX+yY}{\varphi}$, $\alpha = \varphi$ and $t = A(z)X + B(z)Y$. We have $u \wedge t = \frac{xB(z)-yA(z)}{\varphi}Z$ thus we take $w = Z$ and $\rho = \frac{xB(z)-yA(z)}{\varphi}$. Taking $v = w \wedge u = \frac{-yX+xY}{\varphi}$, we obtain a basis (u, v, w) satisfying (2). We get:

$$\begin{aligned}\exp\left(xX + yY\right)\exp\left(\epsilon\left(A(z)X + B(z)Y\right)\right) &= \exp\Bigg(xX + yY + \epsilon\Bigg(A(z)X + B(z)Y + \\ &\rho\Bigg(-\frac{-yX + xY}{\varphi} + \frac{\varphi}{2}\cot\left(\frac{\varphi}{2}\right)\frac{-yX + xY}{\varphi} + \frac{\varphi}{2}Z\Bigg)\Bigg)\Bigg) + \mathcal{O}(\epsilon^2) \\ &= \exp\left(\left(x + \epsilon C\right)X + \left(y + \epsilon D\right)Y + \epsilon E\cdot Z\right)\end{aligned}$$

  with $C = A(z) + \rho y\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)$, $D = B(z) + x\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)$, $E = \rho\frac{\varphi}{2}$.

- In cylindrical coordinates we can write:

$$\exp\left(\left(x + \epsilon C\right)X + \left(y + \epsilon D\right)Y + \epsilon EZ\right) = \exp\left(x(\epsilon)X + y(\epsilon)Y\right)\exp\left(\epsilon\gamma(\epsilon)Z\right).$$

We use the first equality of lemma 2.1 with $u = \frac{x(\epsilon)X+y(\epsilon)Y}{\varphi(\epsilon)}$, $\alpha = \varphi(\epsilon) = \sqrt{x(\epsilon)^2 + y(\epsilon)^2}$, $v = Z$ and $\beta = \gamma(\epsilon)$. We also have $w = \frac{y(\epsilon)X - x(\epsilon)Y}{\varphi(\epsilon)}$. Then we obtain:

$$\begin{aligned}
\exp\left(x(\epsilon)X + y(\epsilon)Y\right)\exp\left(\epsilon\gamma(\epsilon)Z\right) &= \exp\Bigg(x(\epsilon)X + y(\epsilon)Y \\
&\quad + \epsilon\frac{\gamma(\epsilon)\varphi(\epsilon)}{2}\left(\cot\left(\frac{\varphi(\epsilon)}{2}\right)Z + \frac{y(\epsilon)X - x(\epsilon)Y}{\varphi(\epsilon)}\right)\Bigg) + \mathcal{O}(\epsilon^2) \\
&= \exp\Bigg(\left(x(\epsilon) + \epsilon\frac{\gamma(\epsilon)}{2}y(\epsilon)\right)X \\
&\quad + \left(y(\epsilon) - \epsilon\frac{\gamma(\epsilon)}{2}x(\epsilon)\right)Y + \epsilon\frac{\gamma(\epsilon)\varphi(\epsilon)}{2}\cot\left(\frac{\varphi(\epsilon)}{2}\right)Z\Bigg) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

Thus we take $x(\epsilon), y(\epsilon)$ and $\gamma(\epsilon)$ such that:

$$\begin{cases}
x + \epsilon C = x(\epsilon) + \epsilon\dfrac{\gamma(\epsilon)}{2}y(\epsilon) & \text{(7a)} \\
y + \epsilon D = y(\epsilon) - \epsilon\dfrac{\gamma(\epsilon)}{2}x(\epsilon) & \text{(7b)} \\
E = \dfrac{\varphi(\epsilon)\gamma(\epsilon)}{2}\cot(\dfrac{\varphi(\epsilon)}{2}) & \text{(7c)}
\end{cases}$$

Note that $x(0) = x$, $y(0) = y$ and $\varphi(0) = \varphi$. Finally we get:

$$\begin{aligned}
\exp\left(xX + yY\right)\exp\left(zZ\right)\exp\left(\epsilon(aX + bY)\right) &= \exp\left(x(\epsilon)X + y(\epsilon)Y\right)\exp\left(\epsilon\gamma(\epsilon)Z\right)\exp\left(zZ\right) + \mathcal{O}(\epsilon^2) \\
&= \exp\left(x(\epsilon)X + y(\epsilon)Y\right)\exp\left(z(\epsilon)Z\right) + \mathcal{O}(\epsilon^2) \\
&\quad \text{with } z(\epsilon) = z + \epsilon\gamma(\epsilon) \\
&= \exp\left(\varphi(\epsilon)(\cos(\theta(\epsilon))X + \sin(\theta(\epsilon))Y\right)\exp\left(z(\epsilon)Z\right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

- To obtain the values of the left invariant vector, we now have to study $\theta'(0)$, $\varphi'(0)$ and $z'(0)$.
  - Using (7c), we directly have $z'(0) = \gamma(0) = \frac{2}{\varphi}\tan\left(\frac{\varphi}{2}\right)E = \rho \times \tan\left(\frac{\varphi}{2}\right)$ with

$$\begin{aligned}
\rho = \frac{xB(z) - yA(z)}{\varphi} &= \cos(\theta)B(z) - \sin(\theta)A(z) \\
&= \cos(\theta)\left(a\sin(z) + b\cos(z)\right) - \sin(\theta)\left(a\cos(z) - b\sin(z)\right) \\
&= a\left(\cos(\theta)\sin(z) - \sin(\theta)\cos(z)\right) + b\left(\cos(\theta)\cos(z) + \sin(\theta)\sin(z)\right) \\
&= a\sin(z-\theta) + b\cos(z-\theta).
\end{aligned}$$

    Thus $z'(0) = \tan\left(\frac{\varphi}{2}\right)\left(a\sin(z-\theta) + b\cos(z-\theta)\right)$.
  - Then, summing the squares of (7a) and (7b), we get $\varphi(\epsilon)^2 = x(\epsilon)^2 + y(\epsilon)^2 = x^2 + y^2 + 2\epsilon\left(xC + yD\right) + \mathcal{O}(\epsilon^2)$, hence $2\varphi(0)\varphi'(0) = 2\left(xC + yD\right)$ and

$$\begin{aligned}
\varphi'(0) = \frac{xC + yD}{\varphi} &= \cos(\theta)C + \sin(\theta)D \\
&= \cos(\theta)\left(A(z) + \rho y\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right) + \sin(\theta)\left(B(z) + x\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right) \\
&= \cos(\theta)A(z) + \sin(\theta)B(z) + \rho\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\left(y\cos(\theta) - x\sin(\theta)\right) \\
&= \cos(\theta)A(z) + \sin(\theta)B(z) \\
&= \cos(\theta)\left(a\cos(z) - b\sin(z)\right) + \sin(\theta)\left(a\sin(z) + b\cos(z)\right) \\
&= a\cos(\theta - z) + b\sin(\theta - z).
\end{aligned}$$

– Now note that $\tan(\theta(\epsilon)) = \frac{y(\epsilon)}{x(\epsilon)}$.
Thus $\frac{\theta'(\epsilon)}{\cos^2(\theta(\epsilon))} = \frac{y'(\epsilon)x(\epsilon) - x'(\epsilon)y(\epsilon)}{x(\epsilon)^2}$ and $\theta'(0) = \cos^2(\theta)\frac{y'(0)x - x'(0)y}{r^2\cos^2(\theta)} = \frac{y'(0)x - x'(0)y}{r^2}$.
Using (7a) and then (7b), we get:

$$x'(0) = C - \frac{\gamma(0)}{2}y(0) = C - \frac{\rho}{2}\tan(\frac{\varphi}{2})y$$
$$y'(0) = D + \frac{\gamma(0)}{2}x(0) = D + \frac{\rho}{2}\tan(\frac{\varphi}{2})x.$$

We obtain:

$$\begin{aligned}\theta'(0) &= \frac{\left(D + \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right)x\right)x - \left(C - \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right)y\right)y}{\varphi^2} = \frac{Dx - Cy}{\varphi^2} + \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right)\\ &= \frac{D\cos(\theta) - C\sin(\theta)}{\varphi} + \frac{\rho}{2}\tan\left(\frac{\varphi}{2}\right).\end{aligned}$$

We have:

$$\begin{aligned}D\cos(\theta) - C\sin(\theta) &= \cos(\theta)\left(B(z) + x\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right)\\ &\quad - \sin(\theta)\left(A(z) + \rho y\left(\frac{1}{\varphi} - \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\right)\\ &= \cos(\theta)B(z) - \sin(\theta)A(z) + \rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)(x\cos(\theta) + y\sin(\theta))\\ &= \rho + \varphi\rho\left(-\frac{1}{\varphi} + \frac{1}{2}\cot\left(\frac{\varphi}{2}\right)\right)\\ &= \frac{\varphi\rho}{2}\cot\left(\frac{\varphi}{2}\right).\end{aligned}$$

Hence: $\theta'(0) = \frac{\rho}{2}\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right) = \frac{a\sin(z-\theta) + b\cos(z-\theta)}{2}\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)$.

- Finally, taking $(a,b) = (1,0)$, we get $\bar{X} = \begin{pmatrix}\cos(\theta - z)\\ \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\\ \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\end{pmatrix}$ and, taking $(a,b) = (0,1)$, we get $\bar{Y} = \begin{pmatrix}\sin(\theta - z)\\ \frac{1}{2}\cos(\theta - z)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\\ \tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\end{pmatrix}$.

We can now calculate the subLaplacian operator $L = \frac{1}{2}\left(\bar{X}^2 + \bar{Y}^2\right)$ (also made in [3]).

$$\begin{aligned}\bar{X}^2 &= \cos(\theta - z)\partial_\varphi\left[\cos(\theta - z)\partial_\varphi + \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta + \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\right]\\ &+ \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta\Bigg[\cos(\theta - z)\partial_\varphi\\ &+ \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta + \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\Bigg]\\ &+ \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\left[\cos(\theta - z)\partial_\varphi + \frac{1}{2}\sin(z-\theta)\left(\cot\left(\frac{\varphi}{2}\right) + \tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta + \tan\left(\frac{\varphi}{2}\right)\sin(z-\theta)\partial_z\right]\end{aligned}$$

and

$$\begin{aligned}\bar{Y}^2 &= \sin(\theta-z)\partial_\varphi\left[\sin(\theta-z)\partial_\varphi+\frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right)+\tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta+\tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\right]\\&+\frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right)+\tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta\left[\sin(\theta-z)\partial_\varphi\right.\\&\left.+\frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right)+\tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta+\tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\right]\\&+\tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\left[\sin(\theta-z)\partial_\varphi+\frac{1}{2}\cos(\theta-z)\left(\cot\left(\frac{\varphi}{2}\right)+\tan\left(\frac{\varphi}{2}\right)\right)\partial_\theta+\tan\left(\frac{\varphi}{2}\right)\cos(z-\theta)\partial_z\right].\end{aligned}$$

We obtain:

$$\begin{aligned}\bar{X}^2+\bar{Y}^2 &= \partial^2_{\varphi,\varphi}+\frac{1}{4}\left(\cot\left(\frac{\varphi}{2}\right)+\tan\left(\frac{\varphi}{2}\right)\right)^2\partial^2_{\theta,\theta}+\left(1+\tan^2\left(\frac{\varphi}{2}\right)\right)\partial^2_{\theta,z}\\&+\frac{1}{2}\left(\cot\left(\frac{\varphi}{2}\right)-\tan\left(\frac{\varphi}{2}\right)\right)\partial_\varphi+\tan^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z}\\&=\partial^2_{\varphi,\varphi}+\frac{1}{\sin^2(\varphi)}\partial^2_{\theta,\theta}+\tan^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z}+\frac{1}{\cos^2\left(\frac{\varphi}{2}\right)}\partial^2_{\theta,z}+\cot(\varphi)\partial_\varphi.\end{aligned}$$

2.3 Calculation of the subLaplacian for $SL(2)$

We now consider the case of $SL(2)$. In this case, (X,Y,Z) are satisfying (3). In order to use our previous results, we define $(\tilde{X},\tilde{Y},\tilde{Z}) := (iX,iY,-Z)$. This way we get $(\tilde{X},\tilde{Y},\tilde{Z})$ a basis of $\mathfrak{su}(2)$ satisfying (2). We have to study

$$\exp\left(xX+yY\right)\exp\left(zZ\right)\exp\left(\epsilon\left(aX+bY\right)\right)=\exp\left(-ix\tilde{X}-iy\tilde{Y}\right)\exp\left(-z\tilde{Z}\right)\exp\left(\epsilon\left(-ia\tilde{X}-ib\tilde{Y}\right)\right).$$

In the previous part, as said in the note concerning the two lemmas, the calculation can be made using complex coefficients. Then we obtain:

$$\begin{aligned}\exp(xX+yY)\exp(zZ)\exp(\epsilon(aX+bY)) &= \exp\left(\varphi(\epsilon)\left(\cos\left(\theta(\epsilon)\right)\tilde{X}+\sin\left(\theta(\epsilon)\right)\tilde{Y}\right)\right)\exp\left(z(\epsilon)\tilde{Z}\right)+\mathcal{O}(\epsilon^2)\\&=\exp\left(i\varphi(\epsilon)\left(\cos\left(\theta(\epsilon)\right)\tilde{X}+\sin\left(\theta(\epsilon)\right)\tilde{Y}\right)\right)\exp\left(-z(\epsilon)\tilde{Z}\right)+\mathcal{O}(\epsilon^2),\end{aligned}$$

with $\varphi(0)=-i\varphi$, $\theta(0)=\theta$ and $z(0)=-z$.

We get:

$$\frac{d}{d\epsilon}\left(i\varphi(\epsilon)\right)_{|\epsilon=0}=i\varphi'(0)=i\left(-ia\cos(\theta-(-z))-ib\sin(\theta-(-z))\right)=a\cos(\theta+z)+b\sin(\theta+z);$$

$$\begin{aligned}\frac{d}{d\epsilon}\left(-z(\epsilon)\right)_{|\epsilon=0} &= -z'(0)=-\tan\left(\frac{-i\varphi}{2}\right)\left(-ia\sin(-z-\theta)-ib\cos(-z-\theta)\right)\\&=-\tanh\left(\frac{\varphi}{2}\right)\left(a\sin(z+\theta)-b\cos(\theta+z)\right);\end{aligned}$$

$$\begin{aligned}\theta'(0) &= \frac{-ia\sin(-z-\theta)-ib\cos(\theta-(-z))}{2}\left(\cot\left(\frac{-i\varphi}{2}\right)+\tan\left(\frac{-i\varphi}{2}\right)\right)\\&=-i\frac{-a\sin(z+\theta)+b\cos(\theta+z)}{2}\left(i\coth\left(\frac{\varphi}{2}\right)-i\tanh\left(\frac{\varphi}{2}\right)\right)\\&=\frac{-a\sin(z+\theta)+b\cos(\theta+z)}{2}\left(\coth\left(\frac{\varphi}{2}\right)-\tanh\left(\frac{\varphi}{2}\right)\right).\end{aligned}$$

Thus we obtain $\bar{X} = \begin{pmatrix} \cos(\theta+z) \\ -\frac{\sin(z+\theta)}{2}\left(\coth\left(\frac{\varphi}{2}\right) - \tanh\left(\frac{\varphi}{2}\right)\right) \\ -\tanh\left(\frac{\varphi}{2}\right)\sin(z+\theta) \end{pmatrix}$ and $\bar{Y} = \begin{pmatrix} \sin(\theta+z) \\ \frac{\cos(\theta+z)}{2}\left(\coth\left(\frac{\varphi}{2}\right) - \tanh\left(\frac{\varphi}{2}\right)\right) \\ \tanh\left(\frac{\varphi}{2}\right)\cos(z+\theta) \end{pmatrix}$.

