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Abstract

In this paper we study the asymptotic behaviors of a plate with non-planar top surface in the framework of linear elasticity. For this plate, we give a decomposition of the displacements. We show that every displacement of the plate is the sum of a Kirchhoff-Love displacement and a residual displacement that takes into account the deformations of the fibers of the plate and the shearing. We also prove Korn's type inequalities.

Résumé

Dans cet article, nous étudions, dans le cadre de l'élasticité linéaire, les comportements asymptotiques d'une plaque dont la surface supérieure est non plane. Pour cette plaque, nous donnons une décomposition des déplacements. Nous montrons que tout déplacement de la plaque est la somme d'un déplacement de Kirchhoff-Love et d'un déplacement résiduel qui prend en compte les déformations des fibres de la plaque et le cisaillement. Nous donnons également des inégalités de type Korn.

Key Words: linear elasticity, Kirchhoff-Love displacement, non-planar plate.
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1 Introduction

The first difficulty encountered when studying a plate with a non-planar top surface is the estimation of displacements. The second, as in all problems of linear elasticity of thin structures, is to give a simple expression to the strain limit tensor. In this paper, we start by extending the result obtained in [21] for flat plates: namely that any displacement is the sum of a Kirchhoff-Love displacement and a residual displacement. Here, our proof uses the result of [21] and is based on simple geometrical considerations concerning the lateral boundary and the parameterization of the non-flat part of the plate. This decomposition of a displacement is followed by estimates of each of its terms using the norm of the strain tensor. When investigating the behavior of the strain tensor of a sequence of displacements, these estimates allow to obtain a simple expression for the limit strain tensor only in terms of the limit Kirchhoff-Love displacements and the limit residual displacements. This done, we can give the asymptotic behavior of a non-planar plate made of an isotropic and homogeneous material and also give the asymptotic behavior of this type of plate in the framework of reduction of dimension and homogenization.

Plates with rough surfaces are the subject of an important literature. In [5], we find a Korn type inequality for a rough surface plate and in [10] a Korn type inequality for a structure made of plates whit non-flat surfaces. The asymptotic behavior of a plate with a rough surface is studied in [6, 7, 8]. Other papers [11, 12, 14, 17] study the jonction of a plate with a vertical rod or a family of regularly spaced vertical rods. As a general reference on elasticity, we refer the reader to [1, 3]. For mathematical modeling of plates we refer to [2, 4, 9, 16, 21]. For the periodic unfolding method we refer to [16].

This paper is organized as follows:

• In Section 2 we introduce the main notations and the plate Ω_δ with non-flat top surface.

• In Section 3 we recall some results obtained in [21] concerning the displacements of a plate. Any displacement of a plate of constant thickness is the sum of a Kirchhoff-Love displacement and a residual part (the shearing and the warping). We also recall the estimates of the terms of this decomposition (see Lemma 3.1 and Proposition 3.1).

• In Section 4 we extend the results obtained in [21] to non-flat plates.

• In Section 5 we transform the plate Ω_δ into a fixed domain Ω using a dilation in the plate thickness direction. Then, we introduce the dimension reduction operator Π_δ which acts on functions defined in Ω_δ, this allows to work with functions defined in Ω. In Subsection 5.1, we consider a sequence of displacements $\{u_\delta\}_\delta$ whose strain tensors satisfy

$$\|e(u_\delta)\|_{L^2(\Omega_\delta)} \leq C\delta^{5/2}.$$

In Theorem 5.1 and Lemma 5.1 we give the asymptotic behavior of the sequence $\big\{1/\delta^2\Pi_\delta\big(e(u_\delta)\big)\big\}_\delta$ using the limits of the partial derivatives of the terms of the decomposition of u_δ.
In order to solve problems of dimension reduction and homogenization, we introduce a specific operator $\mathfrak{T}_{\varepsilon\delta}$ that combines dimension reduction and periodic unfolding. The dimension of the periodic cells is ε, we are interested in the case

$$(\varepsilon,\delta)\to(0,0)\qquad\text{and}\qquad\frac{\varepsilon}{\delta}\to\theta\in(0,+\infty). \tag{1.1}$$

In Subsection 5.2, we choose a sequence of displacements $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta}$ satisfying

$$\|e(u_{\varepsilon,\delta})\|_{L^2(\Omega_\delta)}\leq C\delta^{5/2}. \tag{1.2}$$

In Theorem 5.2 we give the asymptotic behavior of the sequence $\big\{1/\delta^2\mathfrak{T}_{\varepsilon\delta}\big(e(u_{\varepsilon,\delta})\big)\big\}_{\varepsilon,\delta}$. In the limit strain tensor the parameter θ appears.

• In Section 6, the plate is submitted to applied body forces. We study the linear elasticity problem posed in the domain Ω_δ. The coefficients of Hooke's law are periodic (period ε) and they also vary strongly according to the thickness of the plate (we make the assumptions (1.1)). The applied body forces are chosen in order that the solution $u_{\varepsilon,\delta}$ of the elasticity problem satisfies (1.2). Thanks to the results of Section 5 we obtain the limit elasticity problem posed in the rescaled and unfolded domain (see Theorem 6.1). Then, as usual in homogenization we introduce the correctors and we give the problem whose solution are the limit membrane displacement $\mathcal{U}_m^\theta$ and the limit bending $\mathcal{U}_3^\theta$. These fields depend continuously on the parameter θ.

• In Section 7 we partially extend the results of Section 4 to plates with strongly oscillating top surfaces.

• The Appendix (Section 8) is devoted to the proofs of technical results used in Section 4.

Throughout this paper, we use the Einstein convention of summation over repeated indices. As a rule, the Greek indices α, β, α' and β' take values in $\{1,2\}$ and the Latin indices i, j, k and l take values in $\{1,2,3\}$. The courant point of $\mathbb{R}^2$ is denoted $x'=(x_1,x_2)$. The courant point of Y is $X'=(X_1,X_2)$ while the courant point of $\mathcal{Y}$ is $X=(X_1,X_2,X_3)$.

2 Notations

Denote

- $\big(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\big)$ the usual orthonormal basis of $\mathbb{R}^3$,
- ω a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary,
- γ a subset of $\partial\omega$ with non null measure,
- ϕ a function belonging to $W^{1,\infty}(\omega)$ and satisfying

$$\forall\big(x',z'\big)\in\overline{\omega}^2\quad|\phi(x')-\phi(z')|\leq K_0|x'-z'|_2,\quad\phi(x')\geq 2C_0,\qquad\forall x'\in\omega,$$

  where $|\cdot|_2$ is the euclidian norm in $\mathbb{R}^2$, K_0 and C_0 are strictly positive constants
- $C_1=\|\phi\|_{L^\infty(\omega)}$,
- $Y=(0,1)^2$, $\mathcal{Y}=Y\times(-C_0,C_1)$,
- $\mathcal{Y}_{x'}=Y\times\big(-C_0,\phi(x')\big)$ for all $x'\in\omega$,
- some spaces

$$\begin{aligned}
&H^1_{per}(Y)\doteq\big\{\chi\in H^1(Y)\mid\chi\ \text{ is 1-periodic with respect to } X_1 \text{ and } X_2\big\},\\
&H^1_{per,0}(Y)\doteq\Big\{\chi\in H^1_{per}(Y)\mid\int_Y\chi(X')dX'=0\Big\},\\
&H^2_{per}(Y)\doteq\big\{\chi\in H^2(Y)\mid\chi \text{ and } \nabla\chi\ \text{ are 1-periodic with respect to } X_1 \text{ and } X_2\big\},\\
&H^2_{per,0}(Y)\doteq H^2_{per}(Y)\cap H^1_{per,0}(Y),\\
&H^1_{per}(\mathcal{Y})\doteq\big\{\chi\in H^1(\mathcal{Y})\mid\chi\ \text{ is 1-periodic with respect to } X_1 \text{ and } X_2\big\},\\
&H^1_{per,0}(\mathcal{Y})\doteq\Big\{\chi\in H^1(\mathcal{Y})\mid\int_Y\chi(X',X_3)dX'=0\quad\text{for a.e. } X_3\in(-C_0,C_1)\Big\}.
\end{aligned}$$

The plate with a non-planar top surface is the domain

$$\Omega_\delta = \Big\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} \mid -\delta C_0 < x_3 < \delta\phi(x_1, x_2) \Big\}.$$

We also consider

$$\Omega_{0,\delta} = \omega \times (-\delta C_0, \delta C_0) \subset \Omega_\delta$$

the plate with flat top and bottom surfaces.
The plate Ω_δ (resp. $\Omega_{0,\delta}$) is clamped on a part Γ_δ (resp. $\Gamma_{0,\delta}$) of its lateral boundary

$$\Gamma_\delta \doteq \big\{ (x_1, x_2, x_3) \in \gamma \times \mathbb{R} \mid -\delta C_0 < x_3 < \delta\phi(x_1, x_2) \big\}, \qquad (\text{resp. } \Gamma_{0,\delta} \doteq \gamma \times (-\delta C_0, \delta C_0)).$$

3 Some reminders on the decomposition of plate displacements

For every $u \in W^{1,p}(\Omega_{0,\delta})^3$, $1 < p < \infty$, we denote $e_{ij}(u)$ the entries of the strain tensor $e(u)$ of u

$$e_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big), \qquad (i,j) \in \{1,2,3\}^2.$$

In [21] we have proven that any displacement $u \in W^{1,p}(\Omega_{0,\delta})^3$, $1 < p < \infty$, is the sum of a Kirchhof-Love displaceement and a residual one

$$\left.\begin{array}{l} u(x) = \underbrace{\begin{pmatrix} \mathcal{U}_1(x') - x_3 \dfrac{\partial \mathcal{U}_3}{\partial x_1}(x') \\ \mathcal{U}_2(x') - x_3 \dfrac{\partial \mathcal{U}_3}{\partial x_2}(x') \\ \mathcal{U}_3(x') \end{pmatrix}}_{\text{Kirchhoff-Love displacement}} + \underbrace{\widetilde{u}(x)}_{\text{residual displacement}} \\ u(x) = U_{KL}(x) + \widetilde{u}(x) \end{array}\right\} \quad \text{for a.e. } x \text{ in } \Omega_{0,\delta} \tag{3.1}$$

where U_{KL} and $\widetilde{u}$ belong to $W^{1,p}(\Omega_{0,\delta})^3$. We recall that the map $u \in W^{1,p}(\Omega_{0,\delta})^3 \longmapsto U_{KL} \in W^{1,p}(\Omega_{0,\delta})^3$ is continuous and linear (see [21]).
The sum

$$\mathcal{U}_m = \mathcal{U}_1\, \mathbf{e}_1 + \mathcal{U}_2\, \mathbf{e}_2$$

is the membrane displacement, while $\mathcal{U}_3$ represents the bending of the mid-surface of the plate. The residual part $\widetilde{u}$ stands for shearing and warping (the deformation of the fibers $\{x'\} \times (-\delta C_0, \delta C_0)$, see [13, 21]). The residual displacement satisfies the following two conditions:

$$\int_{-C_0\delta}^{C_0\delta} \widetilde{u}_1(x', x_3) dx_3 = \int_{-C_0\delta}^{C_0\delta} \widetilde{u}_2(x', x_3) dx_3 = 0 \qquad \text{for a.e. } x' \in \omega. \tag{3.2}$$

We have

Theorem 3.1 (Theorem 6.1 in [21]). *Let u be in $W^{1,p}(\Omega_{0,\delta})^3$, $1 < p < \infty$. The fields $\mathcal{U}_m = \mathcal{U}_1\mathbf{e}_1 + \mathcal{U}_2\mathbf{e}_2$, $\mathcal{U}_3$ and $\widetilde{u}$ satisfy*

$$\mathcal{U}_m \in W^{1,p}(\omega)^2, \quad \mathcal{U}_3 \in W^{2,p}(\omega), \quad \widetilde{u} \in W^{1,p}(\Omega_{0,\delta})^3$$

and the following estimates:

$$\begin{aligned} &\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)} \le \frac{C}{\delta^{1/p}} \|e(u)\|_{L^p(\Omega_{0,\delta})}, \\ &\|D^2\mathcal{U}_3\|_{L^p(\omega)} \le \frac{C}{\delta^{1+1/p}} \|e(u)\|_{L^p(\Omega_{0,\delta})}, \\ &\|\widetilde{u}\|_{L^p(\Omega_{0,\delta})} + \delta \|\nabla \widetilde{u}\|_{L^p(\Omega_{0,\delta})} \le C\delta \|e(u)\|_{L^p(\Omega_{0,\delta})}. \end{aligned} \tag{3.3}$$

The constants do not depend on δ.

Lemma 3.1 (Lemma 3.4 in [21]). *Suppose that the plate is clamped on $\Gamma_{0,\delta}$. Then, we have*

$$\mathcal{U}_1 = \mathcal{U}_2 = 0 \quad a.e.\ on\ \gamma$$

and

$$\|\mathcal{U}_3\|_{L^p(\gamma)} \leq C\delta^{1-2/p}\|e(u)\|_{L^p(\Omega_{0,\delta})}, \qquad \|\nabla\mathcal{U}_3\|_{L^p(\gamma)} \leq \frac{C}{\delta^{2/p}}\|e(u)\|_{L^p(\Omega_{0,\delta})}. \tag{3.4}$$

The constants do not depend on δ.

As a consequence of the above theorem and lemma, one has

Proposition 3.1 (Proposition 3.5 in [21]). *Let u be a displacement in $W^{1,p}(\Omega_{0,\delta})$, $1 < p < \infty$. Assume the plate clamped on $\Gamma_{0,\delta}$. Then, we have*

$$\begin{aligned}
&\|\mathcal{U}_1\|_{W^{1,p}(\omega)} + \|\mathcal{U}_2\|_{W^{1,p}(\omega)} + \delta\|\mathcal{U}_3\|_{W^{2,p}(\omega)} \leq \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{0,\delta})},\\
&\|u_1\|_{L^p(\Omega_{0,\delta})} + \|u_2\|_{L^p(\Omega_{0,\delta})} + \delta\|u_3\|_{L^p(\Omega_{0,\delta})} \leq C\|e(u)\|_{L^p(\Omega_{0,\delta})},\\
&\sum_{\alpha,\,\beta=1}^{2}\Big\|\frac{\partial u_\beta}{\partial x_\alpha}\Big\|_{L^p(\Omega_{0,\delta})} + \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_{L^p(\Omega_{0,\delta})} \leq C\|e(u)\|_{L^p(\Omega_{0,\delta})},\\
&\sum_{\alpha=1}^{2}\Big(\Big\|\frac{\partial u_3}{\partial x_\alpha}\Big\|_{L^p(\Omega_{0,\delta})} + \Big\|\frac{\partial u_\alpha}{\partial x_3}\Big\|_{L^p(\Omega_{0,\delta})}\Big) \leq \frac{C}{\delta}\|e(u)\|_{L^p(\Omega_{0,\delta})}.
\end{aligned} \tag{3.5}$$

The constants do not depend on δ.

4 Displacements decompositions of the non-planar plate

Let u be a displacement in $W^{1,p}(\Omega_\delta)^3$, $1 < p < \infty$. The restriction $u_{|\Omega_{0,\delta}}$ is a displacement of the plate $\Omega_{0,\delta}$, we decompose it as (3.1). The Kirchhoff-Love displacement U_{KL} can also be considered as a displacement of the plate Ω_δ, so it belongs to $W^{1,p}(\Omega_\delta)^3$. We write

$$u = U_{KL} + \mathfrak{u}. \tag{4.1}$$

The residual part $\mathfrak{u}$ belongs to $W^{1,p}(\Omega_\delta)^3$, it stands for the deformations of the fibers of the plate Ω_δ.
Note that $\mathfrak{u} = \widetilde{u}$ a.e. in $\Omega_{0,\delta}$.

Theorem 4.1. *For every displacement u in $W^{1,p}(\Omega_\delta)^3$, $1 < p < \infty$, the terms $\mathcal{U}$ and $\mathfrak{u}$ of its decomposition given by* (4.1) *satisfy*

$$\begin{aligned}
&\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)} \leq \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|D^2\mathcal{U}_3\|_{L^p(\omega)} \leq \frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|e(\mathfrak{u})\|_{L^p(\Omega_\delta)} \leq C\|e(u)\|_{L^p(\Omega_\delta)}
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\|\mathfrak{u}\|_{L^p(\Omega_\delta)} \leq C\ \delta\ \|e(u)\|_{L^p(\Omega_\delta)},\\
&\|\nabla\mathfrak{u}\|_{L^p(\Omega_\delta)} \leq C\|e(u)\|_{L^p(\Omega_\delta)}
\end{aligned} \tag{4.3}$$

with a constant C independent of δ.

