Some Bourgain-Brézis type solutions via complex interpolation

Eduard Curcă

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May 30, 2023

Abstract

In 2002 Bourgain and Brézis proved that given a vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ such that $\operatorname{div} u = \operatorname{div} v$. We prove several results of a similar nature in which we take into consideration the Fourier support of the solutions. For instance, in the case $d \geq 3$ we prove the following: for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ (where $p \in [2,\infty)$ and $q \in (1,2)$), with $supp\ \widehat{v} \subseteq \mathbb{R}^d \backslash (-\infty,0)^d$, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$, with $supp\ \widehat{v} \subseteq \mathbb{R}^d \backslash (-\infty,0)^d$, such that

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}}.$$

Our arguments rely on a version of the complex interpolation method combined with some ideas of Bourgain and Brézis.

Contents

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1 Introduction

1.1 Overview

Suppose $d \geq 2$ is an integer and consider some compactly supported function $f \in L^d\left(\mathbb{R}^d\right)$. Standard Calderón-Zygmund theory shows that there exists a vector field $u \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{W}^{1,d}\left(\mathbb{R}^d\right)$ such that

$$\operatorname{div} u = f, \tag{1}$$

in the distributions sense on $\mathbb{R}^d$. Indeed, it sufices to set

$$u := \nabla|\nabla|^{-2} f, \tag{2}$$

and to use the fact that the components of ∇u are of the form $R_i R_j u$ where $R_1, ..., R_d$ are the Riesz transforms on $\mathbb{R}^d$ ($\widehat{R_j\varphi}(\xi) = (\xi_j/|\xi|)\widehat{\varphi}(\xi)$, for any Schwartz φ). Since each R_j is a Calderón-Zygmund operator, we easily get that each component of ∇u belongs to $L^d(\mathbb{R}^d)$.

Note that the space $\dot{W}^{1,d}$ does not embed in L^∞ and hence, the solution in $\dot{W}^{1,d}$ provided by the expression (2) may fall outside L^∞ (see for instance the example given by L. Nirenberg in [5, Remark 7, p. 400]). However, as it was shown[1] by Bourgain and Brézis (2002), the fact that (1) admits a (possibly another) solution $u \in L^\infty(\mathbb{R}^d)$ is a direct consequence of the Gagliardo embedding $(W^{1,1}(\mathbb{R}^d) \hookrightarrow L^{d'}(\mathbb{R}^d)$, where $d' := d/(d-1))$ (see [5, Proposition 1]).

Even more, Bourgain and Brézis have proved in [5, Theorem 1] the following striking fact: there exists a solution u to (1) that is *simultaneously* bounded and in the "right" Sobolev space $\dot{W}^{1,d}(\mathbb{R}^d)$. In other words, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ which is a solution to (1). In the general case where $d \geq 2$, this result was proved by an involved approximation argument using the Littlewood-Paley square function. We mention that the complicated construction used in [5] can also be used in more general situations. By similar constructive methods, Bourgain and Brézis proved an analogue existence result for more general underdeterminated Hodge

Keywords: Bourgain-Brézis solutions, Fourier multipliers, Divergence equation.
MSC 2020 classification: 42B15, 42B35, 46B70

[1] The results of Bourgain and Brézis were stated in the case of the torus $\mathbb{T}^d$, however, it is easy to transfer these results on $\mathbb{R}^d$.

systems. Following the ideas in [5] and [6], Bousquet, Mironescu and Russ ([7], 2014) and later Bousquet, Russ, Wang and Yung ([8], 2017) provided generalizations of the Bourgain-Brézis results in the scale of Triebel-Lizorkin spaces. For instance, adapted to the case of the divergence equation, Theorem 2 in [8] gives us:

Theorem 1. *Suppose that $1 < p, q < \infty$ and consider some vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$. Then, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{F}_q^{d/p,p}} \lesssim \|v\|_{\dot{F}_q^{d/p,p}}.$$

Remark 2. *Note that by Calderón-Zygmund theory any (compactly supported) $f \in \dot{F}_q^{d/p-1,p}(\mathbb{R}^d)$ can be written as the divergence of the vector field $v = \nabla |\nabla|^{-2} f \in \dot{F}_q^{d/p,p}(\mathbb{R}^d)$. Hence, since $\dot{F}_2^{1,d} = \dot{W}^{1,d}$, when $p = d$ and $q = 2$, from Theorem 1 above we recover the result of Bourgain and Brézis.*

A similar existence result holds for the scale of Besov spaces, the proof being technically the same as for Theorem 1. Since in this paper we are concerned more with the Besov version, we explicitely state it below:

Theorem 3. *Suppose that $1 < p, q < \infty$ and consider some vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$. Then, there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ that satisfies*

$$\operatorname{div} u = \operatorname{div} v,$$

and such that

$$\|u\|_{L^\infty \cap \dot{B}_q^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}}.$$

Remark 4. *Throughout the paper we will call the given vector v the source and u will be called solution. A similar convention will also be applied to more general equations.*

It is worth noticing that in the special case where $d = 2$ (and hence, $p = 2$) Bourgain and Brézis have found a much simpler proof of their existence result (see [5, Section 4, p. 403]). In this case the proof is by duality and it is nonconstructive. Also, by similar methods, a proof was found by Mazya (2007) for the case $p = q = 2$ of Theorem 1 (or equivalently of Theorem 3) (see [16]). Again, the proof is by duality and strikingly simple. (See also [17] for some related discussions.) However, both approaches, namely that of Bourgain and Brézis in the case $d = 2$ and that of Mazya are based on L^2-Fourier analysis arguments that are unlikely to be extended to the case where $p \neq 2$.

There is yet another situation of a different nature:

Proposition 5. *Let $d \geq 2$ be an integer and consider some $r \in (1, \infty)$. Then, for any vector field $v \in \dot{B}_1^{d/r,r}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_1^{d/r,r}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_1^{d/r,r}} \lesssim \|v\|_{\dot{B}_1^{d/r,r}}.$$

Indeed, since $\dot{B}_1^{d/r,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \cap \dot{B}_1^{d/r,r}(\mathbb{R}^d)$, it suffices to set $u := v$ (see subsection 2.1 for the definition of $\dot{B}_1^{d/r,r}(\mathbb{R}^d)$ we use in this paper). Since Proposition 5 and Theorem 3 have much easier proofs than the constructive proof used in [5] and in [7], it would be interesting to find a way to "interpolate" between Proposition 5 and Theorem 3.

1.2 A naive interpolation strategy

One may try to interpolate in the following way. By the closed range theorem and the lifting property of the Besov spaces we can reformulate Proposition 5 and Theorem 3

$$\|g\|_{\dot{B}^{-d/r+1,r'}_{\infty}} \sim \|\nabla g\|_{\dot{B}^{-d/r,r'}_{\infty}} \lesssim \|\nabla g\|_{L^1+\dot{B}^{-d/r,r'}_{\infty}}\,, \tag{3}$$

and respectively,

$$\|g\|_{\dot{W}^{-d/2,2}} \sim \|\nabla g\|_{\dot{W}^{-d/2,2}} \lesssim \|\nabla g\|_{L^1+\dot{W}^{-d/2,2}}\,, \tag{4}$$

for any Schwartz function g with $\widehat{g}$ vanishing in a neighborhood of 0.

