Long-time asymptotic of the Lifshitz-Slyozov equation with nucleation

Juan Calvo, Erwan Hingant, Romain Yvinec

Long-time asymptotic of the Lifshitz–Slyozov equation with nucleation

Juan Calvo*, Erwan Hingant†, and Romain Yvinec‡

*Universidad de Granada, Avda. Hospicio s/n, 18071, Granada, Spain. Research Unit "Modeling Nature" (MNat). E-mail: juancalvo@ugr.es

†LAMFA, Université de Picardie Jules Verne, CNRS, 33 rue Saint-Leu, 80039 Amiens, France. E-mail: erwan.hingant@u-picardie.fr

‡Université Paris-Saclay, Inria, Centre Inria de Saclay, 91120, Palaiseau, France & PRC, INRAE, CNRS, Université de Tours, 37380, Nouzilly, France. E-mail: romain.yvinec@inrae.fr

November 17, 2023

Abstract

We consider the Lifshitz–Slyozov model with inflow boundary conditions of nucleation type. We show that for a collection of representative rate functions the size distributions approach degenerate states concentrated at zero size for sufficiently large times. The proof relies on monotonicity properties of some quantities associated to an entropy functional. Moreover, we give numerical evidence on the fact that the convergence rate to the goal state is algebraic in time. Besides their mathematical interest, these results can be relevant for the interpretation of experimental data.

2020 Mathematics Subject Classification. *35Q92; 35B40; 35L04 .*

Key words and phrases. Long-time behavior, Lifshitz-Slyozov equation, Entropy functional, Nucleation theory.

1 Introduction

In this work we study the long time behavior of the Lifshitz–Slyozov (LS) model with nucleation boundary conditions. The model reads:

$$\frac{\partial f(t,x)}{\partial t} + \frac{\partial \{(a(x)u(t) - b(x))f(t,x)\}}{\partial x} = 0\,, \quad t > 0\,,\ x \in (0,\infty)\,, \tag{1}$$

with

$$u(t) + \int_0^\infty x f(t,x)\,dx = \rho\,, \quad t > 0\,, \tag{2}$$

for some given $\rho > 0$, subject to the boundary condition

$$((a(x)u(t) - b(x))f(t,x))_{|x=0} = \mathfrak{n}(u(t)), \quad t \in \{s > 0 \ : \ u(s) > \Phi_0\}, \tag{3}$$

where $\Phi_0 = \lim_{x\to 0^+} \frac{b(x)}{a(x)}$, and the initial condition

$$f(0,x) = f^{in}(x), \qquad x \in (0,\infty). \tag{4}$$

The LS model describes the temporal evolution of a mixture of monomers and aggregates that undergo the following interactions: a monomer can join an existing aggregate of size x, with an attachment rate $a(x)$, and a monomer can detach from an existing aggregate of size x, with a detachment rate $b(x)$. The variable x describes the size of the aggregates, so that $f(t,x)$ is the number density of aggregates at time t, whereas $u(t)$ stands for monomer concentration at time t. Equation (2) simply encodes the fact that the total mass ρ is preserved.

Depending on the specific rates $a(x)$ and $b(x)$, the model may or may not need a boundary condition. No boundary condition is needed for the original Lifshitz–Slyozov version of the model [19], and this is also the case for the various instances of "Ostwald ripening" that have been analyzed in the literature, e.g. [6, 7, 9, 15, 22–24]. In this article we are interested in situations where the kinetic rates are such that a boundary condition is needed to make sense of the model. Classical examples are power-law rates like $a(x) \propto x^{\alpha}$ and $b(x) \propto x^{\beta}$ with $0 \le \alpha \le \beta \le 1$. To our knowledge, only the special power-law case with equal power ($\alpha = \beta$) has been considered in the literature, see [6] with zero Dirichlet boundary conditions. For the main applications we have in mind the nucleation rate $\mathfrak{n}$ in equation (3) follows a mass action kinetics, that is $\mathfrak{n}(u) \propto u^{i_0}$ with $i_0 \in \mathbb{N}^*$.

The boundary condition (3) encodes the creation of new aggregates from the available pool of monomers. We understand this as an effective description of a nucleation process, that may be described by a more detailed discrete model. Connections between the discrete-size Becker–Döring model and the continuous-size Lifshitz–Slyozov model are indeed well known [8, 16, 30]. A boundary condition of the form (3) has been deduced from appropriate scaling limits of the Becker–Döring model in the case of a second order nucleation kinetics, see [8, 12] for details. Higher order mass action kinetics can also be used to describe the nucleation process in a phenomenological way [1, 27, 33]; such boundary conditions arise as scaling limits of modified versions of the Becker–Döring model. It is also interesting to note that some nucleation boundary conditions were derived in [12] that do not follow mass action kinetics. This can be relevant for the description of protein polymerization phenomena. All in all, this is a first step towards a number of important applications in the science of materials and in the field of neurodegenerative diseases [27–29, 31].

We prove below that under generic conditions, when the function $\frac{b}{a}$ is strictly increasing, solutions will concentrate at vanishing aggregate sizes, while the concentration of available monomers drops down to the activation threshold Φ_0 in equation (3). Under more specific hypotheses we can get more precise information about the temporal rates at which this dynamics takes place. We complement this with a numerical investigation. The general picture that emerges points to the fact that the concentration dynamics is quite slow, actually taking place with algebraic rates (that depend on α, β and $\mathfrak{n}$). These results are in line with previously known results for the case of outgoing characteristics, which indicate that degenerate steady states are approached at an algebraic rate, see [5, 9].

Since the transient behavior for equations (1)–(3) spans a wide temporal scale, this model is thus suitable for comparison with experimental data, e.g. those originating in protein polymerization experiments *in vitro*. It would be interesting to investigate whether potential asymptotic profiles, after suitable normalisation, could be universal or not, in comparison to the outgoing case [5, 9]. Note also that transient oscillatory behavior, for specific initial conditions, cannot be ruled out by our results, see e.g. [18]. However, if we are interested in very long time scales the concentration behavior will take over and then it seems advisable to introduce corrections to the model (e.g. size-diffusion corrections, coagulation-fragmentation operators...) in order to get a more realistic goal state, see for instance [3, 8, 10, 11, 14, 20, 21, 30, 32]. Similarly, if persistent oscillations are required, one should consider extra mechanisms, see e.g. [13, 25, 26].

2 Statement of the problem and results

Here below $\mathbb{R}_+ = (0, +\infty)$ and $(1 + x)dx$ denotes the measure with density $x \longmapsto (1 + x)$ with respect to the Lebesgue measure on $\mathbb{R}_+$. The space $L^1(\mathbb{R}_+, (1 + x)dx)$ denotes the space of integrable function w.r.t. the former measure. This space might be endowed with the weak topology denoted by w whose convergence is characterized against bounded functions. Finally, $\mathcal{C}([0, \infty), w - L^1(\mathbb{R}_+, (1 + x)dx))$ is the set of continuous functions from $[0, +\infty)$ into $L^1(\mathbb{R}_+, (1 + x)dx)$ equipped with its weak topology that is, for such f,

$$t \longmapsto \int_0^\infty f(t, x)\varphi(x)(1 + x)dx\,,$$

is continuous on $[0, \infty)$ for all $\varphi \in L^\infty(0, \infty)$. Here and in the sequel L^p refers to the standard Lebesgue spaces and $W^{1,p}$ refers to the standard Sobolev spaces. We may add a subscript to it, so that the integration variable is made clear. $\mathcal{C}^k$ is the space of continuous function whose k^{th} derivatives are continuous, and $\mathcal{C}^k_c$ stands for its subspace consisting of compactly supported functions.

In the remainder the rates a, b and $\mathfrak{n}$ are assumed nonnegative and continuous functions on $[0, \infty)$ and such that

$$\forall x > 0\,,\; a(x) > 0 \quad \text{and} \quad \Phi(x) := \frac{b(x)}{a(x)} \to_{x\to 0^+} \Phi_0\,. \tag{H0}$$

We deal with global solutions for which the boundary condition is defined for all time, namely solutions defined in the following sense:

Definition 2.1. Let $0 \leq f^{\mathrm{in}} \in L^1(\mathbb{R}_+, (1+x)dx)$ and $\rho > 0$. We say that $0 \leq f \in \mathcal{C}([0,\infty), w - L^1(\mathbb{R}_+, (1+x)dx))$ is a (global) solution to the LS equation with nucleation (1)-(4) provided that:

1. For all $t \geq 0$ and for every $\varphi \in \mathcal{C}^0([0,\infty))$ such that $\varphi' \in L^\infty(0,\infty)$, we have

$$\int_0^\infty \varphi(x) f(t,x)\, dx = \int_0^\infty \varphi(x) f^{\mathrm{in}}(x)\, dx + \int_0^t \int_0^\infty (a(x)u(s) - b(x))\varphi'(x) f(s,x)\, dx\, ds + \int_0^t \varphi(0)\mathfrak{n}(u(s))\, dt\,. \quad (5)$$