Proof. The estimates $(4.2)_{1,2}$ are the consequences of those in Theorem 3.1 and the fact that $\|e(u)\|_{L^2(\Omega_{0,\delta})} \leq \|e(u)\|_{L^2(\Omega_\delta)}$ since $\Omega_{0,\delta} \subset \Omega$. We have

$$e(U_{KL}) = \begin{pmatrix} e_{11}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1^2} & e_{12}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2} & 0)\\ e_{12}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2} & e_{22}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_2^2} & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

Since U_{KL} is a affine function with respect to x_3, the estimate $(4.2)_{1,2}$ and the fact that $\mathfrak{u} = u - U_{KL}$ lead to $(4.2)_3$. The proof of (4.3) is postponed in Subsection 8.2 of the Appendix. □

If the plate is clamped on Γ_δ then we have the following Korn's type inequalities.

Proposition 4.1. *Let u be a displacement in $W^{1,p}(\Omega_\delta)$, $1<p<\infty$. Assume the plate clamped on Γ_δ. Then, we have*

$$\begin{aligned}
&\|u_1\|_{L^p(\Omega_\delta)}+\|u_2\|_{L^p(\Omega_\delta)}+\delta\|u_3\|_{L^p(\Omega_\delta)}\le C\|e(u)\|_{L^p(\Omega_\delta)},\\
&\sum_{\alpha,\,\beta=1}^{2}\Big\|\frac{\partial u_\beta}{\partial x_\alpha}\Big\|_{L^p(\Omega_\delta)}+\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_{L^p(\Omega_\delta)}\le C\|e(u)\|_{L^p(\Omega_\delta)},\\
&\sum_{\alpha=1}^{2}\Big(\Big\|\frac{\partial u_3}{\partial x_\alpha}\Big\|_{L^p(\Omega_\delta)}+\Big\|\frac{\partial u_\alpha}{\partial x_3}\Big\|_{L^p(\Omega_\delta)}\Big)\le\frac{C}{\delta}\|e(u)\|_{L^p(\Omega_\delta)}.
\end{aligned}\tag{4.4}$$

The constants do not depend on δ.

Proof. Estimates (4.4) are the consequences of the decomposition (4.1) of u and the estimates (4.2)-(4.3). □

From now on, in Sections 5 and 6, every $u\in W^{1,p}(\Omega_\delta)^3$ will be decomposed as the sum of a Kirchhoff-Love displacement and a residual displacement.

5 Asymptotic behavior of a sequence of displacements

The set of admissible displacements is denoted $\mathbb{D}_\delta$

$$\mathbb{D}_\delta\doteq\big\{u\in H^1(\Omega_\delta)^3\mid u=0\ \text{ on }\ \Gamma_\delta\big\}.\tag{5.1}$$

We rescale Ω_δ to Ω in the $\mathbf{e}_3$ direction. We set

$$\begin{aligned}
&\Omega\doteq\Big\{(x',X_3)\in\omega\times\mathbb{R}\mid\ -C_0<X_3<\phi(x')\Big\},\\
&\Omega_1\doteq\omega\times(-C_0,C_1),\qquad\Omega\subset\Omega_1.
\end{aligned}$$

We denote

$$\Xi_\varepsilon\doteq\Big\{\xi\in\mathbb{Z}^2\mid\varepsilon(\xi+Y)\subset\omega\Big\},\qquad\widehat{\omega}_\varepsilon\doteq\text{Interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}\varepsilon(\xi+\overline{Y})\Big),\qquad\Lambda_\varepsilon\doteq\omega\setminus\overline{\widehat{\omega}_\varepsilon}.$$

Definition 5.1 (The dimension reduction operator Π_δ)**.** *For every measurable function ψ on Ω_δ, we define the measurable function $\Pi_\delta(\psi)$ on Ω as follows:*

$$\Pi_\delta(\psi)(x',X_3)=\psi(x',\delta X_3)\quad\text{ for a.e. }\ (x',X_3)\in\Omega.$$

For every $f\in L^1(\Omega_\delta)$ and $g\in L^2(\Omega_\delta)$, one has

$$\int_{\Omega_\delta}f\,dx=\delta\int_\Omega\Pi_\delta(f)\,dx'dX_3,\qquad\|\Pi_\delta(g)\|_{L^2(\Omega)}=\frac{1}{\delta^{1/2}}\|g\|_{L^p(\Omega_\delta)}.$$

Moreover, if $f\in H^1(\Omega_\delta)$, one has ($\alpha\in\{1,2\}$)

$$\Pi_\delta\Big(\frac{\partial f}{\partial x_3}\Big)=\frac{1}{\delta}\frac{\partial\Pi_\delta(f)}{\partial X_3},\qquad\Pi_\delta\Big(\frac{\partial f}{\partial x_\alpha}\Big)=\frac{\partial\Pi_\delta(f)}{\partial x_\alpha}\qquad\text{a.e. in }\ \Omega.$$

We recall below the definition of the classical unfolding operator (see [16, Chapter 1]).

Definition 5.2 (The unfolding operator $\mathcal{T}_\varepsilon$)**.** *For every measurable function g on ω, we define the measurable function $\mathcal{T}_\varepsilon(g)$ on $\omega\times Y$ as follows:*

$$\mathcal{T}_\varepsilon(g)(x',X')=\begin{cases}g\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]+\varepsilon(X_1\mathbf{e}_1+X_2\mathbf{e}_2)\Big)^1\\ \qquad\qquad\qquad\qquad\text{for a.e. }\ (x',X')=(x_1,x_2,X_1,X_2)\in\widehat{\omega}_\varepsilon\times Y,\\ 0\qquad\qquad\qquad\qquad\text{for a.e. }\ (x',X')=(x_1,x_2,X_1,X_2)\in\Lambda_\varepsilon\times Y.\end{cases}$$

[1]For $x'\in\mathbb{R}^2$, $[x']\in\mathbb{Z}^2$ is the integer par of x'.

The unfolding operator with parameters will also be denoted $\mathcal{T}_\varepsilon$.

Definition 5.3. *For every measurable function ψ on Ω_1, we define the measurable function $\mathcal{T}_\varepsilon(\psi)$ on $\omega \times \mathcal{Y}$ as follows:*

$$\mathcal{T}_\varepsilon(\psi)(x',X) = \begin{cases} \psi\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big] + \varepsilon(X_1\mathbf{e}_1 + X_2\mathbf{e}_2), X_3\Big) & \\ \qquad\qquad\qquad\qquad \textit{for a.e.} \quad (x',X) = (x',X') = (x_1,x_2,X_1,X_2,X_3) \in \widehat{\omega}_\varepsilon \times \mathcal{Y}, \\ 0 \qquad\qquad\qquad\qquad \textit{for a.e.} \quad (x',X) = (x',X') = (x_1,x_2,X_1,X_2,X_3) \in \Lambda_\varepsilon \times \mathcal{Y}. \end{cases}$$

Regarding the properties of the unfolding operator $\mathcal{T}_\varepsilon$ with or without parameters, we refer the reader to [16, Subsections 1.1 and 1.5].
We will denote $\mathfrak{T}_{\varepsilon\delta} = \mathcal{T}_\varepsilon \circ \Pi_\delta$ the dimension reduction and unfolding operator which acts on measurable function on Ω_δ.
Note that if ψ does not depend on x_3 then $\mathfrak{T}_{\varepsilon\delta}(\psi) = \mathcal{T}_\varepsilon(\psi)$.

5.1 First case: only dimension reduction

Let $\{u_\delta\}_\delta$ be a sequence of displacements belonging to $\mathbb{D}_\delta$ and satisfying

$$\|e(u_\delta)\|_{L^2(\Omega_\delta)} \leq C\delta^{5/2}. \tag{5.2}$$

The constant does not depend on δ.
We decompose u_δ (see (4.1)). The terms of its decomposition satisfy (see (4.4), (4.2) and (4.3))

$$\begin{aligned} &\|\mathcal{U}_{\delta,\alpha}\|_{H^1(\omega)} + \delta\|\mathcal{U}_{\delta,3}\|_{H^2(\omega)} \leq C\delta^2, \\ &\|\mathfrak{u}_\delta\|_{L^2(\Omega_\delta)} \leq C\ \delta^{7/2}, \qquad \|\nabla\mathfrak{u}_\delta\|_{L^2(\Omega_\delta)} \leq C\delta^{5/2}. \end{aligned} \tag{5.3}$$

From which and the properties of Π_δ we obtain

$$\|\Pi_\delta(\mathfrak{u}_\delta)\|_{L^2(\Omega)} + \Big\|\frac{\partial\Pi_\delta(\mathfrak{u}_\delta)}{\partial X_3}\Big\|_{L^2(\Omega)} \leq C\delta^3, \qquad \Big\|\frac{\partial\Pi_\delta(\mathfrak{u}_\delta)}{\partial x_\alpha}\Big\|_{L^2(\Omega)} \leq C\delta^2. \tag{5.4}$$

Denote

$$\begin{aligned} H^1_\gamma(\omega) &\doteq \big\{\varphi \in H^1(\omega) \mid \varphi = 0 \text{ a.e. on } \gamma\big\}, \\ H^2_\gamma(\omega) &\doteq \big\{\varphi \in H^2(\omega) \mid \varphi = 0 \text{ and } \nabla\varphi = 0 \text{ a.e. on } \gamma\big\}, \\ \mathbb{D}_M(\omega) &\doteq H^1_\gamma(\omega)^2 \times H^2_\gamma(\omega), \\ \mathbb{W}(\Omega) &\doteq \Big\{\Phi \in L^2(\Omega)^3 \mid \frac{\partial\Phi}{\partial X_3} \in L^2(\Omega)^3\Big\}. \end{aligned}$$

We equip $\mathbb{W}(\Omega)$ with the norm

$$\forall\Phi \in \mathbb{W}(\Omega), \qquad \|\Phi\|_{\mathbb{W}(\Omega)} = \sqrt{\|\Phi\|^2_{L^2(\Omega)} + \Big\|\frac{\partial\Phi}{\partial X_3}\Big\|^2_{L^2(\Omega)}}.$$

The strain tensor of a displacement $u \in H^1(\Omega_\delta)^3$, decomposed as (4.1) is

$$\begin{aligned} e(u) = &\begin{pmatrix} e_{11}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1^2} & e_{12}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2} & 0) \\ e_{12}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2} & e_{22}(\mathcal{U}) - x_3\dfrac{\partial^2\mathcal{U}_3}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} e_{11}\big(\mathfrak{u}\big) & e_{12}\big(\mathfrak{u}\big) & \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{u}_3}{\partial x_1} + \dfrac{\partial\mathfrak{u}_1}{\partial x_3}\Big) \\ e_{12}\big(\mathfrak{u}\big) & e_{22}\big(\mathfrak{u}\big) & \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{u}_3}{\partial x_1} + \dfrac{\partial\mathfrak{u}_2}{\partial x_3}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{u}_3}{\partial x_1} + \dfrac{\partial\mathfrak{u}_1}{\partial x_3}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{u}_3}{\partial x_1} + \dfrac{\partial\mathfrak{u}_2}{\partial x_3}\Big) & \dfrac{\partial\mathfrak{u}_3}{\partial x_3} \end{pmatrix}. \end{aligned} \tag{5.5}$$

The following theorem is proved in [21].

Theorem 5.1 (Theorem 4.2 in [21]). *Let $\{u_\delta\}_\delta$ be a sequence of displacements belonging to $\mathbb{D}_\delta$ and satisfying*

$$\|e(u_\delta)\|_{L^2(\Omega_\delta)} \leq C\delta^{5/2}.$$

There exist a subsequence of $\{\delta\}$, still denoted $\{\delta\}$, and functions $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3) \in \mathbb{D}_M(\omega)$ and $\mathfrak{U} \in \mathbb{W}(\Omega)$ such that

$$\begin{aligned}
&\frac{1}{\delta^2}\mathcal{U}_{\delta,\alpha} \rightharpoonup \mathcal{U}_\alpha \quad \text{weakly in } H^1_\gamma(\omega) \text{ and strongly in } L^2(\omega),\\
&\frac{1}{\delta}\mathcal{U}_{\delta,3} \rightharpoonup \mathcal{U}_3 \quad \text{weakly in } H^2(\omega) \text{ and strongly in } H^1(\omega),\\
&\frac{1}{\delta^3}\Pi_\delta(\mathfrak{u}_\delta) \rightharpoonup \mathfrak{U} \quad \text{weakly in } \mathbb{W}(\Omega),\\
&\frac{1}{\delta^2}\frac{\partial \Pi_\delta(\mathfrak{u}_\delta)}{\partial x_\alpha} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega)^3.
\end{aligned} \tag{5.6}$$

Moreover, one has

$$\begin{aligned}
&\frac{1}{\delta^2}\Pi_\delta(u_{\delta,\alpha}) \rightharpoonup \mathcal{U}_\alpha - X_3\frac{\partial \mathcal{U}_3}{\partial x_\alpha} \quad \text{weakly in } H^1(\Omega),\\
&\frac{1}{\delta}\Pi_\delta(u_{\delta,3}) \longrightarrow \mathcal{U}_3 \quad \text{strongly in } H^1(\Omega).
\end{aligned} \tag{5.7}$$

Note that the limit displacement in the rescaled plate Ω is of Kirchhoff-Love type.
For every $\mathcal{V} = \big(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\big) \in H^1_\gamma(\omega)^2 \times H^2_\gamma(\omega)$ we define the symmetric tensor $E_M(\mathcal{V})$ by

$$E_M(\mathcal{V}) \doteq \begin{pmatrix} e_{11}(\mathcal{V}) - X_3\,\dfrac{\partial^2 \mathcal{V}_3}{\partial x_1^2} & e_{12}(\mathcal{V}) - X_3\,\dfrac{\partial^2 \mathcal{V}_3}{\partial x_1 \partial x_2} & 0\\ e_{12}(\mathcal{V}) - X_3\,\dfrac{\partial^2 \mathcal{V}_3}{\partial x_1 \partial x_2} & e_{22}(\mathcal{V}) - X_3\,\dfrac{\partial^2 \mathcal{V}_3}{\partial x_2^2} & 0\\ 0 & 0 & 0 \end{pmatrix}$$

and for every $\mathfrak{V} \in \mathbb{W}(\Omega)$, the tensor $E_w(\mathfrak{V})$ by

$$E_w(\mathfrak{V}) \doteq \begin{pmatrix} 0 & 0 & \dfrac{1}{2}\dfrac{\partial \mathfrak{V}_1}{\partial X_3}\\ 0 & 0 & \dfrac{1}{2}\dfrac{\partial \mathfrak{V}_2}{\partial X_3}\\ \dfrac{1}{2}\dfrac{\partial \mathfrak{V}_1}{\partial X_3} & \dfrac{1}{2}\dfrac{\partial \mathfrak{V}_2}{\partial X_3} & \dfrac{\partial \mathfrak{V}_3}{\partial X_3} \end{pmatrix}$$

Now, we can give the limit of the rescaled strain tensor of the sequence $\{u_\delta\}_\delta$. As a consequence of Theorem 5.1 we have

Lemma 5.1. *Let $\{u_\delta\}_\delta$ be the sequence of displacements belonging to $\mathbb{D}_\delta$ introduced in Theorem 5.1. One has*

$$\frac{1}{\delta^2}\Pi_\delta\big(e(u_\delta)\big) \rightharpoonup E_M(\mathcal{U}) + E_w(\mathfrak{U}) \text{ weakly in } L^2(\Omega)^{3\times 3}. \tag{5.8}$$

Proof. The displacement $\Pi_\delta(u_\delta)$ belongs to $H^1(\Omega)^3$ and its strain tensor $\Pi_\delta\big(e(u_\delta)\big)$ is given a.e. in Ω by

$$\begin{aligned}
\Pi_\delta\big(e(u_\delta)\big) = &\begin{pmatrix} e_{11}(\mathcal{U}_\delta) - X_3\,\delta\dfrac{\partial^2 \mathcal{U}_{\delta,3}}{\partial x_1^2} & e_{12}(\mathcal{U}_\delta) - X_3\,\delta\dfrac{\partial^2 \mathcal{U}_{\delta,3}}{\partial x_1 \partial x_2} & 0)\\ e_{12}(\mathcal{U}_\delta) - X_3\,\delta\dfrac{\partial^2 \mathcal{U}_{\delta,3}}{\partial x_1 \partial x_2} & e_{22}(\mathcal{U}_\delta) - X_3\,\delta\dfrac{\partial^2 \mathcal{U}_{\delta,3}}{\partial x_2^2} & 0\\ 0 & 0 & 0 \end{pmatrix}\\
+ &\begin{pmatrix} e_{11}\big(\Pi_\delta(\mathfrak{u}_\delta)\big) & e_{12}\big(\Pi_\delta(\mathfrak{u}_\delta)\big) & \dfrac{1}{2}\Big(\dfrac{\partial \Pi_\delta(\mathfrak{u}_{\delta,3})}{\partial x_1} + \dfrac{1}{\delta}\dfrac{\partial \Pi_\delta(\mathfrak{u}_{1,\delta})}{\partial X_3}\Big)\\ e_{12}\big(\Pi_\delta(\mathfrak{u}_\delta)\big) & e_{22}\big(\Pi_\delta(\mathfrak{u}_\delta)\big) & \dfrac{1}{2}\Big(\dfrac{\partial \Pi_\delta(\mathfrak{u}_{\delta,3})}{\partial x_1} + \dfrac{1}{\delta}\dfrac{\partial \Pi_\delta(\mathfrak{u}_{2,\delta})}{\partial X_3}\Big)\\ \dfrac{1}{2}\Big(\dfrac{\partial \Pi_\delta(\mathfrak{u}_{\delta,3})}{\partial x_1} + \dfrac{1}{\delta}\dfrac{\partial \Pi_\delta(\mathfrak{u}_{1,\delta})}{\partial X_3}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial \Pi_\delta(\mathfrak{u}_{\delta,3})}{\partial x_1} + \dfrac{1}{\delta}\dfrac{\partial \Pi_\delta(\mathfrak{u}_{2,\delta})}{\partial X_3}\Big) & \dfrac{1}{\delta}\dfrac{\partial \Pi_\delta(\mathfrak{u}_{\delta,3})}{\partial X_3} \end{pmatrix}.
\end{aligned} \tag{5.9}$$

Then, due to the convergences in Theorem 5.1 we get (5.8). □

5.2 Second case: dimension reduction and homogenization simultaneously

Let $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta}$ be a sequence of displacements belonging to $\mathbb{D}_\delta$ and satisfying

$$\|e(u_{\varepsilon,\delta})\|_{L^2(\Omega_\delta)} \leq C\delta^{5/2}. \tag{5.10}$$

The constant does not depend on ε and δ.
We decompose $u_{\varepsilon,\delta}$ as (4.1). Then, we extend the displacement $\mathfrak{u}_{\varepsilon,\delta}$ using the extension operator $\mathcal{P}_\delta$ (see Lemma 8.2 in Appendix). The extended displacement $\mathcal{P}_\delta(\mathfrak{u}_{\varepsilon,\delta})$ will be still denoted $\mathfrak{u}_{\varepsilon,\delta}$. Due to Lemma 8.2 in Appendix, the estimates $(5.3)_{2,3}$ and the properties of the reduction of dimension and unfolding operators, we have

$$\|\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta})\|_{L^2(\omega\times\mathcal{Y})} + \big\|\nabla_X\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta})\big\|_{L^2(\omega\times\mathcal{Y})} \leq C\delta^3. \tag{5.11}$$

The constant does not depend on ε and δ.