For each Banach function space Y on $\mathbb{R}^d$ denote by $\mathcal{G}(Y)$ the space of all vector fields in Y that are gradients, i.e.,

$$\mathcal{G}(Y) := \{g \in Y \mid \operatorname{curl} g = 0\}\,.$$

With this notation one can view (3) and (4) as the embeddings

$$\mathcal{G}(L^1 + \dot{B}^{-d/r,r'}_{\infty}) \hookrightarrow \dot{B}^{-d/r+1,r'}_{\infty},$$

and respectively,

$$\mathcal{G}(L^1 + \dot{W}^{-d/2,2}) \hookrightarrow \dot{W}^{-d/2+1,2}.$$

By complex interpolation (we may consider the real interpolation as well), we conclude that, for any $\theta \in (0,1)$,

$$(\mathcal{G}(L^1 + \dot{B}^{-d/r,r'}_{\infty}), \mathcal{G}(L^1 + \dot{W}^{-d/2,2}))_\theta \hookrightarrow (\dot{B}^{-d/r+1,r'}_{\infty}, \dot{W}^{-d/2+1,2})_\theta. \tag{5}$$

The right hand side of (5) can be easily computed explicitly (see for instance [2, Chapter 6]):

$$(\dot{B}^{-d/r+1,r'}_{\infty}, \dot{W}^{-d/2+1,2})_\theta = \dot{B}^{-d/p+1,p'}_{q'},$$

where $1/p = (1-\theta)/r + \theta/2$ and $1/q = (1-\theta)/1 + \theta/2$. Now we would like to have

$$\mathcal{G}(L^1 + \dot{B}^{-d/p,p'}_{q'}) \hookrightarrow (\mathcal{G}(L^1 + \dot{B}^{-d/r,r'}_{\infty}), \mathcal{G}(L^1 + \dot{W}^{-d/2,2}))_\theta, \tag{6}$$

and combining this with (5), we would get via the closed range theorem the fact that for any vector field $v \in \dot{B}^{d/p,p}_{q}(\mathbb{R}^d)$ there exists another vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}^{d/p,p}_{q}(\mathbb{R}^d)$ of the same divergence as v. However, computing explicitly the left hand side of (5) or proving (6) only by interpolation theory is a quite difficult task. Naively we may have the following strategy for proving (6). We can observe that, since we have the embeddings $L^1, \dot{B}^{-d/r,r'}_{\infty} \hookrightarrow L^1 + \dot{B}^{-d/r,r'}_{\infty}$ and $L^1, \dot{W}^{-d/2,2} \hookrightarrow L^1 + \dot{W}^{-d/2,2}$, we can conclude by interpolation that

$$L^1, \dot{B}^{-d/p,p'}_{q'} \hookrightarrow (L^1 + \dot{B}^{-d/r,r'}_{\infty}, L^1 + \dot{W}^{-d/2,2})_\theta$$

or equivalently

$$L^1 + \dot{B}^{-d/p,p'}_{q'} \hookrightarrow (L^1 + \dot{B}^{-d/r,r'}_{\infty}, L^1 + \dot{W}^{-d/2,2})_\theta.$$

Consequently,

$$\mathcal{G}(L^1 + \dot{B}^{-d/p,p'}_{q'}) \hookrightarrow \mathcal{G}(L^1 + \dot{B}^{-d/r,r'}_{\infty}, L^1 + \dot{W}^{-d/2,2}_{\theta}).$$

Hence, in order to obtain (6) it would be sufficient to have

$$\mathcal{G}((L^1 + \dot{B}^{-d/r,r'}_{\infty}, L^1 + \dot{W}^{-d/2,2})_\theta) \hookrightarrow (\mathcal{G}(L^1 + \dot{B}^{-d/r,r'}_{\infty}), \mathcal{G}(L^1 + \dot{W}^{-d/2,2}))_\theta,$$

or, equivalently, since the other embedding is trivial,

$$\mathcal{G}((L^1+\dot{B}^{-d/r,r'}_{\infty},L^1+\dot{W}^{-d/2,2})_\theta)=(\mathcal{G}(L^1+\dot{B}^{-d/r,r'}_{\infty}),\mathcal{G}(L^1+\dot{W}^{-d/2,2}))_\theta. \tag{7}$$

We can further reformulate this fact as

$$\mathcal{N}\cap(Y_0,Y_1)_\theta=(\mathcal{N}\cap Y_0,\mathcal{N}\cap Y_1)_\theta\,, \tag{8}$$

where $Y_0=L^1+\dot{B}^{-d/r,r'}_{\infty}$, $Y_1=L^1+\dot{W}^{-d/2,2}$ and $\mathcal{N}$ is the spaces of the fields in Y_0+Y_1 that are gradients, i.e, $\mathcal{N}:=\mathcal{G}(Y_0+Y_1)$. The difficulty of proving (7) consists in the fact that, for the general situation when Y_0, Y_1, $\mathcal{N}$ are Banach spaces the question whether or not (8) holds does not have yet a satisfactory answer. If we replace the complex interpolation method in (8) with the real K-method of interpolation, then, (8) may be false for some particular choice of the spaces Y_0, Y_1, $\mathcal{N}$ (see for instance [15]). In the case of the complex interpolation method it seems that even less is known when (8) is valid.

This naive interpolation strategy seems inappropriate to prove (6) or even a weaker statement like

$$\mathcal{G}(L^1+\dot{B}^{-d/p,p'}_{1})\hookrightarrow(\mathcal{G}(L^1+\dot{B}^{-d/r,r'}_{\infty}),\mathcal{G}(L^1+\dot{W}^{-d/2,2}))_\theta,$$

which corresponds to the following existence result:

Proposition 6. *Let $d\geq 2$ be an integer and consider some parameters $p\in(2,\infty)$ and $q\in(1,2)$. Then, for any vector field $v\in\mathcal{S}'(\mathbb{R}^d)\cap\dot{B}^{d/p,p}_{q}(\mathbb{R}^d)$ there exists a vector field $u\in L^\infty(\mathbb{R}^d)\cap\dot{B}^{d/p,p}_{\infty}(\mathbb{R}^d)$ such that*

$$\operatorname{div}u=\operatorname{div}v,$$

and

$$\|u\|_{L^\infty\cap\dot{B}^{d/p,p}_{\infty}}\lesssim\|v\|_{\dot{B}^{d/p,p}_{q}}.$$

1.3 The main results

In this paper we take into consideration the spectrum of solutions. In what follows, the spectrum of a tempered distribution v is its Fourier support, i.e., $spec(v):=supp\ \widehat{\triangle v}$ (see Remark 14 for a more general definition). Adapted to the case of the divergence equation, our main result reads:

Theorem 7. *Let $d\geq 3$ be an integer and consider the set $\Delta:=\mathbb{R}^d\backslash(-\infty,0)^d$. Consider some parameters $p\in[2,\infty)$ and $q\in(1,2)$. Then, for any vector field $v\in\mathcal{S}'(\mathbb{R}^d)\cap\dot{B}^{d/p,p}_{q}(\mathbb{R}^d)$, with $spec(v)\subseteq\Delta$ there exists a vector field $u\in L^\infty(\mathbb{R}^d)\cap\dot{B}^{d/p,p}_{2}(\mathbb{R}^d)$, with $spec(u)\subseteq\Delta$ such that*

$$\operatorname{div}u=\operatorname{div}v, \tag{9}$$

and

$$\|u\|_{L^\infty\cap\dot{B}^{d/p,p}_{2}}\lesssim\|v\|_{\dot{B}^{d/p,p}_{q}}.$$

In the case where $d=2$ our method does not provide solutions with the spectrum in Δ. Nevertheless, one can obtain solutions with the spectrum in a different type of sets. For each $\delta\in(0,\pi/4)$ let C_δ be the symmetric cone

$$C_\delta:=\{(\xi_1,\xi_2)\in\mathbb{R}^2\ \mid\ |\xi_1|\leq(\tan\delta)\,|\xi_2|\}.$$

With this notation we have:

Theorem 8. *Consider the numbers $\delta \in (0, \pi/8)$, $\varepsilon \in (0, 1]$ and some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^2) \cap \dot{B}_q^{2/p,p}(\mathbb{R}^2)$, with $spec(v) \subseteq C_\delta$, there exists a vector field $u \in L^\infty(\mathbb{R}^2) \cap \dot{B}_2^{2/p,p}(\mathbb{R}^2)$, with $spec(u) \subseteq C_{(1+\varepsilon)\delta}$, such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{2/p,p}} \lesssim \|v\|_{\dot{B}_q^{2/p,p}}.$$

When compared with Theorem 3 one can observe that Theorem 7 (or Theorem 8) has two major drawbacks. First, we are not alowed to take $p < 2$ or $q \geq 2$ as parameters for the space $\dot{B}_q^{d/p,p}$ on the source side. Secondly, for the space in which we obtain the solution we lose some control of the "third parameter". In other words, we would prefer to obtain $L^\infty \cap \dot{B}_q^{d/p,p}$ for the solution space instead of $L^\infty \cap \dot{B}_2^{d/p,p}$ which is a slighlty larger space. (See however, Lemma 35 for a "perfect" version of our results in the case $p = q = 2$.) On the other hand it is unlikely that one can deduce Theorem 7 directly from Theorem 3. Indeed, given a vector field $v \in \dot{B}_q^{d/p,p}$, with $spec(v) \subseteq \Delta$, by Theorem 3 one can find some vector field $u \in L^\infty \cap \dot{B}_q^{d/p,p}$ such that $\operatorname{div} u = \operatorname{div} v$, however, not necessarly with $spec(u) \subseteq \Delta$. It is not obvious that one can obtain a solution u with $spec(u) \subseteq \Delta$ by direct methods: suppose P_Δ is the Fourier projection on Δ, i.e., $P_\Delta = I - P_+$, where P_+ is the Riesz projection and I is the identity operator. We have $P_\Delta v = v$ and we can write

$$\operatorname{div} P_\Delta u = \operatorname{div} v.$$

However, since P_D is not bounded on L^∞, we may *not* have $P_\Delta u \in L^\infty$, i.e., $P_\Delta u$ is *not* in general a candidate for a solution. The same observation applies to or Theorem 8. To our knowledge, except for the method we give in this paper, there is no other method in the literature able to prove results like Theorem 7 or Theorem 8.