As before, we can calculate the subLaplacian operator

$$\begin{aligned} L = \frac{1}{2}(\bar{X}^2 + \bar{Y}^2) &= \partial^2_{\varphi,\varphi} + \frac{1}{4}\left(\coth\left(\frac{\varphi}{2}\right) - \tanh\left(\frac{\varphi}{2}\right)\right)^2 \partial^2_{\theta,\theta} + \left(1 - \tanh^2\left(\frac{\varphi}{2}\right)\right)\partial^2_{\theta,z} \\ &\quad + \frac{1}{2}\left(\coth\left(\frac{\varphi}{2}\right) + \tanh\left(\frac{\varphi}{2}\right)\right)\partial_\varphi + \tanh^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z} \\ &= \partial^2_{\varphi,\varphi} + \frac{1}{\sinh^2(\varphi)}\partial^2_{\theta,\theta} + \tanh^2\left(\frac{\varphi}{2}\right)\partial^2_{z,z} + \frac{1}{\cosh^2\left(\frac{\varphi}{2}\right)}\partial^2_{\theta,z} + \coth(\varphi)\partial_\varphi. \end{aligned}$$

2.4 Geometrical interpretation of the Brownian motion in $SU(2)$

Using the subLaplacian operator, we can now define the Brownian motion $\mathbb{B}_t$ in $SU(2)$. Using cylindrical coordinates, there exists φ_t, θ_t and z_t three real diffusion processes, such that:

$$\mathbb{B}_t = \exp(\varphi_t(\cos(\theta_t)X + \sin(\theta_t)Y))\exp(z_t Z) \text{ and } \begin{cases} \langle d\varphi_t, d\varphi_t\rangle = dt \\ \langle d\theta_t, d\theta_t\rangle = \frac{1}{\sin^2(\varphi_t)}dt \\ \langle dz_t, dz_t\rangle = \tan^2\left(\frac{\varphi_t}{2}\right)dt \\ \langle d\theta_t, dz_t\rangle = \frac{1}{2\cos^2\left(\frac{\varphi_t}{2}\right)}dt \\ \langle d\varphi_t, d\theta_t\rangle = \langle d\varphi_t, dz_t\rangle = 0 \\ Drift(d\varphi_t) = cot(\varphi_t)dt \\ Drift(d\theta_t) = 0 \\ Drift(dz_t) = 0. \end{cases}$$

Then, taking B^1_t and B^2_t two real Brownian motions, we get:

$$\begin{cases} d\varphi_t = dB^1_t + \cot(\varphi_t)dt \\ d\theta_t = \frac{1}{\sin(\varphi_t)}dB^2_t \\ dz_t = \tan\left(\frac{\varphi_t}{2}\right)dB^2_t \end{cases}.$$

Using the first two equations, we get the infinitesimal generator of the diffusion (φ_t, θ_t):

$$\frac{1}{2}\left(\partial^2_{\varphi,\varphi} + \frac{1}{\sin^2(\varphi)}\partial^2_{\theta,\theta} + \cot(\varphi)\partial_\varphi\right).$$

That is exactly the Laplace Beltrami operator on the sphere S^2 in spherical coordinates (φ, θ). Then, (φ_t, θ_t) are the polar coordinates of the Brownian motion $\Pi_1(\mathbb{B}_t)$ on S^2.
If we denote by A_t the area swept by the path of (φ_t, θ_t) on S^2 with respect with the north pole defined by the polar coordinates, that is the quantity $A_t = \int_0^t (1 - \cos(\varphi_s))d\theta_s$, we get:

$$dA_t = \frac{1-\cos(\varphi_t)}{\sin(\varphi_t)}dB^2_t = \tan\left(\frac{\varphi_t}{2}\right)dB^2_t = dz_t.$$

Remark. *Note that this swept area is a signed value. For example, if θ_t is decreasing for all t, then we get $A_t < 0$.*

This prove Theorem (1.3) for $SU(2)$. Let us now deal with the distance between two Brownian motions on $SU(2)$ by proving Proposition (1.4).

Proof of Proposition (1.4) for $SU(2)$. We first need a geometric estimation of $d_{cc}(x,y)$ for $x,y \in SU(2)$. We denote (φ^x,θ^x,z^x) (resp. (φ^y,θ^y,z^y)) the cylindrical coordinates of x (resp. y). As said before, we have estimations for $d_{cc}(0,(\varphi,\theta,z))$, with (φ,θ,z) the cylindrical coordinates of $x^{-1}y$. Thus, we need to compute (φ,θ,z). We have:

$$x^{-1}y = \big(\exp\big(\varphi^x(\cos(\theta^x)X+\sin(\theta^x)Y)\big)\exp(z^xZ)\big)^{-1}\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ).$$

Considering the matrix form of X and Y, we get:

$$\begin{aligned}\exp(\varphi(\cos(\theta)X+\sin(\theta)Y))^{-1} &= \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) & e^{i\theta}\sin\left(\frac{\varphi}{2}\right)\\ -e^{-i\theta}\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)\end{pmatrix}^{-1} = \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) & -e^{i\theta}\sin\left(\frac{\varphi}{2}\right)\\ e^{-i\theta}\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)\end{pmatrix}\\ &= \exp(-\varphi(\cos(\theta)X+\sin(\theta)Y)).\end{aligned}$$

Then, using lemma 2.1 as in the first part, we get for $\varphi\in[0,\pi]$, $\theta\in[0,2\pi[$ and $z\in]-2\pi,2\pi]$

$$\begin{aligned}\exp(zZ)\exp(\varphi(\cos(\theta)X+\sin(\theta)Y)) &= \exp\Big(\varphi\big((\cos(\theta)\cos(z)-\sin(\theta)\sin(z))X\\ &\quad+(\cos(\theta)\sin(z)+\sin(\theta)\cos(z))Y\big)\Big)\exp(zZ)\\ &= \exp\big(\varphi(\cos(z+\theta)X+\sin(\theta+z)Y)\big)\exp(zZ).\end{aligned}$$

Thus:

$$\begin{aligned}x^{-1}y &= \big(\exp\big(\varphi^x(\cos(\theta^x)X+\sin(\theta^x)Y)\big)\exp(z^xZ)\big)^{-1}\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ)\\ &= \exp(-z^xZ)\exp\big(-\varphi^x(\cos(\theta^x)X+\sin(\theta^x)Y)\big)\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ)\\ &= \exp\big(-\varphi^x(\cos(\theta^x-z^x)X+\sin(\theta^x-z^x)Y)\big)\exp(-z^xZ)\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ)\\ &= \exp\big(-\varphi^x(\cos(\theta^x-z^x)X+\sin(\theta^x-z^x)Y)\big)\\ &\times\exp\big(\varphi^y(\cos(\theta^y-z^x)X+\sin(\theta^y-z^x)Y)\big)\exp\big((z^y-z^x)Z\big)\\ &= \exp\big(\varphi(\cos(\theta)X+\sin(\theta)Y)\big)\exp(zZ)\exp\big((z^y-z^x)Z\big)\end{aligned}$$

with $\exp\big(\varphi(\cos(\theta)X+\sin(\theta)Y)\big)\exp(zZ)$ equal to the matricial product:

$$\begin{pmatrix}\cos\left(-\frac{\varphi^x}{2}\right) & e^{i(\theta^x-z^x)}\sin\left(-\frac{\varphi^x}{2}\right)\\ -e^{-i(\theta^x-z^x)}\sin\left(-\frac{\varphi^x}{2}\right) & \cos\left(-\frac{\varphi^x}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\varphi^y}{2}\right) & e^{i(\theta^y-z^x)}\sin\left(\frac{\varphi^y}{2}\right)\\ -e^{-i(\theta^y-z^x)}\sin\left(\frac{\varphi^y}{2}\right) & \cos\left(\frac{\varphi^y}{2}\right)\end{pmatrix}.$$

Then we get:

$$\begin{cases}\cos\left(\frac{\varphi}{2}\right)e^{i\frac{z}{2}} = \cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)+e^{i(\theta^x-\theta^y)}\sin\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right) & \text{(8a)}\\ e^{i\left(\theta-\frac{z}{2}\right)}\sin\left(\frac{\varphi}{2}\right) = \cos\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right)e^{i(\theta^y-z^x)}-\sin\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)e^{i(\theta^x-z^x)}. & \text{(8b)}\end{cases}$$

We can now try to get a geometric interpretation on the values φ and z.

- We begin with φ: we are going to prove that φ is the usual distance on S^2 between $\Pi_1(x)$ and $\Pi_1(y)$. Let us denote ρ this distance.
  Indeed, we note that:

$$\begin{aligned}\cos(\rho(\Pi_1(x),\Pi_1(y))) &= <\Pi_1(x),\Pi_1(y)>_{\mathbb{R}^3}\\ &= \sin(\theta^x)\sin(\theta^y)\sin(\varphi^x)\sin(\varphi^y)+\cos(\theta^x)\cos(\theta^y)\sin(\varphi^x)\sin(\varphi^y)+\cos(\varphi^x)\cos(\varphi^y)\\ &= \cos(\varphi^x)\cos(\varphi^y)+\sin(\varphi^x)\sin(\varphi^y)\cos(\theta^x-\theta^y).\end{aligned}$$

From (8a), we get

$$\begin{aligned}
\cos^2\left(\frac{\varphi}{2}\right) &= \left(\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right) + \cos\left(\theta^x-\theta^y\right)\sin\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right)\right)^2 \\
&+ \sin^2(\theta^x-\theta^y)\sin^2\left(\frac{\varphi^x}{2}\right)\sin^2\left(\frac{\varphi^y}{2}\right) \\
&= \cos^2\left(\frac{\varphi^x}{2}\right)\cos^2\left(\frac{\varphi^y}{2}\right) + \sin^2\left(\frac{\varphi^x}{2}\right)\sin^2\left(\frac{\varphi^y}{2}\right) \\
&+ 2\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)\cos(\theta^x-\theta^y)\sin\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right) \\
&= \frac{(1+\cos(\varphi^x))(1+\cos(\varphi^y))}{4} + \frac{(1-\cos(\varphi^x))(1-\cos(\varphi^y))}{4} + \frac{\sin(\varphi^x)\sin(\varphi^y)\cos(\theta^x-\theta^y)}{2} \\
&= \frac{1+\cos(\varphi^x)\cos(\varphi^y)}{2} + \frac{\sin(\varphi^x)\sin(\varphi^y)\cos(\theta^x-\theta^y)}{2} \\
&= \frac{1+\cos(\rho(\Pi_1(x),\Pi_1(y)))}{2} = \cos^2\left(\frac{\rho(\Pi_1(x),\Pi_2(y))}{2}\right).
\end{aligned}$$

We obtain the announced result as φ and $\rho(\Pi_1(x),\Pi_1(y))$ live in $[0,\pi]$.
Moreover, we get that

$$\cos(\varphi) = \cos(\varphi^x)\cos(\varphi^y) + \sin(\varphi^x)\sin(\varphi^y)\cos(\theta^x-\theta^y). \tag{9}$$

- We will now deal with z. Let us denote $\mathcal{A}$ the area of the spherical triangle with vertices $\Pi_1(x)$, $\Pi_1(y)$ and N, the north pole. The lengths of the opposite sides are respectively φ^y, φ^x and R . Using an equivalent of the Heron formula for spherical triangle (see[8]), we have:

$$\cos\left(\frac{\mathcal{A}}{2}\right) = \frac{1}{4\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)}\left(1+\cos(\varphi^x)+\cos(\varphi^y)+\cos(\varphi)\right).$$

Because of (8a) and (9), we get:

$$\begin{aligned}
\cos\left(\frac{z}{2}\right) &= \frac{1}{\cos\left(\frac{\varphi}{2}\right)}\left(\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right) + \cos\left(\theta^x-\theta^y\right)\sin\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right)\right) \\
&= \frac{1}{\cos\left(\frac{\varphi}{2}\right)}\left(\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right) + \frac{\cos(\varphi)-\cos(\varphi^x)\cos(\varphi^y)}{\sin(\varphi^x)\sin(\varphi^y)}\sin\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right)\right) \\
&= \frac{1}{4\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)}\left(4\cos^2\left(\frac{\varphi^x}{2}\right)\cos^2\left(\frac{\varphi^y}{2}\right) + \cos(\varphi) - \cos(\varphi^x)\cos(\varphi^y)\right) \\
&= \frac{1}{4\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)}\left(\left(1+\cos(\varphi^x)\right)\left(1+\cos(\varphi^y)\right) + \cos(\varphi) - \cos(\varphi^x)\cos(\varphi^y)\right) \\
&= \frac{1}{4\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi^x}{2}\right)\cos\left(\frac{\varphi^y}{2}\right)}\left(1+\cos(\varphi^x)+\cos(\varphi^y)+\cos(\varphi)\right).
\end{aligned}$$

Thus we get $|z| \equiv \mathcal{A} \mod (4\pi)$.

Moreover, still using (8a), we get that $\sin\left(\frac{z}{2}\right) = \sin(\theta^x-\theta^y)\frac{\sin\left(\frac{\varphi^x}{2}\right)\sin\left(\frac{\varphi^y}{2}\right)}{\cos\left(\frac{\varphi}{2}\right)}$, and so $z<0$ if and only if $\theta^x < \theta^y$.