Below, we proceed simultaneously to the dimension reduction and the homogenization of the sequence $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta}$. *From now on, we assume that*

$$(\varepsilon,\delta) \longrightarrow (0,0) \qquad \text{and} \qquad \frac{\varepsilon}{\delta} \longrightarrow \theta \in (0,+\infty).$$

We denote

$$\begin{aligned}\boldsymbol{\Omega} &\doteq \Big\{(x',X) \in \omega\times\mathcal{Y} \mid -C_0 < X_3 < \phi(x')\Big\},\\ \mathbb{W}(\boldsymbol{\Omega}) &\doteq \Big\{\mathfrak{V} \in L^2(\boldsymbol{\Omega})^3 \mid \nabla_X\mathfrak{V} \in L^2(\boldsymbol{\Omega})^{3\times 3}\\ &\qquad\qquad \text{and } \mathfrak{V} \text{ is 1-periodic with respect to } X_1 \text{ and } X_2\Big\}.\end{aligned}$$

We endow $\mathbb{W}(\boldsymbol{\Omega})$ with the norm

$$\forall\mathfrak{V} \in \mathbb{W}(\boldsymbol{\Omega}), \qquad \|\mathfrak{V}\|_{\mathbb{W}(\boldsymbol{\Omega})} = \sqrt{\|\mathfrak{V}\|^2_{L^2(\boldsymbol{\Omega})} + \|\nabla_X\mathfrak{V}\|^2_{L^2(\boldsymbol{\Omega})}}.$$

For every $\mathfrak{V} \in \mathbb{W}(\boldsymbol{\Omega})$, we define the 3×3 symmetric tensor $E^\theta_w(\mathfrak{V}) \in L^2(\boldsymbol{\Omega})^{3\times 3}$ by

$$E^\theta_w(\mathfrak{V}) \doteq \begin{pmatrix} \dfrac{\partial\mathfrak{V}_1}{\partial X_1} & * & * \\ \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{V}_2}{\partial X_1} + \dfrac{\partial\mathfrak{V}_1}{\partial X_2}\Big) & \dfrac{\partial\mathfrak{V}_2}{\partial X_2} & * \\ \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{V}_3}{\partial X_1} + \theta\dfrac{\partial\mathfrak{V}_1}{\partial X_3}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial\mathfrak{V}_3}{\partial X_2} + \theta\dfrac{\partial\mathfrak{V}_2}{\partial X_3}\Big) & \theta\dfrac{\partial\mathfrak{V}_3}{\partial X_3} \end{pmatrix}$$

and the function $\mathcal{M}(\mathfrak{V})$ by

$$\mathcal{M}(\mathfrak{V})(x') = \frac{1}{\phi(x')+C_0}\int_{\mathcal{Y}_{x'}} \mathfrak{V}(x',X)dX, \qquad \text{for a.e. } x' \in \omega.$$

$\mathcal{M}(\mathfrak{V})$ belongs to $L^2(\omega)^3$, the measure of $\mathcal{Y}_{x'}$ is $\phi(x')+C_0$. The operator $\mathcal{M}$ is continuous and linear from $\mathbb{W}(\boldsymbol{\Omega})$ into $L^2(\omega)^3$.

Theorem 5.2. *Let $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta}$ be a sequence of displacements belonging to $\mathbb{D}_\delta$ and satisfying* (5.10). *There exist a subsequence of $\{\varepsilon,\delta\}$, still denoted $\{\varepsilon,\delta\}$, and functions $\mathcal{U} = (\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3) \in \mathbb{D}_M(\omega)$ and $\widehat{\mathfrak{U}} \in \mathbb{W}(\boldsymbol{\Omega})$ satisfying $\mathcal{M}(\widehat{\mathfrak{U}}) = 0$ a.e. in ω such that*

$$\begin{aligned}&\frac{1}{\delta^2}\mathcal{U}_{\varepsilon,\delta,\alpha} \rightharpoonup \mathcal{U}_\alpha \quad \textit{weakly in } H^1_\gamma(\omega) \textit{ and strongly in } L^2(\omega),\\ &\frac{1}{\delta}\mathcal{U}_{\varepsilon,\delta,3} \rightharpoonup \mathcal{U}_3 \quad \textit{weakly in } H^2(\omega) \textit{ and strongly in } H^1(\omega),\end{aligned} \tag{5.12}$$

Moreover, one has

$$\begin{aligned}&\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}(u_{\varepsilon,\delta,\alpha}) \longrightarrow \mathcal{U}_\alpha - X_3\frac{\partial\mathcal{U}_3}{\partial x_\alpha} \quad \textit{strongly in } L^2(\omega;H^1(\mathcal{Y})),\\ &\frac{1}{\delta}\mathfrak{T}_{\varepsilon\delta}(u_{\varepsilon,\delta,3}) \longrightarrow \mathcal{U}_3 \quad \textit{strongly in } L^2(\omega;H^1(\mathcal{Y})).\end{aligned} \tag{5.13}$$

Furthermore

$$\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e(u_{\varepsilon,\delta})\big) \rightharpoonup E_M(\mathcal{U}) + E_w^\theta(\widehat{\mathfrak{U}}) \text{ weakly in } L^2(\omega\times\mathcal{Y})^{3\times 3}. \tag{5.14}$$

Proof. There exist a subsequence of $\{\varepsilon,\delta\}$, still denoted $\{\varepsilon,\delta\}$ and functions $\mathcal{U}=(\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3)\in\mathbb{D}_M(\omega)$, $\widehat{\mathcal{U}}_m\in L^2(\omega;H^1_{per,0}(Y))^2$, $\widehat{\mathcal{U}}_3\in L^2(\omega;H^2_{per,0}(Y))^2$ and $\mathring{\mathfrak{U}}\in L^2\big(\omega;H^1_{per}(\mathcal{Y})\big)^3$ such that convergences (5.12) hold and $((\alpha,\beta)\in\{1,2\}^2)$

$$\begin{aligned}
&\frac{1}{\delta^2}\mathcal{T}_\varepsilon\big(\mathcal{U}_{\varepsilon,\delta,m}\big)\rightharpoonup\mathcal{U}_m \quad \text{weakly in } L^2(\omega;H^1(Y))^2,\\
&\frac{1}{\delta^2}\mathcal{T}_\varepsilon\Big(\frac{\partial\mathcal{U}_{\varepsilon,\delta,m}}{\partial x_\alpha}\Big)\rightharpoonup\frac{\partial\mathcal{U}_m}{\partial x_\alpha}+\frac{\partial\widehat{\mathcal{U}}_m}{\partial X_\alpha} \quad \text{weakly in } L^2(\mathbf{\Omega})^2,\\
&\frac{1}{\delta}\mathcal{T}_\varepsilon\big(\mathcal{U}_{\varepsilon,\delta,3}\big)\longrightarrow\mathcal{U}_3 \quad \text{strongly in } L^2(\omega;H^2(Y)),\\
&\frac{1}{\delta}\mathcal{T}_\varepsilon\Big(\frac{\partial\mathcal{U}_{\varepsilon,\delta,3}}{\partial x_\alpha}\Big)\rightharpoonup\frac{\partial\mathcal{U}_3}{\partial x_\alpha} \quad \text{weakly in } L^2(\omega\times Y),\\
&\frac{1}{\delta}\mathcal{T}_\varepsilon\Big(\frac{\partial^2\mathcal{U}_{\varepsilon,\delta,3}}{\partial x_\alpha\partial x_\beta}\Big)\rightharpoonup\frac{\partial\mathcal{U}_3}{\partial x_\alpha\partial x_\beta}+\frac{\partial^2\widehat{\mathcal{U}}_3}{\partial X_\alpha\partial X_\beta} \quad \text{weakly in } L^2(\omega\times Y),
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
&\mathfrak{T}_{\varepsilon\delta}\Big(\frac{1}{\delta^3}\mathfrak{u}_{\varepsilon,\delta}\Big)\rightharpoonup\mathring{\mathfrak{U}} \quad \text{weakly in } \mathfrak{L}^2(\omega\times\mathcal{Y})^3,\\
&\varepsilon\mathfrak{T}_{\varepsilon\delta}\Big(\frac{1}{\delta^3}\frac{\partial\mathfrak{u}_{\varepsilon,\delta}}{\partial x_\alpha}\Big)=\frac{\partial}{\partial X_\alpha}\mathfrak{T}_{\varepsilon\delta}\Big(\frac{1}{\delta^3}\mathfrak{u}_{\varepsilon,\delta}\Big)\rightharpoonup\frac{\partial\mathring{\mathfrak{U}}}{\partial X_\alpha} \quad \text{weakly in } L^2(\omega\times\mathcal{Y})^3,\\
&\delta\mathfrak{T}_{\varepsilon\delta}\Big(\frac{1}{\delta^3}\frac{\partial\mathfrak{u}_{\varepsilon,\delta}}{\partial x_3}\Big)=\frac{\partial}{\partial X_3}\mathfrak{T}_{\varepsilon\delta}\Big(\frac{1}{\delta^3}\mathfrak{u}_{\varepsilon,\delta}\Big)\rightharpoonup\frac{\partial\mathring{\mathfrak{U}}}{\partial X_3} \quad \text{weakly in } L^2(\omega\times\mathcal{Y})^3.
\end{aligned} \tag{5.16}$$

Now, we have

$$\mathfrak{T}_{\varepsilon\delta}\big(e(u_{\varepsilon,\delta})\big)=\begin{pmatrix} e_{11}(\mathcal{U}_{\varepsilon,\delta})-X_3\,\delta\dfrac{\partial^2\mathcal{U}_{\delta,3}}{\partial x_1^2} & e_{12}(\mathcal{U}_{\varepsilon,\delta})-X_3\,\delta\dfrac{\partial^2\mathcal{U}_{\delta,3}}{\partial x_1\partial x_2} & 0)\\ e_{12}(\mathcal{U}_{\varepsilon,\delta})-X_3\,\delta\dfrac{\partial^2\mathcal{U}_{\delta,3}}{\partial x_1\partial x_2} & e_{22}(\mathcal{U}_{\varepsilon,\delta})-X_3\,\delta\dfrac{\partial^2\mathcal{U}_{\delta,3}}{\partial x_2^2} & 0\\ 0 & 0 & 0\end{pmatrix}$$
$$+\begin{pmatrix} \dfrac{1}{2\varepsilon}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,1})}{\partial X_1} & \dfrac{1}{2\varepsilon}\Big(\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,2})}{\partial X_1}+\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,1})}{\partial X_2}\Big) & \dfrac{1}{2}\Big(\dfrac{1}{\varepsilon}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,3})}{\partial X_1}+\dfrac{1}{\delta}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,1})}{\partial X_3}\Big)\\ \dfrac{1}{2\varepsilon}\Big(\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,1})}{\partial X_2}+\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,2})}{\partial X_1}\Big) & \dfrac{1}{\varepsilon}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,2})}{\partial X_2} & \dfrac{1}{2}\Big(\dfrac{1}{\varepsilon}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,3})}{\partial X_1}+\dfrac{1}{\delta}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{2,\delta})}{\partial X_3}\Big)\\ \dfrac{1}{2}\Big(\dfrac{1}{\varepsilon}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,3})}{\partial X_1}+\dfrac{1}{\delta}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,1})}{\partial X_3}\Big) & \dfrac{1}{2}\Big(\dfrac{1}{\varepsilon}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,3})}{\partial X_1}+\dfrac{1}{\delta}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{2,\delta})}{\partial X_3}\Big) & \dfrac{1}{\delta}\dfrac{\partial\mathfrak{T}_{\varepsilon\delta}(\mathfrak{u}_{\varepsilon,\delta,3})}{\partial X_3}\end{pmatrix}.$$

We divide by δ^2 and pass to the limit using (5.15) and $(5.16)_{2,3}$. This gives

$$\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e(u_{\varepsilon,\delta})\big)\rightharpoonup E_M(\mathcal{U})+\begin{pmatrix} e_{11,X'}(\widehat{\mathcal{U}}_m)-X_3\dfrac{\partial^2\widehat{\mathcal{U}}_3}{\partial X_1^2} & e_{12,X'}(\widehat{\mathcal{U}}_m)-X_3\dfrac{\partial^2\widehat{\mathcal{U}}_3}{\partial X_1\partial X_2} & 0)\\ e_{12,X'}(\widehat{\mathcal{U}}_m)-X_3\dfrac{\partial^2\widehat{\mathcal{U}}_3}{\partial X_1\partial X_2} & e_{22,X'}(\widehat{\mathcal{U}}_m)-X_3\dfrac{\partial^2\widehat{\mathcal{U}}_3}{\partial X_2^2} & 0\\ 0 & 0 & 0\end{pmatrix}$$
$$+\frac{1}{\theta}\begin{pmatrix} \dfrac{\partial\mathring{\mathfrak{U}}_1}{\partial X_1} & * & *\\ \dfrac{1}{2}\Big(\dfrac{\partial\mathring{\mathfrak{U}}_2}{\partial X_1}+\dfrac{\partial\mathring{\mathfrak{U}}_1}{\partial X_2}\Big) & \dfrac{\partial\mathring{\mathfrak{U}}_2}{\partial X_2} & *\\ \dfrac{1}{2}\Big(\dfrac{\partial\mathring{\mathfrak{U}}_3}{\partial X_1}+\theta\dfrac{\partial\mathring{\mathfrak{U}}_1}{\partial X_3}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial\mathring{\mathfrak{U}}_3}{\partial X_2}+\theta\dfrac{\partial\mathring{\mathfrak{U}}_2}{\partial X_3}\Big) & \theta\dfrac{\partial\mathring{\mathfrak{U}}_3}{\partial X_3}\end{pmatrix} \quad \text{weakly in } L^2(\omega\times\mathcal{Y})^{3\times 3}.$$

where $e_{\alpha\beta,X'}(\widehat{\mathcal{U}}_m)=\frac{1}{2}\Big(\dfrac{\partial\widehat{\mathcal{U}}_\alpha}{\partial X_\beta}+\dfrac{\partial\widehat{\mathcal{U}}_\beta}{\partial X_\alpha}\Big)$. Set

$$\widehat{\mathfrak{U}}=\Big(\widehat{\mathcal{U}}_1-X_3\frac{\partial\widehat{\mathcal{U}}_3}{\partial X_1}\Big)\mathbf{e}_1+\Big(\widehat{\mathcal{U}}_2-X_3\frac{\partial\widehat{\mathcal{U}}_3}{\partial X_2}\Big)\mathbf{e}_2+\theta\widehat{\mathcal{U}}_3\,\mathbf{e}_3+\frac{1}{\theta}\big(\mathring{\mathfrak{U}}-\mathcal{M}(\mathring{\mathfrak{U}})\big).$$

Observe that $\mathcal{M}(\widehat{\mathfrak{U}}) = 0$ a.e. in ω. This leads to (5.14). Convergences (5.13) are the consequences of (5.12) and the properties of $\mathfrak{T}_{\varepsilon\delta}$. □

Lemma 5.2. *For every function $\widehat{\mathfrak{V}} \in \mathbb{W}(\mathbf{\Omega})$, one has ($\alpha \in \{1,2\}$)*

$$\begin{cases} \theta \geq 1, & \|\nabla_X \widehat{\mathfrak{V}}_\alpha\|_{L^2(\mathbf{\Omega})} + \dfrac{1}{\theta}\Big\|\dfrac{\partial \widehat{\mathfrak{V}}_3}{\partial X_\alpha}\Big\|_{L^2(\mathbf{\Omega})} + \theta\Big\|\dfrac{\partial \widehat{\mathfrak{V}}_3}{\partial X_3}\Big\|_{L^2(\mathbf{\Omega})} \leq C\|E_w^\theta(\widehat{\mathfrak{V}})\|_{L^2(\mathbf{\Omega})}, \\ \theta \leq 1, & \Big\|\dfrac{\partial \widehat{\mathfrak{V}}}{\partial X_\alpha}\Big\|_{L^2(\mathbf{\Omega})} + \theta\Big\|\dfrac{\partial \widehat{\mathfrak{V}}}{\partial X_3}\Big\|_{L^2(\mathbf{\Omega})} \leq C\|E_w^\theta(\widehat{\mathfrak{V}})\|_{L^2(\mathbf{\Omega})}. \end{cases} \tag{5.17}$$

The constant does not depend on θ.