In fact, when $d \geq 3$, we will prove a more general result than Theorem 7. Our methods alow us to work with more general Fourier multipliers than the usual derivatives. In order to formulate our result we first need some preparations.

Let $\sigma \in C^2(\mathbb{R}^d, \mathbb{R})$ be a function. We consider the following properties (that may or not be satisfied by σ):

(P1) The function σ satisfies the estimate

$$|\nabla^\alpha \sigma(\xi)| \lesssim |\xi|^{1-|\alpha|},$$

on $\mathbb{R}^d$, for any multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 2$;

(P2) The function σ is odd in the variable ξ_1 and even in any other variable, i.e.,

$$\sigma(\epsilon_1 \xi_1, \epsilon_2 \xi_2, ..., \epsilon_d \xi_d) = \epsilon_1 \sigma(\xi_1, \xi_2, ..., \xi_d),$$

on $\mathbb{R}^d$, for any signs $\epsilon_1, ..., \epsilon_d \in \{-1, 1\}$.

Introduce the new functions $\sigma_1, ..., \sigma_d$ defined by

$$\sigma_j(\xi_1, \xi_2, ..., \xi_d) := \sigma(\xi_j, \xi_2, ...\xi_{j-1}, \xi_1, \xi_{j+1}..., \xi_d),$$

on $\mathbb{R}^d$, for any index $j \in \{1, 2, ..., d\}$.

Consider some half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$ and a family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$. We say that the function G_j is *adapted* to the half-space D_j if there exists a rotation R on $\mathbb{R}^d$ (depending on j) and a function $\sigma : \mathbb{R}^d \to \mathbb{R}$ (depending on j) satisfying (P1), (P2) such that $D_j = R(U)$, where $U := \mathbb{R}^{d-1} \times (0, \infty)$, and

$$G_j = (\sigma_1 \circ R, ..., \sigma_{d-1} \circ R).$$

We say that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is *adapted* to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$, if for each $j \in \{1, ..., d\}$ the function G_j is adapted to D_j and

$$\sum_{j=1}^{d} |G_j(\xi)| \, \mathbf{1}_{D_j}(\xi) \sim |\xi| \, \mathbf{1}_D(\xi), \tag{10}$$

on $\mathbb{R}^d$, where $D := \cup_{j=1}^d D_j$.

Let us recall now some standard notation concerning the Fourier multipliers. To a scalar valued function $m \in L^1_{loc}(\mathbb{R}^d \backslash \{0\}, \mathbb{R})$ we associate the Fourier multiplier $m(\nabla)$ defined by the relation

$$\widehat{m(\nabla)f}(\xi) := m(\xi)\widehat{f}(\xi),$$

on $\mathbb{R}^d$, for any Schwartz function f whose Fourier transform $\widehat{f}$ is compactly supported and vanishing in a neighborhood of 0. In most of the cases one can extend the meaning of $m(\nabla)$ as follows. Let us denote by $\mathcal{S}_{c,\sharp}$ the space of all Schwartz function f whose Fourier transform $\widehat{f}$ is compactly supported and vanishing in a neighborhood of 0. Suppose E and F are some Banach function spaces on $\mathbb{R}^d$ such that $\mathcal{S}_{c,\sharp}$ is dense in E and

$$\|m(\nabla)f\|_F \lesssim \|f\|_E,$$

for any $f \in \mathcal{S}_{c,\sharp}$. Then, by linearity and density $m(\nabla)$ can be uniquely extended to a bounded operator $m(\nabla) : E \to F$ (see also Remark 13). We will often say that m is the symbol of the Fourier multiplier $m(\nabla)$.

To a vector valued function $G : \mathbb{R}^d \to \mathbb{R}^{d-1}$, with $G = (G^1, ..., G^{d-1})$, where $G^1, ..., G^{d-1} : \mathbb{R}^d \to \mathbb{R}$ are scalar functions of polynomial growth, we associate the vector-valued Fourier multiplier

$$G(\nabla) := (G^1(\nabla), ..., G^{d-1}(\nabla)).$$

In other words, if $f \in \mathcal{S}_{c,\sharp}$, by $G(\nabla)f$ we mean

$$G(\nabla)f := (G^1(\nabla)f, ..., G^{d-1}(\nabla)f).$$

Suppose $u^1, ..., u^{d-1} \in \mathcal{S}_{c,\sharp}$ and let u be the $(d-1)$-vector field[2] $u := (u^1, ..., u^{d-1})$. By $G(\nabla) \cdot u$ we mean

$$G(\nabla) \cdot u := G^1(\nabla)u^1 + ... + G^{d-1}(\nabla)u^{d-1}.$$

Now we can formulate our generalisation of Theorem 7.

[2]In this paper the distributions with $d-1$ components will be called $(d-1)$-vector fields.

Theorem 9. *Let $d \geq 3$ be an integer and consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Suppose that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is adapted to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$.*

Then, for any system of $(d-1)$-vector fields $(v_j)_{j=1,..,d}$ with $v_j \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ and $spec(v_j) \subseteq D_j$, there exists a system of $(d-1)$-vector fields $(u_j)_{j=1,..,d}$, with $u_j \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$ and $spec(u_j) \subseteq D_j$, such that

$$\sum_{j=1}^{d} G_j(\nabla) \cdot u_j = \sum_{j=1}^{d} G_j(\nabla) \cdot v_j, \tag{11}$$

and

$$\sum_{j=1}^{d} \|u_j\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \sum_{j=1}^{d} \|v_j\|_{\dot{B}_q^{d/p,p}}.$$

In this paper equations such as (9) or (11) will be called *divergence-like* equations.

In subsection 5.1 we will see that Theorem 9 easily implies Theorem 7.

1.4 About the proofs

Our proofs of Theorem 9, Theorem 7 and Theorem 8 are based on two ingredients:

1. The $\mathcal{W}$-method of interpolation. This is the key method that we are using throughout the paper. Let (A_0, A_1) and (B_0, B_1) be Banach couples and $T : A_0 + A_1 \to B_0 + B_1$ be a linear operator such that $T(B_j) \hookrightarrow T(A_j)$, for any $j = 0, 1$. Suppose we want to see under which conditions on the spaces involved and the operator T we have

$$T\left(F_\theta^1(B_0, B_1)\right) \hookrightarrow T\left(F_\theta^2(A_0, A_1)\right), \tag{12}$$

for some θ-interpolation functors F_θ^1, F_θ^2. One can say that, in some sense, we "interpolate" linear equations or that we preserve some form of surjectivity of the operator T. In order to give reasonable sufficient conditions for (12) to hold for some convenient interpolation functors we introduce a variant of the complex interpolation method which will be called the $\mathcal{W}$-method[3]. In our case F_θ^1 will be given by the usual complex method of interpolation and F_θ^2 will be given by our $\mathcal{W}$-method. Roughly speaking this method consists in the following. Suppose (A_0, A_1) is a compatible couple of Banach spaces. In order to define the interpolation space of the couple (A_0, A_1) via the $\mathcal{W}$-method we use the three lines lemma on the strip as in the standard complex method of interpolation. However, instead of quantifying the endpoint regularity of the analytic functions involved via the norms $L^\infty(\mathbb{R}, A_j)$ $(j = 0, 1)$ we use slightly more complicated quantities that depend on some prescribed pair of Banach spaces (X_0, X_1). In this way, for each $\theta \in (0, 1)$, we obtain an interpolation space that will be denoted by $(A_0, X_0 \mid A_1, X_1)_\theta$. The efficiency of the $\mathcal{W}$-method relies (between other facts) on properly choosing the spaces X_0, X_1.