2. For all $t \geq 0$,

$$u(t) := \rho - \int_0^\infty x f(t,x) dx > \Phi_0.$$

Note that for any smooth test function φ, a solution as defined above satisfies the following moment equation:

$$\frac{d}{dt}\int_0^\infty \varphi(x) f(t,x) = \varphi(0)\mathfrak{n}(u(t)) + \int_0^\infty (a(x)u(t) - b(x))\varphi'(x) f(t,x)\,. \quad (6)$$

Our main hypothesis relies on the monotonicity of Φ. Namely, we suppose that

$$\Phi \text{ is strictly increasing}\,. \quad \text{(H1)}$$

We shall assume some technical hypotheses as well: First, for any $\varepsilon > 0$,

$$a'\,,\; b' \in L^\infty(\varepsilon, \infty)\,, \quad \text{(H2)}$$

which entails the existence of a constant $C > 0$ such that, for all $x \geq 0$,

$$a(x) + b(x) \leq C(1+x)\,. \quad (7)$$

Next, we suppose

$$\inf_{x\in(1,\infty)} a(x) > 0\,, \text{ and } \frac{1}{a} \in L^1(0,1)\,, \quad \text{(H3)}$$

and that there exists a constant $C > 0$ such that, for all $x \geq 0$,

$$b(x)\Phi(x) \geq \tfrac{1}{C}\min(1, x^2)\,. \quad \text{(H4)}$$

Concerning the nucleation rate, we assume that there exist two constants $c > 0$ and $k_0 \geq 1$ such that for all $z \geq 0$

$$\mathfrak{n}(z) \geq c z^{k_0}\,. \quad \text{(H5)}$$

Finally, we will assume that the initial condition f^{in} belongs to $L^1(\mathbb{R}_+, (1+x)dx)$, is non-negative and moreover

$$\int_0^\infty \left(\int_0^x \Phi(z) dz + x^2 \right) f^{\mathrm{in}}(x) dx < \infty\,. \quad \text{(H6)}$$

The well-posedness of equations (1)-(4) was studied in [4] under some additional conditions of a technical nature -which are probably non-optimal. At any rate, taking (H1) for granted we know for sure that solutions (should they exist) will be global in time with $u(t) > \Phi_0$ for all times. Therefore, during the rest of the document we will only concentrate on the set of assumptions needed for our analysis of the long time behavior.

Some comments are in order concerning our set of hypotheses. Besides ensuring well-posedness, hypothesis (H1) is the natural counterpart to the typical Ostwald Ripening phenomena where, $\Phi(x) = x^{-1/3}$ for which the long-time asymptotics have been studied e.g. [9]. Note that hypothesis (H2) is not that demanding; actually, blow-up phenomena may take place for strictly superlinear rates [6]. Hypothesis (H3), has been required in connection with the well-posedness of the problem, so that characteristics go back to $x = 0$ in finite time and render the boundary condition relevant. Condition (H4) is purely technical and does not imply a strong restriction, recall the power law rates introduced in Section 1, namely $a(x) \propto x^{\alpha}$ and $b(x) \propto x^{\beta}$ with $0 \le \alpha \le \beta \le 1$. Indeed, such rates satisfy all our hypotheses. Concerning the nucleation rate, any mass action kinetics can be considered under hypothesis (H5). Finally, hypothesis (H6) imposes just a mild technical requirement on the initial condition.

In order to ascertain the temporal evolution of the system, the number of aggregates constitutes an important quantity that is defined as

$$M_0(t) := \int_0^{\infty} f(t,x)\, dx\,.$$

According to the boundary condition (3), this evolves in time via

$$\frac{dM_0}{dt} = \mathfrak{n}(u)\,.$$

To discuss concentration phenomena, we define $\mathcal{M}_\rho^+([0,+\infty))$ the set of nonnegative Radon measures on $[0,+\infty)$ with total variation less or equal to ρ. This space can be equipped with the weak topology whose convergence corresponds to the convergence against any continuous and bounded function.

Our main results in this document describe concentration phenomena for the long-time evolution of the LS equation with nucleation boundary conditions:

Theorem 2.2. *Under hypotheses* (H0-H6)*, any global solution in the sense of Definition 2.1 satisfies*

- $\lim_{t\to+\infty} M_0(t) = +\infty$,
- $\lim_{t\to+\infty} u(t) = \Phi_0$,
- $\lim_{t\to+\infty} xf(t,x)dx = (\rho - \Phi_0)\delta_0$, *weakly in* $\mathcal{M}_\rho^+([0,\infty))$.

The proof relies on a Lyapunov functional, which we introduce in Section 3. Right after that we proceed with the proof of Theorem 2.2 in Section 4.

Note that Theorem 2.2 is a generalisation of an earlier result proved in [3] for $a(x) = 1$ and $b(x)$ such that $0 < c_1 \leq b'(x) \leq c_2$ for some constants c_1 and c_2. In that particular case, the rates can be computed explicitly. Those are algebraic and depend on the specific form of the nucleation rate, but are nevertheless quite slow. For more general coefficients, our method of proof cannot provide specific estimates and therefore we cannot ascertain the timescales over which the average aggregate size tends to zero. However, we performed numerical simulations, whose results suggest that this algebraic trend is actually what we should expect generically. This is discussed in Section 5. Our results are complemented with an analysis of the case of constant Φ, which requires a separate treatment. Its long time behavior is analyzed in Section 6 and expands on the results given in [6, 7].

One of the main points underlying the previous results is to discriminate whether the system will be able to fuel nucleation reactions to the extent that the number of fragments grows without control. We actually show that this is the case for a representative number of situations. Since the total mass is preserved, this suggests that the average aggregate size becomes smaller and smaller, which is an instance of *dust formation*.

3 Lyapunov functional

We shall introduce a Lyapunov functional in the vein of [3,9]. For k a continuous and positive function on $[0, +\infty)$ with continuous derivative and f a solution of equations (1)–(4), we define for all $t \geq 0$,

$$H_k(t) = \int_0^\infty k(x) f(t, x) dx + K(u(t)), \tag{8}$$

with $K(v) = \int_0^v k' \circ \Phi^{-1}(z) dz$. This makes sense since Φ is monotonous. The functional H_k is a Lyapunov functional and its time derivative D_k is called its dissipation, as the following result makes clear.

Proposition 3.1. *Assume $0 \leq k \in \mathcal{C}^1([0, +\infty))$ is convex with $k(0) = 0$ and f is a solution in the sense of Definition 2.1. If $H_k(0) < \infty$, then $t \longmapsto H(t)$ is non-increasing, non-negative and for all $t \geq 0$*

$$H_k(t) + \int_s^t D_k(s) ds \leq H_k(s), \qquad \forall\, 0 \leq s < t, \tag{9}$$

where

$$0 \leq D_k(t) = \int_0^\infty \left(K'(u(t)) - K'(\Phi(x))\right) \left(u(t) - \Phi(x)\right) a(x) f(t, x) dx,$$

belongs to $L^1_t(0, \infty)$.

Proof. Let $R > 0$, $k_R(x) = k(x)$ for $x < R$ and $k_R(x) = k'(R)(x - R) + k(R)$ for $x \geq R$. Notice that k_R can be used as a test function in (5). Moreover, k_R is convex. We construct H_{k_R} via formula (8) and we compute the dissipative part, which is nonnegative because $K' = k_R' \circ \Phi^{-1}$ is increasing. Then we conclude by Fatou's lemma. □

As a straightforward consequence with $k(x) = \frac{1}{2}x^2$ we get the following result:

Corollary 3.2. *Assume hypothesis* (H6) *and let f be a solution in the sense of Definition 2.1. We define*

$$H(t) = \frac{1}{2}\int_0^\infty x^2 f(t,x)dx + \Psi(u(t)), \tag{10}$$

with $\Psi(v) = \int_0^v \Phi^{-1}(z)dz$. We have that $H(0) < \infty$ and $t \longmapsto H(t)$ is non-increasing, non-negative and the dissipation part is given, for all $t \geq 0$, by

$$D(t) = \int_0^\infty \left(\Phi^{-1}(u(t)) - \Phi^{-1}(\Phi(x))\right)(u(t) - \Phi(x))\, a(x) f(t,x)dx\,.$$

Remark 3.3. There are other useful choices of the function k. We will make use of $k(x) = \int_0^x \Phi(y)dy$ which leads to $k' \circ \Phi^{-1} = Id$ and then

$$\frac{d}{dt}H_k \leq -\int_0^\infty (u(t) - \Phi(x))^2 a(x) f(t,x)dx \in L_t^1(0,\infty). \tag{11}$$

We may also take $k(x) = x^\eta$ with $\eta \geq 1$ to control moments of the form

$$M_\eta(t) := \int_0^\infty x^\eta f(t,x)dx\,.$$

Indeed, for any moment greater than 1 which is initially finite, we have a uniform bound for every positive time.

4 Proof of the main result

4.1 The number of fragments diverges

We prove first a generic result showing that a shattering phenomenon takes place on long time intervals: the number of fragments diverges.