Finally, $d_{cc}^2(x,y) \sim \rho^2(\Pi_1(x),\Pi_1(y)) + |\tilde{\mathcal{A}}|$, with $\tilde{\mathcal{A}} \equiv z_y - z_x + \text{sign}(\theta_x-\theta_y)\mathcal{A} \mod (4\pi)$, $\tilde{\mathcal{A}} \in]-2\pi, 2\pi]$. Now, considering our Brownian motions, we get $d_{cc}(\mathbb{B}_t^x, \mathbb{B}_t^y) \sim \rho(\Pi_1(\mathbb{B}_t^x), \Pi_1(\mathbb{B}_t^y)) + |\tilde{\mathcal{A}}_t|$ with $\tilde{\mathcal{A}}_t \equiv z_t^y - z_t^x + \text{sign}(\theta_t^x-\theta_t^y)\mathcal{A}_t \mod (4\pi)$, $\tilde{\mathcal{A}}_t \in]-2\pi, 2\pi]$ and with $\mathcal{A}_t$ the area of the spherical triangle of vertices $\Pi_1(\mathbb{B}_t^x), \Pi_1(\mathbb{B}_t^y)$ and N. We can also give a geometric interpretation of the quantity $z_t^y - z_t^x + \text{sign}(\theta_t^x-\theta_t^y)\mathcal{A}_t$. In fact $z_t^y + \text{sign}(\theta_t^x-\theta_t^y)\mathcal{A}_t - z_t^x - \text{sign}(\theta_0^x-\theta_0^y)\mathcal{A}_0$ is the signed swept area of a loop starting from y, following:

- the path $s \mapsto \Pi_1(\mathbb{B}^y_s), s \leq t$ from $\Pi_1(y)$ to $\Pi_1(\mathbb{B}^y_t)$;
- the geodesic joining $\Pi_1(\mathbb{B}^y_t)$ to $\Pi_1(\mathbb{B}^x_t)$;
- the path $s \mapsto \Pi_1(\mathbb{B}^x_s), s \leq t$ joining $\Pi_1(\mathbb{B}^x_t)$ to $\Pi_1(x)$;
- finally the geodesic joining $\Pi_1(x)$ to $\Pi_1(y)$.

Thus, up to a constant, the quantity $z^y_t - z^x_t + \text{sign}(\theta^x_t - \theta^y_t)\mathcal{A}_t$ is this signed area with sign changing when the paths of $\Pi_1(\mathbb{B}^y_t)_t$ and $\Pi_1(\mathbb{B}^x_t)_t$ are crossing. □

2.5 Geometrical interpretation of the Brownian motion in $SL(2,\mathbb{R})$

As for $SU(2)$, the Brownian motion $\mathbb{B}_t$ in $SL(2,\mathbb{R})$ can been characterized by three real diffusion processes φ_t, θ_t and z_t satisfying:

$$\mathbb{B}_t = \exp(\varphi_t(\cos(\theta_t)\bar{X} + \sin(\theta_t)\bar{Y}))\exp(z_t\bar{Z}) \text{ and } \begin{cases} \langle d\varphi_t, d\varphi_t\rangle = dt \\ \langle d\theta_t, d\theta_t\rangle = \frac{1}{\sinh^2(\varphi_t)}dt \\ \langle dz_t, dz_t\rangle = \tanh^2\left(\frac{\varphi_t}{2}\right)dt \\ \langle d\theta_t, dz_t\rangle = \frac{1}{2\cosh^2\left(\frac{\varphi_t}{2}\right)}dt \\ \langle d\varphi_t, d\theta_t\rangle = \langle d\varphi_t, dz_t\rangle = 0 \\ Drift(d\varphi_t) = coth(\varphi_t)dt \\ Drift(d\theta_t) = 0 \\ Drift(dz_t) = 0. \end{cases}$$

As before, taking B^1_t and B^2_t two real independent Brownian motions, we get: $\begin{cases} d\varphi_t = dB^1_t + \coth(\varphi_t)dt \\ d\theta_t = \frac{1}{\sinh(\varphi_t)}dB^2_t \\ dz_t = \tanh\left(\frac{\varphi_t}{2}\right)dB^2_t \end{cases}$.

We obtain an analogous interpretation as for $SU(2)$. We remark that the diffusion (φ_t, θ_t) admits the Laplace Beltrami operator on the hyperbolic plane $\mathbf{H}^2$ in the polar coordinates (φ, θ) as infinitesimal generator, that is:

$$\frac{1}{2}\left(\partial^2_{\varphi,\varphi} + \frac{1}{\sinh^2(\varphi)}\partial^2_{\theta,\theta} + \coth(\varphi)\partial_\varphi\right).$$

We also use the fact that the area A_t swept by the path of (φ_t, θ_t) on $\mathbf{H}^2$ with respect of the pole defined by the polar coordinates, is $A_t = \int_0^t(\cosh(\varphi_s) - 1)d\theta_s$. We have indeed:

$$dA_t = \frac{\cosh(\varphi_t) - 1}{\sinh(\varphi_t)}dB^2_t = \tanh\left(\frac{\varphi_t}{2}\right)dB^2_t = dz_t$$

As before this swept area is a signed value depending of the sign of $d\theta_t$. This prove Theorem (1.3) for $SL(2)$.

We can now prove the geometric interpretation for the distance between two Brownian motions on $SL(2)$ contained in Proposition (1.4).

Proof of proposition (1.4) for $SL(2)$. We proceed as for $SU(2)$. We denote $(\varphi^x, \theta^x, z^x)$ (resp. $(\varphi^y, \theta^y, z^y)$)

the cylindrical coordinates of x (resp. y). This time we get

$$\begin{aligned}\exp(\varphi(\cos(\theta)X+\sin(\theta)Y))^{-1} &= \begin{pmatrix}\cosh\left(\frac{\varphi}{2}\right)+\sinh\left(\frac{\varphi}{2}\right)\cos(\theta) & -\sinh\left(\frac{\varphi}{2}\right)\sin(\theta)\\ -\sinh\left(\frac{\varphi}{2}\right)\sin(\theta) & \cosh\left(\frac{\varphi}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\cos(\theta)\end{pmatrix}^{-1}\\ &= \begin{pmatrix}\cosh\left(\frac{\varphi}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\cos(\theta) & \sinh\left(\frac{\varphi}{2}\right)\sin(\theta)\\ \sinh\left(\frac{\varphi}{2}\right)\sin(\theta) & \cosh\left(\frac{\varphi}{2}\right)+\sinh\left(\frac{\varphi}{2}\right)\cos(\theta)\end{pmatrix}\\ &= \exp(-\varphi(\cos(\theta)X+\sin(\theta)Y)).\end{aligned}$$

Using again Lemma 2.1 but with $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$, we get:

$$\begin{aligned}\exp(zZ)\exp(\varphi(\cos(\theta)X+\sin(\theta)Y)) &= \exp\left(-z\tilde{Z}\right)\exp\left(-i\varphi\left(\cos(\theta)\tilde{X}+\sin(\theta)\tilde{Y}\right)\right)\\ &= \exp\left(-i\varphi\left(\cos(\theta-z)\tilde{X}+\sin(\theta-z)\tilde{Y}\right)\right)\exp\left(-z\tilde{Z}\right)\\ &= \exp\big(\varphi(\cos(\theta-z)X+\sin(\theta-z)Y)\big)\exp(zZ).\end{aligned}$$

Thus:

$$\begin{aligned}x^{-1}y &= \big(\exp\big(\varphi^x(\cos(\theta^x)X+\sin(\theta^x)Y)\big)\exp(z^xZ)\big)^{-1}\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ)\\ &= \exp(-z^xZ)\exp\big(-\varphi^x(\cos(\theta^x)X+\sin(\theta^x)Y)\big)\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ)\\ &= \exp\big(-\varphi^x(\cos(\theta^x+z^x)X+\sin(\theta^x+z^x)Y)\big)\exp(-z^xZ)\exp\big(\varphi^y(\cos(\theta^y)X+\sin(\theta^y)Y)\big)\exp(z^yZ)\\ &= \exp\big(-\varphi^x(\cos(\theta^x+z^x)X+\sin(\theta^x+z^x)Y)\big)\\ &\times\exp\big(\varphi^y(\cos(\theta^y+z^x)X+\sin(\theta^y+z^x)Y)\big)\exp\big((z^y-z^x)Z\big)\\ &= \exp\big(\varphi(\cos(\theta)X+\sin(\theta)Y)\big)\exp(zZ)\exp\big((z^y-z^x)Z\big)\end{aligned}$$

If we denote $M=\exp\big(\varphi(\cos(\theta)X+\sin(\theta)Y)\big)\exp(zZ)$, we have:

$$M=\begin{pmatrix}\cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right)+\sinh\left(\frac{\varphi}{2}\right)\cos\left(\theta+\frac{z}{2}\right) & -\cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\sin\left(\theta+\frac{z}{2}\right)\\ \cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\sin\left(\theta+\frac{z}{2}\right) & \cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right)-\sinh\left(\frac{\varphi}{2}\right)\cos\left(\theta+\frac{z}{2}\right)\end{pmatrix}$$

The matrix M is also equal to the product:

$$\exp\big(-\varphi^x(\cos(\theta^x+z^x)X+\sin(\theta^x+z^x)Y)\big)\exp\big(\varphi^y(\cos(\theta^y+z^x)X+\sin(\theta^y+z^x)Y)\big).$$

In particular we get:

$$\begin{aligned}
M_{1,1} &= \cosh\left(\frac{\varphi^x}{2}\right)\cosh\left(\frac{\varphi^y}{2}\right) + \cosh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\cos(\theta^y + z^x) \\
&\quad - \cosh\left(\frac{\varphi^y}{2}\right)\sinh\left(\frac{\varphi^x}{2}\right)\cos(\theta^x + z^x) - \sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\cos(\theta^y - \theta^x) \\
M_{1,2} &= -\cosh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\sin(\theta^y + z^x) - \sinh\left(\frac{\varphi^x}{2}\right)\cosh\left(\frac{\varphi^y}{2}\right)\cos(\theta^x + z^x) \\
&\quad + \sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\sin(\theta^y - \theta^x) \\
M_{2,1} &= -\cosh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\sin(\theta^y + z^x) - \sinh\left(\frac{\varphi^x}{2}\right)\cosh\left(\frac{\varphi^y}{2}\right)\cos(\theta^x + z^x) \\
&\quad - \sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\sin(\theta^y - \theta^x) \\
M_{2,2} &= \cosh\left(\frac{\varphi^x}{2}\right)\cosh\left(\frac{\varphi^y}{2}\right) - \cosh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\cos(\theta^y + z^x) \\
&\quad + \cosh\left(\frac{\varphi^y}{2}\right)\sinh\left(\frac{\varphi^x}{2}\right)\cos(\theta^x + z^x) - \sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\cos(\theta^y - \theta^x)
\end{aligned}$$

Thus:

$$\begin{cases}
\cosh\left(\frac{\varphi}{2}\right)\cos\left(\frac{z}{2}\right) = \frac{M_{1,1}+M_{2,2}}{2} = \cosh\left(\frac{\varphi^x}{2}\right)\cosh\left(\frac{\varphi^y}{2}\right) - \sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\cos(\theta^y - \theta^x) & \text{(10a)} \\
\cosh\left(\frac{\varphi}{2}\right)\sin\left(\frac{z}{2}\right) = \frac{M_{2,1}+M_{1,2}}{2} = -\sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)\sin(\theta^y - \theta^x). & \text{(10b)}
\end{cases}$$

- This time, the distance between $\Pi_2(x)$ and $\Pi_2(y)$ is characterized by

$$\cosh(\rho(\Pi_2(x),\Pi_2(y))) = \cosh(\varphi^x)\cosh(\varphi^y) + \sinh(\varphi^x)\sinh(\varphi^y)\cos(\theta^x - \theta^y).$$

  Using the identity $\cosh^2(\frac{\varphi}{2}) = \frac{1+\cosh(\varphi)}{2}$ we obtain

$$\cosh(\varphi) = \cosh(\varphi^x)\cosh(\varphi^y) + \sinh(\varphi^x)\sinh(\varphi^y)\cos(\theta^x - \theta^y) \tag{11}$$

  and so: $\varphi = \rho(\Pi_2(x),\Pi_2(y))$

- We now deal with z. Using the same notations as before (this time M is the pole used to define polar coordinates on $\mathbf{H}$) and the equivalent of the Heron formula for hyperbolic triangles, we have: $\cos\left(\frac{\mathcal{A}}{2}\right) = \frac{1}{4\cosh\left(\frac{\varphi}{2}\right)\cosh\left(\frac{\varphi^x}{2}\right)\cosh\left(\frac{\varphi^y}{2}\right)}(1 + \cosh(\varphi^x) + \cosh(\varphi^y) + \cosh(\varphi)) = \cos\left(\frac{z}{2}\right)$ and thus $z = \text{sign}(z)\mathcal{A}$. Moreover, we have $\sin\left(\frac{z}{2}\right) = \sin(\theta^x - \theta^y)\frac{\sinh\left(\frac{\varphi^x}{2}\right)\sinh\left(\frac{\varphi^y}{2}\right)}{\cosh\left(\frac{\varphi}{2}\right)}$, and so $z < 0$ if and only if $\theta^x < \theta^y$.

Finally, $d_{cc}^2(x,y) \sim \rho^2\left(\Pi_2(x),\Pi_2(y)\right) + \sqrt{|\tilde{\mathcal{A}}|}$ with $\tilde{\mathcal{A}} \equiv z_y - z_x + \text{sign}(\theta_x - \theta_y)\mathcal{A} \mod (4\pi)$ and $\tilde{\mathcal{A}} \in]-2\pi, 2\pi]$. And, for Brownian motions on $SL(2)$, we get $d_{cc}(\mathbb{B}_t^x, \mathbb{B}_t^y) \sim \rho(\Pi_2(\mathbb{B}_t^x),\Pi_2(\mathbb{B}_t^y)) + \sqrt{|\tilde{\mathcal{A}}_t|}$ with $\tilde{\mathcal{A}}_t \equiv z_t^y - z_t^x + \text{sign}(\theta_t^x - \theta_t^y)\mathcal{A}_t \mod (4\pi)$ and $\mathcal{A}_t$ the area of the hyperbolic triangle of vertices $\Pi_2(\mathbb{B}_t^x), \Pi_2(\mathbb{B}_t^y)$ and N. As before the quantity $z_t^y - z_t^x + \text{sign}(\theta_t^x - \theta_t^y)\mathcal{A}_t$ is equal up to a constant to the signed area delimited by the paths $s \mapsto p(\mathbb{B}_s^x), s \le t$, $s \mapsto \Pi_2(\mathbb{B}_s^y), s \le t$ and the geodesics joining $\Pi_2(x)$ to $\Pi_2(y)$ and $\Pi_2(\mathbb{B}_t^x)$ to $\Pi_2(\mathbb{B}_t^y)$ respectively. □

3 Couplings on manifolds of constant curvature k

Our aim, now, is to use these results to find a good way to define couplings of Brownian motions on $SU(2)$ and $SL(2)$. As the Brownian motion on $SU(2)$ (resp. $SL(2)$) is entirely determined by a Brownian motion on S^2 (resp. $\mathbf{H}^2$), to define a coupling on this Lie group, we only need to define a coupling on S^2 (resp. $\mathbf{H}^2$). Note that, in the case of the Heisenberg group we have the same remark but considering a Brownian motion on $\mathbb{R}^2$. Thus we are going to describe a coupling model on manifolds of dimension 2 with constant curvature k, having, for $k=1$, coupling on S^2, for $k=-1$, coupling on $\mathbf{H}^2$ and even, for $k=0$, coupling on $\mathbb{R}^2$.

3.1 Description of a coupling model

Let M be a manifold of dimension 2 with constant curvature k. Let us start with a way to describe Brownian motions using Itô depiction in a frame in the sense of Emery (see [7] for a basic introduction).