When the spaces X_j have the UMD property, under some additional embedding assumptions concerning the Banach spaces involved, the $\mathcal{W}$-method of interpolation preserves the "surjectivity" of operators (i.e., (12) holds). The main requirements for applying the $\mathcal{W}$-method are twofold:

(i) On one hand one needs to verify some embedding conditions for the domains and the co-domains of the operator. We give some simple necessary conditions that are easy to formulate, however not sharp. We also mention that, in the absence of any such conditions, it is not possible to preserve surjectivity (see the examples in the second part of subsection 3.3).

[3]Here, "$\mathcal{W}$" stands for "weak".

(ii) On the other hand explicitly computing the space $(A_0, X_0 \mid A_1, X_1)_\theta$ seems to be difficult in practice. However, there are particular situations in which we can embed $(A_0, X_0 \mid A_1, X_1)_\theta$ in some convenient space. More precisely, when A_j are of the form $A_j = A \cap X_j$, for some Banach space A, we have

$$(A_0, X_0 \mid A_1, X_1)_\theta = (A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow A \cap (X_0, X_1)_\theta .$$

For instance, when we have $X_0 = \dot{B}_2^{d/r,r}$, $X_1 = \dot{B}_2^{d/2,2}$ and $A = L^\infty$, the above embedding becomes

$$(L^\infty \cap \dot{B}_2^{d/r,r}, \dot{B}_2^{d/p,p} \mid L^\infty \cap \dot{B}_2^{d/2,2}, \dot{B}_2^{d/2,2})_\theta \hookrightarrow L^\infty \cap (\dot{B}_2^{d/r,r}, \dot{B}_2^{d/2,2})_\theta.$$

By using only the result of Mazya (Theorem 3 in the case $p = q = 2$) and the $\mathcal{W}$-method together with the embedding $\dot{B}_1^{d/p,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ we easily obtain the following:

Theorem 10. *Let $d \geq 2$ be an integer and consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{B}_q^{d/p,p}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{B}_2^{d/p,p}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{B}_2^{d/p,p}} \lesssim \|v\|_{\dot{B}_q^{d/p,p}} .$$

One can even obtain an analogue of Theorem 10 for a class of Lorentz-Sobolev spaces (for definitions see subsection 2.2). Namely, by using only the result of Mazya and the $\mathcal{W}$-method, together with some standard facts in the theory of Lorentz spaces, we easily obtain the following:

Theorem 11. *Let $d \geq 2$ be an integer and consider some parameters $p \in [2, \infty)$ and $q \in (1, 2)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{W}^{d/p} L^{p,3/2}(\mathbb{R}^d)$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{d/p} L^{p,2}(\mathbb{R}^d)$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{W}^{d/p} L^{p,2}} \lesssim \|v\|_{\dot{W}^{d/p} L^{p,q}} .$$

The conditions $p \geq 2$ and $q < 2$ in Theorem 10, Theorem 11, as well as in Theorem 7 and Theorem 9, are induced by some technical limitations of the $\mathcal{W}$-method (see subsection 5.2).

2. The Bourgain-Brézis technique. In [5, Section 4, p. 403] Bourgain and Brézis proved the torus analogue of Theorem 3 in the case where $p = q = d = 2$. They conluded the existence of solutions for the divergence equation by duality. Namely, they proved that (see [5, Lemma 2, p. 403])

$$\|u\|_{L^2(\mathbb{T}^2)} \lesssim \|\nabla u\|_{L^1(\mathbb{T}^2) + W^{-1,2}(\mathbb{T}^2)} , \tag{13}$$

for any $u \in L^1(\mathbb{T}^2)$, with $\widehat{u}(0) = 0$. In order to obtain this, they used the following key estimate (see [5, (4.20), p. 405]):

$$\left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{n_1 n_2}{|n|^4} \sin n_1\theta_1 \sin n_2\theta_2 \right| \leq C,$$

uniformly in $\theta_1, \theta_2 \in \mathbb{T}$, for some numerical constant $C > 0$. By convexity this allows us to write

$$\left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{n_1 n_2}{|n|^4} \widehat{F_1}(n) \widehat{F_2}(n) \right| \leq C \|F_1\|_{L^1(\mathbb{T}^2)} \|F_2\|_{L^1(\mathbb{T}^2)} ,$$

for any $F_1, F_2 \in L^1(\mathbb{T}^2)$. Thanks to this bilinear estimate, after decomposing ∇u in the space $L^1(\mathbb{T}^2) + W^{-1,2}(\mathbb{T}^2)$, we can deal with the space $L^1(\mathbb{T}^2)$ in (13).

We use the technique introduced by Bourgain and Brézis in [5, Section 4, p. 403] and we prove a version of Theorem 9 (and Theorem 8) in the case where the source space is $\dot{W}^{d/2,2}$. As we will see, thanks to this technique we are able to work with more general Fourier multipliers than derivatives. Also, it is this technique that allows us to gain some control on the Fourier spectrum of solutions. The results obtained by this method are "perfect" in the sense that that the source space is $\dot{W}^{d/2,2}$ and the solution space is $L^\infty \cap \dot{W}^{d/2,2}$; there is no loss of regularity in the third parameter. The drawback of this technique is the fact that it does not apply to the case where $p \neq 2$.

As in the case of Theorem 10, we can easily obtain Theorem 9 using the $\mathcal{W}$-method. This time however, insted of using Mazya's result we use the more general results that we obtain via the Bourgain-Brézis technique. Using the properties of Lorentz spaces we can give a Lorentz-Sobolev version of Theorem 9. In fact, our methods will provide more general results. On one hand the function spaces we work with can be more general than those in the statements of our final results (see for instance Theorem 37). One the other hand, the conditions imposed on Fourier multipliers and the Fourier spectrum of the solutions can be more general. Also, by using the technique of Mazya, one can easily obtain a version of Theorem 3 in the case $p = q = 2$ that concerns general Hodge systems. Combining this result with the $\mathcal{W}$-method one can obtain an analogue of Theorem 10 for Hodge systems. We will not consider however, such issues here. In this paper, we limit ourselves to some model situations that are easier to describe.

Notation. Throughout the paper we use mainly standard notation. For instance, we often use the symbols $\lesssim$ and $\sim$. For two nonnegative variable quantities a and b we write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$. For simplicity we denote by $spec(f)$ the Fourier spectrum of a tempered distribution f; in other words, $spec(f) = supp\ \widehat{f}$. Everywhere in this paper $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions. When X is a function space on $\mathbb{R}^d$ and $u = (u_1, ..., u_d)$ is a vector filed on $\mathbb{R}^d$ where each u_j belongs to X, we write $u \in X$ instead of $u \in X^d$. A similar convention will be made for the $(d-1)$-vector fields. Other notation will be introduced when needed.

2 Function spaces

In this section we quickly recall the definition and some properties of some standard function spaces.

2.1 Sobolev and Besov spaces

Let $\mathcal{S}_\sharp$ be the space of all Schwartz functions f on $\mathbb{R}^d$ such that $\widehat{f}$ vanishes in a neighborhood of 0. When $1 < p < \infty$ and $\alpha \in \mathbb{R}$ the homogeneous space $\dot{W}^{\alpha,p}(\mathbb{R}^d)$ is obtained by completion of $\mathcal{S}_\sharp$ under the norm

$$\|f\|_{\dot{W}^{\alpha,p}} := \||\nabla|^\alpha f\|_{L^p}.$$

We can see that we can also define the above homogeneous spaces $\dot{W}^{\alpha,p}$ by completion of the the normed function spaces $\dot{W}^{\alpha,p}_c(\mathbb{R}^d)$. Here, $\dot{W}^{\alpha,p}_c(\mathbb{R}^d)$ is the space of all the compactly supported functions whose $\dot{W}^{\alpha,p}$-norm is finite. The spaces $\dot{W}^{\alpha,p}$ as defined here are complete.