Proposition 4.1. *Assume hypotheses* (H0, H2,H4-H6)*. Then, for every solution in the sense of Definition 2.1, there holds that* $\lim_{t\to+\infty} M_0(t) = +\infty$.

Proof. Taking **1** as a test function in (5) we have

$$M_0(t) = M_0(0) + \int_0^t \mathfrak{n}(u(s))ds\,,$$

thus M_0 is monotonically increasing. The result follows directly in the case $\Phi_0 > 0$ by hypothesis (H5). Therefore, we provide a proof for $\Phi_0 = 0$. We argue by contradiction.

Suppose M_0 is bounded above independently of time. By equation (5) with $\varphi(x) = x$ and noticing inequality (7) and $0 \le u(t) \le \rho$, we deduce

$$u'(t) = -u(t)\int_0^\infty a(x)f(t,x) + \int_0^\infty b(x)f(t,x)dx \in L_t^\infty(0,\infty)\,.$$

This entails $u \in W^{1,\infty}(0,\infty)$. Moreover, using Cauchy-Schwarz's inequality,

$$|u'(t)| \le \left(\int_0^\infty a(x)f(t,x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty a(x)(u(t)-\Phi(x))^2 f(t,x)dx\right)^{\frac{1}{2}}\,.$$

By (H6) and Proposition 3.1 with $k(x) = \int_0^x \Phi(y)dy$ -see also Remark 3.3, we have

$$D_k(t) = \int_0^\infty a(x)(u(t)-\Phi(x))^2 f(t,x)dx \in L_t^1(0,\infty)\,. \tag{12}$$

Thus, with inequality (7), we deduce that $u' \in L^2(0,\infty)$.

Next, we notice that $u^{k_0} \in L^1(0,\infty)$; this is due to hypothesis (H5) and

$$c\int_0^\infty u^{k_0}(t)dt \le \int_0^\infty \mathfrak{n}(u(t))dt \le \limsup_{t\to\infty} M_0(t) - M_0(0),$$

together with the supposed bound on the 0^{th}-order moment. Let $p > \max(2,k_0)$, we have

$$\int_0^\infty u^p(t)dt = \int_0^\infty u^{p-k_0}(t)u^{k_0}(t) \le \rho^{p-k_0}\|u^{k_0}\|_{L^1(0,\infty)}\,,$$

thus $u^p \in L^1(0,\infty)$. Moreover $(u^p)' = pu'u^{p-1}$ belongs to $L^1(0,\infty)$ because

$$\int_0^\infty |u'u^{p-1}|dt \le \left(\int_0^\infty |u'|^2\right)^{\frac{1}{2}}\int_0^\infty u^{2(p-1)}dt \le \|u'\|_{L^2}\int_0^\infty u^{2(p-1)-k_0+k_0}dt\,.$$

But $2p-2-k_0 > 0$, thus $u^{2(p-1)-k_0} \le \rho^{2(p-1)-k_0}$ and therefore $(u^p)'$ belongs to $L^1(0,\infty)$ because u^{k_0} does. This implies that $u^p \in W^{1,1}(0,\infty)$ and hence $u(t)^p \to 0$ as $t \to +\infty$.

We now turn to the dissipation part, equation (12). There exists a sequence of times $t_n \to +\infty$ such that $D_k(t_n) \to 0$ by integrability. Actually the dissipation reads

$$D_k(t) = u^2(t)\int_0^\infty a(x)f(t,x)\,dx + \int_0^\infty b(x)\Phi(x)f(t,x)\,dx - 2u(t)\int_0^\infty b(x)f(t,x)\,dx\,.$$

Using inequality (7) and the definition of solution, the first and last integrals are continuous and bounded in time. Together with the fact that $u(t) \to 0$ as $t \to \infty$ we have that

$$\lim_{n\to+\infty}\int_0^\infty b(x)\Phi(x)f(t_n,x)dx = 0.$$

Thanks to hypothesis (H4) and Cauchy–Swartz's inequality we get

$$\int_0^\infty xf(t,x)dx \le \left(\int_0^\infty b(x)\Phi(x)f(t,x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty C(1+x^2)f(t,x)dx\right)^{\frac{1}{2}}\,.$$

Since the 0^{th}-order moment is bounded and the second order moment as well by Corollary 3.2, we obtain

$$\lim_{n\to+\infty}\int_0^\infty xf(t_n,x)dx = 0.$$

This contradicts the fact that $u(t_n) = \rho - \int_0^\infty xf(t_n,x)dx \to 0$. $\square$

Since mass is preserved, the divergence of M_0 implies that the average aggregate size tends to zero.

4.2 Concentration behavior for the mass density

We can use the dissipation to extract some information on the long-time asymptotic. We proceed by standard Lasalle's invariance principle arguments, proving that the orbits are relatively compact and we identify trajectories in the ω-limit set to be time independent. In fact we shall work out the argument from scratch because we lack the continuity of our Lyapunov functional.

In this section we assume that our hypotheses (H0-H5) hold true and we take f a solution in the sense of Definition 2.1 with initial data f^{in} satisfying hypothesis (H6). We let $\{t_n\}$ an arbitrary increasing sequence of times with $\lim_{n\to+\infty} t_n = +\infty$. Let $T>0$ arbitrary, we define for $t\in[0,T]$

$$f^n(t,x) = f(t+t_n,x) \text{ and } \mu_t^n(dx) = xf^n(t,x)dx\,.$$

The measures μ_t^n are bounded nonnegative Radon measures on $(0,+\infty)$ with $\mu_t^n((0,+\infty)) = \rho$. Let H be given by (10) and $H^n(t) = H(t+t_n)$.

Let $\mathcal{M}_\rho^+(0,+\infty)$ the set of nonnegative Radon measures on $(0,+\infty)$ with mass less or equal to ρ. The topology induced by the dual of $\mathcal{C}_c^0(0,\infty)$ on $\mathcal{M}_\rho^+(0,\infty)$ is called vague topology. We denote this space with such topology by $v-\mathcal{M}_\rho^+(0,\infty)$. This space is metrizable and compact [2], with metric

$$d(\mu,\nu) = \sum_{k\geq 0} 2^{-k}\min(1,|\langle\mu,\varphi_k\rangle - \langle\nu,\varphi_k\rangle|)\,,$$

for all μ, ν in $\mathcal{M}_\rho^+(0,\infty)$ where $(\varphi_k)_{k\geq 0}\in\mathcal{C}_c^\infty(0,\infty)$ is dense in $\mathcal{C}_c^0(0,\infty)$ and $\langle\mu,\varphi\rangle = \int\varphi\,d\mu$ denotes the duality pairing.

Lemma 4.2. *$\{\mu^n\}$ is relatively sequentially compact in $\mathcal{C}^0([0,T]), v-\mathcal{M}_\rho^+(0,\infty))$.*

Proof. Let $\varphi\in\mathcal{C}_c^\infty(0,\infty)$, we have by (5) that

$$\frac{d}{dt}\int_0^\infty \varphi(x)xf(t,x)dx = \int_0^\infty (x\varphi)'a(x)(u(t)-\Phi(x))f(t,x)dx\,.$$

As φ is compactly supported and u is bounded by ρ, there exists a constant $C_\varphi > 0$, such that $|(x\varphi)'a(x)(u(t) - \Phi(x))| \leq C_\varphi x$, thus for all $t \geq 0$,

$$\left| \frac{d}{dt} \int_0^\infty \varphi(x) x f(t,x) dx \right| \leq C_\varphi \rho \, .$$

Hence, for any $t_0 \in [0,T]$,

$$\limsup_{t \to t_0} \sup_n d(\mu_t^n, \mu_{t_0}^n) \leq \limsup_{t \to t_0} \sum_{k \geq 0} 2^{-k} \min(1, C_{\varphi_k} \rho |t - t_0|) = 0 \ ,$$

and thus the sequence is equicontinuous on $[0,T]$. We conclude by the Arzelà-Ascoli theorem. □

Lemma 4.3. *There exist $u \in \mathcal{C}^0([0,T])$ with $0 \leq u \leq \rho$ and a subsequence of $\{u^n\}$ which converges to u pointwise on $[0,T]$.*

Proof. Notice that by standard truncation arguments with equations (5), (7) and hypothesis (H6),

$$\left| \frac{d}{dt} \int_0^\infty x^2 f(t,x) dx \right| \leq K(\rho + 1) \int_0^\infty x(1+x) f(t,x) dx \, .$$

The latter is uniformly bounded in time. By Arzelà-Ascoli we may extract a subsequence of $t \longmapsto \int_0^\infty x^2 f^n(t,x) dx$ which converges uniformly on $[0,T]$. Moreover, by the monotonicity and nonegativity of $H(t)$ it has a limit H^∞ as $t \to \infty$ and H^n converges uniformly on $[0,T]$ to H^∞. This proves that $\Psi(u^n(t)) = H^n(t) - \int_0^\infty x^2 f^n(t,x) dx$ converges uniformly to a function $\overline{\Psi}(t)$ on $[0,T]$. Note that $\Psi = \int_0^x \Phi^{-1}(y) dy$ is continuous and strictly increasing (because $\Phi^{-1} > 0$), thus $u^n(t)$ converges to $u(t) := \Psi^{-1}(\overline{\Psi}(t))$ pointwise. □

Remark 4.4. We might replace x^2 above by $x^{1+\eta}$, provided that $(a(x) + b(x)) x^\theta \leq Cx$ for some $\theta \geq 0$ and $C > 0$.

