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# Hazardous waste transportation: a cost allocation analysis

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**Abstract** This paper studies hazardous waste transportation problems. Due to their dangerous nature, the transportation of these waste implies a risk of incident having irreversible consequences on the environment. This problem has lead to a body of legal statutes that monitor the generation, storage and transportation of hazardous waste. Assuming that the transport of hazardous waste is done in a cooperative manner on a transport network, this paper investigates how to share the cost of maintaining and operating such network among the involved agents. We analyze the hazardous transportation problem from the viewpoint of axiomatic analysis. We consider several axioms that are interpreted through different environmental law principles and provide a characterization of a new allocation rule: the responsibility rule. Then we show that the responsibility rule coincides with the multi-choice Shapley value of an appropriate multi-choice game.

*Keywords:* Hazardous waste, Transportation network, Cost allocation, Multi-choice games.

*JEL Codes:* C71, Q53, R42.

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## 1 Introduction

Over the last decades, the generation of hazardous waste from industrial activities has been steadily increasing. Tons of such waste are traded both internationally and locally, involving massive waste movements. Because of the nature of such waste, threatening incident may occur during transportation causing damage that may be irreversible. To address the problems of hazardous waste, a variety of environmental regulations have emerged, establishing a body of legal statutes that monitor their generation and transportation. To cite a few: the “Basel convention” regulates the international movements of hazardous waste and their disposal; the “Comprehensive Environmental Response, Compensation, and Liability Act” (CERCLA or Superfund Act) regulates the storage sites of hazardous waste in the United States; the “Treaty on the Functioning of the European Union” regulates the movements of hazardous waste within its borders. Each of these regulations is based on a preventive approach that aims to implement appropriate measures to reduce the risk associated with hazardous waste.

In this paper, we consider a finite set of agents involved in a hazardous waste transportation network. The network is modeled by a directed sink tree graph divided into several portions, and each agent has a certain amount of waste it wishes to ship. We suppose that a central authority

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sets preventive measures to mitigate the environmental risks associated with the hazardous waste involved in the network. Such preventive measures entails a maintenance cost for each portion of the network, which depends on the amount of waste passing through this portion. Two questions then arise: which agent is liable for the risk on the network? How should the cost of maintaining the network be allocated among its users?

To answer these questions, we define a hazardous waste transportation problem as a tuple consisting of a directed sink tree graph, a vector of maximal amounts of waste and a list of cost functions (one for each portion of the network). The sink of the tree is a waste treatment facility where the total waste is shipped. The vector of maximal amounts of waste corresponds to the total amounts of waste each agent is shipping to the facility. The cost function of each portion describes the maintenance cost of the portion when a certain amount of waste is conveyed to it. An allocation rule is defined as a mapping that associates to each hazardous waste transportation problem a list of payoffs describing the cost share of each agent for each of its amount of waste.

We analyze the hazardous transportation problem from the viewpoint of axiomatic analysis. We consider several axioms that are derived from different environmental law principles. These principles, that we further discuss in Section 3.2, can be distinguished into two subsets. The first subset is specific to environmental issues and contains: the prevention and precautionary principles, which set the duty of agents to take appropriate measures to prevent environmental risk; the polluter pays principle, which asserts that the cost associated with a pollution hazard should be born by those causing the hazard. The second subset of principles intersects with some tort law principles. It contains different liability regimes that set out the terms under which an agent is liable for certain hazards. Inspired by the Basel convention, we consider the strict liability, the joint liability and the several liability regimes. This allows us to clearly define which agent is liable for the risk caused on which portion of the network. Accordingly, we consider several axioms that can be interpreted with these environmental law principles. The first axiom is a classical Efficiency axiom widely used in allocation problems. In our framework, this axiom requires that the total maintenance cost of the hazardous waste transportation network should be fully covered by the whole set of agents. The second axiom is related to the several liability principle. Several liability is a legal doctrine that depicts an agent liable in proportion to the risks it causes. Accordingly, we consider the axiom of Independence of higher waste amounts, which requires that the cost share of an agent for any amount of waste does not depend on a higher waste amount of any agent, including its own. This implies that an agent does not pay for the risk associated with waste it is not able to generate. Then, we consider the axiom of Path consistency, which requires that an agent's cost share does not depend on the costs of portions not used by that agent. We relate this axiom to the strict liability whereby an agent's liability is tied to the portion it uses. Finally, we introduce the axiom of Upstream solidarity for a cost increase, which requires that each agent located upstream a portion should be equally impacted if the marginal cost of this portion varies. This axiom complements the joint liability principles by removing any ambiguity that may arise under this liability regime.

We show that there is a unique allocation rules satisfying Efficiency, Independence of higher waste amounts, Path consistency and Upstream solidarity for a cost increase (Proposition 4). This rule, called the responsibility rule (Theorem 1), can be described as follows. Consider any portion  $k$  and suppose that each agent liable for this portion is shipping the same amount of waste  $j$ . If an agent is not able to ship this amount then it ships its maximum. In this case, we say that the agents form a  $j$ -synchronized waste profile. Then, the agents liable for the portion  $k$ , and that

ship the amount  $j$ , share equally the variation in cost between shipping the  $j$ -synchronized and shipping the  $(j - 1)$ -synchronized waste profiles. By successively applying this procedure on each portion for which an agent is liable, we obtain its payoff (for its waste amount  $j$ ) according to the responsibility rule.

Finally, we analyze hazardous waste transportation problems by adopting a cooperative game theoretic approach. Specifically, we define a multi-choice game from a hazardous waste transportation problem. We show that the responsibility rule coincides with the multi-choice Shapley value, introduced in Lowing and Techer [2022], of such multi-choice game (Theorem 3).

## Related literature

This work is closely related to the wide literature on cost sharing problems in networks. Specifically, our model generalizes the model of cleaning a polluted river introduced in Ni and Wang [2007]. The authors consider a polluted river as a line divided into several segments. The cost of cleaning each segment is exogenously given. Ni and Wang [2007] propose and characterize two allocation rules: the local responsibility rule and the upstream responsibility rule. Further, they show that both rules coincide with the Shapley value of an appropriate cooperative TU-game. This model has been extended by Dong et al. [2012] by considering a polluted river network modeled by a directed sink tree. Recently, van den Brink et al. [2018] study the polluter river network problem and show that a polluted river network can be interpreted as a permission tree structure. They provide different allocation rules based on solutions for TU-games with a permission structure. Based on the polluter pays principle, Gómez-Rúa [2013] propose new allocation rules for sharing the cost of a polluted river inspired by properties of water taxes. Alcalde-Unzu et al. [2015] consider a transfer rate that measures the proportion of pollutant transferred from one segment to another. They show that uncertainty on this transfer rate can be reduced by using information from the cleaning costs. This allows to define some limits to the liability of agents. Moreover they propose and characterize the Upstream responsibility rule that takes into account these limits. Although our model can be related to Ni and Wang [2007] and Dong et al. [2012], our study differs from them on two points. First, we allow the agents to have multiple amounts of waste. Second, we explicitly consider that the amounts of waste flow from one point of network to another. This leads to a cumulative effect of the waste in the cost of maintaining a portion. Such effect is not taken into account by Dong et al. [2012]. Two recent studies that rely on multi-choice cooperative game concepts are closely related to ours. Lowing [2023a] studies the cost allocation problem in which a set of consumer is part of a natural gas distribution network that they jointly use. The author provides an axiomatic study in this context and characterizes the Connection rule, the Uniform rule and their convex combinations. The author also highlights their relations with the multi-choice Shapley value, the multi-choice Equal division value and the multi-choice egalitarian Shapley value.<sup>1</sup> To our knowledge, the closest study to ours is that of Béal et al. [2023]. The authors further study the model of hazardous waste transportation and introduce a new allocation rule, the Liability rule. They also provide several axiomatic characterizations and show the relation with a new extension of the Shapley value to multi-choice games. A comparable axiomatic study is also provided, the authors shows that the Liability rules differs from the responsibility rule with respect to two axioms only.

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<sup>1</sup>Since the Connection rule coincides with multi-choice Shapley value of an appropriate multi-choice game defined from a gas distribution problem, it is direct that it also coincides with the responsibility rule studied in this paper.

This work can also be related to the literature on transportation of hazardous materials. This literature mainly focuses on the reduction of threatening incident that may arise from the transportation of hazardous materials. Using an operation research point of view, the approach is more about routing-scheduling and network design than on cost allocation. A recent survey of network design problem can be found in Mohri et al. [2021]. Therefore, this paper can be considered as a complementary study, by considering the cost allocation point of view.