Proof. Consider the domains $\mathcal{Y}_{x'}$, $x' \in \omega$. Note that

$$Y \times (-C_0, C_0) \subset \mathcal{Y}_{x'} \subset Y \times (-C_0, C_1) \qquad \forall x' \in \omega.$$

Proceeding as in [16, Chapter 11] we obtain that for every $\boldsymbol{\psi} \in H^1_{per}(\mathcal{Y}_{x'})^3$ ($\alpha \in \{1,2\}$)

$$\begin{cases} \theta \geq 1, & \|\nabla_X \boldsymbol{\psi}_\alpha\|_{L^2(\mathcal{Y}_{x'})} + \dfrac{1}{\theta}\Big\|\dfrac{\partial \boldsymbol{\psi}_3}{\partial X_\alpha}\Big\|_{L^2(\mathcal{Y}_{x'})} + \theta\Big\|\dfrac{\partial \boldsymbol{\psi}_3}{\partial X_3}\Big\|_{L^2(\mathcal{Y}_{x'})} \leq C\|E_w^\theta(\boldsymbol{\psi})\|_{L^2(\mathcal{Y}_{x'})}, \\ \theta \leq 1, & \Big\|\dfrac{\partial \boldsymbol{\psi}}{\partial X_\alpha}\Big\|_{L^2(\mathcal{Y}_{x'})} + \theta\Big\|\dfrac{\partial \boldsymbol{\psi}}{\partial X_3}\Big\|_{L^2(\mathcal{Y}_{x'})} \leq C\|E_w^\theta(\boldsymbol{\psi})\|_{L^2(\mathcal{Y}(x'))} \end{cases} \tag{5.18}$$

where $H^1_{per}(\mathcal{Y}_{x'})^3$ is the subspace of $H^1(\mathcal{Y}_{x'})^3$ containing the periodic functions with respect to X_1 and X_2. The constant C does not depend on θ and x'.
Now, let $\widehat{\mathfrak{V}}$ be in $\mathbb{W}(\mathbf{\Omega}) \cap \mathcal{C}^1(\overline{\omega} \times \overline{\mathcal{Y}})^3$. We apply the above result to the function $\widehat{\mathfrak{V}}(x', \cdot) \in H^1_{per}(\mathcal{Y}_{x'})^3$. This leads to (5.17). Then, a density argument gives the result for every function in $\mathbb{W}(\mathbf{\Omega})$. □

Corollary 5.1. *For every $\theta \in (0, +\infty)$ and for every $\boldsymbol{\psi} \in H^1_{per}(\mathcal{Y}_{x'})^3$ we have*

$$\|\nabla_X \boldsymbol{\psi}\|_{L^2(\mathcal{Y}_{x'})} \leq C\Big(\theta + \frac{1}{\theta}\Big)\|E_w^\theta(\boldsymbol{\psi})\|_{L^2(\mathcal{Y}_{x'})}. \tag{5.19}$$

The constant does not depend on θ and x'.

Lemma 5.3. *There exist two strictly positive constants c and C, independent of θ, such that for every $\mathcal{V} = (\mathcal{V}_m, \mathcal{V}_3) \in H^1(\omega)^2 \times H^2(\omega)$ and $\widehat{\mathfrak{V}} \in \mathbb{W}(\mathbf{\Omega})$ satisfying $\mathcal{M}(\widehat{\mathfrak{V}}) = 0$ a.e. in ω, we have*

$$\begin{aligned} c\Big(\|\mathcal{V}_m\|_{H^1(\omega)} + \|\mathcal{V}_3\|_{H^2(\omega)} + \frac{\theta}{1+\theta^2}\|\widehat{\mathfrak{V}}\|_{\mathbb{W}(\mathbf{\Omega})}\Big) &\leq \|E_M(\mathcal{V}) + E_w^\theta(\widehat{\mathfrak{V}})\|_{L^2(\mathbf{\Omega})} \\ &\leq C\big(\|\mathcal{V}_m\|_{H^1(\omega)} + \|\mathcal{V}_3\|_{H^2(\omega)} + (1+\theta)\|\widehat{\mathfrak{V}}\|_{H^1(\mathbf{\Omega})}\big). \end{aligned} \tag{5.20}$$

Proof. First, due to the periodicity of the fields $\widehat{\mathfrak{V}}_m$ and $\widehat{\mathfrak{V}}_3$, we have

$$\|E_M(\mathcal{V}) + E_w^\theta(\widehat{\mathfrak{V}})\|^2_{L^2(\mathbf{\Omega})} = \|E_M(\mathcal{V})\|^2_{L^2(\mathbf{\Omega})} + \|E_w^\theta(\widehat{\mathfrak{V}})\|^2_{L^2(\mathbf{\Omega})}.$$

Then, since the components of $E_M(\mathcal{V})$ are affine functions with respect to X_3, there exist two strictly positive constants c_M and C_M such that

$$\begin{aligned} c_M \sum_{\alpha,\beta=1}^{2} \Big(\|e_{\alpha\beta}(\mathcal{V}_m\|^2_{L^2(\omega)} + \Big\|\frac{\partial^2 \mathcal{V}_3}{\partial x_\alpha \partial x_\beta}\Big\|^2_{L^2(\omega)}\Big) \\ \leq \|E_M(\mathcal{V})\|^2_{L^2(\mathbf{\Omega})} \leq C_M \sum_{\alpha,\beta=1}^{2} \Big(\|e_{\alpha\beta}(\mathcal{V}_m\|^2_{L^2(\omega)} + \Big\|\frac{\partial^2 \mathcal{V}_3}{\partial x_\alpha \partial x_\beta}\Big\|^2_{L^2(\omega)}\Big). \end{aligned}$$

Then, the boundary conditions together with the Poincaré inequality yield

$$c\big(\|\mathcal{V}_m\|_{H^1(\omega)} + |\mathcal{V}_3|_{H^2(\omega)}\big) \leq \|E_M(\mathcal{V})\|_{L^2(\mathbf{\Omega})} \leq C\big(\|\mathcal{V}_m\|_{H^1(\omega)} + |\mathcal{V}_3|_{H^2(\omega)}\big).$$

From the definition of the matrix $E_w^\theta(\widehat{\mathfrak{V}})$, there exists a constant C_w independent of θ such that

$$\|E_w^\theta(\widehat{\mathfrak{V}})\|_{L^2(\mathbf{\Omega})} \le C_w(1+\theta)\|\nabla_X \widehat{\mathfrak{V}}\|_{L^2(\mathbf{\Omega})}.$$

Now, (5.19) leads to

$$\frac{\theta}{1+\theta^2}\|\nabla_X \widehat{\mathfrak{V}}(x',\cdot)\|_{L^2(\mathcal{Y}_{x'})} \le C\|E_w^\theta(\widehat{\mathfrak{V}})(x',\cdot)|_{L^2(\mathcal{Y}_{x'})} \qquad \text{for a.e. } x' \in \omega.$$

Observe that for every $x' \in \omega$, the domains $\mathcal{Y}_{x'}$ are star-shaped with respect to balls whose radii are $\inf\{1/2, 2C_0\}$ and have a diameter less than $2+2C_1$. Hence, [13, Proposition 2.2] gives the following Poincaré-Wirtinger inequality

$$\|\widehat{\mathfrak{V}}(x',\cdot) - \mathcal{M}(\widehat{\mathfrak{V}})(x')\|_{L^2(\mathcal{Y}_{x'})} \le C\|\nabla_X \widehat{\mathfrak{V}}(x',\cdot)\|_{L^2(\mathcal{Y}_{x'})}. \qquad \text{for a.e. } x' \in \omega$$

where the constant does not depend on $x' \in \omega$.
Thus

$$\frac{\theta}{1+\theta^2}\|\widehat{\mathfrak{V}}\|_{\mathbb{W}(\mathbf{\Omega})} \le C\|E_w^\theta(\widehat{\mathfrak{V}})\|_{L^2(\mathbf{\Omega})}.$$

Finally we obtain (5.20). □

6 The linear elasticity problem in Ω_δ

For $1 \le i,j,k,l \le 3$, let a_{ijkl} be in $L^\infty(\mathcal{Y})$ and satisfy the symmetry conditions

$$a_{ijkl}(X) = a_{jikl}(X) = a_{klij}(X) \qquad \text{for a.e. } X \in \mathcal{Y},$$

as well as the coercivity condition

$$a_{ijkl}\tau_{ij}\tau_{kl} \ge c_0\,\tau_{ij}\tau_{ij} \qquad \text{a.e. in } \mathcal{Y} \tag{6.1}$$

for every symmetric matrix $\tau = (\tau_{ij})$ of order 3, where c_0 is a strictly positive constant.
The coefficients $a_{ijkl}^{\varepsilon,\delta}$ of the Hooke tensor are then given by

$$a_{ijkl}^{\varepsilon,\delta}(x) = a_{ijkl}\Big(\Big\{\frac{x'}{\varepsilon}\Big\}, \frac{x_3}{\delta}\Big) \quad \text{for a.e. } x \in \Omega_\delta$$

where $\{x'\}$ is the fractional part of x'[2].
The constitutive law of the materials is the relation between the strain tensor and the stress tensor,

$$\sigma_{ij}^{\varepsilon,\delta}(v) = a_{ijkl}^{\varepsilon,\delta}\, e_{kl}(v), \qquad \forall v \in \mathbb{D}_\delta. \tag{6.2}$$

For simplicity, we consider only applied body forces f_δ.
The displacement $u_{\varepsilon,\delta} \in \mathbb{D}_\delta$ is the solution of the following elasticity problem:

$$\begin{cases} \displaystyle\int_{\Omega_\delta} \sigma_{ij}^{\varepsilon,\delta}(u_{\varepsilon,\delta})\, e_{ij}(v)\, dx = \int_{\Omega_\delta} f_\delta \cdot v\, dx \\ \forall v \in \mathbb{D}_\delta. \end{cases} \tag{6.3}$$

We make the assumption that the applied body forces f_δ are of the form

$$f_\delta(x) = \big(\delta^2 f_\alpha(x') + \delta x_3 g_\alpha(x')\big)\mathbf{e}_\alpha + \delta^3 f_3(x')\mathbf{e}_3 \quad \text{for a.e. } x \in \Omega_\delta, \tag{6.4}$$

where $f = (f_1, f_2, f_3)$ is in $L^2(\omega)^3$ and $g = (g_1, g_2)$ in $L^2(\omega)^2$.
Now, for every $u \in \mathbb{D}_\delta$ decompose as (4.1) one has

$$\begin{aligned} \int_{\Omega_\delta} f_\delta \cdot U_{KL}\, dx &= \delta^3 \int_\omega f_\alpha\, \mathcal{U}_\alpha\, (\phi + C_0) dx' - \frac{\delta^4}{3}\int_\omega g_\alpha \frac{\partial \mathcal{U}_3}{\partial x_\alpha}\, (\phi^3 + C_0^3) dx' \\ &\quad + \frac{\delta^3}{2}\int_\omega g_\alpha\, \mathcal{U}_\alpha\, (\phi^2 - C_0^2) dx' - \frac{\delta^4}{2}\int_\omega f_\alpha \frac{\partial \mathcal{U}_3}{\partial x_\alpha}\, (\phi^2 - C_0^2) dx' \\ &\quad + \delta^4 \int_\omega f_3\, \mathcal{U}_3\, (\phi + C_0) dx', \\ \Big|\int_{\Omega_\delta} f_\delta \cdot \mathfrak{u}\, dx\Big| &\le C\delta^3\big(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\big)\|e(u)\|_{L^2(\Omega_\delta)}. \end{aligned}$$

[2]Remind that for a.e. $x' \in \mathbb{R}^2$, we have $x' = [x'] + \{x'\}$ with $[x'] \in Z^2$ and $\{x'\} \in Y$.

From the estimates (3.5) and those above, we obtain an upper bound of the right-hand side of (6.3)

$$\Big|\int_{\Omega_\delta} f_\delta \cdot u\, dx\Big| \le C\delta^{5/2}\big(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\big)\|e(u)\|_{L^2(\Omega_\delta)}.$$

Applying the above estimate for $u = u_{\varepsilon,\delta}$ taken as test function in (6.3), gives

$$\|e(u_{\varepsilon,\delta})\|_{L^2(\Omega_\delta)} \le C\delta^{5/2}\big(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\big). \tag{6.5}$$

6.1 The unfolded limit problem

In this subsection, we will investigate the asymptotic behavior of the sequence $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta}$, the solution to problem (6.3).

Theorem 6.1. *Let $u_{\varepsilon,\delta}$ be the solution to problem (6.3). There exist $\mathcal{U}^\theta = (\mathcal{U}_1^\theta, \mathcal{U}_2^\theta, \mathcal{U}_3^\theta) \in \mathbb{D}_M(\omega)$ and $\widehat{\mathfrak{U}}^\theta \in \mathbb{W}(\mathbf{\Omega})$ satisfying $\mathcal{M}(\widehat{\mathfrak{U}}^\theta) = 0$ a.e. in ω such that ($\alpha \in \{1,2\}$)*

$$\begin{aligned}
&\frac{1}{\delta^2}\mathcal{U}_{\varepsilon,\delta,\alpha} \longrightarrow \mathcal{U}_\alpha^\theta \quad \text{strongly in } H^1_\gamma(\omega),\\
&\frac{1}{\delta}\mathcal{U}_{\varepsilon,\delta,3} \rightharpoonup \mathcal{U}_3^\theta \quad \text{weakly in } H^2(\omega) \text{ and strongly in } H^1(\omega),\\
&\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}(u_{\varepsilon,\delta,\alpha}) \longrightarrow \mathcal{U}_\alpha^\theta - X_3\frac{\partial \mathcal{U}_3^\theta}{\partial x_\alpha} \quad \text{strongly in } L^2(\omega; H^1(\mathcal{Y})),\\
&\frac{1}{\delta}\mathfrak{T}_{\varepsilon\delta}(u_{\varepsilon,\delta,3}) \longrightarrow \mathcal{U}_3^\theta \quad \text{strongly in } L^2(\omega; H^1(\mathcal{Y}))
\end{aligned} \tag{6.6}$$

and

$$\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e(u_{\varepsilon,\delta})\big) \longrightarrow E_M(\mathcal{U}^\theta) + E_w^\theta(\widehat{\mathfrak{U}}^\theta) \text{ strongly in } L^2(\mathbf{\Omega})^{3\times 3}. \tag{6.7}$$

The pair $(\mathcal{U}^\theta, \widehat{\mathfrak{U}}^\theta) \in \mathbb{D}_M(\omega) \times \mathbb{W}(\mathbf{\Omega})$ is the unique solution to

$$\begin{cases}
\displaystyle\int_{\mathbf{\Omega}} a_{ijkl}\big(E_{M,ij}(\mathcal{U}^\theta) + E^\theta_{w,ij}(\widehat{\mathfrak{U}}^\theta)\big)\big(E_{M,kl}(\mathcal{V}) + E^\theta_{w,kl}(\widehat{\mathfrak{V}})\big)\, dx' dX\\
\displaystyle= \int_\omega f\cdot\mathcal{V}\,(\phi + C_0)dx' - \frac{1}{3}\int_\omega g_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\,(\phi^3 + C_0^3)dx'\\
\displaystyle+ \frac{1}{2}\int_\omega \Big(g_\alpha\,\mathcal{V}_\alpha - f_\alpha\frac{\partial \mathcal{V}_3}{\partial x_\alpha}\Big)(\phi^2 - C_0^2)dx' \qquad \forall (\mathcal{V}, \widehat{\mathfrak{V}}) \in \mathbb{D}_M(\omega)\times\mathbb{W}(\mathbf{\Omega}).
\end{cases} \tag{6.8}$$

The solution to the above problem satisfies the following estimate:

$$\|\mathcal{U}_m^\theta\|_{H^1(\omega)} + \|\mathcal{U}_3^\theta\|_{H^2(\omega)} + \frac{\theta}{1+\theta^2}\|\widehat{\mathfrak{U}}^\theta\|_{\mathbb{W}(\mathbf{\Omega})} \le C\big(\|f\|_{L^2(\omega)} + \|g\|_{L^2(\omega)}\big). \tag{6.9}$$

The constant C does not depend on θ.