Proposition 4.5. *The measure $x f(t,x) dx$ converges to 0 in $v - \mathcal{M}_\rho^+(0,+\infty)$ as $t \to +\infty$.*

Proof. By the previous two Lemmas we can extract a subsequence such that μ^n converges to some μ in $\mathcal{C}^0([0,T]), v - \mathcal{M}_\rho^+(0,\infty))$ and u^n converges poinwise to some $u \in \mathcal{C}^0([0,T])$. Given that $H^n(t)$ converges to H^∞ for all $t \in [0,T]$, we use Corollary 3.2 and equation (9) to deduce that

$$\lim_{n \to +\infty} \int_0^T \int_0^\infty \left(\Phi^{-1}(u^n(t)) - \Phi^{-1}(\Phi(x))\right) \left(u^n(t) - \Phi(x)\right) a(x) f^n(t,x) dx dt = 0 \, .$$

Let $m \geq 1$ and $K_m = [\frac{1}{m}, m]$ and define χ_m a nonnegative continuous function with compact support in $(0,\infty)$, equal to a positive constant on K_m and such that $\chi_m(x) \leq a(x)/x$ for all $x > 0$. This is possible by the continuity and positivity of a. Hence,

$$\lim_{n \to +\infty} \int_0^T \int_0^\infty \left(\Phi^{-1}(u^n(t)) - \Phi^{-1}(\Phi(x))\right)\right) \left(u^n(t) - \Phi(x)\right) \chi_m(x) \mu_t^n(dx) dt = 0 \, . \qquad (13)$$

Expanding the product, we easily see that space integrals converge uniformly in time on $[0,T]$. Moreover, u^n converges pointwise and is bounded, hence the above limit can be interchanged with the time integrals and we conclude that

$$\int_0^T \int_{K_m} \left(\Phi^{-1}(u(t)) - \Phi^{-1}(\Phi(x))\right)(u(t) - \Phi(x))\,\chi_m(x)\mu_t(dx)dt = 0\,,$$

for all $m \geq 1$. Thus, a.e. $t \in [0,T]$ we have the following measure equality

$$\left(\Phi^{-1}(u(t)) - \Phi^{-1}(\Phi(x))\right)(u(t) - \Phi(x))\,\mu_t(dx) = 0\,.$$

As u is continuous, the above equality is achieved for all $t \in [0,T]$. Noticing that Φ^{-1} is increasing, for all $t \in [0,T]$, either $\mu_t = 0$ or both $u(t) > \Phi_0$ and there exists $m_t : [0,T] \to (0,\rho]$ such that $\mu_t = m_t\delta_{\Phi^{-1}(u(t))}$.

We should prove the second alternative could not occur. Indeed, let t_0 such that $\mu_{t_0} = m_{t_0}\delta_{x_0}$ with $x_0 = \Phi^{-1}(u(t_0)) > 0$ -because $u(t_0) > \Phi_0$. For all $\varphi \in \mathcal{C}_c^0(0,\infty)$,

$$\lim_{x\to+\infty} \int_0^\infty \varphi(x)f^n(t_0,x)dx = m_{t_0}\frac{\varphi(x_0)}{x_0}\,,$$

which proves that $f^n(t_0,x)dx$ converges in $\mathcal{M}_\rho^+(0,\infty)$ and contradicts that the 0^{th}-moment $M_0(t_0+t_n)$ goes to ∞ as $n\to\infty$, by Prop. 4.1.

Hence, we have shown that $\lim_{n\to\infty}\mu_t^n = 0$. Note that we have $\mu_t^n = xf(t+t_n,x)dx$. Taking $t=0$ we deduce that

$$\lim_{n\to\infty} xf(t_n,x)dx = \lim_{n\to\infty}\mu_0^n = 0\,, \qquad \text{vaguely in } \mathcal{M}_\rho^+((0,+\infty))\,,$$

and this holds true for any sequence $t_n \to \infty$. Therefore, we have proved that $xf(t,x)dx$ has a limit as $t\to\infty$ and this limit is 0. □

We conclude that the limit measure concentrates at the origin.

Lemma 4.6. *There exists $m > 0$ and a subsequence (not relabelled) such that*

$$\lim_{n\to\infty} xf(t_n,x)dx = m\delta_0\,, \qquad \textit{weakly in } \mathcal{M}_\rho^+([0,+\infty))\,.$$

Proof. Note that $\nu^n = xf(t_n,x)dx$ defines a sequence of measures on $[0,+\infty)$, such that $\nu^n_{|(0,+\infty)} = \mu^n$. The sequence $\{\nu^n\}$ is bounded and thus admits a subsequence which converges $v-\mathcal{M}_\rho^+([0,+\infty))$; recall that this is the dual of $\mathcal{C}_c^0([0,\infty))$. By the previous results, the limit ν verifies that $\nu_{|(0,+\infty)} = 0$ as measures. Thus $\nu = m\delta_0$ for some $m \geq 0$. We recall that $\langle\nu^n,x\rangle$ is uniformly bounded in n; this control allows to improve the convergence to $\langle\nu^n,\varphi\rangle \to \langle\nu,\varphi\rangle$ for all $\varphi \in \mathcal{C}_b([0,+\infty))$, the set of continuous and bounded functions. □

It remains to identify m which will be the result of the next section.

4.3 Identification of the concentrated mass

This paragraph is devoted to the proof that $u(t)$ approaches the critical value Φ_0; by mass conservation, the concentrated mass m in Lemma 4.6 is therefore $\rho - \Phi_0$. This will conclude the proof of Theorem 2.2. In this section we still assume that our Hypotheses (H0-H5) hold true and we take f a solution in the sense of Definition 2.1 with initial datum f^{in} satisfying hypothesis (H6).

Lemma 4.7. *We have that* $\lim_{t\to+\infty} \int_0^\infty x^2 f(t,x)dx = 0.$

Proof. By hypothesis (H6) and a (refined) de La Vallé Poussin's lemma [17] there exists $\beta \in \mathcal{C}^1([0,\infty))$, nonegative, increasing, convex, such that $\lim_{x\to\infty} \beta(x)/x = +\infty$ and

$$\int_0^\infty \beta(x) x f^{\text{in}}(x)dx < \infty\,.$$

According to Remark 3.3 we may use the Lyapunov functional (8) with $k(x) = \beta(x)x$, which is convex too, to deduce that

$$\sup_{t>0} \int_0^\infty \beta(x)xf(t,x)dx < \infty.$$

Let a sequence of times $t_n \nearrow \infty$; Lemma 4.6 ensures that there exists a subsequence (not relabelled) such that $xf(t_n,x)dx \to m\delta_0$ weakly. Thus, we let $\chi_R = \min(x,R)$, so that

$$\begin{aligned} \int_0^\infty x^2 f(t_n,x)dx &\le \int_0^\infty \chi_R(x)xf(t_n,x)dx + \int_R^\infty x^2 f(t_n,x)dx \\ &\le \int_0^\infty \chi_R(x)xf(t_n,x)dx + \sup_{z>R}\left(\frac{z}{\beta(z)}\right)\sup_{t>0}\int_0^\infty \beta(x)xf(t,x)dx\,. \quad (14) \end{aligned}$$

We take the limit $n \to +\infty$; the first term on the right hand side goes to 0 since $xf(t_n,x)dx \to m\delta_0$ and $\chi_R(0) = 0$. Finally we take the limit $R \to +\infty$ and the remaining term goes to 0. Since this is true for all sequences, we get the full limit. □

Remark 4.8. Here again we might deal with $x^{1+\eta}$ instead of x^2 as in Remark 4.4.

Lemma 4.9. *There holds that* $\lim_{t\to+\infty} u(t) = \Phi_0$.

Proof. The convergence of $H(t)$ as $t \to \infty$ together with the previous lemma entails that $\Psi(u(t))$ converges to some constant c. By the (strict) monotonicity of Ψ and its continuity, Ψ^{-1} is also continuous and thus $u(t)$ converges to $\theta := \Psi^{-1}(c)$. Assume that $\theta > \Phi_0$. We will prove a contradiction. We introduce the function $A(x) = \int_0^x \frac{1}{a}$, well-defined thanks to hypotheses (H0) and (H3). Letting $A^n(x) = \int_{1/n}^x \frac{1}{a}$ for $x > 1/n$ and $A^n(x) = 0$ otherwise, we get that A^n is continuous and $A^{n\prime} \in L^\infty$. Thus we can use A^n as a test function in the moment equation (5) and pass to the limit ($A^n \le A$) to get

$$\frac{d}{dt}\int_0^\infty A(x)f(t,x)dx = u(t)M_0(t) - \int_0^\infty \Phi(x)f(t,x)dx\,.$$

This is justified since the function Φ is continuous on $[0,\varepsilon]$, thus bounded, and there exists $K_\varepsilon > 0$ such that $\Phi(x) = b(x)/a(x) \le K_\varepsilon x$ for all $x > \varepsilon$ thanks to hypothesis (H0) and inequality (7).