This paper is organized as follows. The next section introduces the notation and provides a brief presentation of directed graphs. Section 3 introduces the hazardous waste transportation problem. Section 3.1 presents the model, and Section 3.2 discusses the different environmental principles. Our axiomatic analysis is contained in Section 3.3, while Section 4 develops the multi-choice cooperative game approach. Finally, Section 5 concludes. All the proofs are collected in an Appendix.

## 2 Preliminaries

Denote by  $\mathbb{N}$  the set of natural integers containing 0, and fix  $K \in \mathbb{N}$ . Given an integer  $x \leq K$ , denote by  $[[0, x]]$  the set  $\{0, 1, \dots, x\}$  of successive integers from 0 to  $x$ , i.e.,  $[[0, x]] = \{z \in \mathbb{N} : 0 \leq z \leq x\}$ .

### 2.1 Directed (tree) graph

A directed graph (henceforth *digraph*) is a pair  $g = (N, E)$  where  $N$  is a finite set of nodes (representing the set of agents) and  $E \subseteq N \times N$  is a binary irreflexive relation. This binary relation  $E$  induces a collection of ordered pairs, i.e.,  $E \subseteq \{(i, i') \in N \times N : i \neq i'\}$ . The pair  $(i, i') \in E$  is interpreted as: there exists a directed link from  $i$  to  $i'$  in  $g$ . For each node  $i \in N$ , define  $U(i) = \{i' \in N : (i', i) \in E\}$  the set of *direct predecessors* of  $i$  in  $g$ . The set  $U^{-1}(i) = \{i' \in N : (i, i') \in E\} = \{i' \in N : i \in U(i')\}$  denotes the set of *direct successors* of  $i$  in  $g$ . Given a digraph  $g$ , a (directed) *path* from  $i$  to  $i'$  is a sequence of distinct nodes  $(i_1, \dots, i_h)$  such that  $i_1 = i$  and for each  $t = 1, \dots, h - 1$ ,  $i_{t+1} \in U^{-1}(i_t)$ , and  $i_h = i'$ . For each  $i \in N$ , define  $\hat{U}(i) = \{i' \in N : \text{there exists a path from } i' \text{ to } i\}$  as the set of agents located *upstream*  $i$  in  $g$ . Similarly, for each  $i \in N$  define  $\hat{U}^{-1}(i) = \{i' \in N : i \in \hat{U}(i')\}$  as the set of agents located *downstream*  $i$  in  $g$ . Given a set of agents  $S \subseteq N$ , we denote by  $U(S) = \bigcup_{i \in S} U(i)$  and  $U^{-1}(S) = \bigcup_{i \in S} U^{-1}(i)$  the set of predecessors, respectively successors, of agents in  $S$ . Similarly, for  $S \subseteq N$  we denote  $\hat{U}(S) = \bigcup_{i \in S} \hat{U}(i)$  and  $\hat{U}^{-1}(S) = \bigcup_{i \in S} \hat{U}^{-1}(i)$ . For convenience, we denote  $\hat{U}_S := \hat{U}(S) \cup S$ , and  $\hat{U}_S^{-1} := \hat{U}^{-1}(S) \cup S$ , and  $\hat{U}_k$  instead of  $\hat{U}_{\{k\}}$ .

Let  $g = (N, E)$  be a digraph. A directed path  $(i_1, \dots, i_h)$ , with  $h > 2$ , is a cycle in  $g$  if  $(i_h, i_1) \in E$ . A digraph  $g$  is called *acyclic* if there is no cycle in  $g$ . A digraph  $g$  is a *sink tree* if there is exactly one node  $i_d \in N$  such that  $U^{-1}(i_d) = \emptyset$ ,  $\hat{U}(i_d) = N \setminus \{i_d\}$ , and for each  $i \in N \setminus \{i_d\}$ ,  $|U^{-1}(i)| = 1$ . Observe that a sink tree is an acyclic directed graph. Denote by  $\mathcal{E}$  the collection of all irreflexive digraphs and by  $\bar{\mathcal{E}}$  the collection of all directed sink trees.

### 3 Hazardous waste transportation problem

#### 3.1 The model

Consider a *transportation network* connecting a finite set of agents  $N$  (directly or indirectly) to a special node  $d$  called the *delivery node*. Such transportation network structure is modeled by a sink tree  $g = (N \cup \{d\}, E)$ , where  $d$  is the sink of the tree. Each element  $(i, i') \in E$  is called a *portion*. Since  $g$  is a sink tree, each agent  $i \in N$  has exactly one direct successor in  $g$ . When no confusion arises, we simply denote a portion  $(i, i')$  by  $i$ . All the notation introduced for sink trees remain valid for a hazardous waste transportation network.

Each agent  $i \in N$  is endowed with a given amount of hazardous waste  $w_i \in \mathbb{N}$ ,  $0 < w_i \leq K$ , to be treated by a treatment facility. We assume that the treatment facility is located at the delivery node  $d$ . Each agent is able to ship any amount  $j \leq w_i$  of waste. However, the actual amount shipped is  $w_i$ . Consider the product set  $W = \prod_{i \in N} [[0, w_i]]$ . An element  $s = (s_i)_{i \in N} \in W$  is referred to a *waste profile*, which indicates each agent's amounts of waste. The profile of maximal amounts of waste is given by  $w = (w_i)_{i \in N}$ . Given the set of agents  $N$ , the profile of maximal amounts of waste  $w$ , and an amount  $j \leq \max_{i \in N} w_i$ , define  $Q(j)$  as the set of agents able to ship  $j$  amounts of waste, i.e.,  $Q(j) := \{i \in N : w_i \geq j\}$ . For any two distinct amounts of waste  $j, j'$  such that  $j \geq j'$ , we have that  $Q(j) \subseteq Q(j')$ . Moreover, given a waste amount  $j \leq \max_{i \in N} w_i$ , define the  $j$ -synchronized waste profile,  $(j \wedge w_i)_{i \in N}$ , in which each agent send the waste amount  $j$ . If an agent is unable to ship  $j$ , then it sends its maximal amounts of waste to the treatment facility.

**Example 1.** Let  $N = \{1, 2, 3, 4, 5\}$  be the set of agents involved in the transport of hazardous waste and  $d$  the delivery node. Consider the hazardous waste transportation network  $g = (N \cup \{d\}, E)$  where  $E = \{(1, d), (3, 1), (2, 1), (5, 2), (4, 2)\}$ , depicted in Figure 1. The direct successor of agent 3 is  $U^{-1}(3) = \{1\}$ . The set of direct predecessors of agent 2 is  $U(2) = \{4, 5\}$ . The set of agents located upstream agent 1 is  $\hat{U}(1) = N \setminus \{1\}$ .

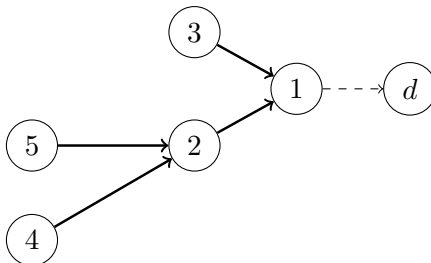


Figure 1: Hazardous waste transportation network.

Shipping any amount of waste through a portion incurs a risk, which in turn carries a cost. This cost is considered as the expense of maintaining the network and mitigating the risks associated with the amount of waste. For each portion  $i \in N$ , let  $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the cost function of this portion. We assume that, for each  $i \in N$  the cost function  $C_i$  is non-negative and non-decreasing over  $\mathbb{R}_+$ . Moreover, we use the convention  $C_i(0) = 0$  for any  $i \in N$ . For each portion  $i \in N$ , it is assumed that the cost function  $C_i$  depends on the total amounts of waste passing through the portion, i.e., the sum of the waste sent by all agents located upstream  $i$  including its own waste. Thus, the cost of transporting the waste profile  $w$  through a portion  $i$  is given by  $C_i(\sum_{k \in \hat{U}_i} w_k)$ . The

total cost of maintaining the network is then given by

$$\sum_{i \in N} C_i \left( \sum_{k \in \hat{U}_i} w_k \right). \quad (1)$$

Let  $\mathcal{C}$  be the set of all cost functions. Denote by  $C = (C_i)_{i \in N} \in \mathcal{C}^N$  a profile of cost functions. For each  $i \in N$ , we denote by  $C_i^0 \in \mathcal{C}$  the null cost function, i.e.,  $C_i^0(x) = 0$  for all  $x \in \mathbb{R}_+$ . A *hazardous waste transportation problem* on a fixed agent set  $N$  and delivery node  $d$  is then a triplet

$$(g, w, C) \in \bar{\mathcal{E}} \times W \times \mathcal{C}^N.$$

Let  $\mathcal{P}^N = \bar{\mathcal{E}} \times W \times \mathcal{C}^N$  be the set of all hazardous waste transportation problems on  $N$ . Notice that polluted river network problems, introduced in Dong et al. [2012], can be viewed as a subclass of hazardous waste transportation problem where  $w_i = 1$  for each  $i \in N$  and each cost function is a constant function.<sup>2</sup>

Consider any hazardous waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . The question that arises is how to allocate the total maintenance cost of the hazardous waste transportation network. An *allocation rule*  $f$  on  $\mathcal{P}^N$  assigns a non-negative incremental payoff (or cost share)  $f_{ij}(g, w, C) \in \mathbb{R}_+$  to each agent  $i \in N$  for each of its waste amount  $0 < j \leq w_i$ . Then, for each  $i \in N$  and each  $j \leq w_i$  the incremental payoff  $f_{ij}(g, w, C) \in \mathbb{R}_+$  is interpreted as the variation of payoff when agent  $i$  increases its transported waste from  $j - 1$  to  $j$ . Such interpretation is consistent with the fact that agent are actually shipping the amount  $w_i$ .