We continue by briefly recalling the definition of the Besov spaces (we do not define here the Triebel-Lizorkin spaces; see [24] for details). Consider a radial function $\Phi \in C^\infty_c(\mathbb{R}^d)$ such that

supp $\Phi \subset B(0,2)$ and $\Phi \equiv 1$ on $B(0,1)$. For $k \in \mathbb{Z}$ we define the operators P_k, acting on the space of tempered distributions on $\mathbb{R}^d$, by the relation

$$\widehat{P_k f}\,(\xi) := \left(\Phi\left(\frac{\xi}{2^k}\right) - \Phi\left(\frac{\xi}{2^{k-1}}\right)\right)\widehat{f}\,(\xi)\,, \tag{14}$$

for any Schwartz function f on $\mathbb{R}^d$. The operators P_k will be called *Littlewood-Paley "projections"* adapted to $\mathbb{R}^d$. For any Schwartz function f we have that

$$f = \sum_{k\in\mathbb{Z}} P_k f,$$

in the sense of tempered distributions. The homogeneous Besov space $\dot{B}^{\alpha,p}_q(\mathbb{R}^d)$ (with $1 \le p, q \le \infty$ and α a real number) is obtained by completion of $\mathcal{S}_\sharp$ under the norm

$$\|f\|_{\dot{B}^{\alpha,p}_q} := \left(\sum_{j\in\mathbb{Z}} 2^{\alpha k q}\,\|P_k f\|^q_{L^p}\right)^{1/q}.$$

We have $\dot{B}^{\alpha,2}_2(\mathbb{R}^d) = \dot{W}^{\alpha,2}(\mathbb{R}^d)$ with equivalent norms.

The main advantage of our definition of the homogeneous Besov spaces is the fact that, whenever $\alpha_0 - d/p_0 = \alpha_1 - d/p_1$ and $\alpha_1 > \alpha_0$ we have the embedding

$$\dot{B}^{\alpha_1,p_1}_{q_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\alpha_0,p_0}_{q_0}(\mathbb{R}^d), \tag{15}$$

for any $q_0, q_1 \in [1,\infty)$ with $q_0 \le q_1$.

Note that we have the following dilation properties:

$$\|f(\lambda\cdot)\|_{\dot{B}^{\alpha,p}_q} \sim \lambda^{\alpha-d/p}\,\|f\|_{\dot{B}^{\alpha,p}_q}\,, \tag{16}$$

for any $f \in \dot{B}^{\alpha,p}_q(\mathbb{R}^d)$ respectively and any $\lambda > 0$. In particular, when $\alpha = d/p$ the spaces $\dot{B}^{\alpha,p}_q$ have the same scaling property as L^∞. In what follows the spaces of the form $\dot{B}^{d/p,p}_q$ will be called critical. It is worth recalling here, that, by a direct application of the Bernstein inequalities we get the embedding $\dot{B}^{d/p,p}_1(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. When $q > 1$ the critical spaces $\dot{B}^{d/p,p}_q$ do not embed in L^∞.

Remark 12. *Note that the spaces $\dot{B}^{d/p,p}_q$ (with $q > 1$) as defined here contain elements that are not tempered distributions. However, when $\alpha < d/p$ the elements of the space $\dot{B}^{\alpha,p}_q$ are all tempered distributions (see for instance [1, Remark 2.26, p. 68] or [3]).*

Remark 13. *Since the operator* $\mathrm{div} : \mathcal{S}_\sharp \cap \dot{B}^{d/p,p}_q \to \dot{B}^{d/p-1,p}_q$ *is linear and bounded (here, $\mathcal{S}_\sharp \cap \dot{B}^{d/p,p}_q$ is endowed with the norm induced by $\dot{B}^{d/p,p}_q$), by density of $\mathcal{S}_\sharp \cap \dot{B}^{d/p,p}_q$ in $\dot{B}^{d/p,p}_q$ it extends uniquely to an operator* $\mathrm{div} : \dot{B}^{d/p,p}_q \to \dot{B}^{d/p-1,p}_q$. *Similar facts hold for other spaces and other operators. In this way we can remove from the hypotheses of Theorem 7, Theorem 8 and Theorem 9 the fact that the source v belongs to $\mathcal{S}'$. For instance (9) will be understood as an equality of two elements in $\dot{B}^{d/p-1,p}_2$. This formulation is the one that we prefer throughout the paper.*

Remark 14. *One can define the spectrum of an element $v \in \dot{B}^{d/p,p}_q$ in the following way. Note that the operator $\triangle : \mathcal{S}_\sharp \cap \dot{B}^{d/p,p}_q \to \dot{B}^{d/p-2,p}_q$ is linear and bounded. Hence, by density of $\mathcal{S}_\sharp \cap \dot{B}^{d/p,p}_q$ in $\dot{B}^{d/p,p}_q$ it extends uniquely to an operator $\triangle : \dot{B}^{d/p,p}_q \to \dot{B}^{d/p-2,p}_q$. According to Remark 12 all the elements of $\dot{B}^{d/p-2,p}_q$ are tempered distributions. Now, if $v \in \dot{B}^{d/p,p}_q$, then $\triangle v \in \dot{B}^{d/p-2,p}_q$ is a tempered distribution and we can define the spectrum of v as $spec(v) := supp\ \widehat{\triangle v}$. This observation will be applied for other function spaces as well.*

It is easy to see that any space of the form $\dot{B}^{\alpha,p}_q$ with $p, q \in (1,\infty)$ is embedded in $l^\alpha_q(L^p)$ and hence it has the UMD property (see for instance [20]).

2.2 Lorentz-Sobolev spaces

Consider[4] some parameters $p \in (1,\infty)$, $q \in [1,\infty]$ and $\alpha \geq 0$. The homogeneous Lorentz-Sobolev spaces $\dot{W}^{\alpha}L^{p,q}(\mathbb{R}^d)$ is the completion of the normed space of Schwartz functions f on $\mathbb{R}^d$ under the norm

$$\|f\|_{\dot{W}^{\alpha}L^{p,q}} := \||\nabla|^{\alpha} f\|_{L^{p,q}},$$

where $L^{p,q}$ is the usual Lorentz space of parameters p and q.

Remark 15. *One can easily adapt the Remarks 12, 13 and 14 to the case of the Lorentz-Sobolev spaces.*

Many of the embedding properties of the Besov and Triebel-Lizorkin spaces hold for the Lorentz-Sobolev spaces (see for instance [23] for detailes). We mention below some properties of Lorentz-Sobolev spaces that will be needed in the proof of Theorem 11. All of them are direct consequences of well-known facts from the theory of Lorentz spaces.

Lemma 16. *For any $r \in [2,\infty)$, we have that*

$$\dot{W}^{d/2,2}(\mathbb{R}^d) \hookrightarrow \dot{W}^{d/r}L^{r,2}(\mathbb{R}^d).$$

Proof. It is a consequence of the improved Sobolev embedding (see for instance [26, Theorem 2.10.2, p. 98]) that

$$\|f\|_{L^{r,2}} \lesssim \left\| |\nabla|^{d(1/2-1/r)} f \right\|_{L^2},$$

for any Schwartz function f on $\mathbb{R}^d$. This can be rewritten as

$$\|f\|_{\dot{W}^{d/r}L^{r,2}} = \left\| |\nabla|^{d/r} f \right\|_{L^{r,2}} \lesssim \left\| |\nabla|^{d/2} f \right\|_{L^2} = \|f\|_{\dot{W}^{d/2,2}},$$

obtaining that $\dot{W}^{d/2,2} \hookrightarrow \dot{W}^{d/r}L^{r,2}$. □

Lemma 17. *For any $r \in (1,\infty)$, we have that*

$$\dot{W}^{d/r}L^{r,1}(\mathbb{R}^d) \hookrightarrow L^{\infty}(\mathbb{R}^d).$$

Proof. For any Schwartz function f on $\mathbb{R}^d$ we have that

$$\left\| I_{d/r} * f \right\|_{L^{\infty}} \lesssim \|f\|_{L^{r,1}}, \tag{17}$$

where $I_{d/r}(x) = |x|^{d/p-d} = |x|^{-d/p'}$, for any $x \in \mathbb{R}^d$. Indeed, using [13, Theorem 1.4.17 (v), p. 52], we have

$$\int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d/r'}} dy \leq \left\| I_{d/r} \right\|_{(L^{r,1})^*} \|f\|_{L^{r,1}} = \left\| I_{d/r} \right\|_{L^{r',\infty}} \|f\|_{L^{r,1}},$$

and we can easily see that

$$\left\| I_{d/r} \right\|_{L^{r',\infty}} = \sup_{\lambda>0} \lambda \left| \left\{ x \in \mathbb{R}^d \mid |x| < (1/\lambda)^{r'/d} \right\} \right|^{1/r'} \sim 1.$$

Hence, (17) holds. We can reformulate (17) as

$$\|f\|_{L^{\infty}} \lesssim \left\| |\nabla|^{d/r} f \right\|_{L^{r,1}} = \|f\|_{\dot{W}^{d/r}L^{r,1}},$$

This shows that $\dot{W}^{d/r}L^{r,1} \hookrightarrow L^{\infty}$. □

[4]The results of this subsection will be used only in the proof of Theorem 11.