Proof. Step 1. First of all, the convergences.
There exist a subsequence of $\{\varepsilon,\delta\}$, still denoted $\{\varepsilon,\delta\}$, and $\mathcal{U}^\theta = (\mathcal{U}_1^\theta, \mathcal{U}_2^\theta, \mathcal{U}_3^\theta) \in \mathbb{D}_M(\omega)$, $\widehat{\mathfrak{U}}^\theta \in \mathbb{W}(\mathbf{\Omega})$ such that the convergences (5.12)-(5.13) and (5.14) hold (see Theorem 5.2).
Step 2. The limit problem.
Let $\mathcal{V}$ be in $\mathbb{D}_M(\omega)$ and $\widehat{\mathfrak{V}} \in H^1(\omega\times\mathcal{Y})^3 \cap \mathbb{W}(\mathbf{\Omega})$ such that $\widehat{\mathfrak{V}} = 0$ on $\gamma\times\mathcal{Y}$ and $\mathcal{M}(\widehat{\mathfrak{V}}) = 0$. We choose as test displacement

$$v_{\varepsilon,\delta}(x) = \frac{1}{\delta^5}\Big[\Big(\delta\mathcal{V}_1(x') - x_3\frac{\partial \mathcal{V}_3}{\partial x_1}(x')\Big)\mathbf{e}_1 + \Big(\delta\mathcal{V}_2(x') - x_3\frac{\partial \mathcal{V}_3}{\partial x_2}(x')\Big)\mathbf{e}_1 + \mathcal{V}_3(x')\mathbf{e}_3 + \delta^3\widehat{\mathfrak{V}}\Big(x', \Big\{\frac{x'}{\varepsilon}\Big\}, \frac{x_3}{\delta}\Big)\Big]$$

for a.e. x in Ω_δ.

A straightforward calculation gives

$$\delta^4\mathfrak{T}_{\varepsilon\delta}\big(e(v_{\varepsilon,\delta})\big) \longrightarrow E_M(\mathcal{V}) + E_w^\theta(\widehat{\mathfrak{V}}) \quad \text{strongly in } L^2(\mathbf{\Omega})^{3\times 3}$$

and

$$\int_{\Omega_\delta} f_\delta \cdot v_{\varepsilon,\delta}\, dx \longrightarrow \int_\omega f \cdot \mathcal{V}\,(\phi + C_0)dx' - \frac{1}{3}\int_\omega g_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\,(\phi^3 + C_0^3)dx' + \frac{1}{2}\int_\omega \big(g_\alpha\,\mathcal{V}_\alpha - f_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\big)(\phi^2 - C_0^2)dx'.$$

Taking $v_{\varepsilon,\delta}$ as test displacement in (6.3), then transforming with $\mathfrak{T}_{\varepsilon\delta}$ and passing to the limit using the convergences (5.14) and those above regarding the test displacement lead to (6.8) with the pair $(\mathcal{V}, \widehat{\mathfrak{V}})$.
Then, since the space of the fields $\widehat{\mathfrak{V}}$ in $H^1(\omega \times \mathcal{Y})^3 \cap \mathbb{W}(\mathbf{\Omega})$ such that $\widehat{\mathfrak{V}} = 0$ on $\gamma \times \mathcal{Y}$ is dense in $\mathbb{W}(\mathbf{\Omega})$, problem (6.8) is satisfied for every $\widehat{\mathfrak{V}}$ in $\mathbb{W}(\mathbf{\Omega})$.

Step 3. Uniqueness of the solution of problem (6.8).

From the inequalities in Lemma 5.3, we get that the bilinear form over $\mathbb{D}_M(\omega) \times \mathbb{W}(\mathbf{\Omega})$

$$\big((\mathcal{V}, \widehat{\mathfrak{V}}),\, (\mathcal{W}, \widehat{\mathfrak{W}})\big) \in \big(\mathbb{D}_M(\omega) \times \mathbb{W}(\mathbf{\Omega})\big)^2 \longmapsto \int_{\mathbf{\Omega}} a_{ijkl}\big(E_{M,ij}(\mathcal{V}) + E^\theta_{w,ij}(\widehat{\mathfrak{V}})\big)\big(E_{M,kl}(\mathcal{W}) + E^\theta_{w,kl}(\widehat{\mathfrak{W}})\big)\, dx'dX$$

is coercive and bounded. So, problem (6.8) admits a unique solution. Hence, the whole sequences in (6.6) and (6.7) converge towards their limits. As a consequence of Lemma 5.3, we obtain the estimate (6.9).

Step 4. Strong convergence of the rescaled strain tensor.

Take $u_{\varepsilon,\delta}$ as test displacement in (6.3), then transform using $\mathfrak{T}_{\varepsilon\delta}$ and divide by δ^4.
We first have

$$\lim_{(\varepsilon,\delta)\to(0,0)} \frac{1}{\delta^4}\int_{(\widehat{\omega}_\varepsilon \times \mathcal{Y})\cap \mathbf{\Omega}} \mathfrak{T}_{\varepsilon\delta}(f_\delta)\cdot \mathfrak{T}_{\varepsilon\delta}(u_{\varepsilon,\delta})\, dx'dX = \int_\omega f\cdot \mathcal{U}^\theta\,(\phi+\psi)dx' - \frac{1}{3}\int_\omega g_\alpha \frac{\partial \mathcal{U}^\theta_3}{\partial x_\alpha}\,(\phi^3+\psi^3)dx' + \frac{1}{2}\int_\omega \big(g_\alpha\,\mathcal{U}^\theta_\alpha - f_\alpha\frac{\partial \mathcal{U}^\theta_3}{\partial x_\alpha}\big)(\phi^2-\psi^2)dx'.$$

Then, the weak lower semi-continuity of the left-hand side of (6.8), the convergences (5.15)-(5.16) together with (5.14) yield

$$\begin{aligned}
&\int_{\mathbf{\Omega}} a_{ijkl}\big(E_{M,ij}(\mathcal{U}^\theta) + E_{w,ij}(\widehat{\mathfrak{U}}^\theta)\big)\big(E_{M,kl}(\mathcal{U}^\theta) + E_{w,kl}(\widehat{\mathfrak{U}}^\theta)\big)\, dx'dX\\
&\le \liminf_{(\varepsilon,\delta)\to(0,0)} \Big(\int_{(\widehat{\omega}_\varepsilon\times\mathcal{Y})\cap\mathbf{\Omega}} a_{ijkl}\,\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e_{ij}(u_{\varepsilon,\delta})\big)\,\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e_{kl}(u_{\varepsilon,\delta})\big)dx'dX\Big)\\
&\le \limsup_{(\varepsilon,\delta)\to(0,0)} \Big(\int_{(\widehat{\omega}_\varepsilon\times\mathcal{Y})\cap\mathbf{\Omega}} a_{ijkl}\,\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e_{ij}(u_{\varepsilon,\delta})\big)\,\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e_{kl}(u_{\varepsilon,\delta})\big)dx'dX\Big)\\
&\le \limsup_{(\varepsilon,\delta)\to(0,0)} \frac{1}{\delta^5}\int_{\Omega_\delta} \sigma^{\varepsilon,\delta}_{ij}(u_{\varepsilon,\delta})\, e_{ij}(u_{\varepsilon,\delta})\, dx = \limsup_{(\varepsilon,\delta)\to(0,0)} \frac{1}{\delta^5}\int_{\Omega_\delta} f_\delta\cdot u_{\varepsilon,\delta}\, dx\\
&= \int_\omega f\cdot\mathcal{U}^\theta\,(\phi+C_0)dx' - \frac{1}{3}\int_\omega g_\alpha\frac{\partial\mathcal{U}^\theta_3}{\partial x_\alpha}\,(\phi^3+C_0^3)dx' + \frac{1}{2}\int_\omega\big(g_\alpha\,\mathcal{U}^\theta_\alpha - f_\alpha\frac{\partial\mathcal{U}^\theta_3}{\partial x_\alpha}\big)(\phi^2-C_0^2)dx'.
\end{aligned}$$

Hence, the above inequalities are equalities. This proves the strong convergence (6.7) of the strain tensor.

Step 4. Last strong convergences.

From (3.2), we obtain

$$\frac{1}{\delta^2}\varepsilon_{\alpha\beta}(\mathcal{U}_{\varepsilon,\delta}) = \frac{1}{2C_0}\int_Y\int_{-C_0}^{C_0}\frac{1}{\delta^2}\mathfrak{T}_{\varepsilon\delta}\big(e_{\alpha\beta}(u_{\varepsilon,\delta})\big)(\cdot,X)\, dX, \quad \text{a.e. in } \omega \tag{6.10}$$

The strong convergence (6.7) implies

$$\frac{1}{\delta^2}\varepsilon_{\alpha\beta}(\mathcal{U}_{\varepsilon,\delta}) \longrightarrow e_{\alpha\beta}(\mathcal{U}^\theta) \quad \text{strongly in } L^2(\omega). \tag{6.11}$$

Hence, the convergence $(6.6)_1$ is strong. □

6.2 The correctors

Denote

$$\mathbf{M}^{11}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\qquad \mathbf{M}^{12}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\qquad \mathbf{M}^{22}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

and for all $x'\in\omega$

$$H^1_{per}(\mathcal{Y}_{x'})=\Big\{\boldsymbol{\chi}\in H^1(\mathcal{Y}_{x'})\mid \boldsymbol{\chi}\ \text{ is 1-periodic with respect to } X_1 \text{ and } X_2\Big\}.$$

Now, we will express the warping $\widehat{\mathfrak{U}}^\theta$ in terms of the partial derivatives of $\mathcal{U}^\theta_m$ and $\mathcal{U}^\theta_3$.
Below, we introduce the warping correctors $\chi^{\alpha\beta}_{m,\theta}$, $\chi^{\alpha\beta}_{b,\theta}$ belonging to $\mathbb{W}(\mathbf{\Omega})$. They are the solution to the following variational problems:

$$\begin{cases}\chi^{\alpha\beta}_{m,\theta}(x',\cdot)\in H^1_{per}(\mathcal{Y}_{x'})^3\ \text{ such that } \mathcal{M}(\chi^{\alpha\beta}_{m,\theta})(x')=0 \text{ and}\\ \displaystyle\int_{\mathcal{Y}_{x'}} a_{ijkl}\left(\mathbf{M}^{\alpha\beta}_{ij}+E^\theta_{w,ij}(\chi^{\alpha\beta}_{m,\theta})\right)E^\theta_{w,kl}(\widehat{\mathfrak{W}})\,dX=0,\quad \forall\widehat{\mathfrak{W}}\in H^1_{per}(\mathcal{Y}_{x'})^3,\\ \chi^{\alpha\beta}_{b,\theta}(x',\cdot)\in H^1_{per}(\mathcal{Y}_{x'})^3\ \text{ such that } \mathcal{M}(\chi^{\alpha\beta}_{b,\theta})(x')=0 \text{ and}\\ \displaystyle\int_{\mathcal{Y}_{x'}} a_{ijkl}\left(X_3\mathbf{M}^{\alpha\beta}_{ij}+E^\theta_{w,ij}(\chi^{\alpha\beta}_{b,\theta})\right)E^\theta_{w,kl}(\widehat{\mathfrak{W}})\,dX=0,\quad \forall\widehat{\mathfrak{W}}\in H^1_{per}(\mathcal{Y}_{x'})^3\end{cases}\qquad\text{for a.e. } x' \text{ in } \omega. \tag{6.12}$$

The above problems (6.12) admit unique solutions. Observe that

$$\chi^{\alpha\beta}_{m,\theta}=\chi^{\beta\alpha}_{m,\theta}\qquad\text{and}\qquad\chi^{\alpha\beta}_{b,\theta}=\chi^{\beta\alpha}_{b,\theta}.$$

Due to Lemma 8.2, the correctors $\chi^{\alpha\beta}_{m,\theta}$, $\chi^{\alpha\beta}_{b,\theta}$ are the restrictions to $\mathbf{\Omega}$ of functions belonging to $L^\infty(\omega;H^1(\mathcal{Y}))^3$. Moreover, we have

$$\big\|E^\theta_w(\chi^{\alpha\beta}_{m,\theta})(x',\cdot)\big\|_{L^2(\mathcal{Y}_{x'})}+\big\|E^\theta_w(\chi^{\alpha\beta}_{b,\theta})(x',\cdot)\big\|_{L^2(\mathcal{Y}_{x'})}\le C \tag{6.13}$$

where the constant is independent of $\theta\in(0,+\infty)$ and $x'\in\omega$.

Thanks to these correctors, we express the field $\widehat{\mathfrak{U}}^\theta$. We obtain

$$\widehat{\mathfrak{U}}^\theta=\chi^{\alpha\beta}_{m,\theta}\,e_{\alpha\beta}(\mathcal{U}^\theta)-\chi^{\alpha\beta}_{b,\theta}\,\frac{\partial^2\mathcal{U}^\theta_3}{\partial x_\alpha\partial x_\beta},\qquad\text{a.e. in }\mathbf{\Omega}. \tag{6.14}$$

Now, in order to obtain certain properties of the homogenized problem, we modify the correctors. First, observe that in the problems given by (6.12), the variable x' is just a parameter. So, we change the microscopic variable $X\in\mathcal{Y}_{x'}$ and the open set $\mathcal{Y}_{x'}$ to obtain problems posed in a fixed domain.
We set

$$X_1=X_1,\qquad X_2=X_2,\qquad X_3'=\frac{X_3+C_0}{2(\phi(x')+C_0)}+\frac{X_3-\phi(x')}{2(\phi(x')+C_0)},\qquad \boldsymbol{\mathcal{Y}}=Y\times(-1/2,1/2),$$
$$\text{and}\qquad \vartheta=\theta(C_0+\phi(x')).$$

Thus

$$\overline{\chi^{\alpha\beta}_{m,\vartheta}}(x',\overline{X})=\chi^{\alpha\beta}_{m,\theta}\Big(x',X_1,X_2,\big(C_0+\phi(x')\big)X_3'+\frac{\phi(x')-C_0}{2}\Big),$$
$$\overline{\chi^{\alpha\beta}_{b,\vartheta}}(x',\overline{X})=\chi^{\alpha\beta}_{b,\theta}\Big(x',X_1,X_2,\big(C_0+\phi(x')\big)X_3'+\frac{\phi(x')-C_0}{2}\Big)$$

where $\overline{X}=(X_1,X_2,X_3')$ is the courant point in $\boldsymbol{\mathcal{Y}}$.
These new correctors belong to $L^\infty\big(\omega;H^1_{per}(\boldsymbol{\mathcal{Y}})\big)^3$ and are the solution to the following variational problems:

$$\begin{cases}\overline{\chi^{\alpha\beta}_{m,\vartheta}}(x',\cdot)\in H^1_{per}(\boldsymbol{\mathcal{Y}})^3\ \text{ such that } \mathcal{M}(\overline{\chi^{\alpha\beta}_{m,\vartheta}})(x')=0 \text{ and}\\ \displaystyle\int_{\boldsymbol{\mathcal{Y}}}\overline{a_{ijkl}}(x',\cdot)\left(\mathbf{M}^{\alpha\beta}_{ij}+\mathcal{E}^\vartheta_{w,ij}(\overline{\chi^{\alpha\beta}_{m,\vartheta}})\right)\mathcal{E}^\vartheta_{w,kl}(\overline{\mathfrak{W}})\,d\overline{X}=0,\quad\forall\overline{\mathfrak{W}}\in H^1_{per}(\boldsymbol{\mathcal{Y}})^3,\\ \overline{\chi^{\alpha\beta}_{b,\vartheta}}(x',\cdot)\in H^1_{per}(\boldsymbol{\mathcal{Y}})^3\ \text{ such that } \mathcal{M}(\overline{\chi^{\alpha\beta}_{b,\vartheta}})(x')=0 \text{ and}\\ \displaystyle\int_{\boldsymbol{\mathcal{Y}}}\overline{a_{ijkl}}(x',\cdot)\left(X_3\mathbf{M}^{\alpha\beta}_{ij}+\mathcal{E}^\vartheta_{w,ij}(\overline{\chi^{\alpha\beta}_{b,\vartheta}})\right)\mathcal{E}^\vartheta_{w,kl}(\overline{\mathfrak{W}})\,d\overline{X}=0,\ \ \forall\overline{\mathfrak{W}}\in H^1_{per}(\boldsymbol{\mathcal{Y}})^3\end{cases}\qquad\text{for a.e. } x' \text{ in } \omega \tag{6.15}$$

where for every $\overline{\mathfrak{V}} \in H^1_{per}(\boldsymbol{\mathcal{Y}})^3$ the symmetric tensor $\mathcal{E}^\vartheta_w(\overline{\mathfrak{V}}) \in L^2(\boldsymbol{\mathcal{Y}})^{3\times 3}$ is given by

$$\mathcal{E}^\vartheta_w(\overline{\mathfrak{V}}) \doteq \begin{pmatrix} \dfrac{\partial \overline{\mathfrak{V}}_1}{\partial X_1} & * & * \\ \dfrac{1}{2}\Big(\dfrac{\partial \overline{\mathfrak{V}}_2}{\partial X_1} + \dfrac{\partial \overline{\mathfrak{V}}_1}{\partial X_2}\Big) & \dfrac{\partial \overline{\mathfrak{V}}_2}{\partial X_2} & * \\ \dfrac{1}{2}\Big(\dfrac{\partial \overline{\mathfrak{V}}_3}{\partial X_1} + \vartheta\dfrac{\partial \overline{\mathfrak{V}}_1}{\partial X_3'}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial \overline{\mathfrak{V}}_3}{\partial X_2} + \vartheta\dfrac{\partial \overline{\mathfrak{V}}_2}{\partial X_3'}\Big) & \vartheta\dfrac{\partial \overline{\mathfrak{V}}_3}{\partial X_3'} \end{pmatrix}$$

and where

$$\overline{a_{ijkl}}(x', \overline{X}) = a_{ijkl}\Big(X_1, X_2, \big(C_0 + \phi(x')\big)X_3' + \frac{\phi(x') - C_0}{2}\Big) \quad \text{for a.e. } (x', \overline{X}) \text{ in } \omega \times \boldsymbol{\mathcal{Y}}.$$