### 3.2 Principles from environmental legislation

The transport of hazardous waste is regulated by a set of legal rules and statutes established both at international and national levels. In this section, we present some principles invoked in the environmental issues and specifically in the transport of hazardous waste. We provide a brief interpretation of each principle within our model. While some principles can be related to tort law principles, others are very specific to environmental issues and deserve a specific interpretation.

**The prevention principle** The prevention principle is the most prevalent principle in environmental law. This principle is at the heart of the preventive approach to the environment, which aims at anticipating, minimizing and preventing environmental hazards. Indeed, the damage caused by pollution tends to be very expensive and even has irreversible consequences. The implementation of preventive measures is therefore crucial in the protection of the environment. Regarding hazardous waste management and transportation, the preventive approach is advocated by the Basel convention. Notably, the convention requires that anyone involved in the management or transportation of hazardous waste to take appropriate measures to prevent any potential pollution from the hazardous waste. In this sense, the prevention principle is the key principles underlying our model.

**The polluter pays principle** The polluter pays principle ensures that those causing pollution must bear the cost it generates. This principle, introduced by Arthur Cecil Pigou in the 1920's, aims at internalizing the external costs generated by polluting activities. It can be thought from two

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<sup>2</sup>We refer the reader to van den Brink et al. [2018] for a recent study of polluted river network problems.

complementary standpoints. First, this principle has a curative function by making the polluter liable in case of environmental damage. From this perspective, it can be seen as lying at the intersection between tort law principles and environmental statutes by assigning the liability of a damage to the polluter. This principle also has a preventive function, assigning the cost of prevention and precautionary measures to agents involved in environmentally hazardous activities. This is reflected in the recommendations on the pollution control made by the OECD or the European Union.<sup>3</sup>

**Strict liability** Strict liability is a legal doctrine according to which an agent is liable regardless of its intent. Thus, in case of a damage, this legal regime does not require justification of fault or negligence from the damaging agent. This doctrine is enforced in situations considered as inherently dangerous. It is required by the Basel convention, the European Environmental liability Directive (2004/35/EC), or the CERCLA (Superfund act). It complements the polluter-pays principle by defining the conditions ruling the liability for environmental hazard.<sup>4</sup> In our framework, the agents' liability is based on the risk associated with the hazardous waste. Since an agent's waste causes a risk to the portion it uses, that agent is liable for that portion.

**Joint liability** Joint liability is a second legal doctrine ruling situations with combined causes of damage. According to this doctrine, each agent involved in a damage is responsible for it, and can therefore be charged for it. In our framework, this principle considers that the agents causing a risk on the same portion of the network are jointly liable for this portion.

**Several liability** The several liability doctrine is another legal doctrine that holds an agent liable in proportion to its contribution to a damage. This doctrine can be found in the Basel convention asserting that each liable agent is liable in proportion to the contribution made by its waste to a damage. In our model, this doctrine hold the agents liable in proportion to their contribution to the risk incurred on the network.

### 3.3 Axiomatic analysis

In this section, we provide an axiomatic analysis on the class of hazardous waste transportation problems. We discuss several axioms for allocation rules inspired by the environmental principles introduced in Section 3.2.

We first consider the Efficiency axiom widely used in axiomatic analysis in economics. This axiom can be related to a major concern in most of the hazardous waste litigation, which is the damage remediation.<sup>5</sup> In our framework, this concern focuses on paying the cost of maintaining the network, leaving room for how to allocate this cost among the agents. Accordingly, the Efficiency axiom requires that the total cost for maintaining the network to be fully borne by all agents.

**Efficiency** For any  $(g, w, C) \in \mathcal{P}^N$ ,

$$\sum_{i \in N} \sum_{j \leq w_i} f_{ij}(g, w, C) = \sum_{i \in N} C_i \left( \sum_{k \in \dot{U}_i} w_k \right).$$

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<sup>3</sup>See OECD recommendations 1972, and the Treaty of Functioning of the European Union, Article 192(1).

<sup>4</sup>We refer the reader to Landes and Posner [1980] for a study in economic analysis of law.

<sup>5</sup>For instance, see the hazardous waste litigation in *United States v. Mosanto* (1988).



The second axiom can be thought as limiting an agent's cost share according to its waste's contribution to the risk of damage. In this sense, this axiom is weakly related to the several liability regime. This liability regime relies on the assumption that there exists a reasonable basis for determining each agent's contribution to the risk it causes to the network. In our framework, since each agent's waste is of the same nature, we use the waste amounts of each agent to define such a basis. Accordingly, the axiom of Independence of higher waste amounts axiom requires that an agent's incremental payoff for a given amount of waste  $j$  does not depend on any amount higher than  $j$ . This axiom is related to the Independence of maximal activity level introduced in Lowing and Techer [2022]. Regarding hazardous waste transportation problems, this axiom ensures that no one subsidizes a higher waste amount from any agent (including its own).

**Independence of higher waste amounts** For any  $i \in N$ , and any  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall j < w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, (j \wedge w_k)_{k \in N}, C).$$

Next, we consider an intuitive axiom that relates an agent's cost share to its distance from the delivery node. Relying on the several liability regime, we consider a second basis for determining the agents' contribution to the risk on the network. We argue that the greater the number of portions used to ship a given amount of waste, the greater the risk, and therefore the cost, incurred by the network. Distance consistency then requires that the farther an agent is from the delivery node, the higher its cost share for a given waste amount  $j$ .

**Distance consistency.** For any  $(g, w, C) \in \mathcal{P}^N$ , any  $i \in N$ ,  $j \leq w_i$  and  $k \in \hat{U}^{-1}(i) \cap Q(j)$ ,

$$f_{ij}(g, w, C) \geq f_{kj}(g, w, C).$$

The next axiom is inspired by the polluter pays principle when considering the strict liability regime. One interpretation of this liability regime (see Posner [1973]) implies that an agent insures the network against the risk it causes. In our framework, this risk is tied to the portion an agent uses. Conversely, an agent does not cause any risk on a portion it does not use. The next axiom is based on the above interpretation of strict liability, taking into account the portions an agent is liable for. Path consistency requires that an agent's cost share is independent of the cost of portions not used by that agent. This axiom is closely related to the axiom of Independence of upstream costs introduced in Dong et al. [2012] in the context of polluted river problems.

**Path consistency.** For any  $i \in N$  and any  $(g, w, C) \in \mathcal{P}^N$ ,  $(g, w, C') \in \mathcal{P}^N$  such that for each  $h \in \hat{U}_i^{-1}$ ,  $C_h = C'_h$ ,

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, w, C').$$

Actually, Path consistency can be related to several axioms invoked in the literature on cost sharing problems in networks. To cite a few, one may consider the axioms of: Independence of unused edges introduced by Bergantiños et al. [2019] in the context of energy networks; Individual independence of outside changes introduced by Sudhölter and Zarzuelo [2017] in the context of highway problems; No free riding introduced by Gopalakrishnan et al. [2021] in the context of allocating responsibility for emissions in a supply chain; Marginal damage independence introduced by Ferey and Dehez [2016] and Oishi et al. [2022] in the context of liability problems.

The next axiom is based on a fairness principle describing how the cost share of certain agents vary when only one cost function varies. Under the joint liability regime, each agent located

upstream of a portion is liable for the cost incurred on this portion. However, the extent to which an agent should be held liable remains to be determined. Upstream solidarity for a cost increase then requires that agents located upstream of a portion be equally impacted by a cost increase of this portion. Specifically, the axiom hypothesizes an increase in the marginal cost of a portion with respect to each amount of waste passing through this portion. Then it requires that the cost share of each agent responsible for this portion should be equally impacted.