Lemma 18. *Suppose $p_0, p_1, q_0, q_1 \in [1,\infty)$ and $\alpha_0, \alpha_1 \geq 0$. Then, for any $\theta \in (0,1)$ we have*

$$(\dot{W}^{\alpha_0} L^{p_0,q_0}(\mathbb{R}^d), \dot{W}^{\alpha_1} L^{p_1,q_1}(\mathbb{R}^d))_\theta = \dot{W}^{\alpha} L^{p,q}(\mathbb{R}^d), \tag{18}$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$.

Proof. This can be proved by Stein's method of interpolation (see for instance [13, Theorem 1.3.7, p. 37]) as follows. Note that the function $\xi \to |\xi|^{it}$ defined on $\mathbb{R}^d \setminus \{0\}$ satisfies

$$\left|\nabla^k |\xi|^{it}\right| \leq C(1+|t|)^{d+2} |\xi|^{-k},$$

a.e. in $\xi \in \mathbb{R}^d \setminus \{0\}$, for any $t \in \mathbb{R}$ and any nonnegative integer $k \leq d+2$, where C is a positive constant depending only on d. It follows that (see [18, Theorem 8.2 , p. 197]) for any $a \in (1,\infty)$, the norm of the operator $|\nabla|^{it} : L^a \to L^a$ satisfies

$$\left\| |\nabla|^{it} \right\|_{L^a \to L^a} \lesssim_a C(1+|t|)^{d+2}.$$

This implies, via the real method of interpolation (see for instance [2, Theorem 5.2.1 (2), p. 109]), that for any $a \in (1,\infty)$ and any $b \in [1,\infty]$, we have

$$\left\| |\nabla|^{it} \right\|_{L^{a,b} \to L^{a,b}} \lesssim_{a,b} C(1+|t|)^{d+2}. \tag{19}$$

Let us consider the analytic family of operators $(T_z)_{z\in S}$ with

$$T_z := |\nabla|^{(1-z)\alpha_0 + z\alpha_1},$$

for all $z \in S$. Thanks to (19), the analytic family $(T_z)_{z\in S}$ satisfies the hypothesis of [13, Theorem 1.3.7, p. 37]. Hence, we get

$$T_\theta(\dot{W}^{\alpha_0} L^{p_0,q_0}, \dot{W}^{\alpha_1} L^{p_1,q_1})_\theta \hookrightarrow (L^{p_0,q_0}, L^{p_1,q_1})_\theta = L^{p,q},$$

and, in a similar way (applying Stein's method for the family $(T_{-z})_{z\in S}$),

$$T_{-\theta}(L^{p,q}) = T_{-\theta}(L^{p_0,q_0}, L^{p_1,q_1})_\theta \hookrightarrow (\dot{W}^{\alpha_0} L^{p_0,q_0}, \dot{W}^{\alpha_1} L^{p_1,q_1})_\theta.$$

Hence,

$$T_\theta(\dot{W}^{\alpha_0} L^{p_0,q_0}, \dot{W}^{\alpha_1} L^{p_1,q_1})_\theta = L^{p,q},$$

and (18) is proven. □

The spaces $\dot{W}^{\alpha} L^{p,q}$ have scaling properties that are similar to those of the Besov spaces (see (16)). In particular, the spaces $\dot{W}^{d/p} L^{p,q}$ have the same scaling as L^∞. As we have seen in Lemma 17 we have $\dot{W}^{d/p} L^{p,1} \hookrightarrow L^\infty$. However, when $q > 1$ the critical spaces $\dot{W}^{d/p} L^{p,q}$ do not embed in L^∞.

Let us see that the spaces $\dot{W}^{\alpha} L^{p,q}$ have the UMD property when $p, q \in (1,\infty)$. For this is sufficient to see that $L^{p,q}$ has the UMD property. Consider some $p_0, p_1 \in (1,\infty)$ such that $p_0 < p < p_1$. Since, L^{p_0} and L^{p_1} are UMD spaces, by Burkholder's theorem (see [9]) the Hilbert transform is bounded on $L^2(\mathbb{T}, L^{p_0})$ and $L^2(\mathbb{T}, L^{p_1})$ respectively. Hence, the Hilbert transform is bounded on the space

$$\begin{aligned} (L^2(\mathbb{T}, L^{p_0}), L^2(\mathbb{T}, L^{p_1}))_{\eta,q} &= L^2(\mathbb{T}, (L^{p_0}, L^{p_1})_{\eta,q}) \\ &= L^2(\mathbb{T}, L^{p,q}), \end{aligned}$$

where $\eta \in (0,1)$ is such that $1/p = (1-\eta)/p_0 + \eta/p_1$ (see for instance [2, Theorem 5.6.2, p. 123]). By Bourgain's theorem ([4]), we get that $L^{p,q}$ has the UMD property.

2.3 Some quotient spaces

Let Y be a Banach function space on $\mathbb{R}^d$ and let $D \subseteq \mathbb{R}^d$ be a measurable set. Relative to the set D we define the closed subspace Y_D of Y by

$$Y_D := \{f \in Y \mid spec(f) \subseteq D\},$$

the norm being the one induced by Y. For simplicity we will denote the quotient space Y/Y_{D^c} by Y/D. In the case where $f \in Y$ is a Schwartz function, we define its Y/D-seminorm by

$$\|f\|_{Y/D} := \inf_{f^- \in Y_{D^c}} \left\|f + f^-\right\|_Y.$$

In this paper we will work with quotient spaces of the form $\dot{W}^{\alpha,2}/D$ and $(L^1 + \dot{W}^{\alpha,2})/D$. One can easily see that for $Y = \dot{W}^{\alpha,2}$ or $Y = L^1 + \dot{W}^{\alpha,2}$ we have the following norming property

$$\sup_{\|f\|_{Y/D} \leq 1} \langle f, g \rangle = \|g\|_{Y^*},$$

for any $g \in Y_D^*$, where the supremum is taken over all Schwartz functions f with $\|f\|_{Y/D} \leq 1$.

In the case where Y is the Sobolev space $\dot{W}^{\alpha,2}$ it is easy to compute the seminorm induced by Y/D. Namely, let us see that for any $u \in \mathcal{S}_\sharp$ and any measurable set $D \subseteq \mathbb{R}^d$ we have

$$\|u\|_{\dot{W}^{\alpha,2}/D} = \left(\int_{\mathbb{R}^d} |\xi|^{2\alpha} \mathbf{1}_D(\xi) \left|\widehat{u}(\xi)\right|^2 d\xi\right)^{1/2}. \tag{20}$$

Indeed, we have

$$\begin{aligned}
\|u\|^2_{\dot{W}^{\alpha,2}/D} &= \inf_{v \in \dot{W}^{\alpha,2}_{D^c}} \int_{\mathbb{R}^d} |\xi|^{2\alpha} \left|\widehat{u}(\xi) + \widehat{v}(\xi)\right|^2 d\xi \\
&= \int_D |\xi|^{2\alpha} \left|\widehat{u}(\xi)\right|^2 d\xi + \inf_{v \in \dot{W}^{\alpha,2}_{D^c}} \int_{D^c} |\xi|^{2\alpha} \left|\widehat{u}(\xi) + \widehat{v}(\xi)\right|^2 d\xi \\
&= \int_D |\xi|^{2\alpha} \left|\widehat{u}(\xi)\right|^2 d\xi.
\end{aligned}$$

We recall that, by P_D we denote the Fourier projection on the set D, i.e., we have

$$\widehat{P_D f}(\xi) = \mathbf{1}_D(\xi) \widehat{f}(\xi),$$

for any $\xi \in \mathbf{R}^d$ and any Schwartz function f. Note that, in the case where $D = (0,\infty)^d$ the operator $P_{(0,\infty)^d}$ is the Riesz projection. In this case we will write P_+ in the place of $P_{(0,\infty)^d}$.

3 The $\mathcal{W}$-method of complex interpolation

In this section we introduce a variant of the complex interpolation and we prove several of its properties. We call this new method of interpolation the $\mathcal{W}$-method and, as stated in the introduction (see subsection 1.4), this will be used in the proof of Theorem 9, Theorem 7, Theorem 8, Theorem 11. We mainly study here only the properties of the $\mathcal{W}$-method that are used in the proof of our main results. In subsection 3.1 we show that the $\mathcal{W}$-method is indeed an interpolation method. However, we ignore some issues specific to the interpolation methods in general such as

computing dual interpolation spaces or reiteration theorems. These problems do not concern us here.