As $\theta > \Phi_0$, we may find $\delta, t_0 > 0$ such that $u(t) > \Phi_0 + 2\delta$ for all $t > t_0$. Next, we may find $\epsilon > 0$ such that $\sup_{x\in[0,\varepsilon]} \Phi(x) < \delta + \Phi_0$. Then

$$\int_0^\infty \Phi(x) f(t,x)dx \le (\delta + \Phi_0)\, M_0(t) + K_\varepsilon \rho\,,$$

and collecting both estimates we arrive to

$$\frac{d}{dt}\int_0^\infty A(x)f(t,x)dx \ge \delta M_0(t) - K_\varepsilon\rho \qquad \text{for all } t > t_0.$$

Since $M_0(t) \to +\infty$ we derive that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^\infty A(x)f(t,x)dx = +\infty.$$

But, with hypothesis (H3), there exists K_A such that $A(x) \le K_A(1+x)$ for all $x > 0$, thus

$$\int_0^\infty A(x)f(t,x)dx \le K_A \int_0^\infty (1+x)f(t,x)dx \le K_A M_0(t) + K_A\rho\,.$$

Given that $M_0(t)/t$ is bounded we have a contradiction. Hence $u(t) \to \Phi_0$ as $t \to \infty$. $\square$

As a consequence of the previous result and mass conservation, we have $m = \rho - \Phi_0$ in the representation of the limit measure $\nu = m\delta_0$. This concludes the proof of Theorem 2.2.

Corollary 4.10. *Under the same hypotheses of the theorem, we have that all moments $M_\theta(t)$ with $\theta \in [0,1)$ diverge as $t \to \infty$.*

This follows from Lemma 4.7, by interpolating the first moment between moments of order θ and two.

5 Long time behavior for power-law rates and numerics

5.1 Rate of convergence for special cases

The purpose of this section is to provide more detailed information of the long time behavior in specific power-law cases. Here we assume $a(x) = ax^\alpha$ and $b(x) = bx^\beta$ with $a, b > 0$ and $0 \le \alpha < \beta \le 1$. Then $\Phi(x) = \frac{b}{a}x^{\beta-\alpha}$ verifies $\Phi_0 = 0$ and is monotonically increasing.

We may define the classical moments of f as

$$M_k(t) := \int_0^\infty x^k f(t,x)\,dx, \quad k \in \mathbb{R}^+.$$

In the case of power-law kinetic rates, the time derivative of the classical moments reads

$$\frac{dM_k}{dt} = kau(t)M_{k+\alpha-1}(t) - kbM_{k+\beta-1}(t)\,.$$

In what follows we consider some particular choices of the exponents α and β for which more specific information can be given (this works also as a guide to numerical conjectures for the general case, see Section 5.2 below).

Lemma 5.1 (Case $\alpha = 0$). *Assume that $\mathfrak{n}(u) = u^{i_0}$ -a bound from above by a power law works the same way. Then there exists some $C > 0$ such that, for advanced t*

$$M_0(t) \le Ct^{\frac{1}{1+i_0\beta}}, \quad u(t) \le Ct^{-\frac{\beta}{1+i_0\beta}}\,.$$

Proof. First we prove that $u(t)M_0^\beta(t) \le C$ for every $t \ge 0$. For that aim, let $y(t) = u(t)M_0^\beta(t)$. We readily compute

$$y' = (bM_\beta - uaM_0)M_0^\beta + \beta u\mathfrak{n}(u)M_0^{\beta-1}\,.$$

Here we can use that moment interpolation yields the estimate

$$M_\beta(t) \le M_1^\beta(t)M_0^{1-\beta}(t) \le \rho^\beta\, M_0^{1-\beta}(t).$$

Therefore,

$$y' \le \varphi(t) + \, b\rho^\beta M_0 - auM_0M_0^\beta = M_0(b\rho^\beta - ay) + \varphi(t)\,,$$

with $\varphi(t) \to 0$ as $t \to \infty$. This implies that $y(t)$ belongs to $[0, \rho^\beta b/a]$ for advanced times.

Thanks to our assumption on $\mathfrak{n}$ we have

$$M_0' = \frac{u^{i_0}M_0^{\beta i_0}}{M_0^{\beta i_0}} \le \frac{\rho^{\beta i_0}(b/a)^{i_0}}{M_0^{\beta i_0}}\,,$$

and therefore

$$\frac{d}{dt}M_0^{1+i_0\beta} \le (1+i_0\beta)\rho^{i_0\beta}(b/a)^{i_0}.$$

Our statements follow easily from here. □

We expect to have equality in the previous estimates. Actually, in the general case $\alpha < \beta$ we conjecture that:

$$M_0(t) \sim t^{\frac{1}{1+i_0(\beta-\alpha)}}, \quad u(t) \sim t^{-\frac{\beta-\alpha}{1+i_0(\beta-\alpha)}}\,. \tag{15}$$

for advanced t. Likewise, we expect that $u(t)M_0^{\beta-\alpha}(t)$ will have a finite limit. Compare with the numerical simulations below.

We prove another partial result along the same lines.

Lemma 5.2 (case $\beta = 1$). *There holds that*

$$\lim_{t\to\infty} u(t)M_\alpha(t) = b\rho/a.$$

Proof. In this specific case we have

$$\frac{dM_1}{dt} = auM_\alpha - bM_1.$$

This is integrated as

$$M_1(t) = M_1(0)e^{-bt} + ae^{-bt}\int_0^t u(\tau)M_\alpha(\tau)e^{b\tau}\,d\tau\,.$$

Since $u(t) \to 0$ as $t \to \infty$, we have that $M_1 \to \rho$ as $t \to \infty$. Now we argue by contradiction. Assume that $\liminf_{t\to\infty} uM_\alpha > b\rho/a$. Then, given $\epsilon > 0$ there exists some $T > 0$ such that $uM_\alpha > b\rho/a + \epsilon$ for $t \geq T$. Therefore, for each $t > T$,

$$\begin{aligned} M_1(t) &\geq M_1(0)e^{-bt} + ae^{-bt}\int_0^T u(\tau)M_\alpha(\tau)e^{b\tau}\,d\tau + a(b\rho/a+\epsilon)e^{-bt}\int_T^t e^{b\tau}\,d\tau \\ &= e^{-bt}\left(M_1(0) + a\int_0^T u(\tau)M_\alpha(\tau)e^{b\tau}\,d\tau\right) + e^{-bt}a(b\rho/a+\epsilon)\frac{e^{bt}-e^{bT}}{b}\,. \end{aligned}$$

Taking the limit $t \to \infty$ we obtain $\lim_{t\to\infty} M_1(t) \geq \rho + a\epsilon/b$, which is a contradiction. We can prove in a similar way that $\limsup_{t\to\infty} uM_\alpha < b\rho/a$ leads to a a contradiction. Thus our statement follows. $\square$

5.2 Numerical experiments and discussions

To approach numerically the Lifshitz-Slyozov equation we use a standard finite volume scheme with an upwind approximation of the fluxes. The behaviour of the solutions is depicted in Figures 1 to 3.

Figure 1 shows time evolution of the distribution for two distinct initial conditions with rates given by $\alpha = 1/3$ and $\beta = 2/3$, see details in the figure's legend. Note that we have no explicit solution at hand and the rate of convergence is unknown in this case. It seems that, roughly speaking, the particular details of the initial condition are lost as time advances and the concentration behaviour that ensues seems to follow a universal profile. Figure 2 shows the rates of convergence of u, and divergence of M_0, to be polynomial. We compare with the conjecture (15) and the results agree. In fact, this is robust according to various set of coefficients (results not shown). To further capture the limiting profile, we plot in Figure 3 the tail distribution $F(t,x) = \int_x^\infty f(t,x)dx$, while normalising the mass and the front speed. Specifically, choosing $x \longmapsto 1/(1+M_0(t))F(t, x/(1+M_0(t))$, we observe that several initial conditions lead to a similar asymptotic profile. However, we emphasize that our numerical scheme is not specifically designed to robustly capture a potential asymptotic profile, as it has been shown in [5] in the outgoing case. Using a sufficiently refined mesh, we hope that at least we capture correctly the rates of convergence.