**Upstream solidarity for a cost increase** For any  $i \in N$ , any  $(g, w, C), (g, w, C') \in \mathcal{P}^N$  such that

$$\forall x \in \mathbb{N}, \quad C_i(x) - C_i((x-1) \vee 0) \geq C'_i(x) - C'_i((x-1) \vee 0),$$

and, for each  $k \in N \setminus \{i\}$ ,  $C_k = C'_k$ , it holds that: for each  $j \in [[0, \max_{l \in \hat{U}_i} w_l]]$ , and each  $l, l' \in \hat{U}_i \cap Q(j)$ ,

$$f_{lj}(g, w, C) - f_{l'j}(g, w, C) = f_{lj}(g, w, C') - f_{l'j}(g, w, C').$$

Upstream solidarity for a cost increase can be seen as a strong application of the principle of solidarity discussed in Thomson [2016]. Since the agents are jointly liable for a portion, any change in the cost function has the same impact on each liable agent's cost share.

Below, we analyze the implication of these axioms. We first show that the combination of Efficiency and Path consistency implies that the cost share of an agent is null whenever the maintenance cost of each portion it is liable for is null.

**Proposition 1.** *Let  $f$  be an allocation rule on  $\mathcal{P}^N$ . If  $f$  satisfies Efficiency and Path consistency, then for any  $i \in N$ , any  $(g, w, C) \in \mathcal{P}^N$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C_k^0$  it holds that*

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = 0.$$

The proof of Proposition 1 is provided in the Appendix. Recall that the cost function is related to the risk involved by the transportation of an amount of waste. Accordingly, it describes the (monetary) impact of a given amount of waste passing through a portion. Thus, Proposition 1 complements our interpretation of the strict liability regime provided above. It implies that an agent's cost share should be zero if its wastes generate no risk on the network.

The next result shows the implication of combining Efficiency and Independence of higher waste amounts.<sup>6</sup>

**Proposition 2.** *Let  $f$  be an allocation rule on  $\mathcal{P}^N$ . If  $f$  satisfies Efficiency and Independence of higher waste amounts, then*

$$\forall j \leq \max_{i \in N} w_i, \quad \sum_{k \in Q(j)} f_{kj}(g, w, C) = \sum_{i \in N} \left( C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) - C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) \right).$$

Proposition 2 implies that the cost allocation rule considers each difference between two successive  $j$ -synchronized waste profile. This is a direct implication of the axiom of Independence of higher waste amounts. The proof of this result is provided in the Appendix.

Below, we show that any allocation rule that satisfies Efficiency, Independence of higher waste amounts, Path consistency, and Upstream solidarity for a cost increase also satisfies Distance consistency.

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<sup>6</sup>This result is related to Proposition 2 in Lowing and Techer [2022].

**Proposition 3.** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , the combination of Efficiency, Independence of higher waste amounts, Path consistency, and Upstream solidarity for a cost increase implies Distance consistency.*

We now introduce the main result of this section. We show that the combination of Efficiency, Independence of higher waste amounts, Path consistency and Upstream cost solidarity yields at most one allocation rule on  $\mathcal{P}^N$ .

**Proposition 4.** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , there is at most one allocation rule satisfying Efficiency, Independence of higher waste amounts, Path consistency and Upstream cost solidarity.*

### 3.4 The responsibility rule

Below, we propose a methods to allocate the total maintenance cost of the hazardous waste transportation network. The *responsibility rule* allocates the variation in cost of portion  $k$  between the  $j$  and  $(j - 1)$ -synchronized waste profiles equally among the agents located upstream of this portion.

Formally, for each hazardous waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ , the responsibility rule,  $f^R$ , is given by

$$\forall i \in N, \forall j \leq w_i,$$

$$f_{ij}^R(g, w, C) = \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j - 1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|}. \quad (2)$$

Observe that formula (2) relies on the implicit assumption that each agent's waste is of the same nature and therefore has the same impact on a portion. Moreover, one can provide a specific allocation process underlying the responsibility rule. Suppose that all agents agree to ship at the same moment the same amount of waste. Then, the variation of cost generated by this shipment on each portion is divided equally among the agents liable for the portion. The responsibility rule can be computed by repeating this reasoning for each amount of waste.

The next result shows that the responsibility rule is the unique allocation rule that matches our interpretation of the environmental principles presented in Section 3.2. Thus, the responsibility rule ensures that each agent pays a fair part of the cost for maintaining the hazardous waste network according to its liability for the risk it poses to the network.

**Theorem 1.** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , an allocation rule satisfies Efficiency, Independence of higher waste amounts, Path consistency, and Upstream solidarity for a cost increase if and only if it is the responsibility rule.*

The proof of Theorem 1 is provided in the Appendix.

**Logical independence** The axioms invoked in Theorem 1 are logically independent, as shown by the following alternative allocation rules on  $\mathcal{P}^N$ .

- The allocation rule  $f$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = 0,$$

satisfies all the axioms except Efficiency.

- The allocation rule  $f$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = \sum_{k \in \hat{U}_i} \frac{C_k(\sum w_l)}{\sum_{l \in \hat{U}_k} w_l},$$

satisfies all the axioms except Independence of higher waste amounts.

- The allocation rule  $f$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = \sum_{k \in N} \frac{C_k(\sum j \wedge w_l) - C_k(\sum (j-1) \wedge w_l)}{|Q(j)|},$$

satisfies all the axioms except Path consistency.

- Take any  $(g, w, C) \in \mathcal{P}^N$ . For each  $i \in N$  and each  $j \leq w_i$  fix an arbitrary  $\alpha^{ij} \in \mathbb{R}_{++}$ . The value  $f^\alpha$  defined as: for each  $(g, w, C) \in \mathcal{P}^N$ ,

$$\forall i \in N, \forall j \leq w_i,$$

$$f_{ij}^\alpha(g, w, C) = \sum_{k \in \hat{U}_i^{-1}} \frac{\alpha^{ij}}{\sum_{h \in \hat{U}_k \cap Q(j)} \alpha^{hj}} \times \left[ C_k(\sum_{k \in \hat{U}_k} j \wedge w_k) - C_k(\sum_{k \in \hat{U}_k} (j-1) \wedge w_k) \right],$$

satisfies all the axioms except Upstream solidarity for a cost increase.

## 4 A cooperative game-theoretical interpretation

In this section, we analyze the class of hazardous waste transportation problems by adopting a cooperative game theoretic approach. We first provide the formal material on multi-choice cooperative games introduced by Hsiao and Raghavan [1992], then we construct a multi-choice game from a hazardous waste transportation problem and focus on the multi-choice Shapley value [Lowing and Techer, 2022] of this game.

### 4.1 Multi-choice games

Let  $N = \{1, \dots, n\}$  be the finite and fixed *set of agents*. Each agent  $i \in N$  has a finite set of pairwise distinct activity levels  $M_i := \{0, \dots, m_i\}$  such that  $m_i \leq K$ . For each agent  $i \in N$ , the set  $M_i$  is linearly ordered from the lowest activity level 0 ( $i$  does not participate) to the maximal activity level  $m_i$ . Denote by  $Q(j) \subseteq N$  the set of *agents able to play the activity level  $j$* . Formally, the set  $Q(j)$  is defined as

$$Q(j) = \{i \in N : m_i \geq j\}.$$

Without loss of generality, we assume that  $Q(1) = N$ . Let  $\mathcal{M}$  be the cartesian product  $\prod_{i \in N} M_i$ . Each element  $s = (s_1, \dots, s_n) \in \mathcal{M}$  specifies a participation profile for agents and is referred to as a (multi-choice) coalition. Thus, a coalition indicates each agent's activity level. Then,  $m = (m_1, \dots, m_n) \in \mathcal{M}$  is the agents' maximal participation profile that plays the role of the grand coalition, whereas  $\mathbf{0} = (0, \dots, 0)$  plays the role of the empty coalition. Let  $s \in \mathcal{M}$  be a multi-choice coalition. We denote by  $(s_{-i}, k)$  the coalition where all agents except  $i$  play at levels defined in  $s$  while  $i$  plays at  $k \in M_i$ . The set  $\mathcal{M}$  endowed with the usual binary relation  $\leq$  on  $\mathbb{R}^n$  induces a (complete) lattice with greatest element  $m$  and least element  $\mathbf{0}$ . For any two coalitions  $a, b \in \mathcal{M}$ ,  $a \vee b$  and  $a \wedge b$  denote their least upper bound and their greatest lower bound over  $\mathcal{M}$ , respectively. We use the notation  $M_i^+ = M_i \setminus \{0\}$  for each  $i \in N$  and  $M^+ = \bigcup_{i \in N} (\{i\} \times M_i^+)$ . A pair  $(i, j) \in M^+$  represents an agent and one of its activity levels. A (cooperative) *multi-choice game* on  $N$  is a pair  $(m, v)$  where  $v : \mathcal{M} \rightarrow \mathbb{R}$  is a characteristic function, such that  $v(\mathbf{0}) = 0$ , that specifies the worth  $v(s)$  when agents participate at profile  $s$ . Denote by  $\mathcal{G}$  the set of multi-choice games  $(m, v)$  on  $N$  such that  $m_i \leq K$  for each  $i \in N$ . Notice that standard cooperative transferable utility games (or simply TU-games) can be viewed as a subclass of multi-choice games satisfying  $m = (1, \dots, 1)$ .