Proposition 3.1. *We consider X_t a semi-martingale on M and $(e_1(t), e_2(t))$ a TM-valued continuous semi-martingale forming an orthonormal basis of vector fields along $(X_t)_t$. We suppose that all the processes are in the same filtration. If X_t is a Brownian motion on M, there exists (U_1, U_2) a Brownian motion on $\mathbb{R}^2$ such that:*

$$d^\nabla X_t = dU_1(t)e_1(t) + dU_2(t)e_2(t). \tag{12}$$

Conversely, if X_t satisfies (12) with (U_1, U_2) a Brownian motion on $\mathbb{R}^2$, then X_t is a Brownian motion on M.

Proof. Let us begin with the direct implication. For $i \in \{1,2\}$, taking U_i such that $dU_i(t) = \langle e_i^*(t), d^\nabla X_t\rangle$, with (e_1^*, e_2^*) the dual basis of (e_1, e_2) in T^*M, we directly get that U_i are martingales. Moreover as X_t is a Brownian motion and $(e_1, e_2)(t)$ forms an orthonormal basis of $T_{X_t}M$, for $i, j \in \{1,2\}$:

$$dU_i(t) \cdot dU_j(t) = (e_i^* \otimes e_j^*)(t)(dX_t, dX_t) = Tr(e_i^* \otimes e_j^*)(t)dt = \delta_{i,j}dt.$$

We obtain the expected result.

For the converse implication, as $\int_0^\cdot \langle \phi(X_t), d^\nabla X_t\rangle = \int_0^\cdot \langle \phi(X_t), e_1^*(t)\rangle dU_1(t) + \int_0^\cdot \langle \phi(X_t), e_2^*(t)\rangle dU_2(t)$ is a martingale for all $\phi \in \Gamma(T^*M)$, we obtain that X_t is a martingale. Then for f a smooth function on M, by Itô's formula, we get:

$$d(f(X_t)) = df(X_t) \cdot d^\nabla X_t + \frac{1}{2}\,\mathrm{Hess}(f)(X_t)(d^\nabla X_t, d^\nabla X_t)$$

with:

$$\begin{aligned}
\mathrm{Hess}(f)(X_t)(d^\nabla X_t, d^\nabla X_t) &= \sum_{i,j} \langle \mathrm{Hess}(f)(X_t), (e_i^* \otimes e_j^*)(t)\rangle (e_i^* \otimes e_j^*)(t)(d^\nabla X_t, d^\nabla X_t) \\
&= \sum_{i,j} \langle \mathrm{Hess}(f)(X_t), (e_i^* \otimes e_j^*)(t)\rangle \langle e_i^*(t), d^\nabla X_t\rangle \cdot \langle e_j^*(t), d^\nabla X_t\rangle \\
&= \sum_{i,j} \langle \mathrm{Hess}(f)(X_t), (e_i^* \otimes e_j^*)(t)\rangle dU_i(t) \cdot dU_j(t) \\
&= Tr(\mathrm{Hess}(f))(X_t)dt = \Delta f(X_t)dt.
\end{aligned}$$

Thus, X_t is a Brownian motion on M. □

We can now define couplings (X_t, Y_t) of Brownian motions on M using two dimensional Brownian motions $(U_1(t), U_2(t))$ and $(V_1(t), V_2(t))$ adapted to a common filtration such that we have relation (1):

$$d^\nabla X_t = dU_1(t)e_1^X(t) + dU_2(t)e_2^X(t) \text{ and } d^\nabla Y_t = dV_1(t)e_1^Y(t) + dV_2(t)e_2^Y(t)$$

with $e_i^X(t)$ and $e_i^Y(t)$ defined as follow. We consider the continuous stochastic process $R_t = \rho(X_t, Y_t)$ with ρ the Riemannian distance. We suppose that $0 < R_t < i(M)$ for all $t \in I$, with $i(M)$ the injectivity radius of M and I an open interval on $\mathbb{R}^+$. Then for all $t \in I$ we can define:

- $e_1^X(t) = \frac{\exp_{X_t}^{-1}(Y_t)}{R_t}$ a unitary vector on $T_{X_t}M$;
- $e_2^X(t)$ such that $(e_1^X(t), e_2^X(t))$ is a direct orthonormal basis on $T_{X(t)}M$;
- $(e_1^Y(t), e_2^Y(t))$ parallel transport of $(e_1^X(t), e_2^X(t))$ along the geodesic joining X_t and Y_t. This define a direct orthonormal basis on $T_{Y_t}M$.

We can use the following lemma to prove the first part of Proposition (1.5):

Lemma 3.2. *Let x, $y \in M$, $r = \rho(x,y)$ with $0 < r < \pi$, $u \in T_xM$ and $v \in T_yM$. We define $e_1^x = \frac{\exp_x^{-1}(y)}{r} \in T_xM$, $e_1^y \in T_yM$ the parallel transport of e_1^x along the geodesic joining x and y, and e_2^x (resp. e_2^y) such that (e_1^x, e_2^x) (resp. (e_1^y, e_2^y)) is a basis of T_xM (resp T_yM). Then, we have:*

$$d\rho_{(x,y)}(u,v) = v_1 - u_1$$

and

$$\mathrm{Hess}(\rho)_{(x,y)}((u,v),(u,v)) = \begin{cases} \sqrt{k}(u_2^2+v_2^2)\cot(\sqrt{k}r) - 2\sqrt{k}u_2v_2\frac{1}{\sin(\sqrt{k}r)}, & \text{if } k>0 \\ \sqrt{-k}(u_2^2+v_2^2)\coth(\sqrt{-k}r) - 2\sqrt{-k}u_2v_2\frac{1}{\sinh(\sqrt{-k}r)}, & \text{if } k<0 \\ \frac{2u_2v_2}{r}, & \text{if } k=0. \end{cases}$$

with $u_i = \langle u, e_i^x\rangle$ and $v_i = \langle v, e_i^y\rangle$ for $i \in \{1,2\}$.

Note that, below, we will use the expression of the case $k > 0$ as the general expression of the Hessian as it induced the two other cases by direct computation in the case $k < 0$ and taking the limit in the case $k = 0$.

Proof of Proposition (1.5), first part. Using this lemma and applying Ito's formula, we directly get:

$$\begin{aligned} dR_t &= d\rho_{(X_t,Y_t)}(d^\nabla X_t, d^\nabla Y_t) + \frac{1}{2}\mathrm{Hess}(\rho)(X_t,Y_t)((d^\nabla X_t, d^\nabla Y_t),(d^\nabla X_t, d^\nabla Y_t)) \\ &= dV_1(t) - dU_1(t) + \frac{1}{2}\Big(\sqrt{k}\big(dU_2(t)\cdot dU_2(t) + dV_2(t)\cdot dV_2(t)\big)\cot(\sqrt{k}R_t) - \frac{2\sqrt{k}}{\sin(\sqrt{k}R_t)}dU_2(t)\cdot dV_2(t)\Big) \\ &= dV_1(t) - dU_1(t) + \frac{1}{2}\Big(2\sqrt{k}\cot(\sqrt{k}R_t)dt - \frac{2\sqrt{k}}{\sin(\sqrt{k}R_t)}dU_2(t)\cdot dV_2(t)\Big) \\ &= dV_1(t) - dU_1(t) + \sqrt{k}\Big(\cot(\sqrt{k}R_t)dt - \frac{1}{\sin(\sqrt{k}R_t)}dU_2(t)\cdot dV_2(t)\Big). \end{aligned} \tag{13}$$

□

Remark. *In fact, by describing the covariant derivative of $(e_i^X(t))_t$ (resp. $(e_i^Y(t))_t$) along X_t (resp. Y_t) we can define our orthonormal basis as the solution of a system of stochastic differential equations depending on U_1, U_2, V_1, V_2 and R_t. In the case where no singularities appear in this system, our coupling keeps sense even if $R_t \in \{0, i(M)\}$. Then we can consider R_t as a signed distance and the results of Proposition (1.5) are still true.*

Let us now prove Lemma (3.2).

Proof of Lemma (3.2). Let x, $y \in M$, $r = \rho(x,y)$, $u \in T_xM$ and $v \in T_yM$. We are going to study the first and second derivatives of functions $E(t) := \frac{1}{2}\rho(\exp_x(tu), \exp_y(tv))^2$.
We define $(s,t) \in [0,1] \times [0,+\infty[\mapsto c(s,t) \in M$ such that:

- $s \mapsto c(s,t)$ is a geodesic starting at $\exp_x(tu)$ and ending at $\exp_y(tv)$ for all $t \geq 0$;
- $t \mapsto c(0,t)$ is a geodesic starting at x and ending at $\exp_x(tu)$;
- $t \mapsto c(1,t)$ is a geodesic starting at y and ending at $\exp_y(tv)$.

In particular, $s \mapsto J(s,t) := \partial_t c(s,t)$ is a Jacobi field for all $t \geq 0$. We define $e_1(s,t) := \frac{\partial_s c(s,t)}{\rho(\exp_x(tu),\exp_y(tv))}$ and $e_2(s,t)$ such that $(e_1(s,t), e_2(s,t))$ is an orthonormal basis of $T_{c(s,t)}M$ for each s,t. In particular $e_1(0,0) = e_1^x$ and $e_1(1,0) = e_1^y$. For the covariant derivative calculus, we will use the notation ∇_s for $\nabla_{\partial_s c(s,t)}$ and ∇_t for $\nabla_{\partial_t c(s,t)}$. Note that, for t constant, $e_i(s,t)$ are defined by parallel transport along c and so $\nabla_s e_i(s,t) = 0$. We will denote $r(t) = \rho(\exp_x(tu), \exp_y(tv)) = ||\partial_s c(s,t)||$, $u_i(s) := \langle J(0,t), e_i(0,t)\rangle$, $u_i := u_i(0)$ and $v_i(s) := \langle J(1,t), e_i(1,t)\rangle$, $v_i := v_i(0)$ for $i \in \{1,2\}$.

Since $s \mapsto c(s,t)$ is a geodesic connecting $\exp_x(tu)$ and $\exp_y(tv)$, it has constant speed and $E(t) = \frac{1}{2}||\partial_s c(s,t)||^2$ for all $s \in [0,1]$. Thus, $E(t) = \frac{1}{2}\int_0^1 \langle \partial_s c(s,t), \partial_s c(s,t)\rangle ds$. We derive this function:

$$\begin{aligned}
E'(t) &= \int_0^1 \langle \nabla_t \partial_s c(s,t), \partial_s c(s,t)\rangle ds = \int_0^1 \langle \nabla_s \partial_t c(s,t), \partial_s c(s,t)\rangle ds \text{ since } \nabla \text{ is torsionfree} \\
&= \int_0^1 \partial_s \langle \partial_t c(s,t), \partial_s c(s,t)\rangle ds - \int_0^1 \langle \partial_t c(s,t), \nabla_s \partial_s c(s,t)\rangle ds \\
&= \int_0^1 \partial_s \langle \partial_t c(s,t), \partial_s c(s,t)\rangle ds \text{ since } s \mapsto c(s,t) \text{ is a geodesic;} \\
&= [\langle \partial_t c(s,t), \partial_s c(s,t)\rangle]_0^1.
\end{aligned}$$

We obtain:

$$\begin{aligned}
E'(0) &= \langle \partial_t c(1,t)_{|t=0}, \partial_s c(s,0)_{|s=1}\rangle - \langle \partial_t c(0,t)_{|t=0}, \partial_s c(s,0)_{|s=0}\rangle \\
&= \langle v, re_1(1,0)\rangle - \langle u, re_1(0,0)\rangle = r(v_1 - u_1).
\end{aligned}$$

We can now calculate the second derivative. We only deal with the case $k > 0$ as the two other cases use similar calculations.

$$E''(t) = \int_0^1 \partial_t \langle \nabla_s \partial_t c(s,t), \partial_s c(s,t)\rangle ds = \int_0^1 \langle \nabla_t \nabla_s \partial_t c(s,t), \partial_s c(s,t)\rangle + \langle \nabla_s \partial_t c(s,t), \nabla_t \partial_s c(s,t)\rangle ds.$$

By definition of the curvature tensor R and since $[\partial_t c, \partial_s c] = 0$, we have

$$R(\partial_t c(s,t), \partial_s c(s,t))\partial_t c(s,t) = \nabla_t \nabla_s \partial_t c(s,t) - \nabla_s \nabla_t \partial_t c(s,t).$$

Then:

$$\begin{aligned}
E''(t) &= \int_0^1 \langle \nabla_s \nabla_t \partial_t c(s,t), \partial_s c(s,t)\rangle + \langle R(\partial_t c(s,t), \partial_s c(s,t))\partial_t c(s,t), \partial_s c(s,t)\rangle + \langle \nabla_s \partial_t c(s,t), \nabla_s \partial_t c(s,t)\rangle ds \\
&= \int_0^1 \langle \nabla_s \nabla_t \partial_t c(s,t), \partial_s c(s,t)\rangle - \langle R(J(s,t), \partial_s c(s,t))\partial_s c(s,t), J(s,t)\rangle + ||\nabla_s J(s,t)||^2 dt.
\end{aligned}$$

The same way as previously we get :

$$\begin{aligned}
\int_0^1 \langle \nabla_s \nabla_t \partial_t c(s,t), \partial_s c(s,t)\rangle ds &= \int_0^1 \langle \nabla_s \nabla_t \partial_t c(s,t), \partial_s c(s,t)\rangle + \langle \nabla_t \partial_t c(s,t), \nabla_s \partial_s c(s,t)\rangle ds \\
&= [\langle \nabla_t \partial_t c(s,t), \partial_s c(s,t)\rangle]_0^1.
\end{aligned}$$

For $t = 0, 1$, $t \mapsto c(s,t)$ is a geodesic and so have constant variations, then $\int_0^1 \langle \nabla_s \nabla_t \partial_t c(s,t), \partial_s c(s,t) \rangle ds = 0$. For t fixed, let us study our Jacobi field along $s \mapsto c(s,t)$. Using the basis $(e_1(s,t), e_2(s,t))$, we can write:

$$J(s,t) = j_1(s,t)e_1(s,t) + j_2(s,t)e_2(s,t).$$

Since $\nabla_s e_i(s,t) = 0$, we have:

$$\begin{aligned}\nabla_s J(s,t) &= \partial_s j_1(s,t)e_1(s,t) + \partial_s j_2(s,t)e_2(s,t) \text{ and} \\ \nabla_s \nabla_s J(s,t) &= \partial^2_{ss} j_1(s,t)e_1(s,t) + \partial^2_{ss} j_2(s,t)e_2(s,t).\end{aligned}$$

Thus:

$$\begin{aligned}E''(t) &= \int_0^1 -\langle R\big(J(s,t), \partial_s c(s,t)\big)\partial_s c(s,t), J(s,t)\rangle + ||\nabla_s J(s,t)||^2 ds \\ &= \int_0^1 -\langle R\big(j_1(s,t)e_1(s,t) + j_2(s,t)e_2(s,t), r(t)e_1(s,t)\big)r(t)e_1(s,t), J(s,t)\rangle ds \\ &+ \int_0^1 (\partial_s j_1(s,t))^2 + (\partial_s j_2(s,t))^2 ds \\ &= \int_0^1 -kr(t)^2 j_2(s,t)^2 + (\partial_s j_1(s,t))^2 + (\partial_s j_2(s,t))^2 ds.\end{aligned}$$

By property of Jacobi fields, we have

$$\begin{aligned}\nabla_s \nabla_s J(s,t) &= -R(J(s,t), \partial_s c(s,t))\partial_s c(s,t) \\ &= -r(t)^2 (j_1(s,t)R(e_1(s,t), e_1(s,t))e_1(s,t) + j_2(s,t)R(e_2(s,t), e_1(s,t))e_1(s,t)) \\ &= -kr(t)j_2(s,t)e_2(s,t).\end{aligned}$$

We obtain $\begin{cases} \partial^2_{ss} j_1(s,t) = 0 \\ j_1(0,t) = u_1(t) \\ j_1(1,t) = v_1(t) \end{cases}$ and $\begin{cases} \partial^2_{ss} j_2(s,t) = -kr(t)^2 j_2(s,t) \\ j_2(0,t) = u_2(t) \\ j_2(1,t) = v_2(t) \end{cases}$ and so $j_1(s,t) = u_1(t)(1-s) + v_1(t)s$ and $j_2(s,t) = a\cos(\sqrt{k}r(t)s) + b\sin(\sqrt{k}r(t)s)$. We have, $a = j_2(0,t) = u_2(t)$ and $b = \frac{j_2(1,t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}$ so

$$j_2(s,t) = u_2(t)\cos(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\sin(\sqrt{k}r(t)s).$$

Moreover, $\partial_s j_2(s,t) = \sqrt{k}r(t)\left(-u_2(t)\sin(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\cos(\sqrt{k}r(t)s)\right)$ and $\partial_s j_1(s,t) = v_1(t) - u_1(t)$.