An important aspect is the relation of the $\mathcal{W}$-method with the classical complex method. We do not know in general how to compute efficiently the interpolation spaces obtained via the $\mathcal{W}$-method. However, as we will see in subsection 3.2 the space obtained via the $\mathcal{W}$-method is, in many "convenient" cases, the same as the space obtained via the classical complex method.

The main feature of the $\mathcal{W}$-method is that one can use it to "interpolate" linear equations. It is one of the main ingredients that enter in the proof of our main results and it is the final goal of this section.

3.1 Construction of the interpolation space

We describe here the $\mathcal{W}$-method and prove some basic properties. The proofs we give are straightforward adaptations of those that correspond to the classical complex interpolation as found in [2, Chapter 4]. Following the general presentation in [2, Chapter 4] let us introduce now the $\mathcal{W}$-method.

For the beginning, fix two Banach spaces X_0 and X_1 and suppose (A_0, A_1) is a Banach couple. Let $\mathcal{F}^2 = \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$ be the linear space of all bounded continuous functions f with values in $A_0 + A_1$, defined on the strip

$$S := \{z \in \mathbb{C} \mid 0 \leq \Re z \leq 1\},$$

that are analytic in the open strip

$$S_0 := \{z \in \mathbb{C} \mid 0 < \Re z < 1\},$$

and moreover, such that $f(j + it) \in A_j$ for any $j = 0, 1$ and any $t \in \mathbb{R}$, and

$$\|f\|_{\mathcal{F}^2} := \max_{j=0,1} \sup_{\|\Lambda_j\| \leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j + it)\|_{X_j}^2 \, dt \right)^{1/2} < \infty, \tag{21}$$

where, for each $j = 0, 1$, $\Lambda_j : A_j \to X_j$ are linear bounded operators. One can easily verify that $\|\cdot\|_{\mathcal{F}^2}$ defines a norm on $\mathcal{F}^2$.

Fix $0 < \theta < 1$. Consider the linear space $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ defined by

$$\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1) := \left\{a \in A_0 + A_1 \mid a = f(\theta), \text{ for some } f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)\right\}.$$

and define, for each $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$, the quantity

$$\|a\|_\theta := \inf \left\{\|f\|_{\mathcal{F}^2} \mid a = f(\theta),\ f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)\right\}.$$

Lemma 19. *The mapping $a \to \|a\|_\theta$ is a norm on $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$.*

In order to prove Lemma 19 we rely on the following basic fact (and at least implicitely well-known):

Lemma 20. *Fix some $1 \leq p < \infty$ and let Z be a Banach space. Suppose $F : S \to Z$ is a bounded continuous function which is analytic in S_0 such that the functions $t \to F(j + it)$ belong to the space $L^p(\mathbb{R}, Z)$.*

Then, for any $z \in S_0$, we have

$$F(z) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(it)}{it-z}dt + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(1+it)}{1+it-z}dt. \tag{22}$$

In particular, for any $\theta \in (0,1)$,

$$\|F(\theta)\|_Z \lesssim_{\theta,p} \max_{j=0,1}\left(\int_{\mathbb{R}} \|F(j+it)\|_Z^p \, dt\right)^{1/p}. \tag{23}$$

Proof of Lemma 20. Fix some $z \in S_0$. Consider some arbitrary $R > 0$ and the curve γ_R given by the boundary of the rectangle $[0,1]\times[-R,R]$, oriented anti-clockwise. For R sufficeintly large we have $z \in [0,1]\times[-R,R]$. By Cauchy's formula we get

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{F(\zeta)}{\zeta - z}d\zeta,$$

and we can rewrite this as

$$\begin{aligned} F(z) &= -\frac{1}{2\pi i}\int_{-R}^{R} \frac{F(it)}{it-z}dt + \frac{1}{2\pi i}\int_{-R}^{R} \frac{F(1+it)}{1+it-z}dt \\ &\quad + \frac{1}{2\pi i}\int_0^1 \frac{F(x+iR)}{x+iR-z}dx - \frac{1}{2\pi i}\int_0^1 \frac{F(x-iR)}{x-iR-z}dx. \end{aligned} \tag{24}$$

Note that, the functions $t \to F(j+it)/(j+it-\theta)$ belong to the space $L^1(\mathbb{R}, Z)$. Indeed, by Hölder's inequality (since we always have $p' > 1$) we can write

$$\begin{aligned} \int_{\mathbb{R}} \left\| \frac{F(j+it)}{j+it-z} \right\|_Z dt &\leq \left(\int_{\mathbb{R}} \frac{1}{|j+it-z|^{p'}}dt\right)^{1/p'} \left(\int_{\mathbb{R}} \|F(j+it)\|_Z^p \, dt\right)^{1/p} \\ &\lesssim_{\theta,z} \left(\int_{\mathbb{R}} \|F(j+it)\|_Z^p \, dt\right)^{1/p} < \infty, \end{aligned} \tag{25}$$

with the natural modification in the case where $p' = \infty$.

Also,

$$\left\| \frac{1}{2\pi i}\int_0^1 \frac{F(x \pm iR)}{x \pm iR - z}dx \right\|_Z \leq \frac{1}{|x \pm iR - z|}\|F\|_{L^\infty(S,Z)} \to 0, \tag{26}$$

when $R \to \infty$.

Using (24), (26), letting $R \to \infty$ and using the dominated convergence theorem, we get the representation formula (22). Using (22) and (25), for $z = \theta$, we obtain (23). □

Proof of Lemma 19. Clearly, $\|\cdot\|_\theta$ is a seminorm on $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$. It remains to see that, if $\|a\|_\theta = 0$, for some $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$, then $a = 0$. We prove this by showing that

$$\|a\|_{A_0+A_1} \lesssim_\theta \|a\|_\theta, \tag{27}$$

for all $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$. For this purpose fix $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ and consider a functional $\lambda \in (A_0+A_1)^*$, with $\|\lambda\| = 1$, such that $\|a\|_{A_0+A_1} \leq 2\lambda(a)$. Consider also a function $f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$, such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}^2} \leq 2\|a\|_\theta$.

Let us define, for each $j = 0, 1$, the linear operators $\Lambda_j : A_j \to X_j$ by

$$\Lambda_j(a_j) = \lambda(a_j)e_j, \quad \text{for any } a_j \in A_j,$$

where $e_j \in X_j$ are some fixed vectors with $\|e_j\|_{X_j} = 1$. Clearly, for any $j = 0, 1$,

$$\|\Lambda_j(a_j)\|_{X_j} = |\lambda(a_j)| \leq \|a_j\|_{A_0+A_1} \leq \|a_j\|_{A_j},$$

for any $a_j \in A_j$, and we get

$$\|\Lambda_j\| \leq 1.$$

Using this observation and introducing the function $F : S \to \mathbb{C}$ defined by $F(z) := \lambda(f(z))$, one can write,

$$\begin{aligned} \max_{j=0,1} \left(\int_{\mathbb{R}} |F(j+it)|^2 \, dt \right)^{1/2} &= \max_{j=0,1} \left(\int_{\mathbb{R}} |\lambda(f(j+it))|^2 \, dt \right)^{1/2} \\ &= \max_{j=0,1} \left(\int_{\mathbb{R}} \|\Lambda_j(f(j+it))\|_{X_j}^2 \, dt \right)^{1/2} \\ &\leq \|f\|_{\mathcal{F}^2} \leq 2\|a\|_\theta < \infty. \end{aligned} \tag{28}$$

This shows, in particular, that the functions $t \to F(j+it)$ belong to the space $L^2(\mathbb{R}, \mathbb{C})$. We also see immediately that F is bounded, continuous on S and analytic in S_0. Hence, by applying Lemma 20 for $Z = \mathbb{C}$ and $p = 2$ (more precisely (23)), using (28), we get

$$\|a\|_{A_0+A_1} \leq 2\lambda(a) = 2F(\theta) \lesssim \max_{j=0,1} \left(\int_{\mathbb{R}} |F(j+it)|^2 \, dt \right)^{1/2} \lesssim \|a\|_\theta,$$

which proves (27). □

Now, thanks to Lemma 19, we can define the interpolation space $(A_0, X_0 \mid A_1, X_1)_\theta$ as being the completion of the normed space $(\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1), \|\cdot\|_\theta)$.