An *incremental payoff vector* for the game  $(m, v)$  is an element  $x \in \mathbb{R}^{|M^+|}$ , where  $x_{ij} \in \mathbb{R}$  is the incremental payoff received by the pair  $(i, j) \in M^+$ . A *value*  $f$  on  $\mathcal{G}$  is a function that assigns a unique incremental payoff vector  $f(m, v)$  to each  $(m, v) \in \mathcal{G}$ . One solution for multi-choice games is the *multi-choice Shapley value* (see Lowing and Techer [2022]), which assigns to each pair in  $M^+$  a weighted average of its marginal contributions. Formally, the multi-choice Shapley value is defined as, for each  $(m, v) \in \mathcal{G}$ , each pair  $(i, j) \in M^+$

$$\varphi_{ij}(m, v) = \sum_{(j-1 \wedge m_k)_{k \in N} \leq s \leq (j \wedge m_k)_{k \in N}} \frac{|\mathcal{S}(s)|! (|Q(j)| - |\mathcal{S}(s)| - 1)!}{|Q(j)!} \left[ v(s + e_i) - v(s) \right], \quad (3)$$

where  $\mathcal{S}(s) = \{i \in N : s_i > 0\}$  denotes the *support* of the coalition  $s$ . Moreover, Lowing and Techer [2022] characterize the multi-choice Shapley value by considering the following axioms for values on  $\mathcal{G}$ . A value  $\phi$  on  $\mathcal{G}$  satisfies:

**Efficiency** if for each  $(m, v) \in \mathcal{G}$ ,

$$\sum_{i \in N} \sum_{j \in M_i^+} \phi_{ij}(m, v) = v(m).$$

**Independence of higher activity levels** if for each  $(m, v) \in \mathcal{G}$ ,

$$\forall (i, j) \in M^+, \quad \phi_{ij}(m, v) = \phi_{ij}((j \wedge m_k)_{k \in N}, v).$$

**Linearity** if for each  $(m, v), (m, w) \in \mathcal{G}$ , each  $a \in \mathbb{R}$ ,

$$\phi(m, av + w, ) = a\phi(m, v) + \phi(m, w).$$

**Null pair** if for each  $(m, v) \in \mathcal{G}$  and each null pair  $(i, j) \in M^+$ ,

$$\phi_{ij}(m, v) = 0.$$

**Equal treatment for equal pairs** if for each  $(m, v) \in \mathcal{G}$  and two distinct equal pairs  $(i, j)(i', j) \in M^+$ ,

$$\phi_{ij}(m, v) = \phi_{i'j}(m, v).$$

**Theorem 2 (Lowing and Techer [2022]).** *A value  $\phi$  on  $\mathcal{G}$  satisfies Efficiency, Linearity, Independence of higher activity levels, Null pair, and Equal treatment of equal pairs if and only if  $\phi = \varphi$ .*

It can be shown that the multi-choice Shapley value can be characterized by invoking Additivity (for each  $(m, v), (m, w) \in \mathcal{G}$ ,  $\phi(m, v + w, \cdot) = \phi(m, v) + \phi(m, w)$ .) instead of Linearity. Moreover, Lowing and Techer [2022] invoke a weaker condition than Equal treatment of equal pair which requires that two equal pairs should receive a payoff of the same sign.

## 4.2 Hazardous waste transportation game

Consider a waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$ , we define the associated multi-choice game  $(w, v_P)$  where  $w$  is the grand coalition and  $v_P : W \rightarrow \mathbb{R}_+$  is the associated characteristic function. Let  $s \leq w$  be a profile of hazardous waste. Recall that  $\mathcal{S}(s)$  is the set of agents that ship a positive amount of waste in the profile  $s$ . The worth  $v_P(s) \in \mathbb{R}_+$  then describes the total maintenance cost whenever the profile  $s$  is shipped to the delivery node under the considered liability regime. Because the polluter pays principle and the strict liability principle (see Section 3.2) hold an agent liable for the costs of downstream portions, we define  $v_P$  as

$$\forall s \leq w, \quad v_P(s) = \sum_{i \in \hat{U}_{\mathcal{S}(s)}^{-1}} C_i \left( \sum_{k \in \hat{U}_i} s_k \right). \quad (4)$$

Observe that the worth of the grand coalition  $v_P(w)$  is equal to the total cost of maintaining the hazardous waste transportation network defined by (1). Actually, most properties of the cost functions  $C_i$  are conveyed to the multi-choice game  $(w, v_P)$ . Since for each  $i \in N$ ,  $C_i(0) = 0$ , it holds that  $v_P(\mathbf{0}) = 0$ . Thus, for each hazardous waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$ , the associated multi-choice game  $(w, v_P)$  is well-defined. Finally, since each cost function  $C_i$ ,  $i \in N$ , is non-negative and non-decreasing, it follows that each multi-choice game  $(w, v_P)$  is a non-negative and monotone game.

**Remark 1.** Suppose that the maximal waste amount of each agent  $i \in N$  is  $w_i = 1$ . Then, the game  $(w, v_P)$  is a TU-game that coincides with the upstream oriented game associated with a polluted river network problem introduced in Dong et al. [2012]. Following van den Brink et al. [2018], given  $P = (g, w, C) \in \mathcal{P}^N$ , one can interpret the network  $g$  as a permission structure. Moreover, van den Brink et al. [2018] shows that the upstream oriented game associated with a polluted river network problem coincides with the dual of a TU-game with permission structure. However, the definition of the dual of a multi-choice game remains debatable. Thus, it is not so clear that the same relation holds for the whole class  $\mathcal{P}^N$ .<sup>7</sup>

The next result states that the responsibility rule coincides with the multi-choice Shapley value  $\varphi$ .

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<sup>7</sup>For a recent study of multi-choice games with a permission structure, we refer the reader to Lowing [2023b].

**Theorem 3.** *For each hazardous waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$  and its associated multi-choice game  $(w, v_P)$ , it holds that*

$$f^R(P) = \varphi(w, v_P).$$

Lowing and Techer [2022] show that the multi-choice Shapley value is related to the discrete serial cost sharing method, introduced by Moulin and Shenker [1992], as it captures the variation of joint cost caused by a demand increase. Accordingly, it can be shown that the total payoff assigned by the multi-choice Shapley value corresponds to the payoff assigned by the serial discrete cost sharing. Therefore, the responsibility rule can be interpreted as the discrete serial cost sharing applied on the class hazardous waste transportation problem.

## 5 Conclusion

In this paper, we study the problem of sharing the maintenance cost of a hazardous waste transportation network modeled by a directed sink tree. We propose an axiomatic analysis on the class of hazardous waste transportation problems. Specifically, we propose several axioms related to environmental law principles. This allows us to precise the liability of each agent regarding the risk it causes to the network, and to characterize a specific cost sharing method called the responsibility rule. We also show that this method coincides with the multi-choice Shapley of an appropriate multi-choice game.

This work can be extended in several directions. First, it should be noticed that the delivery node,  $d$ , is exogenously given and does not belong to the agent's set. Relaxing this hypothesis would mean to choose a location for the treatment facility among the set of agents. However, due to the nature of the treatment facility, the agents may be reluctant to host it. This gives rise to a so-called NIMBY problem that has been studied by Shapley and Shubik [1969], and Ambec and Kervinio [2016], using cooperative game theory. From a different viewpoint, Sakai [2012] conducts an axiomatic study for sharing the cost associated with a NIMBY facility. To the best of our knowledge, no work has addressed the cost allocation of a NIMBY facility in a network structure using the axiomatic method. We suggest that such work can be conducted by adapting our model.

Moreover, one can consider a model with several treatment facilities each one having a treatment capacity (which may differ from one another). Suppose that the network is modeled by a connected undirected graph. The agents have to coordinate themselves in order to ship each amount of waste with respect to the treatment capacities of the facilities. Then, they split the resulting transportation cost. In this case, the analysis become quite different introducing an optimization problem to on top of the cost allocation. This is left for future research.

Another interesting direction for further research would be to study how the implementation of the responsibility rule may affect the incentive for agents to decrease their generation of waste. This could be an important aspect in meeting the goals of environmental law regarding environmental preservation.