Then, for $t = 0$, we have:

$$\begin{aligned}E''(0) &= \int_0^1 (v_1 - u_1)^2 + kr^2 \left(\left(\sin^2(\sqrt{k}rs) - \cos^2(\sqrt{k}rs)\right)\left(u_2^2 - \frac{(v_2 - u_2\cos(\sqrt{k}))^2}{\sin^2(\sqrt{k}r)}\right)\right) \\ &- 4u_2\frac{v_2 - u_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}\cos(\sqrt{k}rs)\sin(\sqrt{k}rs)\big) ds \\ &= (v_1 - u_1)^2 + kr^2\Bigg(- \frac{\cos(\sqrt{k}r)\sin(\sqrt{k}r)}{\sqrt{k}r}\left(u_2^2 - \frac{(v_2 - u_2\cos(\sqrt{k}r))^2}{\sin^2(\sqrt{k}r)}\right) \\ &- \frac{2}{\sqrt{k}r}u_2\frac{v_2 - u_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}\sin^2(\sqrt{k}r)\Bigg)\end{aligned}$$

$$E''(0) = (v_1-u_1)^2 + \sqrt{k}r\Bigg(-\cos(\sqrt{k}r)\sin(\sqrt{k}r)\left(u_2^2 - \frac{v_2^2+u_2^2\cos^2(\sqrt{k}r) - 2u_2v_2\cos(\sqrt{k}r)}{\sin^2(\sqrt{k}r)}\right)$$
$$- \frac{2}{\sqrt{k}r}\left(u_2v_2 - u_2^2\cos(\sqrt{k}r)\right)\sin(\sqrt{k}rt)\Bigg)$$
$$= (v_1-u_1)^2 + \sqrt{k}r\Bigg(u_2^2(-\cos(\sqrt{k}r)\sin(\sqrt{k}r) + 2\cos(\sqrt{k}r)\sin(\sqrt{k}r) + \frac{\cos^3(\sqrt{k}r)}{\sin(\sqrt{k}r)}) + v_2^2\frac{\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}$$
$$- u_2v_2\left(2\frac{\cos^2(\sqrt{k}r)}{\sin(\sqrt{k}r)} + 2\sin(\sqrt{k}r)\right)\Bigg)$$

Finally $E''(0) = (v_1-u_1)^2 + \sqrt{k}r\left((u_2^2+v_2^2)\cot(\sqrt{k}r) - 2u_2v_2\frac{1}{\sin(\sqrt{k}r)}\right)$ and we obtain, for $u \in T_xM$, $v \in T_yM$:

$$d\left(\frac{1}{2}\rho^2\right)_{(x,y)}(u,v) = r(v_1-u_1)$$

and

$$\mathrm{Hess}\left(\frac{1}{2}\rho^2\right)_{(x,y)}((u,v),(u,v)) = (v_1-u_1)^2 + \sqrt{k}r\left((u_2^2+v_2^2)\cot(\sqrt{k}r) - 2u_2v_2\frac{1}{\sin(\sqrt{k}r)}\right).$$

Finally we get:

$$d\rho_{(x,y)}(u,v) = \frac{d}{ds}_{|s=0}\rho(\exp_s(su),\exp_y(sv)) = \frac{d}{ds}_{|s=0}(\sqrt{2E(s)}) = \frac{E'(0)}{\sqrt{2E(0)}} = (v_1-u_1)$$

and

$$\mathrm{Hess}(\rho)_{(x,y)}((u,v),(u,v)) = \frac{d^2}{ds^2}_{|s=0}\rho(\exp_s(su),\exp_y(sv)) = \frac{d}{ds}_{|s=0}\frac{E'(s)}{\sqrt{2E(s)}} = \frac{E''(0)}{\sqrt{2E(0)}} - \frac{E'(0)^2}{(2E(0))^{\frac{3}{2}}}$$
$$= \frac{(v_1-u_1)^2}{r} + \sqrt{k}\left((u_2^2+v_2^2)\cot(\sqrt{k}r) - 2u_2v_2\left(\frac{1}{\sin(\sqrt{k}r)}\right)\right) - \frac{(v_1-u_1)^2}{r}$$
$$= \sqrt{k}(u_2^2+v_2^2)\cot(\sqrt{k}r) - 2\sqrt{k}u_2v_2\frac{1}{\sin(\sqrt{k}r)}.$$

□

3.2 Swept area

We now focus on the signed swept area $\mathcal{A}_t$ between X_t and Y_t as described before. This way we will be able to compare the last coordinates of the two Brownian motions on $SU(2)$ (resp. $SL(2)$). Note that this signed area stays with constant sign while the two paths of the Brownian motions don't cross. We are first going to find an expression of the area in terms of a function of the covariant derivative of the paths. Let γ_x, $\gamma_y : [0,+\infty[\mapsto M$ be two continuous curves on M starting at x and y respectively. We can consider $(s,t) \in [0,1]\times[0,+\infty[\mapsto c(s,t) \in M$ such that $s \mapsto c(s,t)$ is a geodesic starting at $\gamma_x(t)$ and ending at $\gamma_y(t)$ for all $t \geq 0$. Providing that the distance between $\gamma_x(t)$ and $\gamma_y(t)$ is non null for each t but small enough, we can also define as before the basis $(e_1^x(t), e_2^y(t))$.
Assuming that the map $(s,\zeta) \mapsto c(s,\zeta)$ is a local chart of M, the volume form is given by $\sqrt{\det(G)}d\zeta ds$ where G is the positive definite symmetric matrix representing the metric in these local coordinates. In particular, we have:

$$\sqrt{\det(G)} = ||\partial_s c(s,\zeta))||^2||\partial_\zeta c(s,\zeta))||^2 - \langle\partial_s c(s,\zeta)), \partial_\zeta c(s,\zeta))\rangle^2 = |\det\left(\partial_s c(s,\zeta), \partial_\zeta c(s,\zeta)\right)|.$$

Then, the signed swept area can be formally defined by $\mathcal{A}(t) := \int_0^1 \int_0^t \det(\partial_s c(s,\zeta), \partial_\zeta c(s,\zeta)) d\zeta ds$ where the determinant is calculated in the basis $(e_1(s,\zeta), e_2(s,\zeta))$ of $T_{c(s,\zeta)}M$ for each ζ, t. We get the following lemma:

Lemma 3.3. *Let $u \in T_xM$ and $v \in T_yM$, we have:*

$$d\mathcal{A}_{(x,y)}(u,v) = \frac{1}{\sqrt{k}} \tan\left(\frac{\sqrt{k}r}{2}\right)(u_2 + v_2),$$

$$\operatorname{Hess} \mathcal{A}_{(x,y)}((u,v),(u,v) = \frac{u_2 v_1 - v_2 u_1}{\cos^2(\frac{\sqrt{k}r}{2})} + \tan^2\left(\frac{\sqrt{k}r}{2}\right)(v_2 v_1 - u_2 u_1).$$

Note that, as in Lemma (3.2), the above results are given as a general formula for the three cases $k > 0$, $k < 0$ and $k = 0$.

Proof. Let $\mathcal{A}(t) := t \mapsto \mathcal{A}_{x,y}(\exp_x(tu), \exp_y(tv))$ the swept area following the trajectories of $t \mapsto \exp_x(tu)$ and $t \mapsto \exp_y(tv)$ evaluated at time t. As for the proof of the previous lemma, we will study $\mathcal{A}'(0)$ and $\mathcal{A}''(0)$.
First, using the same notations as the previous proof, we have:

$$\begin{aligned}
\mathcal{A}'(t) &= \int_0^1 \det(\partial_s c(s,t), \partial_t c(s,t)) ds = \int_0^1 \det(r(t)e_1(s,t), J(s,t)) ds = r(t) \int_0^1 j_2(s,t) ds \\
&= r(t)\left(\int_0^1 u_2(t)\cos(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\sin(\sqrt{k}r(t)s) ds\right) \\
&= \frac{1}{\sqrt{k}}\left(u_2(t)[\sin(\sqrt{k}r(t)s)]_0^1 + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}[-\cos(\sqrt{k}r(t)s)]_0^1\right) \\
&= \frac{1}{\sqrt{k}}\left(u_2(t)\sin(\sqrt{k}r(t)) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\left(1 - \cos(\sqrt{k}r(t))\right)\right) \\
&= \frac{u_2(t)\sin^2(\sqrt{k}r(t)) - u_2(t)\cos(\sqrt{k}r(t))(1 - \cos(\sqrt{k}r(t))) + v_2(t)(1 - \cos(\sqrt{k}r(t)))}{\sqrt{k}\sin(\sqrt{k}r(t))} \\
&= (u_2(t) + v_2(t))\frac{1 - \cos(\sqrt{k}r(t))}{\sqrt{k}\sin(\sqrt{k}r(t))} = (u_2(t) + v_2(t))\frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r(t)}{2}\right).
\end{aligned}$$

So we get the covariant derivative and also:

$$\mathcal{A}''(t) = (u_2'(t) + v_2'(t))\frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r(t)}{2}\right) + (u_2(t) + v_2(t))\frac{r'(t)}{2\cos^2(\frac{\sqrt{k}r(t)}{2})}.$$

Note that $r'(0)$ has been calculated previously for the covariant derivative of the distance, so:

$$\mathcal{A}''(0) = (u_2'(0) + v_2'(0))\frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}r}{2})\right) + (u_2 + v_2)\frac{v_1 - u_1}{2\cos^2(\frac{\sqrt{k}r}{2})}.$$

We now just have to compute $u_2'(0)$ and $v_2'(0)$. As $u_2(s) = \langle \partial_t c(0,t), e_2(0,t)\rangle$ and $t \mapsto c(s,t)$ is a geodesic, we have:

$$u_2'(s) = \langle \nabla_t \partial_t c(0,t), e_2(0,t)\rangle + \langle \partial_t c(0,t), \nabla_t e_2(0,t)\rangle = \langle \partial_t c(0,t), \nabla_t e_2(0,t)\rangle.$$

Thus, $u_2'(0) = \langle v, \nabla_t e_2(0,t)_{|t=0}\rangle$ and, in the same way, we have $v_2'(0) = \langle v, \nabla_t e_2(1,t)_{|t=0}\rangle$.

Note that:

$$\begin{aligned}\langle \nabla_t e_2(s,t), e_2(s,t)\rangle &= \frac{1}{2}\partial_t \langle e_2(s,t), e_2(s,t)\rangle = 0 \text{ since } \langle e_2(s,t), e_2(s,t)\rangle = 1,\\ \langle \nabla_t e_2(s,t), e_1(s,t)\rangle &= \partial_t(\langle e_2(s,t), e_1(s,t)\rangle) - \langle e_2(s,t), \nabla_t e_1(s,t)\rangle = -\langle e_2(s,t), \nabla_t e_1(s,t)\rangle\\ &\qquad \text{since } \langle e_2(s,t), e_1(s,t)\rangle = 0.\end{aligned}$$

Then, $\nabla_t e_2(s,t) = -\langle \nabla_t e_1(s,t), e_2(s,t)\rangle e_1(s,t)$. We have:

$$\begin{aligned}\nabla_t e_1(s,t) &= \nabla_t\left(\partial_s c(s,t) \times \frac{1}{r(t)}\right) = -\frac{r'(t)}{r(t)^2}\partial_s c(s,t) + \nabla_t(\partial_s c(s,t)) \times \frac{1}{r(t)}\\ &= -\frac{r'(t)}{r(t)} e_1(s,t) + \frac{\nabla_s J(s,t)}{r(t)}.\end{aligned}$$

Thus:

$$\langle \nabla_t e_1(s,t), e_2(s,t)\rangle = \frac{1}{r(t)}\partial_s j_2(s,t) = \sqrt{k}\left(-u_2(t)\sin(\sqrt{k}r(t)s) + \frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))}\cos(\sqrt{k}r(t)s)\right).$$

We obtain:

$$\nabla_t e_2(0,t) = -\sqrt{k}\frac{v_2(t) - u_2(t)\cos(\sqrt{k}r(t))}{\sin(\sqrt{k}r(t))} e_1(0,t) \text{ and } \nabla_t e_2(1,t) = -\sqrt{k}\frac{v_2(t)\cos(\sqrt{k}r(t)) - u_2(t)}{\sin(\sqrt{k}r(t))} e_1(1,t)$$

and so:

$$u_2'(0) = \sqrt{k}\frac{u_2\cos(\sqrt{k}r) - v_2}{\sin(\sqrt{k}r)}u_1 \text{ and } v_2'(0) = \sqrt{k}\frac{u_2 - v_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}v_1.$$