These coefficients belong to $L^\infty(\omega \times \boldsymbol{\mathcal{Y}})$ and they satisfy the coercivity condition

$$\overline{a_{ijkl}}\tau_{ij}\tau_{kl} \geq c_0\, \tau_{ij}\tau_{ij} \qquad \text{a.e. in } \omega \times \boldsymbol{\mathcal{Y}}.$$

The new correctors satisfy

$$\big\|\mathcal{E}^\vartheta_w(\overline{\chi^{\alpha\beta}_{m,\vartheta}})(x', \cdot)\big\|_{L^2(\boldsymbol{\mathcal{Y}})} + \big\|\mathcal{E}^\vartheta_w(\overline{\chi^{\alpha\beta}_{b,\vartheta}})(x', \cdot)\big\|_{L^2(\boldsymbol{\mathcal{Y}})} \leq C, \qquad \text{for a.e. } x' \in \omega. \tag{6.16}$$

As a consequence of (5.19) and the above estimates, we have

$$\|\overline{\chi^{\alpha\beta}_{m,\vartheta}}(x', \cdot)\|_{H^1(\boldsymbol{\mathcal{Y}})} \leq C\Big(\vartheta + \frac{1}{\vartheta}\Big), \qquad \|\overline{\chi^{\alpha\beta}_{b,\vartheta}}(x', \cdot)\|_{H^1(\boldsymbol{\mathcal{Y}})} \leq C\Big(\vartheta + \frac{1}{\vartheta}\Big), \quad \text{for a.e. } x' \in \omega. \tag{6.17}$$

In the above estimates, the constants do not depend on ϑ and $x' \in \omega$.

6.3 The limit problem in ω

Theorem 6.2. *The field $\mathcal{U}^\theta \in \mathbb{D}_M(\omega)$ is the unique solution to*

$$\begin{cases} \displaystyle\int_\omega A^{m,\theta}_{\alpha\beta\alpha'\beta'}\, e_{\alpha\beta}(\mathcal{U}^\theta)\, e_{\alpha'\beta'}(\mathcal{V})\, dx' + \int_\omega A^{mb,\theta}_{\alpha\beta\alpha'\beta'}\, \Big(e_{\alpha\beta}(\mathcal{U}^\theta)\, \frac{\partial^2 \mathcal{V}_3}{\partial x_{\alpha'}\partial x_{\beta'}} + e_{\alpha\beta}(\mathcal{V})\, \frac{\partial^2 \mathcal{U}^\theta_3}{\partial x_{\alpha'}\partial x_{\beta'}}\Big) dx' \\ \displaystyle + \int_\omega A^{b,\theta}_{\alpha\beta\alpha'\beta'}\, \frac{\partial^2 \mathcal{U}^\theta_3}{\partial x_\alpha \partial x_\beta}\, \frac{\partial^2 \mathcal{V}_3}{\partial x_{\alpha'}\partial x_{\beta'}} dx' = \int_\omega f \cdot \mathcal{V}\, (\phi + C_0) dx' - \frac{1}{3}\int_\omega g_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\, (\phi^3 + C_0^3) dx' \\ \displaystyle + \frac{1}{2}\int_\omega \Big(g_\alpha\, \mathcal{V}_\alpha - f_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\Big)(\phi^2 - C_0^2) dx', \qquad\qquad \forall \mathcal{V} \in \mathbb{D}_M(\omega), \end{cases} \tag{6.18}$$

where

$$\begin{aligned} A^{m,\theta}_{\alpha\beta\alpha'\beta'} &= \int_{\mathcal{Y}_{x'}} a_{ijkl}\big(\mathbf{M}^{\alpha\beta}_{ij} + E^\theta_{w,ij}(\chi^{\alpha\beta}_{m,\theta})\big)\big(\mathbf{M}^{\alpha'\beta'}_{kl} + E^\theta_{w,kl}(\chi^{\alpha'\beta'}_{m,\theta})\big)\, dX, \\ A^{b,\theta}_{\alpha\beta\alpha'\beta'} &= \int_{\mathcal{Y}_{x'}} a_{ijkl}\big(X_3\mathbf{M}^{\alpha\beta}_{ij} + E^\theta_{w,ij}(\chi^{\alpha\beta}_{b,\theta})\big)\big(X_3\mathbf{M}^{\alpha'\beta'}_{kl} + E^\theta_{w,kl}(\chi^{\alpha'\beta'}_{b,\theta})\big)\, dX, \quad \textit{a.e. in } \omega. \\ A^{mb,\theta}_{\alpha\beta\alpha'\beta'} &= \int_{\mathcal{Y}_{x'}} a_{ijkl}\big(\mathbf{M}^{\alpha\beta}_{ij} + E^\theta_{w,ij}(\chi^{\alpha\beta}_{m,\theta})\big)\big(X_3\mathbf{M}^{\alpha'\beta'}_{kl} + E^\theta_{w,kl}(\chi^{\alpha'\beta'}_{b,\theta})\big)\, dX, \end{aligned} \tag{6.19}$$

Observe that due to the symmetry conditions of the a_{ijkl}'s, one has

$$\begin{aligned} &\int_{\mathcal{Y}_{x'}} a_{ijkl}\big(\mathbf{M}^{\alpha\beta}_{ij} + E^\theta_{w,ij}(\chi^{\alpha\beta}_{m,\theta})\big)\big(X_3\mathbf{M}^{\alpha'\beta'}_{kl} + E^\theta_{w,kl}(\chi^{\alpha'\beta'}_{b,\theta})\big)\, dX \\ &\qquad = \int_{\mathcal{Y}_{x'}} a_{ijkl}\big(X_3\mathbf{M}^{\alpha\beta,\theta}_{ij} + E^\theta_{w,ij}(\chi^{\alpha\beta}_{b,\theta})\big)\big(\mathbf{M}^{\alpha'\beta'}_{kl} + E^\theta_{w,kl}(\chi^{\alpha'\beta'}_{m,\theta})\big)\, dX. \end{aligned}$$

Proof of Theorem 6.2. In (6.8), we choose as test displacements $\mathcal{V} \in \mathbb{D}_M(\omega)$ and

$$\widehat{\mathfrak{V}} = \chi^{\alpha\beta}_{m,\theta}\, e_{\alpha\beta}(\mathcal{V}) - \chi^{\alpha\beta}_{b,\theta}\, \frac{\partial^2 \mathcal{V}_3}{\partial x_\alpha \partial x_\beta}.$$

Then, we replace $\widehat{\mathfrak{U}}^\theta$ by its value given by (6.14). That gives (6.18). As usual, thanks to (6.1) and the expressions (6.19) of the bilinear form in the left-hand side of (6.18), we get the coercivity and continuity of this bilinear form. Hence, we have the existence and unicity of the solution to (6.18). □

As a consequence of the estimates (6.13), the coefficients $A^{m,\theta}_{\alpha\beta\alpha'\beta'}$, $A^{b,\theta}_{\alpha\beta\alpha'\beta'}$ and $A^{mb,\theta}_{\alpha\beta\alpha'\beta'}$ belong to $L^\infty\big(\omega\big)$. More precisely, we have

Lemma 6.1. *As functions of the variables (ϑ, x'), all the coefficients $A^{m,\theta}_{\alpha\beta\alpha'\beta'}$, $A^{b,\theta}_{\alpha\beta\alpha'\beta'}$ and $A^{mb,\theta}_{\alpha\beta\alpha'\beta'}$ belong to $L^\infty\big((0,+\infty)\times\omega\big)$. Moreover, for a.e. $x'\in\omega$ the functions $\theta\longmapsto A^{m,\theta}_{\alpha\beta\alpha'\beta'}(x')$, $\theta\longmapsto A^{b,\theta}_{\alpha\beta\alpha'\beta'}$ and $\theta\longmapsto A^{mb,\theta}_{\alpha\beta\alpha'\beta'}$ are Lipschitz continuous on every segment included in $(0,+\infty)$.*

Proof. First, observe that

$$A^{m,\theta}_{\alpha\beta\alpha'\beta'}(x')=\int_{\boldsymbol{\mathcal{Y}}}\overline{a_{ijkl}}(x',\overline{X})\,\big(\mathbf{M}^{\alpha\beta}_{ij}+\mathcal{E}^\vartheta_{w,ij}\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}\big)(x',\overline{X})\big)\,\big(\mathbf{M}^{\alpha'\beta'}_{kl}+\mathcal{E}^\vartheta_{w,kl}\big(\overline{\chi^{\alpha'\beta'}_{m,\vartheta}}\big)(x',\overline{X})\big)\,d\overline{X}.$$

We have similar equalities for the other coefficients.
We have

$$\mathcal{E}^\vartheta_w(\overline{\mathfrak{W}})-\mathcal{E}^{\vartheta'}_w(\overline{\mathfrak{W}})=\frac{\vartheta-\vartheta'}{2}\begin{pmatrix}0&0&\dfrac{\partial\overline{\mathfrak{W}}_1}{\partial X'_3}\\[2ex]0&0&\dfrac{\partial\overline{\mathfrak{W}}_2}{\partial X'_3}\\[2ex]\dfrac{\partial\overline{\mathfrak{W}}_1}{\partial X'_3}&\dfrac{\partial\overline{\mathfrak{W}}_2}{\partial X'_3}&2\dfrac{\partial\overline{\mathfrak{W}}_3}{\partial X'_3}\end{pmatrix}\qquad\forall\overline{\mathfrak{W}}\in H^1_{per}(\boldsymbol{\mathcal{Y}})^3.\tag{6.20}$$

Now, starting from $(6.15)_1$ and taking into account the above equality (6.20) and (5.19), for a.e. $x'\in\omega$ we obtain

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{Y}}}\overline{a_{ijkl}}(x',\cdot)\,\mathcal{E}^\vartheta_{w,ij}\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\,\mathcal{E}^\vartheta_{w,kl}(\overline{\mathfrak{W}})\,d\overline{X}\\&\le C|\vartheta-\vartheta'|\Big(\big\|\mathcal{E}^{\vartheta'}_w(\overline{\mathfrak{W}})\big\|_{L^2(\boldsymbol{\mathcal{Y}})}\|\nabla_{\overline{X}}\overline{\chi^{\alpha\beta}_{m,\vartheta'}}(x',\cdot)\|_{L^2(\mathcal{Y})}+\|\nabla_{\overline{X}}\overline{\mathfrak{W}}\|_{L^2(\mathcal{Y})}\big(1+\big\|\mathcal{E}^{\vartheta'}_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\|_{L^2(\mathcal{Y})}\big)\Big)\\&\le C|\vartheta-\vartheta'|\big\|\mathcal{E}^{\vartheta'}_w(\overline{\mathfrak{W}})\big\|_{L^2(\boldsymbol{\mathcal{Y}})}\Big(\|\nabla_{\overline{X}}\overline{\chi^{\alpha\beta}_{m,\vartheta'}}(x',\cdot)\|_{L^2(\mathcal{Y})}+\vartheta'+\frac{1}{\vartheta'}+\big\|\mathcal{E}^{\vartheta'}_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\|_{L^2(\mathcal{Y})}\Big).\end{aligned}$$

Now, choose $\overline{\mathfrak{W}}=\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)$, due to (6.16) and (6.17) this gives

$$\big\|\mathcal{E}^\vartheta_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|^2_{L^2(\boldsymbol{\mathcal{Y}})}\le C|\vartheta-\vartheta'|\big\|\mathcal{E}^{\vartheta'}_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|_{L^2(\boldsymbol{\mathcal{Y}})}\Big(\vartheta'+\frac{1}{\vartheta'}\Big).$$

Similarly we show that

$$\big\|\mathcal{E}^{\vartheta'}_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|^2_{L^2(\boldsymbol{\mathcal{Y}})}\le C|\vartheta-\vartheta'|\big\|\mathcal{E}^{\vartheta}_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|_{L^2(\boldsymbol{\mathcal{Y}})}\Big(\vartheta+\frac{1}{\vartheta}\Big).$$

Hence

$$\big\|\mathcal{E}^\vartheta_w\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|_{L^2(\boldsymbol{\mathcal{Y}})}\le C|\vartheta-\vartheta'|\Big(\vartheta+\vartheta'+\frac{1}{\vartheta}+\frac{1}{\vartheta'}\Big)$$

and then

$$\big\|\big(\overline{\chi^{\alpha\beta}_{m,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|_{H^1(\boldsymbol{\mathcal{Y}})}\le C|\vartheta-\vartheta'|\Big(\vartheta+\vartheta'+\frac{1}{\vartheta}+\frac{1}{\vartheta'}\Big)^2.$$

As a consequence, for a.e. $x'\in\omega$ the function $\theta\longmapsto A^{m,\theta}_{\alpha\beta\alpha'\beta'}(x')$ is Lipschitz continuous on every segment included in $(0,+\infty)$. In the same way, we prove that

$$\begin{aligned}\big\|\mathcal{E}^\vartheta_w\big(\overline{\chi^{\alpha\beta}_{b,\vartheta}}-\overline{\chi^{\alpha\beta}_{b,\vartheta'}}\big)(x',\cdot)\big\|_{L^2(\boldsymbol{\mathcal{Y}})}&\le C|\vartheta-\vartheta'|\Big(\vartheta+\vartheta'+\frac{1}{\vartheta}+\frac{1}{\vartheta'}\Big),\\\big\|\big(\overline{\chi^{\alpha\beta}_{b,\vartheta}}-\overline{\chi^{\alpha\beta}_{m,\vartheta'}}\big)(x',\cdot)\big\|_{H^1(\boldsymbol{\mathcal{Y}})}&\le C|\vartheta-\vartheta'|\Big(\vartheta+\vartheta'+\frac{1}{\vartheta}+\frac{1}{\vartheta'}\Big)^2.\end{aligned}$$

So, the functions $\theta\longmapsto A^{b,\theta}_{\alpha\beta\alpha'\beta'}$ and $\theta\longmapsto A^{mb,\theta}_{\alpha\beta\alpha'\beta'}$ are Lipschitz continuous on every segment included in $(0,+\infty)$. □

As immediate consequence of the above lemma, we have

Theorem 6.3. *The map $\theta \in (0,+\infty) \longmapsto \mathcal{U}^\theta \in \mathbb{D}_M(\omega)$ is locally Lipchitz continuous for the strong topology of $\mathbb{D}_M(\omega)$.*

Remark 6.1. *As in [16, Chapter 11] we can show that*

- *the map $\theta \longmapsto \mathcal{U}^\theta$ admits a limit $\mathcal{U}^0 \in \mathbb{D}_M(\omega)$[3] when θ goes to 0. This limit corresponds to the case*

$$(\varepsilon,\delta) \to (0,0) \qquad \text{and} \qquad \frac{\varepsilon}{\delta} \to 0.$$

  It can also be obtained if we first homogenize $\varepsilon \to 0$, δ being fixed, and then reduce the dimension $\delta \to 0$.

- *the map $\theta \longmapsto \mathcal{U}^\theta$ admits a limit $\mathcal{U}^\infty \in \mathbb{D}_M(\omega)$[4] when θ goes to $+\infty$. This limit corresponds to the case*

$$(\varepsilon,\delta) \to (0,0) \qquad \text{and} \qquad \frac{\varepsilon}{\delta} \to +\infty.$$

  It can also be obtained if we first reduce the dimension $\delta \to 0$, ε being fixed, and then homogenize $\varepsilon \to 0$.

6.4 The case of an isotropic and homogeneous material

In this subsection we assume that the plate is made of an isotropic and homogeneous material whose Lamé constants are λ and μ. In this case, one has

$$a_{ijkl} = \lambda \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu\big(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}\big)$$

where the $\boldsymbol{\delta}_{ij}$'s are the Kronecker symbols.