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## 6 Appendix

**Proposition 1** *Let  $f$  be an allocation rule on  $\mathcal{P}^N$ . If  $f$  satisfies Efficiency and Path consistency, then for any  $i \in N$ , any  $(g, w, C) \in \mathcal{P}^N$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C_k^0$  it holds that*

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = 0.$$

*Proof.* Let  $f$  be an allocation rule on  $\mathcal{P}^N$  that satisfies Efficiency and Path consistency. Consider the waste transportation problem  $(g, w, C^0) \in \mathcal{P}^N$ . By Efficiency, we have that

$$\sum_{i \in N} \sum_{j \leq w_i} f_{ij}(g, w, C^0) = 0.$$

Since an allocation rule  $f$  assigns a non-negative cost share  $f_{ij}(g, w, C) \in \mathbb{R}_+$  to each agent  $i \in N$  and each  $1 \leq j \leq w_i$ , it holds that

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C^0) = 0.$$

Consider any  $i \in N$  and any waste transportation problem  $(g, w, C)$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C_k^0$ . By Path consistency, one obtains

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, w, C^0) = 0,$$

the desired result. This concludes the proof of the proposition. ■

**Proposition 2** *Let  $f$  be an allocation rule on  $\mathcal{P}^N$ . If  $f$  satisfies Efficiency and Independence of higher waste amounts, then*

$$\forall j \leq \max_{i \in N} w_i, \quad \sum_{k \in Q(j)} f_{kj}(g, w, C) = \sum_{i \in N} \left( C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) - C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) \right).$$

*Proof.* Let  $f$  be an allocation rule as hypothesized and consider any waste amount  $0 < j \leq \max_{i \in N} w_i$ . By Independence of higher waste amounts, we have that

$$\forall i \in Q(j), \quad f_{ij}(g, w, C) = f_{ij}(g, (j \wedge w_k)_{k \in N}, C), \tag{5}$$

$$\forall i \in Q(j-1) \quad f_{ij-1}(g, w, C) = f_{ij-1}(g, ((j-1) \wedge w_k)_{k \in N}, C). \tag{6}$$



By Efficiency and Independence of higher waste amounts, it holds that

$$\sum_{i \in N} \sum_{l=1}^{j \wedge w_i} f_{il}(g, (j \wedge w_k)_{k \in N}, C) = \sum_{i \in N} C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) = \sum_{i \in N} \sum_{l=1}^{j \wedge w_i} f_{il}(g, w, C), \quad (7)$$

$$\sum_{i \in N} \sum_{l=1}^{j-1 \wedge w_i} f_{il}(g, ((j-1) \wedge w_k)_{k \in N}, C) = \sum_{i \in N} C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) = \sum_{i \in N} \sum_{l=1}^{j-1 \wedge w_i} f_{il}(g, w, C). \quad (8)$$

Thus, subtracting (8) to (7), one obtains

$$\sum_{k \in Q(j)} f_{kj}(g, w, C) = \sum_{i \in N} \left( C_i \left( \sum_{l \in \hat{U}_i} j \wedge w_l \right) - C_i \left( \sum_{l \in \hat{U}_i} (j-1) \wedge w_l \right) \right),$$

the desired result. ■

**Proposition 3** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , the combination of Efficiency, Independence of higher waste amounts, Path consistency, and Upstream solidarity for a cost increase implies Distance consistency.*

*Proof.* Let  $f$  be an allocation rule on  $\mathcal{P}^N$  as hypothesized. Take any waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . For each  $k \in N$ , define the cost profile  $\lambda^{k,C} \in \mathcal{C}^N$  such that  $\lambda_k^{k,C} = C_k$ , and  $\lambda_i^{k,C} = C_i^0$  for each  $i \in N \setminus \{k\}$ . Take a waste amount  $j \leq \max_{i \in N} w_i$  such that  $|Q(j)| \geq 2$ , and fix two agents  $i, k \in N$  such that:  $i \in Q(j)$  and  $k \in \hat{U}_i^{-1} \cap Q(j)$ . By Path consistency, we have that

$$\begin{aligned} f_{kj}(g, w, C) &= f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}), \\ f_{ij}(g, w, C) &= f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}). \end{aligned} \quad (9)$$

We show that  $f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C})$ . To do so, consider the waste allocation problems  $(g, w, C^0) \in \mathcal{P}^N$  and  $(g, w, \lambda^{k,C}) \in \mathcal{P}^N$ . Observe that, for each  $h \in N \setminus \{k\}$ ,  $\lambda_h^{k,C} = C_h^0$ . Moreover, since  $\lambda_k^{k,C}$  is a non-decreasing function, it holds that

$$\begin{aligned} \forall s \in W, \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} s_h \right) - \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} (s_h - 1) \vee 0 \right) &\geq C_k^0 \left( \sum_{h \in \hat{U}_k} s_h \right) - C_k^0 \left( \sum_{h \in \hat{U}_k} (s_h - 1) \vee 0 \right) \\ \iff \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} s_h \right) - \lambda_k^{k,C} \left( \sum_{h \in \hat{U}_k} (s_h - 1) \vee 0 \right) &\geq 0. \end{aligned} \quad (10)$$

Therefore, by Upstream solidarity for a cost increase one obtains

$$f_{kj}(g, w, \lambda^{k,C}) - f_{kj}(g, w, C^0) = f_{ij}(g, w, \lambda^{k,C}) - f_{ij}(g, w, C^0).$$

By Proposition 1,  $f_{kj}(g, w, C^0) = f_{ij}(g, w, C^0) = 0$ . Thus, one obtains

$$f_{kj}(g, w, \lambda^{k,C}) = f_{ij}(g, w, \lambda^{k,C}).$$

We apply the same reasoning by adding successively each  $\lambda_h^{h,C}$  where  $h \in \hat{U}_k^{-1}$  to the problem. Then, by successive applications of Upstream solidarity for a cost increase it holds that

$$f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) = f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}). \quad (11)$$

Thus, by (9) and (11), we know that

$$f_{kj}(g, w, C) = f_{kj}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) = f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

It remains to show that

$$f_{ij}(g, w, C) = f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

By Independence of higher waste amounts, this can be written as

$$f_{ij}(g, (j \wedge w_k)_{k \in N}, C) = f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

Consider the directed path from  $i$  to  $k$  denoted by  $(i_1, \dots, i_t)$ , where  $i_1 = i$  and  $i_t = k$ . Pick  $i_{t-1} \in \hat{U}_i^{-1} \setminus \hat{U}_k^{-1}$  located just upstream  $k$  on the path from  $i$  to  $k$ . Consider the hazardous waste problem  $(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C})$ . Observe that, for each  $h' \in N \setminus \{i_{t-1}\}$ ,  $\lambda_{h'}^{i_{t-1},C} = C_{h'}^0$ . Since  $\lambda_{i_{t-1}}^{i_{t-1},C}$  is non-decreasing, by Upstream solidarity for a cost increase, there is  $c_j^{i_{t-1}} \in \mathbb{R}$  such that:

$$\begin{aligned} \forall l \in \hat{U}_{i_{t-1}} \cap Q(j), \\ f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) - f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) = c_j^{i_{t-1}}. \end{aligned} \quad (12)$$

We now show that  $c_j^{i_{t-1}} \geq 0$ . Since  $f$  satisfies Efficiency and Independence of higher waste amounts, by Proposition 2 we have that

$$\begin{aligned} \sum_{l \in Q(j)} f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) &= \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_h} j \wedge w_a \right) \\ &+ \lambda_{i_{t-1}}^{i_{t-1},C} \left( \sum_{a \in \hat{U}_{i_{t-1}}} j \wedge w_a \right) \\ &- \left( \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_h} (j-1) \wedge w_a \right) \right. \\ &\left. + \lambda_{i_{t-1}}^{i_{t-1},C} \left( \sum_{a \in \hat{U}_{i_{t-1}}} (j-1) \wedge w_a \right) \right), \end{aligned} \quad (13)$$

and

$$\begin{aligned} \sum_{l \in Q(j)} f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) &= \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_k} j \wedge w_a \right) \\ &- \sum_{h \in \hat{U}_k^{-1}} \lambda_h^{h,C} \left( \sum_{a \in \hat{U}_k} (j-1) \wedge w_a \right). \end{aligned} \quad (14)$$

Observe that, for  $l \in Q(j) \setminus \hat{U}_{i_{t-1}}$  we have  $\lambda_{l'}^{i_{t-1},C} = C_{l'}^0$ , where  $l' \in \hat{U}_l^{-1}$ . This implies that  $\sum_{h \in \hat{U}_k^{-1}} \lambda_{l'}^{h,C} + \lambda_{l'}^{i_{t-1},C} = \sum_{h \in \hat{U}_k^{-1}} \lambda_{l'}^{h,C}$ . Therefore, by applying Path consistency to  $l \in Q(j) \setminus \hat{U}_{i_{t-1}}$  one obtains

$$f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) = f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