Finally:

$$\begin{aligned}\mathcal{A}''(0) &= \left(\frac{u_2\cos(\sqrt{k}r) - v_2}{\sin(\sqrt{k}r)}u_1 + \frac{u_2 - v_2\cos(\sqrt{k}r)}{\sin(\sqrt{k}r)}v_1\right)\tan\left(\frac{\sqrt{k}r}{2}\right) + (u_2 + v_2)\frac{v_1 - u_1}{2\cos^2(\frac{\sqrt{k}r}{2})}\\ &= \left((u_2u_1 - v_2v_1)\cot(\sqrt{k}r) + (u_2v_1 - v_2u_1)\frac{1}{\sin(\sqrt{k}r)}\right)\tan\left(\frac{\sqrt{k}r}{2}\right) + \frac{u_2v_1 - v_2u_1 - (u_2u_1 - v_2v_1)}{2\cos^2(\frac{\sqrt{k}r}{2})}\\ &= (u_2u_1 - v_2v_1)\left(\cot(\sqrt{k}r)\tan\left(\frac{\sqrt{k}r}{2}\right) - \frac{1}{2\cos^2(\frac{\sqrt{k}r}{2})}\right) + (u_2v_1 - v_2u_1)\left(\frac{\tan(\frac{\sqrt{k}r}{2})}{\sin(\sqrt{k}r)} + \frac{1}{2\cos^2(\frac{\sqrt{k}r}{2})}\right)\\ &= (u_2u_1 - v_2v_1)\left(\frac{\cos(\sqrt{k}r)}{2\cos^2(\frac{\sqrt{k}r}{2})} - \frac{1}{2\cos^2(\frac{\sqrt{k}r}{2})}\right) + (u_2v_1 - v_2u_1)\left(\frac{1}{2\cos^2(\frac{\sqrt{k}r}{2})} + \frac{1}{2\cos^2(\frac{\sqrt{k}r}{2})}\right)\\ &= -(u_2u_1 - v_2v_1)\tan^2\left(\frac{\sqrt{k}r}{2}\right) + (u_2v_1 - v_2u_1)\frac{1}{\cos^2(\frac{\sqrt{k}r}{2})}.\end{aligned}$$

□

For the coupling (X_t, Y_t) defined by (1.5) we can now use Itô's formula to end the proof of Proposition (1.5), finding an equation for the signed swept area, up to a constant, A_t :

Proof of Proposition (1.5), second part.

$$\begin{aligned} dA_t &= \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}R_t}{2}\right)(dU_2(t)+dV_2(t)) + \frac{1}{2}\Bigg(\frac{dU_2(t)\cdot dV_1(t) - dV_2(t)\cdot dU_1(t)}{\cos^2(\frac{\sqrt{k}R_t}{2})} \\ &\quad + \tan^2\left(\frac{\sqrt{k}R_t}{2}\right)(dV_2(t)\cdot dV_1(t) - dU_2(t)\cdot dU_1(t))\Bigg) \\ &= \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}R_t}{2}\right)(dU_2(t)+dV_2(t)) + \frac{1}{2}\left(\frac{dU_2(t)\cdot dV_1(t) - dV_2(t)\cdot dU_1(t)}{\cos^2(\frac{\sqrt{k}R_t}{2})}\right). \end{aligned}$$

We also directly obtain

$$dR_t \cdot dA_t = \frac{1}{\sqrt{k}}\tan\left(\frac{\sqrt{k}R_t}{2}\right)(dV_1(t)\cdot dU_2(t) - dU_1(t)\cdot dV_2(t)).$$

□

3.3 Co-adapted coupling

In this subsection, we will deal with the special case of co-adaptive Brownian motions on M. We first recall the general definition of a co-adapted coupling and then adapt it to our model:

Definition 3.1. *Let μ_t and ν_t the laws in time t of two continuous Markov processes $(X_t)_t$ and $(Y_t)_t$ on M. The process $(\tilde{X}_t, \tilde{Y}_t)_t$ is said a co-adapted coupling of $(X_t)_t$ and $(Y_t)_t$ if:*

- *the processes $\tilde{X}$ and $\tilde{Y}$ lay in a same filtered probability space $(\mathcal{F}_t)_t$ and follow in each time t the probability laws μ_t and ν_t respectively;*
- *for all bounded measurable function f, $z \in M$, s, $t > 0$, the functions $z \mapsto \mathbb{E}[f(\tilde{X}_{t+s}) \mid \mathcal{F}_s,\ \tilde{X}_s = z]$ and $z \mapsto \mathbb{E}[f(X_{t+s}) \mid X_s = z]$ (resp. $z \mapsto \mathbb{E}[f(\tilde{Y}_{t+s}) \mid \mathcal{F}_s,\ \tilde{Y}_s = z]$ and $z \mapsto \mathbb{E}[f(Y_{t+s}) \mid Y_s = z]$) are equals $\mathbb{P}_{X_s}$-a.s. (resp. $\mathbb{P}_{Y_s}$-a.s.).*

Proposition 3.4. *We consider a coupling $(X_t, Y_t)_t$ of Brownian motions on M satisfying (1) in a filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$. The following assertions are equivalent:*

(i) $(X_t, Y_t)_t$ is a co-adapted coupling;

(ii) $U := \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ and $V := \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ form a co-adapted coupling of Brownian motions in $\mathbb{R}^2$;

(iii) Enriching the filtration if needed, there exists a two dimensional Brownian motion W adapted to the filtration and independent of U, and K_t, $\hat{K}_t \in \mathcal{M}_2(\mathbb{R})$ with $K_t K_t^T + \hat{K}_t \hat{K}_t^T = I_2$, K_t, $\hat{K}_t \in \mathcal{F}_t$ such that :

$$dV(t) = K_t dU(t) + \hat{K}_t dW(t). \tag{14}$$

Proof. The equivalence of (ii) and (iii) is a well known fact about couplings in $\mathbb{R}^2$ that can be found in [10] for example.
Let us show that (ii) implies (i). The converse implication may be dealt with the same method, also used in [14]. We denote $\mathcal{G}$ the filtration induced by U and V. As said in a previous remark, our basis are solutions of equations depending of R, U and V. Then, $X_t, Y_t, (e_1^X(t), e_2^X(t))$ and $(e_1^Y(t), e_2^Y(t))$ are adapted to the filtration $\mathcal{G}$. Moreover, X_t and Y_t are Markov processes for this same filtration. Then, this is also true for the filtration $\mathcal{F}$ induced by X_t and Y_t as it is included in $\mathcal{G}$. Thus, for $s, t > 0$, $z \in M$, we have

$$\mathbb{E}[f(X_{t+s})|\mathcal{F}_s, X_s = z] = \mathbb{E}[\mathbb{E}[f(X_{t+s})|\mathcal{G}_s, X_s = z]|\mathcal{F}_s, X_s = z] = \mathbb{E}[P_{s,t}f(z)|\mathcal{F}_s, X_s = z] = P_{s,t}f(z)$$

with $P_{s,t}$ the transition function of X. Thus we obtain a co-adaptive coupling for (X, Y). □

Using (14), we can rewrite our relations for R_t and A_t:

Proposition 3.5. *For a co-adapted coupling satisfying (1) and Proposition (3.4), we have:*

$$\begin{aligned}
dR_t \cdot dR_t &= 2(1-K_{1,1})dt;\\
dR_t &\overset{(m)}{=} \sqrt{k}\cot(\sqrt{k}R_t)dt - \frac{\sqrt{k}}{\sin(\sqrt{k}R_t)}K_{2,2}dt = \sqrt{k}\frac{\cos(\sqrt{k}R_t)-K_{2,2}}{\sin(\sqrt{k}R_t)}dt;\\
dA_t \cdot dA_t &= 2\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(1+K_{2,2})dt;\\
dA_t &\overset{(m)}{=} \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(K_{1,2}-K_{2,1});\\
dR_t \cdot dA_t &= \frac{1}{\sqrt{k}}\tan(\frac{\sqrt{k}R_t}{2})(K_{1,2}-K_{2,1})dt.
\end{aligned}$$

Proof. Let us recall that $dR_t = dV_1(t) - dU_1(t) + \frac{\sqrt{k}}{\sin(\sqrt{t}R_t)}(\cos(\sqrt{t}R_t)dt - dU_2(t)\cdot dV_2(t)$. Thus:

$$\begin{aligned}
dR_t \cdot dR_t &= (dV_1(t)-dU_1(t))\cdot(dV_1(t)-dU_1(t))\\
&= 2dt - 2dU_1(t)\cdot dV_1(t)\\
&= 2dt - 2dU_1(t)\cdot(K_{1,1}dU_1(t)+K_{1,2}dU_2(t)) \text{ since } W \text{ is independant with } U\\
&= 2(1-K_{1,1})dt;\\
Drift(dR_t) &= \frac{\sqrt{k}}{\sin(\sqrt{t}R_t)}\big(\cos(\sqrt{t}R_t)dt - dU_2(t)\cdot(K_{1,2}dU_1(t)+K_{2,2}dU_2(t))\big)\\
&= \sqrt{k}\cot(\sqrt{k}R_t)dt - \frac{\sqrt{k}}{\sin(\sqrt{k}R_t)}K_{2,2}dt = \sqrt{k}\frac{\cos(\sqrt{k}R_t)-K_{2,2}}{\sin(\sqrt{k}R_t)}dt.
\end{aligned}$$

Then, using $dA_t = \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}(dU_2(t)+dV_2(t)) + \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(dU_2(t)\cdot dV_1(t) - dV_2(t)\cdot dU_1(t))$:

$$\begin{aligned}
dA_t \cdot dA_t &= \frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(dU_2(t)+dV_2(t))\cdot(dU_2(t)+dV_2(t))\\
&= \frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(2dt+2dU_2(t)\cdot dV_2(t))\\
&= 2\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}\big(dt+dU_2(t)\cdot(K_{2,1}dU_1(t)+K_{2,2}dU_2(t))\big)\\
&= 2\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(1+K_{2,2})dt;\\
Drift(dA_t) &= \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(dU_2(t)\cdot dV_1(t) - dV_2(t)\cdot dU_1(t))\\
&= \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}\big(dU_2(t)\cdot(K_{1,1}dU_1(t)+K_{1,2}dU_2(t)) - (K_{2,1}dU_1(t)+K_{2,2}dU_2(t))\cdot dU_1(t)\big)\\
&= \frac{1}{2\cos^2(\frac{\sqrt{k}R_t}{2})}(K_{1,2}-K_{2,1}).
\end{aligned}$$

Finally

$$\begin{aligned} dR_t \cdot dA_t &= \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}\big((dV_1(t) - dU_1(t)) \cdot (dU_2(t) + dV_2(t))\big) \\ &= \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}\big(dV_1(t) \cdot dU_2(t) - dU_1(t) \cdot dV_2(t)\big) \\ &= \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}\big((K_{1,1}dU_1(t) + K_{1,2}dU_2(t)) \cdot dU_2(t) - dU_1(t) \cdot (K_{2,1}dU_1(t) + K_{2,2}dU_2(t))\big) \\ &= \frac{\tan(\frac{\sqrt{k}R_t}{2})}{\sqrt{k}}(K_{1,2} - K_{2,1})dt. \end{aligned}$$

□

Example.

- *Synchronous coupling: We take $K_t = I_2$, $\hat{K}_t = 0$, $dV(t) = dU(t)$. We get:*

$$dR_t \cdot dR_t = 0;$$
$$Drift(dR_t) = \sqrt{k}\frac{cos(\sqrt{k}R_t) - 1}{\sin(\sqrt{k}R_t)}dt = -\sqrt{k} \times \tan(\frac{\sqrt{k}R_t}{2})dt.$$

Thus R_t is deterministic and $R_t = \frac{2}{\sqrt{k}} \arcsin(e^{-\frac{kt}{2}} \sin(\frac{\sqrt{k}R_0}{2}))$. We also have:

$$dA_t \cdot dA_t = 4\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}dt \text{ and } Drift(dA_t) = 0.$$

In particular, A_t is a martingale.

- *Reflection coupling: For $K_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{K}_t = 0$, $dV_1(t) = -dU_1(t)$ and $dV_2(t) = dU_2(t)$, we get:*

$$dR_t \cdot dR_t = 4dt \text{ and } Drift(dR_t) = -\sqrt{k} \times \tan(\frac{\sqrt{k}R_t}{2}).$$

As before, we have:

$$dA_t \cdot dA_t = 4\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}dt \text{ and } Drift(dA_t) = 0.$$

- *Perverse coupling: For $K_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{K}_t = 0$, $dV_1(t) = dU_1(t)$ and $dV_2(t) = -dU_2(t)$, we get:*

$$dR_t \cdot dR_t = 0;$$
$$Drift(dR_t) = \sqrt{k}\frac{\cos(\sqrt{k}R_t) + 1}{\sin(\sqrt{k}R_t)}dt = \sqrt{k} \times \cot(\frac{\sqrt{k}R_t}{2}).$$

Thus R_t is deterministic and $R_t = \frac{2}{\sqrt{k}} \arccos(e^{-\frac{kt}{2}} \cos(\frac{\sqrt{k}R_0}{2}))$. We also have A_t constant.

In all these examples, $dR_t \cdot dA_t = 0$.

We also can add a noise to these couplings in order to remove the drift part. In particular, this will permit to obtain a coupling with constant distance between the Brownian motions.

- Synchronous coupling with noise/ fixed-distance coupling:

  Taking $K_t = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\sqrt{k}R_t) \end{pmatrix}$, $\hat{K}_t = \begin{pmatrix} 0 & 0 \\ 0 & \sin(\sqrt{k}R_t) \end{pmatrix}$, we get R_t constant. We also have:

  $$dA_t \cdot dA_t = 2\frac{\tan^2(\frac{\sqrt{k}R_t}{2})}{k}(1+\cos(\sqrt{k}R_t))dt = \frac{4}{k}\sin^2(\frac{\sqrt{k}R_t}{2})dt = \frac{4}{k}\sin^2(\frac{\sqrt{k}R_0}{2})dt;$$
  $$Drift(dA_t) = 0.$$

  Note that A_t is a Brownian motion up to a multiplicative constant.

  We can do the same for the reflection coupling.

- Reflection coupling with noise: For $K_t = \begin{pmatrix} -1 & 0 \\ 0 & \cos(\sqrt{R_t}) \end{pmatrix}$, $\hat{K}_t = \begin{pmatrix} 0 & 0 \\ 0 & \sin(\sqrt{R_t}) \end{pmatrix}$, we get:

  $$dR_t \cdot dR_t = 4dt \text{ and } Drift(dR_t) = 0.$$

  Thus $\frac{1}{2}R_t$ is a Brownian motion. We also have:

  $$dA_t \cdot dA_t = \frac{4}{k}\sin^2(\frac{\sqrt{k}R_t}{2})dt \text{ and } Drift(dA_t) = 0.$$

For these two couplings, $dR_t \cdot dA_t = 0$ too.

Remark.
- *In [14], Pascu and Popescu proved on the sphere that there exists a co-adapted coupling of Brownian motions of deterministic distance function R_t if and only if R is continuous and satisfy the inequality*

  $$-\tan(\frac{R_t}{2})dt \leq dR_t \leq \cot(\frac{R_t}{2}))dt.$$

  In particular the synchronous coupling and perverse coupling described above are the couplings realizing the extrema of this inequality.