6 Long-time behavior for proportional attachment and detachment rates

In this section we discuss the special case of $b(x) = \Phi_0 a(x)$ for some given $\Phi_0 > 0$. Particular instances of this situation for power-law rates (with outflow behavior or zero boundary conditions) have been studied in [6, 7]. Here we investigate the case of nucleation boundary conditions (3). For this section we still assume hypotheses (H0), (H2) and (H3). Note that hypothesis (H1) is replaced by Φ being constant. Here hypothesis (H4) is not required and hypothesis (H5) is replaced by

$$\mathfrak{n} \text{ is Lipschitz continuous on } [\Phi_0, \rho] \text{ and } \mathfrak{n}(z) \geq \mathfrak{n}(\Phi_0) = 0\,, \text{ for all } z \in [0, \rho]\,, \tag{H5'}$$

that is, nucleation cannot fuel anymore at the critical value. Finally, (H6) reduces to just

$$f^{\text{in}} \in L^1(\mathbb{R}_+, (1+x)dx)\,. \tag{H6'}$$

We shall assume morevover that $a \in \mathcal{C}^1(0, \infty)$, which entails that the characteristic curves are well-defined. Therefore, in this case of constant Φ, existence and uniqueness of global solutions is ensured, see [4].

Further, we let

$$A(x) = \int_0^x \frac{1}{a(y)} dy\,,$$

for all $x \geq 0$, which is an increasing $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}_+$ into $\mathbb{R}_+$ with $A(0) = 0$. We denote

$$M_a(t) = \int_0^\infty a(x) f(t, x) dx\,,$$

which is finite for all $t \geq 0$ by (7).

The main result of this section rules out concentration phenomena for the density and provides an explicit rate of convergence for u:

Theorem 6.1. *Assume $a \in \mathcal{C}^1(0, \infty)$ and $b(x) = \Phi_0 a(x)$. Under hypotheses* (H0)*,* (H2)*,* (H3)*,* (H5')*,* (H6')*, any global solution in the sense of Definition 2.1 satisfies:*

- $\lim_{t\to\infty} u(t) = \Phi_0$,
- *There exists $\overline{f} \in L^1(\mathbb{R}_+, (1+x)dx)$ such that*

$$\lim_{t\to+\infty} f(t, \cdot) = \overline{f}\,, \quad w - L^1(\mathbb{R}_+)\,.$$

Indeed, the limit $\overline{f}$ has a representation formula, given at end of the proof of Theorem 6.1, see (22). This representation depends noticeably on the chosen initial condition. Furthermore, if a is non-decreasing in x then M_a is increasing in t and our proof shows that the trend to equilibrium is exponential:

$$0 < u(t) - \Phi_0 \leq (u(0) - \Phi_0) e^{-M_a(0)t}\,.$$

Proof. We proceed in a number of separate steps. During the proof we assume $f^{\text{in}} \neq 0$. There is no loss of generality in so doing, as the given boundary conditions ensure that starting with $f^{\text{in}} = 0$ produces some nonvanising $f(t_0)$ for some $t_0 > 0$ small, which we may take as a new initial condition.

Step 1: Mild formulation. We will represent the solution in terms of characteristics. For that aim we will use several results from [4]. The equation determining the characteristics reads

$$\frac{d}{ds}X(s;t,x) = (u(s) - \Phi_0)a(X(s;t,x)); \quad X(t;t,x) = x. \tag{16}$$

For any given $x > 0$ we can ensure existence and uniqueness of a maximal solution $X(\cdot;t,x)$ on $(\sigma_t(x), \infty)$. Note the following: either $\sigma_t(x) = 0$ and $\lim_{s\to 0} X(s,t,x) > 0$, or $\sigma_t(x) > 0$ and $\lim_{s\to\sigma_t(x)} X(s,t,x) = 0$.

Therefore, for any $s \in (\sigma_t(x), +\infty)$ we can integrate (16) as follows:

$$X(s;t,x) = A^{-1}\left(A(x) + \int_t^s (u(\tau) - \Phi_0)d\tau\right).$$

We define $\gamma(t)$ for all $t \geq 0$ through

$$\gamma(t) := \int_0^t (u(\tau) - \Phi_0)d\tau\,, \text{ and } x_c(t) = A^{-1}(\gamma(t)). \tag{17}$$

The curve $x_c(t)$ corresponds to $X(t;0,0)$. In [4], it is proved for all $t > 0$, σ_t is a $\mathcal{C}^1$-diffeomorphism form $(0, x_c(t))$ into $(0,t)$ and $X(0;t,\cdot)$ is also a diffeomorphism from $(x_c(t),\infty)$ into $(0,\infty)$. These facts provide the following mild formulation, for any bounded $\varphi \in \mathcal{C}^0([0,\infty))$

$$\int_0^\infty f(t,x)\varphi(x)dx = \int_0^t \mathfrak{n}(u(s))\varphi(\sigma_t^{-1}(s))ds + \int_0^\infty f^{in}(x)\varphi(X(t;0,x))dx. \tag{18}$$

Step 2: $\gamma(t)$ is bounded. Note that

$$X(t;0,x) = A^{-1}\left(A(x) + \gamma(t)\right) \geq A^{-1}(\gamma(t)).$$

Hence, by equation (18) with $\varphi(x) = x$,

$$\rho \geq \int_0^\infty xf(t,x)dx \geq \int_0^\infty f^{in}(x)X(t;0,x)dx \geq A^{-1}(\gamma(t))\int_0^\infty f^{\text{in}}(x)dx,$$

which proves our claim.

Step 3: $u(t) \to \Phi_0$ as $t \to \infty$. Note that $\gamma'(t) = u(t) - \Phi_0$ and

$$u'(t) = -(u(t) - \Phi_0)M_a(t).$$

It follows that Φ_0 is a steady state for $u(t)$. Since $u(0) > \Phi_0$ and $M_a(t)$ is continuous and non-negative (with $M_a(t) = 0$ if and only if $f(t) = 0$, that is, if and only if $u(t) = \rho$), we obtain that $u(t)$ decreases and $u(t) > \Phi_0$ for all $t \geq 0$. Thus u has a limit, which is Φ_0.

Step 4: $\gamma(t)$ is strictly increasing and has a limit as $t \to \infty$. The previous steps ensure that γ is (strictly) increasing, because $\gamma'(t) = u(t) - \Phi_0 > 0$, and γ is bounded, so γ has a limit

$$\overline{\gamma} = \lim_{t\to\infty} \gamma(t)\,. \tag{19}$$

Step 5: $x_c(t)$ has a limit as $t \to \infty$. Combining the definition of $x_c(t) = A^{-1}(\gamma(t))$, the continuity of A^{-1} and the limit of γ, we obtain that $x_c(t)$ has a limit,

$$\overline{x}_c = \lim_{t\to\infty} x_c(t) = A^{-1}(\overline{\gamma})\,. \tag{20}$$

Step 6: $X(t;0,\cdot)$ has a limit as $t \to \infty$. We remark that $X(t;0,x) = A^{-1}(A(x) + \gamma(t))$ for $x > 0$ which entails that $X(t;0,x)$ has a limit too, since A is $\mathcal{C}^1$-diffeomorphism, and

$$\overline{X}(x) = \lim_{t\to\infty} X(t;0,x) = A^{-1}(A(x) + \overline{\gamma})\,. \tag{21}$$

This makes clear that $\overline{X}$ is a $\mathcal{C}^1$-diffeomorphism from $(0,\infty)$ into $(\overline{x}_c,\infty)$.

Step 7: σ_t^{-1} has a limit as $t \to \infty$. Recall from Step 1 that $\lim_{s\to\sigma_t(x)} X(s,t,x) = 0$. Using the fact that $\lim_{x\to\sigma_t^{-1}(s)} X(s;t,x) = 0$ -see [4]- we deduce that

$$\sigma_t^{-1}(s) = A^{-1}\left(\int_s^t (u(\tau) - \Phi_0)d\tau\right) = A^{-1}(\gamma(t) - \gamma(s))\,.$$

and thus we have the limit

$$\overline{\sigma}^{-1}(s) = \lim_{t\to\infty} \sigma_t^{-1}(s) = A^{-1}(\overline{\gamma} - \gamma(s))\,.$$

We conclude that $\overline{\sigma}(x) = \gamma^{-1}(\overline{\gamma} - A(x))$ is a $\mathcal{C}^1$-diffeomorphism from $(0,\overline{x}_c)$ into $(0,t)$, with reciprocal $\overline{\sigma}^{-1}$.

Step 8: The limit density. Let $\varphi \in \mathcal{C}^0([0,\infty))$ be bounded; we insert it in equation (18). By the limit in Step 7 and the dominated convergence theorem, we conclude that

$$\begin{aligned}\lim_{t\to\infty} \int_0^\infty f(t,x)\varphi(x)dx &= \int_0^\infty \mathfrak{n}(u(s))\varphi(\overline{\sigma}^{-1}(s))ds + \int_0^\infty f^{in}(x)\varphi(\overline{X}(x))dx \\ &= \int_0^{\overline{x}_c} \mathfrak{n}(u(\overline{\sigma}(x)))|\overline{\sigma}'(x)|\varphi(x)dx + \int_{\overline{x}_c}^\infty f^{in}(\overline{X}^{-1}(x))|\overline{X}^{-1}{}'|\varphi(x)dx \end{aligned} \tag{22}$$

which proves the desired result. □

The result of Theorem 6.1, and in particular formula (22), is in line with [6] in the case $\mathfrak{n} = 0$ and a and b power-law functions, where semi-explicit expressions are available for $\overline{x}_c, \overline{\gamma}$ and $\overline{X}$.