From this observation, and subtracting (13) and (14), one obtains

$$\begin{aligned} &\sum_{l \in \hat{U}_{k'} \cap Q(j)} f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{k',C}) - f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) \\ &= \lambda_{k'}^{k',C} \left( \sum_{a \in \hat{U}_{h'}} j \wedge w_a \right) - \lambda_{k'}^{k',C} \left( \sum_{a \in \hat{U}_{h'}} (j-1) \wedge w_a \right) \\ &= |\hat{U}_{k'} \cap Q(j)| \times c_j^{k'}, \end{aligned} \quad (15)$$

where the last equality comes from (12). Since  $\lambda_{i_{t-1}}^{i_{t-1},C}$  is non-decreasing, it holds that  $c_j^{k'} \geq 0$ . Therefore,

$$\forall l \in \hat{U}_{i_{t-1}} \cap Q(j), f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) \geq f_{lj}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

In particular, it holds that

$$f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C} + \lambda^{i_{t-1},C}) \geq f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}).$$

Repeating the same reasoning by considering each agent that belongs to the path from  $i$  to  $k$ , one obtains

$$f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \geq f_{ij}(g, (j \wedge w_k)_{k \in N}, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}). \quad (16)$$

Thus, by (9), (11), (16), and Independence of higher waste amounts, one concludes that

$$f_{ij}(g, w, C) \stackrel{(9)}{=} f_{ij}(g, w, \sum_{h \in \hat{U}_i^{-1}} \lambda^{h,C}) \stackrel{(16)}{\geq} f_{ij}(g, w, \sum_{h \in \hat{U}_k^{-1}} \lambda^{h,C}) \stackrel{(11)}{=} f_{kj}(g, w, C),$$

the desired result. ■

**Proposition 4** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , there is at most one allocation rule satisfying Efficiency, Independence of higher waste amounts, Path consistency and Upstream cost solidarity.*

*Proof.* Let  $f$  be a solution on  $\mathcal{P}^N$  satisfying Efficiency, Independence of higher waste amounts, Path consistency and Upstream cost solidarity. We show that  $f$  is uniquely determined. Take any waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . For each  $k \in N$ , define the cost profile  $\lambda^{k,C} \in \mathcal{C}^N$  such that  $\lambda_k^{k,C} = C_k$ , and  $\lambda_i^{k,C} = C_i^0$  for each  $i \in N \setminus \{k\}$ . Observe that, for each waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ , the cost profile  $C$  can be written as

$$C = \sum_{k \in N} \lambda^{k,C}.$$

For each  $(g, w, C) \in \mathcal{P}^N$ , set

$$\mathcal{K}(g, w, C) = \{k \in N : C_k \neq C_k^0\}. \quad (17)$$

We proceed by induction on  $|\mathcal{K}(g, w, C)|$ .

**Induction basis.** If  $|\mathcal{K}(g, w, C)| = 0$ , then  $C = C^0$ . By Efficiency and the definition of an allocation rule, it holds that

$$\forall i \in N, \forall j \leq w_i, \quad f_{ij}(g, w, C) = 0.$$

**Induction hypothesis.** Assume that, for each  $(g, w, C)$  such that  $|\mathcal{K}(g, w, C)| = t$ , where  $0 \leq t \leq n - 1$ ,  $f(g, w, C)$  is uniquely determined.

**Induction step.** Let  $(g, w, C) \in \mathcal{P}^N$  be such that  $|\mathcal{K}(g, w, C)| = t + 1$ . Set

$$\bar{T} = \bigcap_{k \in \mathcal{K}(g, w, C)} \hat{U}_k, \quad (18)$$

and for each  $k \in \mathcal{K}(g, w, C)$  set

$$T_k = \hat{U}_k \setminus \bar{T}. \quad (19)$$

Observe that,  $\bar{T}$  is such that there is a unique agent  $k \in \bar{T}$  such that  $C_k \neq C_k^0$ . Indeed, if  $C_k = C_k^0$  for each agent  $k \in \bar{T}$ , then a contradiction arises and  $k \notin \mathcal{K}(g, w, C)$ . Any other agent  $k' \in \bar{T}$  is such that  $C_{k'} = C_{k'}^0$ . Two cases have to be considered with respect to  $\bar{T}$ .

**Case 1.** If  $\bar{T} = \emptyset$ , then for each agent  $i \in N$ , there is  $k \in \mathcal{K}(g, w, C)$  such that  $k \in N \setminus \hat{U}_i^{-1}$ . By Path consistency, it holds that

$$\forall j \leq w_i, \quad f_{ij}(g, w, C) = f_{ij}(g, w, C - \lambda^{k,C}).$$

By the induction hypothesis,  $f_{ij}(g, w, C - \lambda^{k,C})$  is uniquely determined. Thus, for each  $i \in N$  and each  $j \leq w_i$ ,  $f_{ij}(g, w, C)$  is uniquely determined.

**Case 2.** Now, suppose that  $\bar{T} \neq \emptyset$ . For each agent  $i \in N \setminus \bar{T}$ , there is  $k \in \mathcal{K}(g, w, C)$  such that  $k \in N \setminus \hat{U}_i^{-1}$ . By applying the same reasoning as in case 1, for each agent  $i \in N \setminus \bar{T}$  and each

$j \leq w_i$ ,  $f_{ij}(g, w, C)$  is uniquely determined.

Consider the subset of agents  $\bar{T}$ . Take any waste amount  $j \leq K$ . We show that any two agents  $i, i' \in \bar{T} \cap Q(j)$  have the same incremental payoff in  $(g, w, C)$  for the waste amount  $j$ . Pick any  $k \in \mathcal{K}(g, w, C)$ . Notice that  $i, i' \in \hat{U}_k \cap Q(j)$ . By Upstream solidarity for a cost increase and the induction basis, it holds that

$$\begin{aligned} f_{ij}(g, w, \lambda^{k,C}) - f_{ij}(g, w, C^0) &= f_{i'j}(g, w, \lambda^{k,C}) - f_{i'j}(g, w, C^0) \\ f_{ij}(g, w, \lambda^{k,C}) &= f_{i'j}(g, w, \lambda^{k,C}). \end{aligned} \quad (20)$$

Now pick any other  $k' \in \mathcal{K}(g, w, C)$ ,  $k' \neq k$  (if any). Observe that

$$\forall s \in W, \quad \lambda_{k'}^{k',C} \left( \sum_{h \in \hat{U}_{k'}} s_h \right) - \lambda_{k'}^{k',C} \left( \sum_{h \in \hat{U}_{k'}} (s_h - 1) \vee 0 \right) \geq 0,$$

and for each  $h \in N \setminus \{k'\}$ ,  $\lambda_h^{k,C} + \lambda_h^{k',C} = \lambda_h^{k,C}$ . By the definition of  $\bar{T}$ , it holds that  $i, i' \in \hat{U}_{k'}$ . Thus, by Upstream solidarity for a cost increase

$$\begin{aligned} f_{ij}(g, w, \lambda^{k,C} + \lambda^{k',C}) - f_{i'j}(g, w, \lambda^{k,C}) &= f_{i'j}(g, w, \lambda^{k,C} + \lambda^{k',C}) - f_{i'j}(g, w, \lambda^{k,C}) \\ f_{ij}(g, w, \lambda^{k,C} + \lambda^{k',C}) &= f_{i'j}(g, w, \lambda^{k,C} + \lambda^{k',C}), \end{aligned} \quad (21)$$

where the last equation comes from (20). We apply the same reasoning by adding successively each  $\lambda^{k,C}$  where  $k \in \mathcal{K}(g, w, C)$  to the problem. Therefore, by successive applications of Upstream solidarity for a cost increase, it holds that

$$\begin{aligned} f_{ij}(g, w, \sum_{k \in \mathcal{K}(g, w, C)} \lambda^{k,C}) &= f_{i'j}(g, w, \sum_{k \in \mathcal{K}(g, w, C)} \lambda^{k,C}) \\ f_{ij}(g, w, C) &= f_{i'j}(g, w, C). \end{aligned} \quad (22)$$

Thus, there exists  $\bar{c}_j \in \mathbb{R}_+$  such that for each  $i, i' \in \bar{T} \cap Q(j)$ :

$$f_{ij}(g, w, C) = f_{i'j}(g, w, C) = \bar{c}_j. \quad (23)$$

Now, we show that for each agent in  $\bar{T} \cap Q(j)$ ,  $f_{ij}(g, w, C)$  is uniquely determined.