- *In the case where $k = 0$ we find all the expected results we could have for the study of the Heisenberg group. See for example [9, 5].*

4 Successful couplings

We are now going to interest ourselves in the construction of a successful coupling. In a first time, we will study successful coupling of Brownian motions on M, a Riemannian manifold with constant curvature, and their swept areas for the areas liying in $\mathbb{R}$. This will induce a first successful coupling on $SU(2)$. Then, considering that, in $SU(2)$, the last coordinate z_t lies in $\mathbb{R}/]-\frac{\pi}{2}, \frac{\pi}{2}]$, we will change some steps to obtain a more efficient coupling.
Using the previous notations, let us suppose that $k > 0$. As said before our method is based on the idea of Kendall for coupling two dimensional real Brownian motions and their swept areas ([9]). The original idea is to switch between reflection and synchronous coupling, using reflection coupling to make R_t decrease, and synchronous coupling to keep the swept area comparable to R_t^2 and decreasing as well.
Here, in comparison to Kendall's original proposition, we will add a noise during the "synchronous coupling step" as in the previous example in order to keep R_t constant. If not, we would have a strictly positive probability to be trapped in a "synchronous coupling step" without returning in a reflection coupling step. Note that, for $k \to 0$, that is for real two dimensional Brownian motion, fixed-distance coupling (synchronous coupling with noise) and synchronous coupling are the same.

Let us choose κ, $\epsilon > 0$ such that $0 < \epsilon < \kappa$. We denote τ the first time of coupling of the two Brownian motions AND their swept areas, that is:

$$\tau := inf\{t > 0 | R_t = 0 \text{ and } A_t = 0\}.$$

Using perverse or synchronous coupling if needed, we can suppose that $0 < R_0 < i(M)$. Note that for $k > 0$, we get $i(M) = \frac{\pi}{\sqrt{k}}$. Let us suppose that there exists $\eta > 0$ such that $i(M) - \eta > R_0 > 0$. We then define another stopping time:

$$\tau_\eta := \inf\{t > 0 | R_t \geq i(M) - \eta\}.$$

We are going to study first $\tau \wedge \tau_\eta$ instead of τ: this way we will have $R_t < i(M) - \eta$ for all $t < \tau$. Using constant fixed-distance coupling if necessary, we can suppose that $A_0 = 0$ without changing the value of R_0 (indeed, A_t is a changed-time Brownian motion during the fixed-distance coupling). We will construct the coupling as follow on $[0, \tau \wedge \tau_\eta]$:

1. We use reflection coupling until the process $\frac{|A_t|}{R_t^2}$ starting at 0 takes the value κ;
2. While the process $\frac{|A_t|}{R_t^2}$, starting at κ, satisfy $\frac{|A_t|}{R_t^2} > \kappa - \epsilon$ we use fixed-distance coupling;
3. While the process $\frac{|A_t|}{R_t^2}$, starting at $\kappa - \epsilon$, satisfy $\frac{|A_t|}{R_t^2} < \kappa$ we use reflection coupling.

We iterate 2 and 3 until $R_t = A_t = 0$ or $R_t = i(M) - \eta$.

Theorem 4.1. *Under the hypothesis $i(M) - 2\eta > R_0 > 0$, the co-adapted coupling described below satisfies $\tau \wedge \tau_\eta < +\infty$ a.s. for $k > 0$. Moreover, we get $\mathbb{P}(\tau > \tau_\eta) < 1$.*

Proof. We denote $\tau' := \tau \wedge \tau_\eta$. We define $N^{(\epsilon)} : [0, \tau'] \to \{0, 1\}$ such that $N^{(\epsilon)}(t) = 0$ during fixed-distance coupling and $N^{(\epsilon)}(t) = 1$ during reflection coupling. We get, for all $t > 0$:

$$dR_t = 2N^{(\epsilon)}(t)dC_t - \sqrt{k}\tan\left(\frac{\sqrt{k}R_t}{2}\right)N^{(\epsilon)}(t)dt$$

$$dA_t = \frac{2}{\sqrt{k}} \times \frac{\sin(\frac{\sqrt{k}R_t}{2})}{\cos(N^{(\epsilon)}(t)\frac{\sqrt{k}R_t}{2})}d\tilde{C}_t$$

with C_t and $\tilde{C}_t$ two independent real Brownian motions. Note that, during reflection coupling, that is when $N^{(\epsilon)}(t) = 1$, we have $\frac{|A_t|}{R_t^2} \leq \kappa$ whereas, during fixed-distance coupling, that is $N^{(\epsilon)}(t) = 0$, we have $\frac{|A_t|}{R_t^2} \geq \kappa - \epsilon$. As R varies only during reflection couplings, if $R_t \to 0$, we have $R_t^2 \geq \frac{1}{\kappa}|A_t|$, thus $A_t \to 0$. In fact, with this strategy we have $\tau' = \inf\{t \mid R_t \in \{0, i(M) - \eta\}\}$. Then we have to prove that R_t hits 0 or $i(M) - \eta$ in an a.s. finite time. For the following calculation, we take $t < \tau'$ that is $0 < R_t < i(M) - \eta$.

Let us define $\sigma(t) := \int_0^t \frac{4}{R_s^2}ds$ and $K_{\sigma(t)} = log(R_t)$. By Itô's formula, we get: $dK_{\sigma(t)} = \frac{dR_t}{R_t} - \frac{1}{2R_t^2}d\langle R_t, R_t\rangle$. Thus:

$$dK_{\sigma(t)} \cdot dK_{\sigma(t)} = \frac{4N^{(\epsilon)}(t)}{R_t^2}dt = N^{(\epsilon)}(t)d\sigma(t)$$

$$Drift(dK_{\sigma(t)}) = -\frac{\sqrt{k}}{R_t}\tan\left(\frac{\sqrt{k}R_t}{2}\right)N^{(\epsilon)}(t)dt - \frac{2N^{(\epsilon)}(t)}{R_t^2}dt = -N^{(\epsilon)}(t)\left(\frac{\sqrt{k}R_t\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right)d\sigma(t).$$

With this time change, excluding the times when $N^{(\epsilon)}(t) = 0$ (fixed-distance coupling) in which K stays constant, $\sigma \mapsto K_\sigma$ acts like a Brownian motion with negative drift.
Let us also define $W_{\sigma(t)} = \frac{A_t}{R_t^2}$. With Itô's formula, we get:

$$dW_{\sigma(t)} = \frac{1}{R_t^2}dA_t - 2\frac{A_t}{R_t^3}dR_t + \frac{1}{2} \times 6\frac{A_t}{R_t^4}d\langle R_t, R_t\rangle = \frac{1}{R_t^2}dA_t - 2\frac{A_t}{R_t^3}dR_t + 3 \times 4N^{(\epsilon)}\frac{A_t}{R_t^4}dt.$$

Thus we have:

$$\begin{aligned}
Drift(dW_{\sigma(t)}) &= \left(\frac{2\sqrt{k}}{R_t^3}\tan\left(\frac{\sqrt{k}R_t}{2}\right)N^{(\epsilon)}(t)A_t + 3 \times 4N^{(\epsilon)}(t)\frac{A_t}{R_t^4}\right)dt \\
&= N^{(\epsilon)}(t)W_{\sigma(t)}\left(\frac{\sqrt{k}R_t}{2}\tan\left(\frac{\sqrt{k}R_t}{2}\right) + 3\right)\ d\sigma(t), \\
dW_{\sigma(t)} \cdot dW_{\sigma(t)} &= \left(\frac{4sin^2\left(\frac{\sqrt{k}R_t}{2}\right)}{kR_t^4cos^2\left(N^{(\epsilon)}(t)\frac{\sqrt{k}R_t}{2}\right)} + \frac{16N^{(\epsilon)}A_t^2}{R_t^6}\right)dt \text{ as } \langle dC_t, d\tilde{C}_t\rangle = 0 \\
&= \left(\frac{\sin^2\left(\frac{\sqrt{k}R_t}{2}\right)}{kR_t^2\cos^2\left(N^{(\epsilon)}(t)\frac{\sqrt{k}R_t}{2}\right)} + \frac{4N^{(\epsilon)}A_t^2}{R_t^4}\right)d\sigma(t) \\
&= \left(\frac{1}{4}\left(\frac{2}{\sqrt{k}R_t}\right)^2\frac{\sin^2\left(\frac{\sqrt{k}R_t}{2}\right)}{\cos^2\left(N^{(\epsilon)}\frac{\sqrt{k}R_t}{2}\right)} + 4N^{(\epsilon)}(t)W_{\sigma(t)}^2\right)d\sigma(t).
\end{aligned}$$

Finally, $dK_{\sigma(t)} \cdot dW_{\sigma(t)} = \frac{1}{R_t}dR_t \cdot \left(-2\frac{A_t}{R_t^3}dR_t\right) = -2\frac{A_t}{R_t^4} \times 4N^{(\epsilon)}(t)dt = -2N^{(\epsilon)}(t)W_{\sigma(t)}\ d\sigma(t)$.

Let $S := \sigma(\tau')$. For the following part, in order to simplify the notations, we will denote σ instead of $\sigma(t)$ and $N^{(\epsilon)}(\sigma)$ instead of $N^{(\epsilon)}(t)$. We want to show that $\tau' < +\infty$ a.s. As $\int_0^S e^{2K_\sigma}d\sigma = \int_0^\tau 4e^{2log(R_t)}\frac{dt}{R_t^2} = 4\tau'$, this is the same as showing that $\int_0^S e^{2K_\sigma}d\sigma < +\infty$ a.s.
First of all let us note that the number of changes of type of coupling (reflection coupling/fixed-distance coupling) is countable as it is finite on all closed and bounded interval of time $[S_1, S_2]$ such that $S_2 < \tau'$. Actually if we have an infinite number of changes of type of coupling, we can define (considering the time scale induced by σ) two sequences $(\sigma_n^s)_n$ and $(\sigma_n^r)_n$ such that $\sigma_n^s < \sigma_n^r < \sigma_{n+1}^s$ for all $n \geq 0$ and such that $N^{(\epsilon)}(\sigma) = 0$ on $[\sigma_n^s, \sigma_n^r[$ and $N^{(\epsilon)}(\sigma_n^r) = 1$. As $S_2 < \tau'$, we have R_t non nul and W_σ well defined on $[S_1, S_2]$ such that $|W_{\sigma_n^s}| = \kappa$ and $|W_{\sigma_n^r}| = \kappa - \epsilon$. As $(\sigma_n^s)_n$ and $(\sigma_n^r)_n$ converge to the same limit, this leads to a contradiction concerning the two previous equalities as $|W|$ is continuous.

Thus we have a countable number of changes of type of coupling. Using the previous notations, let us denote $[\sigma_n^s, \sigma_n^r[$ the interval during which $N^{(\epsilon)}(\sigma) = 0$ and $[\sigma_n^r, \sigma_{n+1}^s[$ the interval during wich $N^{(\epsilon)}(\sigma) = 1$. The same way we denote t_n^s and t_n^r such that $\sigma(t_n^s) = \sigma_n^s$ and $\sigma(t_n^r) = \sigma_n^r$.
As seen before, R is constant on intervals $[t_n^s, t_n^r[$ and acting as a two times Brownian motion with negative drift out of these intervals. As the time needed to exit a bounded open interval for a real Brownian motion with negative drift is a.s. finite, we have $\sum_{n\geq 0}(t_{n+1}^s - t_n^r) < +\infty$ a.s. Note that:

$$\int_0^S N^{(\epsilon)}(\sigma)e^{2K_\sigma}d\sigma = \sum_{n\geq 0}\int_{\sigma_n^r}^{\sigma_{n+1}^s} e^{2K_\sigma}d\sigma = \sum_{n\geq 0}\int_{\sigma_n^r}^{\sigma_{n+1}^s} R_t^2 d\sigma = \sum_{n\geq 0}\int_{t_n^r}^{t_{n+1}^s} 4dt = 4\sum_{n\geq 0}(t_{n+1}^s - t_n^r).$$

Then, this quantity is a.s. finite and, in order to show that $\tau' < +\infty$, it is enough to show that:

$$\int_0^S (1 - N^{(\epsilon)}) e^{2K_\sigma} d\sigma < +\infty.$$

As R_t and K_σ are constant during fixed-distance coupling, we have:

$$\int_0^S (1 - N^{(\epsilon)}) e^{2K_\sigma} = \sum_{n \geq 0} \int_{\sigma_n^s}^{\sigma_n^r} e^{2K_\sigma} d\sigma = \sum_{n \geq 0} e^{2K_{\sigma_n^s}} (\sigma_n^r - \sigma_n^s).$$

Thus we want to show that $\sum_{n \geq 0} e^{2K_{\sigma_n^s}} (\sigma_n^r - \sigma_n^s) < +\infty$ a.s.

- We will first show the equality:

$$\mathbb{E}\left[\exp\left(-\sum_{n \geq 0} e^{2K_{\sigma_n^s}} (\sigma_n^r - \sigma_n^s) \right) \Big| (K_{\sigma_m^s})_m \right] = \exp\left(-\sum_{n \geq 0} e^{K_{\sigma_n^s}} \times \frac{\frac{\sqrt{k} R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} \times 2\sqrt{2}\epsilon \right). \quad (15)$$

  As our coupling is the fixed-distance one on $[\sigma_n^s, \sigma_n^r]$, there exists V_σ, a real Brownian motion, such that:

$$dW_\sigma = \frac{1}{\sqrt{k} R_t} \sin\left(\frac{\sqrt{k} R_t}{2} \right) dV_\sigma = \frac{1}{\sqrt{k} R_{t_n^s}} \sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right) dV_\sigma.$$

  In particular $|W_\sigma|$ only depends on $R_{t_n^s}$ for $\sigma_n^r \geq \sigma \geq \sigma_n^s$ and so depends on $K_{\sigma_n^s}$ only.
  Then, knowing $\left(K_{\sigma_m^s}\right)_m$, $\sigma_n^r - \sigma_n^s$ is the first hitting time of $k - \epsilon$ by the process $|W_\sigma|$. As $|W_{\sigma_n^s}| = \kappa$, by continuity, W_σ keeps the same sign all along the interval $[\sigma_n^s, \sigma_n^r]$. Then we have:

$$\begin{aligned}
\sigma_n^r - \sigma_n^s &= \inf\{\sigma > 0 \ | W_{\sigma + \sigma_n^s} = \text{sign}(W_{\sigma_n^s})(\kappa - \epsilon)\} \\
&= \inf\{\sigma > 0 \ | (W_{\sigma + \sigma_n^s} - W_{\sigma_n^s}) = \text{sign}(W_{\sigma_n^s})(\kappa - \epsilon - \kappa)\} \\
&= \inf\left\{ \sigma > 0 \ \Big| \frac{\sqrt{k} R_{t_n^s}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} (W_{\sigma + \sigma_n^s} - W_{\sigma_n^s}) = -\frac{\sqrt{k} R_{t_n^s}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} \text{sign}(W_{\sigma_n^s}) \epsilon \right\}.
\end{aligned}$$

  Moreover, on $[\sigma_n^s, \sigma_n^r]$ and conditional on $K_{\sigma_n^s}$, $\frac{\sqrt{k} R_{t_n^s}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} (W_{\sigma + \sigma_n^s} - W_{\sigma_n^s})_\sigma$ is a real Brownian motion starting from 0. Thus, conditional on $K_{\sigma_n^s}$, $\sigma_n^r - \sigma_n^s$ has the same law than T_{a_n} the hitting time of $a_n = \frac{\sqrt{k} R_{t_n^s}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} \epsilon$ by a real Brownian motion starting at 0. Then the Laplace transform gives:

$$\mathbb{E}[\exp(-\mu T_{a_n})] = \exp(-a_n \sqrt{2\mu}) \ \forall \mu > 0 \text{ and } \mathbb{E}[\exp\left(-\mu(\sigma_n^r - \sigma_n^s)\right) | K_{\sigma_n^s}] = \exp(-\frac{\sqrt{k} R_{t_n^s}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} \epsilon \sqrt{2\mu}).$$

  Finally:

$$\mathbb{E}\left[\exp\left(-e^{2K_{\sigma_n^s}} (\sigma_n^r - \sigma_n^s) \right) | (K_{\sigma_m^s})_m \right] = \exp\left(-\frac{\sqrt{k} R_{t_n^s}}{\sin\left(\frac{\sqrt{k} R_{t_n^s}}{2} \right)} \epsilon \sqrt{2} e^{K_{\sigma_n^s}} \right).$$

Furthermore, conditional to $(K_{\sigma^s_m})_m$, $(\sigma^r_n - \sigma^s_n)_n$ are independent. Thus:

$$\mathbb{E}\left[\exp\left(-\sum_{n=0}^{N} e^{2K_{\sigma^s_n}}(\sigma^r_n - \sigma^s_n)\right)\middle|(K_{\sigma^s_m})_m\right] = \exp\left(-\sum_{n=0}^{N}\frac{\frac{\sqrt{k}R_{t^s_n}}{2}}{\sin\left(\frac{\sqrt{k}R_{t^s_n}}{2}\right)}2\epsilon\sqrt{2}e^{K_{\sigma^s_n}}\right).$$

Using the dominated convergence theorem we get the announced equality.