In this subsection, we denote u_δ the solution to problem (6.3). We first have the convergences in Theorem 5.1 and Lemma 5.1.

Here, we easily obtain

$$\frac{\partial \mathfrak{U}_1}{\partial X_3} = \frac{\partial \mathfrak{U}_2}{\partial X_3} 2 = 0, \qquad \frac{\partial \mathfrak{U}_3}{\partial X_3} = \frac{\lambda}{\lambda+2\mu}\Big(- \Big(\frac{\partial \mathcal{U}_1}{\partial x_1} + \frac{\partial \mathcal{U}_2}{\partial x_2}\Big) + X_3 \Delta \mathcal{U}_3\Big) \qquad \text{a.e. in } \Omega.$$

The following theorem is a consequence of Theorem 6.2 and the values of the a_{ijkl}'s.

Theorem 6.4. *The field $\mathcal{U} \in \mathbb{D}_M(\omega)$ is the unique solution to*

$$\begin{cases} \dfrac{E}{1-\nu^2}\displaystyle\int_\omega \Big[(1-\nu)\, e_{\alpha\beta}(\mathcal{U})\, e_{\alpha\beta}(\mathcal{V}) + \nu e_{\alpha\alpha}(\mathcal{U}) e_{\beta\beta}(\mathcal{V})\Big]\,(\phi + C_0)dx' \\ + \dfrac{E}{2(1-\nu^2)}\displaystyle\int_\omega (1-\nu)\Big(e_{\alpha\beta}(\mathcal{U})\,\frac{\partial^2 \mathcal{V}_3}{\partial x_\alpha \partial x_\beta} + e_{\alpha\beta}(\mathcal{V})\,\frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\Big)(\phi^2 - C_0^2)dx' \\ + \dfrac{E}{2(1-\nu^2)}\displaystyle\int_\omega \nu\Big(e_{\alpha\alpha}(\mathcal{U})\,\Delta\mathcal{V}_3 + e_{\alpha\alpha}(\mathcal{V})\,\Delta\mathcal{U}_3\Big)(\phi^2 - C_0^2)dx' \\ + \dfrac{E}{3(1-\nu^2)}\displaystyle\int_\omega \Big[(1-\nu)\frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\,\frac{\partial^2 \mathcal{V}_3}{\partial x_\alpha \partial x_\beta} + \nu \Delta\mathcal{U}_3 \Delta\mathcal{V}_3\Big](\phi^3 + C_0^3)dx' \\ = \displaystyle\int_\omega f\cdot\mathcal{V}\,(\phi + C_0)dx' - \frac{1}{3}\int_\omega g_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\,(\phi^3 + C_0^3)dx' \\ + \dfrac{1}{2}\displaystyle\int_\omega \Big(g_\alpha\,\mathcal{V}_\alpha - f_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha}\Big)(\phi^2 - C_0^2)dx', \qquad \forall \mathcal{V} \in \mathbb{D}_M(\omega) \end{cases} \tag{6.21}$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young modulus and $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ the Poisson constant.

[3] for the strong topology of $\mathbb{D}_M(\omega)$

[4] for the strong topology of $\mathbb{D}_M(\omega)$

Moreover, one has

$$\frac{1}{\delta}\Pi_\delta\big(\sigma_{11}(u_\delta)\big) \longrightarrow \frac{E}{1-\nu^2}\Big[e_{11}(\mathcal{U}) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1^2} + \nu\Big(e_{22}(\mathcal{U}) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_2^2}\Big)\Big] \quad \textit{strongly in } L^2(\Omega),$$
$$\frac{1}{\delta}\Pi_\delta\big(\sigma_{22}(u_\delta)\big) \longrightarrow \frac{E}{1-\nu^2}\Big[e_{22}(\mathcal{U}) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_2^2} + \nu\Big(e_{11}(\mathcal{U}) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1^2}\Big)\Big] \quad \textit{strongly in } L^2(\Omega),$$
$$\frac{1}{\delta}\Pi_\delta\big(\sigma_{12}(u_\delta)\big) \longrightarrow 2\mu\Big[e_{12}(\mathcal{U}) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2}\Big] \quad \textit{strongly in } L^2(\Omega),$$
$$\frac{1}{\delta}\Pi_\delta\big(\sigma_{i3}(u_\delta)\big) \longrightarrow 0 \quad \textit{strongly in } L^2(\Omega).$$

7 Complement. The case of a plate with a strongly oscillating top surface

Let us consider a function ϕ belonging to $W^{1,\infty}(\omega\times Y)$ such that for a.e. $x'\in\omega$ the map $(X_1,X_2)\longmapsto\phi(x',X_1,X_2)$ is 1-periodic with respect to X_1 and X_2 and satisfies

$$|\phi(x',X')-\phi(z',Z')|\le K_0\big(|x'-z'|_2+|X'-Z'|_2\big),\quad \forall\big(x',X',z',Z'\big)\in(\overline{\omega\times Y})^2,$$
$$\phi(x',X')\ge 2C_0,\qquad \forall(x',X')\in\omega\times Y,\qquad C_1=\|\phi\|_{L^\infty(\omega\times Y)},$$

where $|\cdot|_2$ is the euclidian norm in $\mathbb{R}^2$, K_0 and C_0 are strictly positive constants.
Now, the plate whose top surface oscillates strongly is the domain

$$\Omega_{\varepsilon\delta}=\Big\{x=(x_1,x_2,x_3)\in\omega\times\mathbb{R}\mid -\delta C_0<x_3<\delta\phi_\varepsilon(x_1,x_2)\Big\}$$

with

$$\phi_\varepsilon(x_1,x_2)=\phi\Big(x_1,x_2,\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big).$$

Observe that $\Omega_{0,\delta}\subset\Omega_{\varepsilon\delta}$.
As in Section 4, we decompose every displacement u belonging to $W^{1,p}(\Omega_{\varepsilon\delta})^3$ as the sum of a Kirchhoff-Love displacement and a residual displacement

$$u=U_{KL}+\mathfrak{u}.$$

We have

Theorem 7.1. *Let u be a displacement in $W^{1,p}(\Omega_{\varepsilon\delta})^3$, $1<p<\infty$, decomposed as the sum of a Kirchhoff-Love displacement and a residual displacement. Assume that there exists a constant $C^\diamond$ such that*

$$\frac{\delta}{\varepsilon}\le C^\diamond\quad \textit{and}\quad \varepsilon\le 1. \tag{7.1}$$

Then, the terms $\mathcal{U}$ and $\mathfrak{u}$ satisfy

$$\begin{aligned}
&\|e_{\alpha\beta}(\mathcal{U}_m)\|_{L^p(\omega)}\le\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|D^2\mathcal{U}_3\|_{L^p(\omega)}\le\frac{C}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_\delta)},\\
&\|e(\mathfrak{u})\|_{L^p(\Omega_{\varepsilon\delta})}\le C\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})}
\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}
&\|\mathfrak{u}\|_{L^p(\Omega_{\varepsilon\delta})}\le C\,\delta\,\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})},\\
&\|\nabla\mathfrak{u}\|_{L^p(\Omega_{\varepsilon\delta})}\le C\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})}
\end{aligned} \tag{7.3}$$

with a constant C independent of ε and δ.

Proof. The estimates (7.2) are proved in the same way as those of Theorem 4.1.
Now, for simplicity we suppose that the origin O belongs to ω. We transform the plate $\Omega_{\varepsilon\delta}$ by the dilation $x=\varepsilon z$, $z\in\omega_{1/\varepsilon}=1/\varepsilon\omega$. The new plate is

$$\mathbf{\Omega}_{\varepsilon\delta}=\Big\{z=(z_1,z_2,z_3)\in\omega_{1/\varepsilon}\times\mathbb{R}\mid -\frac{\delta}{\varepsilon}C_0<z_3<\frac{\delta}{\varepsilon}\Phi_\varepsilon(z_1,z_2)\Big\}$$

where

$$\Phi_\varepsilon(z') = \phi(\varepsilon z_1, \varepsilon z_2, z_1, z_2) \qquad \forall z' = (z_1, z_2) \in \overline{\omega}_{1/\varepsilon}$$

and satisfies

$$|\Phi_\varepsilon(z') - \Phi_\varepsilon(t')| \le K_0\big(\varepsilon|z' - t'|_2 + |z' - t|_2\big) \le 2K_0|z' - t'|_2, \quad \forall\big(z', t'\big) \in \omega_{1/\varepsilon}.$$

The plate $\mathbf{\Omega}_{\varepsilon\delta}$ has a thickness of order δ/ε, the diameter of the domain $\omega_{1/\varepsilon}$ is of order $1/\varepsilon$. The proof of (7.3) follows the same lines as the proof of (4.3) (see Subsection 8.2 in the Appendix). □

The question now is: what happens if we replace the assumptions (7.1) with the following:

$$\lim_{(\delta,\varepsilon)\to(0,0)} \frac{\delta}{\varepsilon} \to \infty \ ? \tag{7.4}$$

The estimates (7.2) remain valid. The top surface of the plate is now made by small beams. Below we give an example.

Choose $\omega = (0, L)^2$, $\varepsilon > 0$ such that $\dfrac{L}{\varepsilon} \in \mathbb{N}$. Let ψ be the 1-periodic function defined by

$$\psi(t) = \begin{cases} 3t & \text{if } t \in [0, 1/3], \\ 1 & \text{if } t \in [1/3, 2/3], \\ 3(1-t) & \text{if } t \in [2/3, 1]. \end{cases}$$

We define ϕ_ε by

$$\phi_\varepsilon(x_1, x_2) = 2C_0 + \psi\Big(\frac{x_1}{\varepsilon}\Big)\psi\Big(\frac{x_2}{\varepsilon}\Big), \qquad \forall (x_1, x_2) \in \overline{\omega}.$$

Denote

$$\Omega^d_\delta = \omega \times (-\delta C_0, 2\delta C_0), \qquad \Omega^u_{\varepsilon\delta} = \Big\{x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} \mid 2\delta C_0 < x_3 < \delta\phi_\varepsilon(x_1, x_2)\Big\}.$$

Since Ω^d_δ is a plate of thickness $3\delta C_0$ we have

$$\|\mathfrak{u}\|_{L^p(\Omega^d_\delta)} \le C\,\delta\,\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})}, \qquad \|\nabla\mathfrak{u}\|_{L^p(\Omega^d_\delta)} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})}.$$

The constant does not depend on δ. The domain $\Omega^u_{\varepsilon\delta}$ is made of thin beams whose lengths are δ and thickness of order ε. Using the results of [20] we obtain

$$\|\mathfrak{u}_1\|_{L^p(\Omega_{\varepsilon\delta})} + \|\mathfrak{u}_2\|_{L^p(\Omega^u_{\varepsilon\delta})} \le C\frac{\delta^2}{\varepsilon}\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})}, \qquad \|\mathfrak{u}_3\|_{L^p(\Omega^u_{\varepsilon\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})},$$
$$\|\nabla\mathfrak{u}\|_{L^p(\Omega^u_{\varepsilon\delta})} \le C\frac{\delta}{\varepsilon}\|e(u)\|_{L^p(\Omega_{\varepsilon\delta})}.$$

The constants do not depend on δ and ε.

8 Appendix

8.1 Simple star-shaped domains

Lemma 8.1. *Let $B_{\delta R} \subset \omega$ be a domain of diameter less than $3\delta R$, star-shaped with respect to to the disc $D(O, \delta R_1)$. If $R_1 \le R \le \dfrac{C_0}{2K_0}$ then the domain*

$$\mathcal{D}_{\delta R} = \Big\{(x_1, x_2, x_3) \in B_{\delta R} \times \mathbb{R} \mid \ -\delta C_0 < x_3 < \delta\phi(x_1, x_2)\Big\}$$

is star-shaped with respect to the ball $B(O, \delta R_1')$ where

$$R_1' = \min\big\{C_0, R_1\big\}. \tag{8.1}$$

$\mathcal{D}_{\delta R}$ has a diameter less than $\delta(3R + 2C_1)$.

Proof. Let $A = (a_1, a_2, a_3)$ be in $\mathcal{D}_{\delta R}$ such that $a_3 \geq 0$ and $M = \big(x_1, x_2, , \delta\phi(x_1, x_2)\big)$ a point on the top surface. Consider the point A_λ of the interval $[A, M)$

$$A_\lambda = (1-\lambda)A + \lambda M, \qquad \lambda \in [0, 1).$$

One has

$$|\phi(x') - \phi\big((1-\lambda)a' + \lambda x'\big)| \leq K_0(1-\lambda)|x' - a'|_2, \qquad x' = (x_1, x_2), \quad a' = (a_1, a_2).$$

Hence

$$\phi(x') - K_0(1-\lambda)|x' - a'|_2 \leq \phi\big((1-\lambda)a' + \lambda x'\big) \leq \phi(x') + K_0(1-\lambda)|x' - a'|_2.$$

The point A_λ belongs to $\mathcal{D}_{\delta R}$ if

$$(1-\lambda)a_3 + \lambda\delta\phi(x') \leq \delta\phi\big((1-\lambda)a' + \lambda x'\big).$$

A sufficient condition to get $A_\lambda \in \mathcal{D}_{\delta R}$, for all $\lambda \in [0, 1)$, is

$$(1-\lambda)a_3 + \lambda\delta\phi(x') \leq \delta\phi(x') - \delta K_0(1-\lambda)|x' - a'|_2$$

for all $\big(x', a'\big) \in B_{\delta R} \times B_{\delta R}$. So, the above condition becomes

$$a_3 \leq \delta\phi(x') - \delta K_0|x' - a'|_2, \qquad \forall x' \in B_{\delta R}. \tag{8.2}$$

Suppose $a_3 \in [0, \delta C_0]$, we recall that $\phi(x') \geq 2C_0$, hence a sufficient condition is

$$\delta C_0 \leq 2\delta C_0 - 2\delta R K_0.$$

Condition met.
Obviously, if $a_3 \in [-\delta C_0, 0]$, (8.2) is satisfied. We have proved that any interval $[A, M)$ from a point $A \in B(O, \delta R_1')$ to a point M on the top surface is included in $\mathcal{D}_{\delta R}$. Similarly, we can prove that any interval $[A, M)$ from a point $A \in B(O, \delta R_1')$ to a point M on the bottom surface is included in $\mathcal{D}_{\delta R}$. The lemma is proved. □

Let f be a function belonging to $W^{1,\infty}(-a_0, a_0)$, $a_0 > 0$, satisfying

$$|f(t) - f(t')| \leq K_1|t - t'|, \qquad \forall (t, t') \in [-a_0, a_0]^2.$$

For every $x_0 \in [-a_0, a_0]$ and $a \in (0, 2a_0)$ such that $-a_0 \leq x_0 < x_0 + a \leq a_0$, the domains $\mathcal{D}^{top}_{a,x_0}$ and $\mathcal{D}^{bottom}_{a,x_0}$ (see Figure 1) defined by

$$\begin{aligned} \mathcal{D}^{top}_{a,x_0} &\doteq \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_0 < x_1 < x_0 + a, \quad f(x_0) - (2K_1+1)a < x_2 < f(x_1)\big\}, \\ \mathcal{D}^{bottom}_{a,x_0} &\doteq \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_0 < x_1 < x_0 + a, \quad f(x_1) < x_2 < f(x_0) + (2K_1+1)a\big\}, \end{aligned} \tag{8.3}$$

are star-shaped with respect to balls of radius $a/2$ (or less) and have a diameter less than $(3K_1 + 2)a$.

8.2 Proof of (4.3)

8.2.1 Interior estimates

Set $R_0 = \dfrac{C_0}{2K_0}$, $Z = (-R_0, R_0)^2$ and

$$\begin{aligned} &\Xi_\delta \doteq \big\{\xi \in 2R_0\mathbb{Z}^2 \mid \delta\xi + \delta Z \subset \omega\big\}, \qquad \widetilde{\omega}_\delta \doteq \text{Interior}\Big(\bigcup_{\xi\in\Xi_\delta} \delta\xi + \delta\overline{Z}\Big), \\ &\widetilde{\Omega}_\delta \doteq \Big\{(x_1, x_2, x_3) \in \widetilde{\omega}_\delta \times \mathbb{R} \mid \ -\delta C_0 < x_3 < \delta\phi(x_1, x_2)\Big\}, \\ &\boldsymbol{\mathcal{D}}_{0,\delta,\xi} = \big(\delta\xi + \delta Z\big) \times (-\delta C_0, \delta C_0) \subset \mathcal{D}_{\delta,\xi}, \\ &\mathcal{D}_{\delta,\xi} = \Big\{(x_1, x_2, x_3) \in \big(\delta\xi + \delta Z\big) \times \mathbb{R} \mid \ -\delta C_0(x_1, x_2) < x_3 < \delta\phi(x_1, x_2)\Big\}. \end{aligned}$$

Note that the domain $\delta\xi + \delta Z$ is star-shaped with respect to the disc $D(\delta\xi, \delta R_0)$.Then, due to Lemma 8.1, the domain $\mathcal{D}_{\delta,\xi}$ has a diameter less than $\delta(3R_0 + 2C_1)$ and is star-shaped with respect to the ball $B\big(\varepsilon\xi, \delta \min\big\{C_0, R_0\big\}\big)$.