Since  $f$  satisfies Efficiency and Independence of higher waste amounts, by Proposition 2 we have

$$\begin{aligned} \sum_{k \in Q(j)} f_{kj}(g, w, C) &= \sum_{k \in N} \left( C_k \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - C_k \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right) \right) \\ &= \sum_{k \in \mathcal{K}(w, C)} \left( C_k \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - C_k \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right) \right). \end{aligned}$$

Equivalently, one can write

$$\begin{aligned} \sum_{i \in (Q(j) \cap \bar{T})} f_{ij}(g, w, C) &= \sum_{k \in \mathcal{K}(w, C)} \left( C_k \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - C_k \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right) \right) \\ &\quad - \sum_{l \in Q(j) \setminus \bar{T}} f_{lj}(g, w, C). \end{aligned} \quad (24)$$

Recall that  $f_{ij}(g, w, C)$  is uniquely determined for each agent in  $Q(j) \setminus \bar{T}$ . Therefore, by (20) and (24),  $f_{ij}(g, w, C)$  is uniquely determined for each  $i \in \bar{T} \cap Q(j)$ . This concludes the proof of the induction step.  $\blacksquare$

**Theorem 1** *On the class of hazardous waste transportation problems  $\mathcal{P}^N$ , an allocation rule satisfies Efficiency, Independence of higher waste amounts, Path consistency, and Upstream solidarity for a cost increase if and only if it is the responsibility rule.*

*Proof.* By Proposition 4, we know that there is a unique allocation rule that satisfies Efficiency, Independence of higher waste amounts, Path consistency, and Upstream solidarity for a cost increase. It remains to show that  $f^R$  satisfies the axioms invoked in Theorem 1.

Consider any waste transportation problem  $(g, w, C) \in \mathcal{P}^N$ . By the definition of  $f^R$  (see (2)), it holds that

$$\begin{aligned}
\sum_{i \in N} \sum_{j \leq w_i} f_{ij}^R(g, w, C) &= \sum_{j=1}^{\max_{i \in N} w_i} \sum_{i \in Q(j)} f_{ij}^R(g, w, C) \\
&= \sum_{j=1}^{\max_{i \in N} w_i} \sum_{i \in Q(j)} \left[ \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \right] \\
&= \sum_{j=1}^{\max_{i \in N} w_i} \sum_{k \in N} \left[ C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l) \right] \\
&= \sum_{k \in N} C_k(\sum_{l \in \hat{U}(k)} w_l).
\end{aligned}$$

This shows that  $f^R$  satisfies Efficiency.

By the definition of the responsibility rule, the incremental payoff of any agent for its waste amount  $j$  does not depend of any waste amount higher than  $j$ . Therefore,  $f^R$  satisfies Independence of higher waste amounts.

Now consider any agent  $i \in N$  and any two hazardous waste problem  $(g, w, C) \in \mathcal{P}^N$ ,  $(g, w, C') \in \mathcal{P}^N$  such that for each  $k \in \hat{U}_i^{-1}$ ,  $C_k = C'_k$ . It holds that

$$\begin{aligned}
f_{ij}^R(g, w, C) &= \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \\
&= \sum_{k \in \hat{U}_i^{-1}} \frac{C'_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C'_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \\
&= f_{ij}^R(g, w, C'),
\end{aligned}$$

which shows that  $f^R$  satisfies Path consistency.

Next, consider any agent  $i \in N$  and any  $(g, w, C), (g, w, C') \in \mathcal{P}^N$  such that

$$\begin{aligned}
&\forall x \in \mathbb{N}, \\
&C_i(x) - C_i((x-1) \vee 0) \geq C'_i(x) - C'_i((x-1) \vee 0),
\end{aligned}$$

and, for each  $k \in N \setminus \{i\}$ ,  $C_k = C'_k$ . Pick any agent  $l \in \hat{U}_i$ . Observe that for any waste amount  $j \leq w_l$ , we have that

$$f_{l_j}^R(g, w, C) - f_{l_j}^R(g, w, C') = \frac{C_i(\sum_{k \in \hat{U}_i} j \wedge w_k) - C_i(\sum_{k \in \hat{U}_i} (j-1) \wedge w_k)}{|\hat{U}_i \cap Q(j)|} \\ - \frac{C'_i(\sum_{k \in \hat{U}_i} j \wedge w_k) - C'_i(\sum_{k \in \hat{U}_i} (j-1) \wedge w_k)}{|\hat{U}_i \cap Q(j)|},$$

which does not depend on the chosen  $l \in \hat{U}_i$ . Thus,  $f^R$  satisfies Upstream solidarity for a cost increase. This concludes the proof of the theorem.  $\blacksquare$

**Theorem 3** *For each hazardous waste transportation problem  $P = (g, w, C) \in \mathcal{P}^N$  and its associated multi-choice game  $(w, v_P)$ , it holds that*

$$f^R(P) = \varphi(w, v_P).$$

*Proof.* Let  $P = (g, w, C) \in \mathcal{P}^N$  be any hazardous waste transportation problem. For each  $k \in N$ , define  $\lambda^{k,C} = (\lambda_i^{k,C})_{i \in N}$  the profile of cost functions such that  $\lambda_k^{k,C} = C_k$ , and  $\lambda_i^{k,C} = C_i^0$  for each  $i \neq k$ . Observe that, for each  $(g, w, C) \in \mathcal{P}^N$ , the cost profile  $C$  can be written as

$$C = \sum_{k \in N} \lambda^{k,C}.$$

Fix any  $k \in N$  and consider the multi-choice game  $(w, u_P^k)$ , where  $u_P^k$  is the characteristic function defined as

$$\forall s \leq w, \quad u_P^k(s) = \sum_{i \in \hat{U}_{S(s)}^{-1}} \lambda_i^{k,C} (\sum_{l \in \hat{U}_i} s_l), \\ = \begin{cases} \lambda_k^{k,C} (\sum_{l \in \hat{U}_k} s_l) & \text{if } k \in \hat{U}_{S(s)}^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for each  $P = (g, w, C) \in \mathcal{P}^N$  and each  $(w, v_P)$ , it holds that

$$v_P = \sum_{k \in N} u_P^k.$$

By computing the multi-choice Shapley value of the multi-choice game  $(w, u_P^k)$ , one can make the following observations. For each  $i \in N \setminus \hat{U}_k$  and each  $j \leq w_i$ , the pair  $(i, j) \in M^+$  is a null pair in  $(w, u_P^k)$ . Since the multi-choice Shapley value satisfies the null pair axiom, it holds that

$$\forall i \in N \setminus \hat{U}_k, \forall j \leq w_i, \quad \varphi_{ij}(w, u_P^k) = 0.$$

Moreover, any two pairs  $(i, j), (i', j)$ , such that  $i, i' \in \hat{U}_k \cap Q(j)$ , are equal pairs in  $(w, u_P^k)$ . Since the multi-choice Shapley value satisfies Equal treatment of equal pairs, we have that for each  $j \leq \max_{i \in N} w_i$

$$\forall i, i' \in \hat{U}_k \cap Q(j), \quad \varphi_{ij}(w, u_P^k) = \varphi_{i'j}(w, u_P^k).$$

Finally, since the multi-choice Shapley value satisfies Multi-efficiency, for each  $j \leq \max_{i \in N} w_i$  it holds that

$$\begin{aligned} \sum_{i \in \hat{U}_k \cap Q(j)} \phi_{ij}(w, u_P^k) &= u_P^k((j \wedge w_k)_{k \in N}) - u_P^k(((j-1) \wedge w_k)_{k \in N}) \\ &= \lambda_k^{k,C} \left( \sum_{l \in \hat{U}_k} j \wedge w_l \right) - \lambda_k^{k,C} \left( \sum_{l \in \hat{U}_k} (j-1) \wedge w_l \right). \end{aligned}$$

By definition of  $\lambda_k^{k,C}$  and Equal treatment of equal pairs, we have that

$$\phi_{ij}(w, u_P^k) = \begin{cases} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} & \text{if } i \in \hat{U}_k \cap Q(j), \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

Recall that, for each  $P \in \mathcal{P}^N$  and multi-choice game  $(w, v_P)$ , it holds that

$$v_P = \sum_{k \in N} u_P^k.$$

Since the multi-choice Shapley value satisfies Additivity, one concludes that

$$\begin{aligned} \forall i \in N, \forall j \leq w_i, \quad \varphi_{ij}(w, v_P) &= \sum_{k \in \hat{U}_i^{-1}} \frac{C_k(\sum_{l \in \hat{U}_k} j \wedge w_l) - C_k(\sum_{l \in \hat{U}_k} (j-1) \wedge w_l)}{|\hat{U}_k \cap Q(j)|} \\ &= f_{ij}^R(g, w, C). \end{aligned}$$

This concludes the proof of the theorem. ■

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