Acknowledgements: J. C. is partially supported by the spanish MINECO-Feder (grant RTI2018-098850-B-I00) and Junta de Andalucía (grants PY18-RT-2422 and A-FQM-311-UGR18). E. H. and R. Y. have been supported by ECOS-Sud project n. C20E03 (France – Chile) and acknowledge financial support from the Inria Associated team ANACONDA.

References

[1] M. F. Bishop and F. A. Ferrone. Kinetics of nucleation-controlled polymerization. a perturbation treatment for use with a secondary pathway. *Biophys. J.*, 46:631–644, 1984.

[2] H. Brezis. Functional analysis, sobolev spaces and partial differential equations. Springer New York, NY, 2010.

[3] J. Calvo, M. Doumic, and B. Perthame. Long-time asymptotics for polymerization models. *Commun. Math. Phys.*, 363(1):111–137, 2018.

[4] J. Calvo, E. Hingant, and R. Yvinec. The initial-boundary value problem for the Lifshitz-Slyozov equation with non-smooth rates at the boundary. *Nonlinearity*, 34(4):1975–2017, 2021.

[5] J. A. Carrillo and T. Goudon. A numerical study on large-time asymptotics of the Lifshitz–Slyozov system. *J. Sci. Comput.*, 20(1):69–113, 2004.

[6] J.-F. Collet and T. Goudon. On solutions of the Lifshitz-Slyozov model. *Nonlinearity*, 13(4):1239–1262, 2000.

[7] J.-F. Collet, T. Goudon, S. Hariz, F. Poupaud, and A. Vasseur. Some Recent Results on the Kinetic Theory of Phase Transitions. In N. B. Abdallah, I. M. Gamba, C. Ringhofer, A. Arnold, R. T. Glassey, P. Degond, and C. D. Levermore, editors, *Transport in Transition Regimes*, volume 135, pages 103–120, New York, NY, 2004. Springer New York.

[8] J.-F. Collet, T. Goudon, F. Poupaud, and A. Vasseur. The Becker-Döring system and its Lifshitz-Slyozov limit. *SIAM J. Appl. Math.*, 62(5):1488–1500, 2002.

[9] J.-F. Collet, T. Goudon, and A. Vasseur. Some remarks on large-time asymptotic of the Lifshitz-Slyozov equations. *J. Stat. Phys.*, 108(1-2):341–359, 2002.

[10] J. G. Conlon. On a diffusive version of the Lifschitz-Slyozov-Wagner equation. *J. Nonlinear Sci.*, 20(4):463–521, 2010.

[11] J. G. Conlon and A. Schlichting. A non-local problem for the Fokker-Planck equation related to the Becker-Döring model. *Discrete Contin Dyn Syst Ser A*, 39(4):1821–1889, 2019.

[12] J. Deschamps, E. Hingant, and R. Yvinec. Quasi steady state approximation of the small clusters in Becker-Döring equations leads to boundary conditions in the Lifshitz-Slyozov limit. *Commun. Math. Sci.*, 15(5):1353–1384, 2017.

[13] M. Doumic, K. Fellner, M. Mezache, and H. Rezaei. A bi-monomeric, nonlinear Becker–Döring-type system to capture oscillatory aggregation kinetics in prion dynamics. *J. Theor. Biol.*, 480:241–261, 2019.

[14] S. Hariz and J. F. Collet. A modified version of the Lifshitz-Slyozov model. *Appl. Math. Lett.*, 12(1):81–85, 1999.

[15] P. Laurençot. Weak solutions to the Lifshitz-Slyozov-Wagner equation. *Indiana Univ. Math. J.*, 50(3):1319–1346, 2001.

[16] P. Laurençot and S. Mischler. From the Becker-Döring to the Lifshitz-Slyozov-Wagner equations. *J. Stat. Phys.*, 106(5-6):957–991, 2002.

[17] P. Laurençot. Weak Compactness Techniques and Coagulation Equations. In *Evolutionary Equations with Applications in Natural Sciences*, Lecture Notes in Mathematics, pages 199–253. Springer, Cham, 2015.

[18] G. Lauretano and R. Spigler. On the Relation between the Lifshitz–Slyozov and the Lifshitz–Slyozov–Wagner Models for Supersaturated Solutions. *SIAM J. Appl. Math.*, 81(2):507–529, 2021.

[19] I. M. Lifshitz and V. V. Slyozov. The kinetics of precipitation from supersaturated solid solutions. *J. Phys. Chem. Solids*, 19(1):35–50, 1961.

[20] M. Marder. Correlations and Ostwald ripening. *Phys. Rev. A*, 36(2):858–874, 1987.

[21] B. Meerson. Fluctuations provide strong selection in Ostwald ripening. *Phys. Rev. E*, 60(3):3072–3075, 1999.

[22] B. Niethammer. On the evolution of large clusters in the Becker-Döring model. *J. Nonlinear Sci.*, 13(1):115–155, 2003.

[23] B. Niethammer and R. L. Pego. On the initial value problem in the Lifshitz-Slyozov-Wagner theory of Ostwald ripening. *SIAM J. Math. Anal.*, 31(3):467–485, 2000.

[24] B. Niethammer and R. L. Pego. Well-posedness for measure transport in a family of nonlocal domain coarsening models. *Indiana Univ. Math. J.*, 54(2):499–530, 2005.

[25] B. Niethammer, R. L. Pego, A. Schlichting, and J. J. L. Velázquez. Oscillations in a Becker–Döring Model with Injection and Depletion. *SIAM J. Appl. Math.*, 82(4):1194–1219, 2022.

[26] R. L. Pego and J. J. L. Velázquez. Temporal oscillations in Becker–Döring equations with atomization. *Nonlinearity*, 33(4):1812–1846, 2020.

[27] S. Prigent, A. Ballesta, F. Charles, N. Lenuzza, P. Gabriel, L. M. Tine, H. Rezaei, and M. Doumic. An Efficient Kinetic Model for Assemblies of Amyloid Fibrils and Its Application to Polyglutamine Aggregation. *PLoS One*, 7(11):e43273, 2012.

[28] C. A. Ross and M. A. Poirier. Protien aggregation and neurodegenerative disease. *Nat. Med.*, 10:S10–S17, 2004.

[29] E. D. Ross, A. Minton, and R. B. Wickner. Prion domains: sequences, structures and interactions. *Nat. Cell Biol.*, 7(11):1039–1044, 2005.

[30] A. Schlichting. Macroscopic limit of the Becker-Döring equation via gradient flows. *ESAIM, Control Optim. Calc. Var.*, 25:36, 2019.

[31] J. Szavits-Nossan, K. Eden, R. J. Morris, C. E. MacPhee, M. R. Evans, and R. J. Allen. Inherent variability in the kinetics of autocatalytic protein self-assembly. *Phys. Rev. Lett.*, 113(9):098101, 2014.

[32] J. J. L. Velázquez. The Becker-Döring equations and the Lifshitz-Slyozov theory of coarsening. *J. Stat. Phys.*, 92(1-2):195–236, 1998.

[33] W.-F. Xue, S. W. Homans, and S. E. Radford. Systematic analysis of nucleation-dependent polymerization reveals new insights into the mechanism of amyloid self-assembly. *Proc. Natl. Acad. Sci. U.S.A.*, 105(26):8926–8931, 2008.