Now, Theorem 2.3 in [13] gives a rigid displacement $\mathbf{r}_\xi$ ($\mathbf{r}_\xi(x) = \mathbf{a}_\xi + \mathbf{B}_\xi\,(x - \delta\xi)$, $x \in \mathbb{R}^3$, $\mathbf{a}_\xi \in \mathbb{R}^3$, $\mathbf{B}_\xi$ is a 3×3 antisymmetric matrix with constant entries) such that

$$\|\nabla(\mathfrak{u} - \mathbf{r}_\xi)\|^p_{L^p(\mathcal{D}_{\delta,\xi})} \le C^*\|e(\mathfrak{u})\|^p_{L^p(\mathcal{D}_{\delta,\xi})}, \qquad \|\mathfrak{u} - \mathbf{r}_\xi\|^p_{L^p(\mathcal{D}_{\delta,\xi})} \le C^*\delta\|e(\mathfrak{u})\|^p_{L^p(\mathcal{D}_{\delta,\xi})}. \tag{8.4}$$

The constant C^* only depends on the ratio $\dfrac{3R_0 + 2C_1}{\min\{C_0, R_0\}}$.

Hence, we have

$$\begin{aligned} 8\delta^3 R_0^2 C_0|\mathbf{B}_\xi|^p_F = \|\nabla\mathbf{r}_\xi\|^p_{L^p(\boldsymbol{\mathcal{D}}_{0,\xi})} &\le 2^{p-1}\big(\|\nabla\mathfrak{u}\|^p_{L^p(\boldsymbol{\mathcal{D}}_{0,\xi})} + \|\nabla(\mathfrak{u} - \mathbf{r}_\xi)\|^p_{L^p(\mathcal{D}_{\delta,\xi})}\big),\\ 8\delta^3 R_0^2 C_0|\mathbf{a}_\xi|^p_2 = \|\mathbf{r}_\xi\|^p_{L^p(\boldsymbol{\mathcal{D}}_{0,\xi})} &\le 2^{p-1}\big(\|\mathfrak{u}\|^p_{L^p(\boldsymbol{\mathcal{D}}_{0,\xi})} + \|\mathfrak{u} - \mathbf{r}_\xi\|^p_{L^p(\mathcal{D}_{\delta,\xi})}\big)\end{aligned} \tag{8.5}$$

where $|\cdot|_F$ is the Frobenius norm.
Since $\mathbf{r}_\xi$ is a affine function with respect to x_1, x_2 and x_3, we have

$$\|\nabla\mathbf{r}_\xi\|^p_{L^p(\mathcal{D}_{\delta,\xi})} \le C^{**}\|\nabla\mathbf{r}_\xi\|^p_{L^p(\boldsymbol{\mathcal{D}}_{0,\xi})}, \qquad \|\mathbf{r}_\xi\|^p_{L^p(\mathcal{D}_{\delta,\xi})} \le C^{**}\|\mathbf{r}_\xi\|^p_{L^p(\boldsymbol{\mathcal{D}}_{0,\xi})}. \tag{8.6}$$

The constant C^{**} only depends on the ratio $\dfrac{C_1}{C_0}$.
Besides we have

$$\begin{aligned} \|\nabla\mathfrak{u}\|^p_{L^p(\mathcal{D}_{\delta,\xi})} &\le 2^{p-1}\big(\|\nabla\mathbf{r}_\xi\|^p_{L^p(\mathcal{D}_{\delta,\xi})} + \|\nabla(\mathfrak{u} - \mathbf{r}_\xi)\|^p_{L^p(\mathcal{D}_{\delta,\xi})}\big)\\ \|\mathfrak{u}\|^p_{L^p(\mathcal{D}_{\delta,\xi})} &\le 2^{p-1}\big(\|\mathbf{r}_\xi\|_{L^p(\mathcal{D}_{\delta,\xi})} + \|\mathfrak{u} - \mathbf{r}_\xi\|_{L^p(\mathcal{D}_{\delta,\xi})}\big).\end{aligned} \tag{8.7}$$

So, (8.4)-(8.5)-(8.6) and then summing the above inequalities (8.7) ($\xi \in \Xi_\delta$) and due to the estimates of $\widetilde{u}$ (see $(3.3)_3$ and remind that $\mathfrak{u} = \widetilde{u}$ in $\Omega_{0,\delta}$), we get

$$\|\nabla\mathfrak{u}\|^p_{L^p(\widetilde{\Omega}_\delta)} \le \mathbf{C}\|e(\mathfrak{u})\|^p_{L^p(\Omega_\delta)}, \qquad \|\mathfrak{u}\|^p_{L^p(\widetilde{\Omega}_\delta)} \le \mathbf{C}\delta^p\|e(\mathfrak{u})\|^p_{L^p(\Omega_\delta)}. \tag{8.8}$$

The constant $\mathbf{C}$ only depends on K_0, C_0, C_1 and obviously on p.

8.2.2 Boundary layer estimates

The domain $\widetilde{\omega}_\delta$ covers a large part of ω. One has

$$\mathrm{dist}(\widetilde{\omega}_\delta, \partial\omega) \le 3\delta R_0.$$

Since the boundary of ω is Lipshitz, there exist constants a_0 and β strictly positive and a finite number N of local coordinate systems (x_{1n}, x_{2n}) in orthonormal frames $\big(O_n; \mathbf{e}_{1n}, \mathbf{e}_{2n}\big)$ and Lipschitz continuous maps $f_n : [-a_0, a_0] \longrightarrow \mathbb{R}$, $1 \le n \le N$ such that

$$\begin{aligned} \partial\omega &= \bigcup_{n=1}^N \Big\{(x_{1n}, x_{2n}) \in \mathbb{R}^p \mid x_{2n} = f_n(x_{1n}), \quad |x_{1n}| < a_0\Big\},\\ \mathcal{A}_{n,\beta} &\doteq \Big\{(x_{1n}, x_{2n}) \in \mathbb{R}^p \mid f_n(x_{1n}) - \beta < x_{2n} < f_n(x_{1n}), \quad |x_{1n}| < a_0\Big\},\\ \mathcal{A}_\beta &\doteq \bigcup_{n=1}^N \mathcal{A}_{n,\beta} \subset \omega.\end{aligned}$$

One easily shows that there exists $\delta_0 > 0$ such that

$$\omega \setminus \overline{\widetilde{\omega}_\delta} \subset \mathcal{A}_\beta, \qquad \forall\delta \in (0, \delta_0].$$

Without loss of generality, we assume that

$$|f_n(t) - f_n(t')| \le K_1|t - t'|, \qquad \forall (t, t') \in [-a_0, a_0]^2, \qquad \forall n \in \{1, \dots, N.\}$$

Now, set

$$x_k = -a_0 + ka_\delta, \quad k \in \Big\{0, \dots, N_\delta - 1\Big\} \quad \text{and} \quad x_{N_\delta} = a_0 - a_\delta$$

where $N_\delta = \left[\frac{2a_0}{a_\delta}\right]$, $[t] \in \mathbb{N}$ is the integer part of $t \in \mathbb{R}^+$. The strictly positive real number $a_\delta \in (0, 2a_0]$ will be given below by (8.9).
Denote

$$\mathcal{A}_{n,k,\delta} \doteq \Big\{(x_{1n}, x_{2n}) \in \mathbb{R}^p \mid \begin{matrix} x_k < x_{1n} < x_k + a_\delta, \\ f_n(x_{1n}) - \beta < x_{2n} < f_n(x_{1n}), \end{matrix} \Big\}, \quad k \in \{0, \dots, N_\delta\},$$

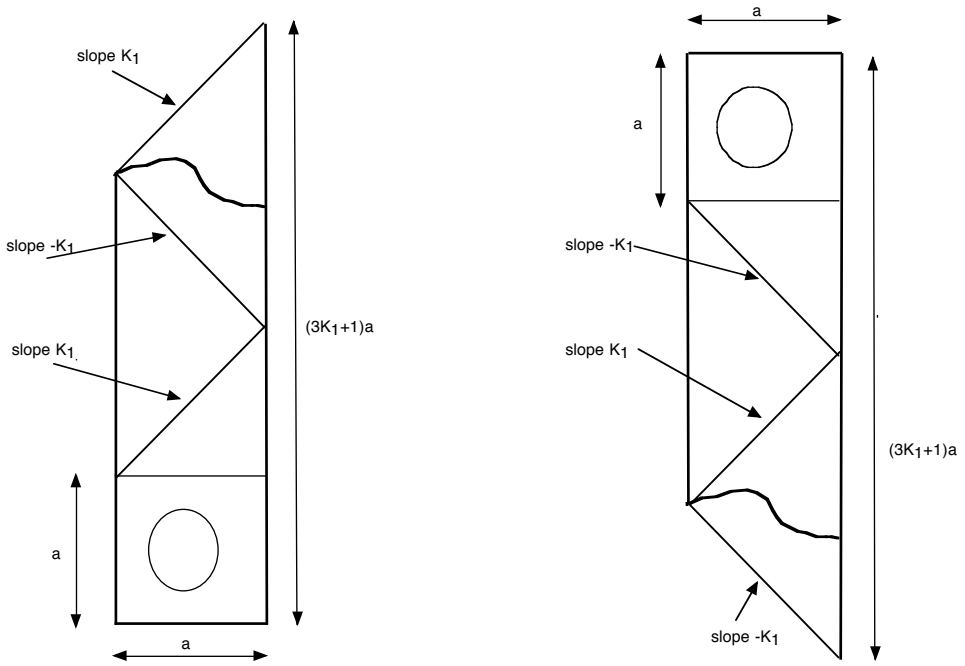

Figure 1: The two domains $\mathcal{D}^{top}_{a,x_0}$ and $\mathcal{D}^{bottom}_{a,x_0}$.

We cover every strip $\mathcal{A}_{n,k,\delta}$ by the domains (see (8.3))

- $\mathcal{D}^{top}_{a,x_0}$ (see Figure 1) where f is replaced by f_n, a by a_δ, x_0 by x_k, $k \in \{0, \dots, N_\delta\}$,
- $\mathcal{D}^{bottom}_{a,x_0}$ (see Figure 1) where f is replaced by $f_n - \beta$, a by a_δ, x_0 by x_k, $k \in \{0, \dots, N_\delta\}$,
- the remaining parts of the strips

$$\Big\{(x_{1n}, x_{2n}) \in (x_k, x_k + a_\delta) \times \mathbb{R} \mid f_n(x_k) + (2K_1 + 1)a_\delta - \beta < x_{2n} < f_n(x_k) - (2K_1 + 1)a_\delta\Big\}$$

  are covered by $\left[\frac{\beta - 2(2K_1+1)a_\delta}{a_\delta}\right] + 1$[5] squares whose edges have length a_δ.

We set

$$a_\delta = \frac{3C_0}{(3K_1 + 2)K_0}\delta, \qquad 0 < \delta \le \inf\Big\{\delta_0, \frac{2(3K_1 + 2)K_0}{3C_0}a_0, \frac{(3K_1 + 2)K_0}{6(2K_1 + 1)C_0}\beta\Big\}. \tag{8.9}$$

Observe that $a_\delta \le 2a_0$, $(3K_1 + 2)a_\delta = 3\delta\frac{C_0}{K_0} \le 3\delta R_0$, $2(2K_1 + 1)a_\delta \le \beta$.
So, the strip $\mathcal{A}_{n,k,\delta}$ is covered by domains star-shaped with respect to balls of radii $a_\delta/2$ and whose diameters are

[5] If this integer is strictly greater than 1, otherwise these parts are empty.

less than $3\delta R_0$. Each point in $\mathcal{A}_{n,k,\delta}$ belongs to at most two domains of the covering.
Then, proceeding as in Subsection 8.2.1, we obtain

$$\|\nabla \mathfrak{u}\|^p_{L^p(\mathcal{B}_{n,k,\delta})} \le \mathbf{C}^* \|e(\mathfrak{u})\|^p_{L^p(\mathcal{B}_{n,k,\delta})}, \qquad \|\mathfrak{u}\|^p_{L^p(\mathcal{B}_{n,k,\delta})} \le \mathbf{C}^* \delta^p \|e(\mathfrak{u})\|^p_{L^p(\mathcal{B}_{n,k,\delta})} \tag{8.10}$$

where

$$\mathcal{B}_{n,k,\delta} = \Big\{(x_1, x_2, x_3) \in \mathcal{A}_{n,k,\delta} \times \mathbb{R} \mid -\delta C_0 < x_3 < \delta\phi(x_1, x_2)\Big\}.$$

The constant $\mathbf{C}^*$ only depends on K_1, K_0, C_0 and δ_0.
Every point contained in $\mathcal{A}_{n,\beta}$ belongs to at most two $\mathcal{A}_{n,k,\delta}$. Then, summing the above inequalities lead to

$$\|\nabla \mathfrak{u})\|^p_{L^p(\mathcal{B}_{n,\beta,\delta})} \le 2\mathbf{C}^* \|e(\mathfrak{u})\|^p_{L^p(\Omega_\delta)}, \qquad \|\mathfrak{u}\|^p_{L^p(\mathcal{B}_{n,\beta,\delta})} \le 2\mathbf{C}^* \delta^p \|e(\mathfrak{u})\|^p_{L^p(\Omega_\delta)} \tag{8.11}$$

where

$$\mathcal{B}_{n,\beta,\delta} = \Big\{(x_1, x_2, x_3) \in \mathcal{A}_{n,\beta} \times \mathbb{R} \mid -\delta C_0 < x_3 < \delta\phi(x_1, x_2)\Big\}.$$

Finally, summing the inequalities (8.8) and (8.11) give

$$\|\nabla \mathfrak{u}\|_{L^p(\Omega_\delta)} \le C^{**} \|e(\mathfrak{u})\|_{L^p(\Omega_\delta)}, \qquad \|\mathfrak{u}\|_{L^p(\Omega_\delta)} \le C^{**} \delta \|e(\mathfrak{u})\|_{L^p(\Omega_\delta)}. \tag{8.12}$$

The constant does not depend on δ, it depends on N, K_1, K_0, C_0, C_1, δ_0 and p.
Note that in the proof of the estimates (4.3), the dimensional parameters a_0, β and the diameter of ω are not involved.

8.3 The extension operator $\mathcal{P}_\delta$

Set

$$\widetilde{\Omega}_\delta \doteq \omega \times (-\delta C_0, \delta C_1).$$

Lemma 8.2. *There exist an extension operator $\mathcal{P}_\delta$ from $W^{1,p}(\Omega_\delta)$, $1 \le p \le \infty$, into $W^{1,p}(\widetilde{\Omega}_\delta)$ such that for all $\Phi \in W^{1,p}(\Omega_\delta)$*

$$\begin{aligned}
&\|\mathcal{P}_\delta(\Phi)\|_{L^p(\widetilde{\Omega}_\delta)} \le \|\Phi\|_{L^p(\Omega_\delta)},\\
&\Big\|\frac{\partial \mathcal{P}_\delta(\Phi)}{\partial x_1}\Big\|_{L^p(\widetilde{\Omega}_\delta)} + \Big\|\frac{\partial \mathcal{P}_\delta(\Phi)}{\partial x_2}\Big\|_{L^p(\omega \times (-C_0, +\infty))} \le C\|\Phi\|_{W^{1,p}(\Omega_\delta)},\\
&\Big\|\frac{\partial \mathcal{P}_\delta(\Phi)}{\partial x_3}\Big\|_{L^p(\widetilde{\Omega}_\delta)} \le C\Big(\|\nabla\Phi\|_{L^p(\Omega_\delta)} + \frac{1}{\delta}\|\Phi\|_{L^p(\Omega_\delta)}\Big).
\end{aligned}$$

The constant C is independent of δ.

Proof. Every measurable function Φ in Ω_δ is extended as follows:

$$\begin{aligned}
&\mathcal{P}_\delta(\Phi)(x', x_3) = \Phi(x', x_3) && \text{for a.e. } x_3 \in \big(-\delta C_0, \delta\phi(x')\big),\\
&\mathcal{P}_\delta(\Phi)(x', x_3) = \Phi\big(x', 2\delta\phi(x') - x_3\big)\Big(1 + \frac{\delta\phi(x') - x_3}{\delta C_0}\Big) && \text{for a.e. } x_3 \in \big(\delta\phi(x'), \delta\phi(x') + \delta C_0\big),\\
&\mathcal{P}_\delta(\Phi)(x', x_3) = 0 && \text{for a.e. } x_3 \in \big(\delta\phi(x') + \delta C_0, +\infty\big)\\
& && \text{and for a.e. } x' \in \omega.
\end{aligned}$$

By construction $\mathcal{P}_\delta(\Phi) = \Phi$ a.e. in Ω_δ. Then, we consider the restriction of $\mathcal{P}_\delta(\Phi) = 0$ to the domain $\widetilde{\Omega}_\delta$. A straightforward calculations gives the estimates of the lemma. □

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