- We will now deal with the quantity $\sum_{n\geq 0}\frac{\frac{\sqrt{k}R_{t^s_n}}{2}}{\sin\left(\frac{\sqrt{k}R_{t^s_n}}{2}\right)}e^{K_{\sigma^s_n}}$ occurring in the previous result.

Let us first notice that $\int_0^{\sigma^s_n} N^{(\epsilon)}(\sigma)d\sigma = \sum_{m=0}^{n-1}(\sigma^s_{m+1} - \sigma^r_m)$. The quantities $\sigma^s_{m+1} - \sigma^r_m$ are the first exit times of the open $]-\kappa,\kappa[$ during reflection times $([\sigma^r_m, \sigma^s_{m+1}])$ by the diffusion W_σ starting at $\pm(\kappa-\epsilon) \in]-\kappa,\kappa[$. Note that, unlike the fixed-distance coupling case, here the sign of W_t can change. During these times we have:

$$\begin{cases} dW_{\sigma(t)} \cdot dW_{\sigma(t)} = \left(\frac{1}{4}\left(\frac{\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{\frac{\sqrt{k}R_t}{2}}\right)^2 + 4W^2_{\sigma(t)}\right)d\sigma(t) \\ Drift(dW_{\sigma(t)}) = W_{\sigma(t)}\left(\frac{\sqrt{k}R_t}{2}\tan\left(\frac{\sqrt{k}R_t}{2}\right) + 3\right)\ d\sigma(t). \end{cases} \tag{16}$$

We now take m some positive integer. We define a new time change: $\zeta_m(\sigma) := \int_0^\sigma dW_{s+\sigma^r_m} \cdot dW_{s+\sigma^r_m}$. As R_t is upper-bounded by $i(M) - \eta$, there exists $0 < M$ a constant such that $\frac{\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{\frac{\sqrt{k}R_t}{2}} < M$ and $\frac{\sqrt{k}R_t}{2}\tan\left(\frac{\sqrt{k}R_t}{2}\right) < M$. Thus,

$$\frac{\sigma}{4} \leq \zeta_m(\sigma) \leq \left(\frac{M^2}{4} + 4\kappa^2\right)\sigma \text{ and } \left|\frac{Drift(d(W^2_\sigma))}{d\zeta_m(\sigma)}\right| \leq 4\kappa(M+3)$$

Then there exists a one dimensional Brownian motion B^m, starting at 0 and independent of ζ_m, such that for all $\sigma \in [0, \sigma^s_{m+1} - \sigma^r_m]$:

$$B^m_{\zeta_m(\sigma)} - 4\kappa(3+M)\zeta_m(\sigma) \leq W_{\sigma+\sigma^r_m} - W_{\sigma^r_m} \leq B^m_{\zeta_m(\sigma)} + 4\kappa(3+M)\zeta_m(\sigma).$$

We now obtain

$$\zeta_m(\sigma^s_{m+1} - \sigma^r_m) \geq \inf\{\zeta > 0 \mid B^m_\zeta + 4\kappa(3+M)\zeta = \kappa - W_{\sigma^r_m}\} \wedge \inf\{\zeta > 0 \mid B^m_\zeta - 4\kappa(3+M)\zeta = -\kappa - W_{\sigma^r_m}\}.$$

As $W_{\sigma^r_m}$ can only take the two values $\kappa - \epsilon$ and $-(\kappa-\epsilon)$, we get $\zeta_m(\sigma^s_{m+1} - \sigma^r_m) \geq T_m$ with:

$$T_m := \inf\{\zeta > 0 \mid B^m_\zeta + 4\kappa(3+M)\zeta = \epsilon\} \wedge \inf\{\zeta > 0 \mid B^m_\zeta - 4\kappa(3+M)\zeta = -\epsilon\}.$$

In particular, we have:

$$\sigma^s_{m+1} - \sigma^r_m \geq \frac{1}{\frac{M^2}{4} + 4\kappa^2}T_m. \tag{17}$$

For all m, $(T_m)_m$ is a sequence of independent and equally distributed variables with nonnegative and finite mean. Then, the strong law of large numbers yields:

$$\left(\frac{M^2}{4} + 4\kappa^2\right)\frac{1}{n}\sum_{m=0}^{n-1}(\sigma^s_{m+1} - \sigma^r_m) \geq \frac{1}{n}\sum_{m=0}^{n-1}T_m \xrightarrow[n\to+\infty]{a.s.} \mathbb{E}[T_0].$$

Thus, for n large enough, we get a.s.:

$$\frac{\int_0^{\sigma_n^s} N^{(\epsilon)}(\sigma)d\sigma}{n} = \frac{1}{n}\sum_{m=0}^{n-1}(\sigma_{m+1}^s - \sigma_m^r) \geq \frac{\mathbb{E}[T_0]}{2(\frac{M^2}{4}+4\kappa^2)} > 0. \tag{18}$$

Moreover, we obtain $\int_0^{\sigma_n^s} N^{(\epsilon)}(\sigma)d\sigma \xrightarrow[n\to+\infty]{a.s} +\infty$

Let us now recall that $K_\sigma = K_0 + \int_0^\sigma N^{(\epsilon)}(s)dC_s - \int_0^\sigma N^{(\epsilon)}\left(\frac{\sqrt{k}R_t\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right)ds$. Thus $K_\sigma - K_0 + \int_0^\sigma N^{(\epsilon)}\left(\frac{\sqrt{k}R_t\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right)ds$ is a Brownian motion for the change of time $\int_0^\sigma N^{(\epsilon)}(s)ds$.

Note that:

$$\frac{1}{2} \leq \frac{\int_0^{\sigma_n^s} N^{(\epsilon)}\left(\frac{\sqrt{k}R_t\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right)ds}{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds}.$$

By the strong law of large number for Brownian motions, we also have:

$$\frac{K_{\sigma_n^s} - K_0 + \int_0^{\sigma_n^s} N^{(\epsilon)}\left(\frac{\sqrt{k}R_t\tan\left(\frac{\sqrt{k}R_t}{2}\right)}{4} + \frac{1}{2}\right)ds}{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds} \xrightarrow[n\to+\infty]{a.s} 0.$$

Finally, a.s. for n large enough, we obtain:

$$\frac{K_{\sigma_n^s}}{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds} \leq -\frac{1}{4}. \tag{19}$$

By combining the results (18) and (19), we get a.s. for n large enough and noting $c_0 = \frac{\mathbb{E}[T_0]}{2\left(\frac{M^2}{4}+4\kappa^2\right)}$:

$$\frac{1}{n}K_{\sigma_n^s} = \frac{K_{\sigma_n^s}}{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds} \times \frac{\int_0^{\sigma_n^s} N^{(\epsilon)}(s)ds}{n} \leq -\frac{c_0}{4} < 0$$

It remains to notice that a.s., for all n, we have $0 < \frac{\sqrt{k}R_{t_n^s}}{2} < \frac{\pi}{2}$. Thus:

$$\frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)} \leq \frac{\pi}{2} \text{ and } \sum_{n\geq M}\frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}e^{K_{\sigma_n^s}} \leq \frac{\pi}{2}\sum_{n\geq M}(e^{-\frac{c_0}{4}})^n < +\infty.$$

Finally we get:

$$\sum_{n\geq 0}\frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}e^{K_{\sigma_n^s}} < +\infty \ a.s. \tag{20}$$

- Using (15) and (20), we have $\mathbb{E}[\exp(-\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s))|(K_{\sigma_m^s})_m] > 0$ a.s.

  Still conditional to $(K_{\sigma_m^s})_m$, since $\exp(-\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)) \geq 0$, the event $\exp(-\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)) > 0$, equivalent to the event $\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s) < +\infty$, has a non-zero probability. As $\left(e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s)\right)_n$

are independent, using the Kolmogorov zero-one law, we get $\mathbb{P}(\sum_{n\geq 0} e^{K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s) < +\infty | (K_{\sigma_m^s})_m) = 1$ a.s.
Finally, $\sum_{m\geq 0} e^{2K_{\sigma_m^s}}(\sigma_m^r - \sigma_m^s) < +\infty$ a.s. and so $\tau' < +\infty$ a.s.

To show that $\mathbb{P}(\tau > \tau_\eta) < 1$, we just need to remark that this event only depends on the evolution of R_t. As R_t is acting as a time changed Brownian motion with negative drift omitting the times where it stays constant, we directly obtain our result. □

We can now give the proof of Theorem (1.6).

Proof of Theorem (1.6). To construct a successful coupling, we now just have to start the coupling described in Theorem 4.1.

(i) If $\tau \wedge \tau_\eta = \tau$, we obtain $R_t = A_t = 0$.

(ii) If $\tau \wedge \tau_\eta = \tau_\eta$, we use a synchronous coupling until $R_t = R_0$ and then a fixed distance coupling until $A_t = 0$ and we re-start the coupling of theorem 4.1.

In step (ii), as R_t is deterministic and decreasing during synchronous coupling, each "synchronous step" will take a constant finite time. It also takes an a.s. finite time to obtain $A_t = 0$ with fixed distance coupling. Thus we repeat the same experiment independently. As the probability that $\tau < \tau_\eta$ is non-zero during the coupling of theorem 4.1, it will take a finite number of change of coupling to be in this event and then have an a.s. finite time of success. Let us remind that, if $R_0 \in \{0, i(M)\}$, we need to use perverse or synchronous coupling before our successful coupling to start with $R_t \in]0, i(M)[$. As R_t is deterministic for perverse and synchronous couplings, this won't change our result. □

Remark. *As announced previously, we can improve this strategy in the case of $SU(2)$: considering $A_t \mod (2\pi)$ instead of a value in $\mathbb{R}$, we can suppose that A_t stays in $]-2\pi, 2\pi]$. This time we can take $\kappa < 2\pi$. This way, fixed-distance coupling is stopped when $W_\sigma \in \{-(\kappa-\epsilon); \kappa-\epsilon\}$ with a non zero probability to reach one or another of these bounds and there will be less time spent in fixed-distance coupling than in theorem (4.1). In particular, for $n \geq 0$, there exists $(V_\sigma)_\sigma$ a Brownian motion starting at 0 such that:*

$$\begin{aligned}
\sigma_n^r - \sigma_n^s &= \inf\left\{\sigma \;\middle|\; sgn(W_{\sigma_n^s})\kappa + \frac{sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}{\sqrt{k}R_{t_n^s}} V_\sigma \in \left\{sgn(W_{\sigma_n^s})(\kappa-\epsilon); sgn(W_{\sigma_n^s})\left(\frac{4\pi}{R_{t_n^s}^2} - (\kappa-\epsilon)\right)\right\}\right\} \\
&= \inf\left\{\sigma \mid sgn(W_{\sigma_n^s})\frac{sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)}{\sqrt{k}R_{t_n^s}} V_\sigma \in \{-\epsilon; \frac{4\pi}{R_{t_n^s}^2} - 2\kappa + \epsilon\}\right\} \\
&< \inf\{\sigma \mid sgn(W_{\sigma_n^s})V_\sigma = a_n\} \text{ with } a_n \text{ defined as before.}
\end{aligned}$$

This way, instead of (15), we obtain:

$$\mathbb{E}[\exp(-\sum_{n\geq 0} e^{2K_{\sigma_n^s}}(\sigma_n^r - \sigma_n^s))|(K_{\sigma_m^s})_m] > \exp\left(-\sum_{n\geq 0} e^{K_{\sigma_n^s}} \times \frac{\frac{\sqrt{k}R_{t_n^s}}{2}}{\sin\left(\frac{\sqrt{k}R_{t_n^s}}{2}\right)} \times 2\sqrt{2}\epsilon\right).$$

In addition to decreasing the time spent in fixed-distance coupling, having bounded values for A_t prevents from using synchronous coupling when R_t is too close of $i(M) = \pi$. Let us explain this.
During reflection coupling, W_σ stays continuous and still satisfies (16). Supposing that $R_t \leq \pi - \eta$ for

all t in $[t_m^r, t_{m+1}^s]$, we still have (17). Else, supposing that $R_t > \pi - \eta$ for some $t \in [t_m^r, t_{m+1}^s]$, we get $|W_{\sigma(t)}| = \frac{|A_t|}{R_t^2} \leq \frac{2\pi}{(\pi-\eta)^2}$. Choosing η, κ and ϵ such that $\frac{2\pi}{(\pi-\eta)^2} < \kappa - \epsilon < 2\pi$, we obtain $|W_{\sigma(t)}| < \kappa - \epsilon$. Thus reflection coupling won't end while R_t stays up to $\pi - \eta$. More, there will exist $\tilde{t}_m^r \in]t_m^r, t_{m+1}^s[$ such that $|W_{\sigma(\tilde{t}_m^r)}| = \kappa - \epsilon$ and $R_t \leq \pi - \eta$ for all t in $[\tilde{t}_m^r, t_{m+1}^s]$. As before, we obtain:

$$\sigma_{m+1}^s - \sigma_m^r > \sigma_{m+1}^s - \sigma(\tilde{t}_m^r) \geq \frac{1}{\frac{M^2}{4} + 4\kappa^2} T_m.$$

Thus the last inequality is always true and we get again (18). Following the rest of the proof of theorem (4.1), we directly obtain $\tau < +\infty$ a.s.

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