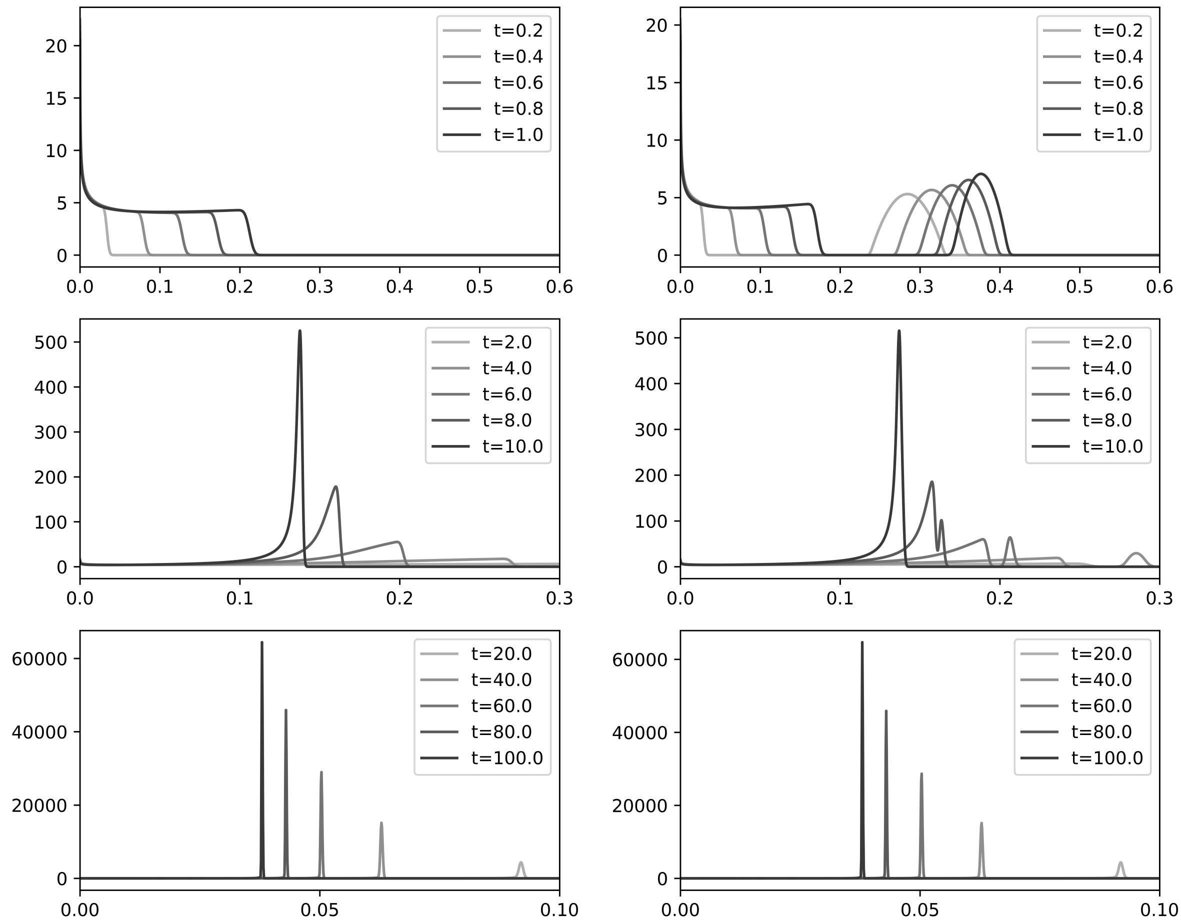

Figure 1. Distribution $f(t,x)$ with respect to x at different times t. The rates are given by $a(x) = x^{1/3}$, $b(x) = x^{2/3}$ and $\mathfrak{n}(z) = z^2$. The total mass is $\rho = 1$. Simulations were performed with a finite volume scheme (upwind), with $\Delta t = 5 \cdot 10^{-5}$ and $\Delta x = 10^{-4}$. Left column: initial condition is $f^{\text{in}} = 0$; Right column: $f^{\text{in}}(x) = (-2000(x-0.2)(x-0.3))_+$.

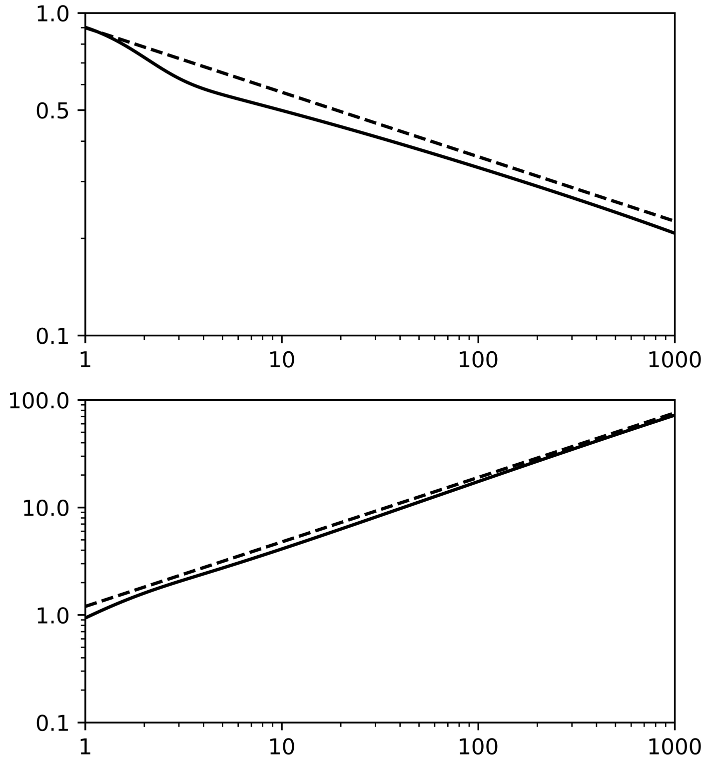

Figure 2. Up: $u(t)$ versus time t in abscissa (straight line) and $t^{-1/5}$ (dashed line). Down: $M_0(t)$ versus time t in abscissa (straight line) and $t^{3/5}$ (dashed line). Graphics are shown in log-log scale. Parameters are the same as Fig. 1 with $f^{\mathrm{in}} = 0$.

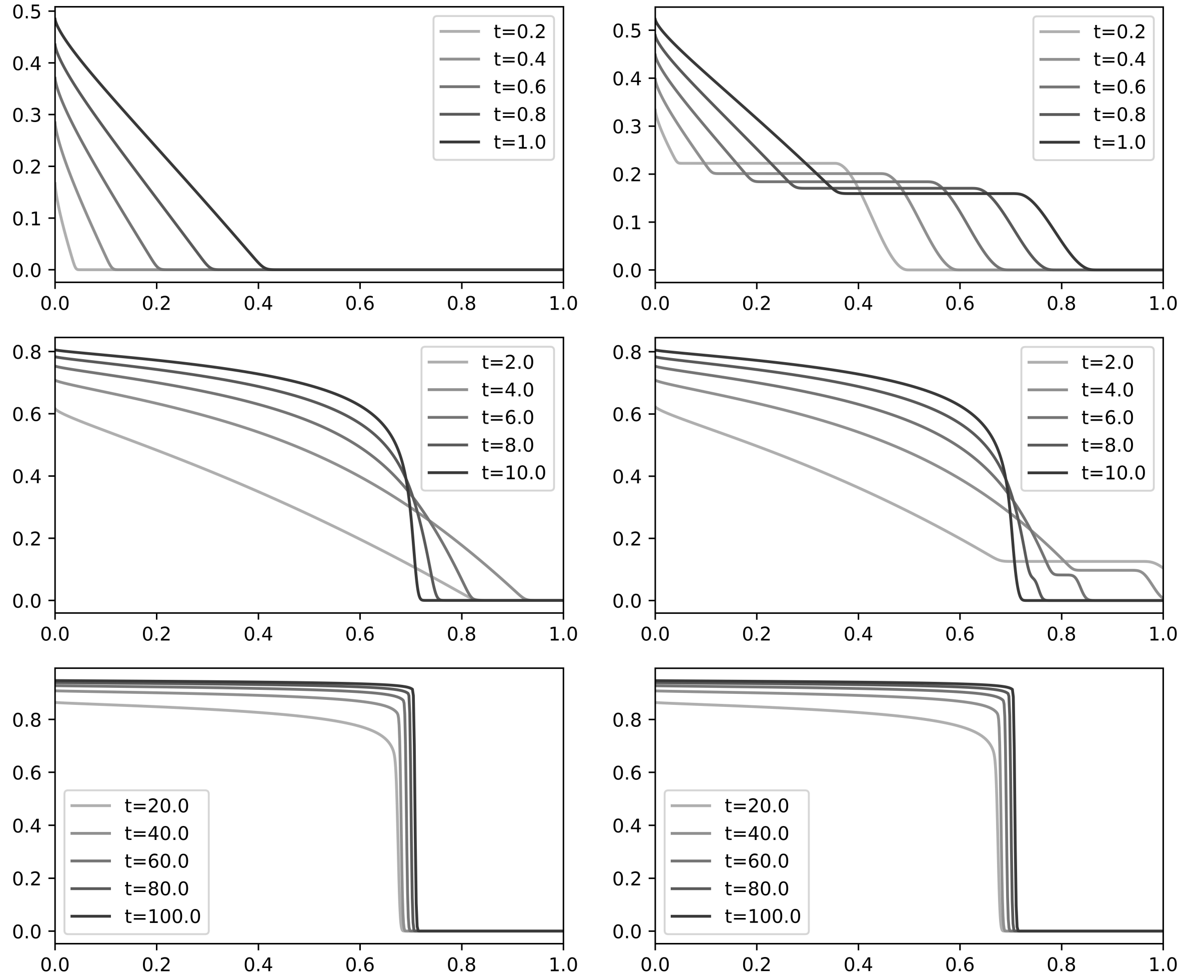

Figure 3. Normalised distribution $x \longmapsto 1/(1+M_0(t))F(t, x/(1+M_0(t))$ with $F(t,x) = \int_x^\infty f(t,x)dx$, with respect to x at different times t. Parameters and simulations are performed as Fig. 1. Left column: initial condition is $f^{\text{in}} = 0$; Right column: $f^{\text{in}}(x) = (-2000(x-0.2)(x-0.3))_+$.