One can easily see that $(A_0, X_0 \mid A_1, X_1)_\theta$ is an intermediate space:

$$A_0 \cap A_1 \hookrightarrow (A_0, X_0 \mid A_1, X_1)_\theta \hookrightarrow A_0 + A_1. \tag{29}$$

The second embedding in (29), follows directly from the inequality (27). In order to see the first embedding, pick $a \in A_0 \cap A_1$ and consider the function $f(z) := \exp(z^2 - \theta^2)\, a$. One can easily check that $f(\theta) = a$, $f \in \mathcal{F}^2$ and

$$\begin{aligned} \|a\|_\theta &\leq \max_{j=0,1} \sup_{\|\Lambda_j\| \leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 \, dt \right)^{1/2} \\ &\leq \max_{j=0,1} \left(\int_{\mathbb{R}} \|f(j+it)\|_{A_j}^2 \, dt \right)^{1/2} \sim_\theta \|a\|_{A_0 \cap A_1}. \end{aligned}$$

This gives us that

$$A_0 \cap A_1 \hookrightarrow \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1) \hookrightarrow (A_0, X_0 \mid A_1, X_1)_\theta.$$

Let us see now that the $\mathcal{W}$-method provides an exact interpolation functor:

Proposition 21. *Consider some Banach spaces X_0, X_1. Let (A_0, A_1), (B_0, B_1) be two Banach couples and $T : A_0 + A_1 \to B_0 + B_1$ be a linear operator such that $T : A_j \to B_j$ is bounded for any $j = 0, 1$, of norm $\|T\|_{j \to j}$.*

Then, the operator

$$T : (A_0, X_0 \mid A_1, X_1)_\theta \to (B_0, X_0 \mid B_1, X_1)_\theta,$$

is bounded and of norm $\|T\|_{\theta \to \theta}$ satisfying

$$\|T\|_{\theta \to \theta} \leq \|T\|_{0 \to 0}^{1-\theta} \|T\|_{1 \to 1}^{\theta}.$$

Proof. Without loss of generality we suppose that $\|T\|_{j\to j} > 0$, for any $j = 0, 1$. For the brevity of notation we denote by $\|\cdot\|_\theta$, $\|\cdot\|_{\mathcal{F}^2}$, $\|\cdot\|'_\theta$ and $\|\cdot\|'_{\mathcal{F}^2}$ the norms on the spaces $(A_0, X_0 \mid A_1, X_1)_\theta$, $\mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$, $(B_0, X_0 \mid B_1, X_1)_\theta$ and $\mathcal{F}^2(B_0, X_0 \mid B_1, X_1)$ respectively. Pick some $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ and fix some $\varepsilon > 0$. Consider a function $f \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$ such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}^2} \leq (1+\varepsilon)\|a\|_\theta$. The function

$$F(z) := \|T\|_{0\to 0}^{z-1} \|T\|_{1\to 1}^{-z} Tf(z)$$

belongs to $\mathcal{F}^2(B_0, X_0 \mid B_1, X_1)$. Indeed, F is bounded and continuous on S with values in B_0+B_1, analytic in S_0 and, for any linear operators $\Lambda'_j : B_j \to X_j$ of norm at most 1, we have

$$\begin{aligned}\left(\int_{\mathbb{R}} \left\|\Lambda'_j F(j+it)\right\|_{X_j}^2 dt\right)^{1/2} &\leq \left(\int_{\mathbb{R}} \left\|\left(\|T\|_{j\to j}^{-1} \Lambda'_j \circ T\right) f(j+it)\right\|_{X_j}^2 dt\right)^{1/2} \\ &\leq \sup_{\|\Lambda_j\|\leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt\right)^{1/2}, \end{aligned} \tag{30}$$

for any $j = 0, 1$, where the supremum is taken over all linear bounded operators $\Lambda_j : A_j \to X_j$ with $\|\Lambda_j\| \leq 1$. Here, we have used the fact that $\|T\|_{j\to j}^{-1} \Lambda'_j \circ T : A_j \to X_j$ is a linear operator of norm at most 1, for any $j = 0, 1$.

Now, by (30), we get

$$\|T\|_{0\to 0}^{\theta-1} \|T\|_{1\to 1}^{-\theta} \|Tf(a)\|'_\theta = \|F(\theta)\|'_\theta \leq \|F\|'_{\mathcal{F}^2} \leq \|f\|_{\mathcal{F}^2} \leq (1+\varepsilon)\|a\|_\theta,$$

and letting $\varepsilon \to 0$ one obtains,

$$\|Ta\|'_\theta \leq \|T\|_{0\to 0}^{1-\theta} \|T\|_{1\to 1}^{\theta} \|a\|_\theta,$$

for any $a \in \mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$. Since, by definition, $\mathcal{C}_\theta(A_0, X_0 \mid A_1, X_1)$ is dense in $(A_0, X_0 \mid A_1, X_1)_\theta$, we get the conclusion. □

3.2 A particular case

In general, computing the interpolation space $(A_0, X_0 \mid A_1, X_1)_\theta$ seems to be a nontrivial task. However, there are some particular cases where an explicit computation is easy.

Let us restrict to the case, where $A_0 = X_0$ and $A_1 = X_1$ and let us denote, for simplicity, the space $(X_0, X_0 \mid X_1, X_1)_\theta$ by $(X_0|X_1)_\theta$. Also, instead of $\mathcal{F}^2(X_0, X_0 \mid X_1, X_1)$ we write $\mathcal{F}^2(X_0|X_1)$ and instead of $\mathcal{C}_\theta(X_0, X_0 \mid X_1, X_1)$ we write $\mathcal{C}_\theta(X_0|X_1)$. In this case, formula (21) becomes

$$\|f\|_{\mathcal{F}^2} = \max_{j=0,1} \left(\int_{\mathbb{R}} \|f(j+it)\|_{X_j}^2 dt\right)^{1/2}. \tag{31}$$

Indeed, for any $j = 0, 1$, we have

$$\sup_{\|\Lambda_j\|\leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt\right)^{1/2} \leq \left(\int_{\mathbb{R}} \|f(j+it)\|_{X_j}^2 dt\right)^{1/2},$$

and

$$\begin{aligned}\sup_{\|\Lambda_j\|\leq 1} \left(\int_{\mathbb{R}} \|\Lambda_j f(j+it)\|_{X_j}^2 dt\right)^{1/2} &\geq \left(\int_{\mathbb{R}} \left\|id_{X_j} f(j+it)\right\|_{X_j}^2 dt\right)^{1/2} \\ &= \left(\int_{\mathbb{R}} \|f(j+it)\|_{X_j}^2 dt\right)^{1/2}, \end{aligned}$$

where $id_{X_j} : X_j \to X_j$ is the identity mapping on X_j.

It turns out that, the space $(X_0|X_1)_\theta$ coincides with the space $(X_0, X_1)_\theta$ obtained via the classical complex interpolation method. The proof of this fact is easy, however we give it below for the sake of completeness.

Proposition 22. *Suppose (X_0, X_1) is a compatible couple of Banach spaces. Then, for any $\theta \in (0,1)$,*

$$(X_0|X_1)_\theta = (X_0, X_1)_\theta,$$

with equivalence of norms.

Remark 23. *Implicitly, the embedding $(X_0|X_1)_\theta \hookrightarrow (X_0, X_1)_\theta$ was already proved and used by Peetre in a different context (see [22, Lemme 1.1]). Both proofs, the one that we give below and Peetre's, are easy consequences of the ideas of Calderón from [10, Section 9.4].*

Proof. Consider some $a \in (X_0, X_1)_\theta \cap \mathcal{C}_\theta(X_0|X_1)$ and let some $f \in \mathcal{F}^2(X_0|X_1)$ be such that $f(\theta) = a$ and

$$\|f\|_{\mathcal{F}^2} \leq 2\|a\|_{(X_0|X_1)_\theta}.$$

By [2, Lemma 4.3.2 (ii), p. 93] (or [10, Section 9.4, (ii)]) we have

$$\|a\|_{(X_0,X_1)_\theta} \leq \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|f(i\tau)\|_{X_0} P_0(\theta,\tau)d\tau\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \|f(1+i\tau)\|_{X_1} P_1(\theta,\tau)d\tau\right)^{\theta}, \tag{32}$$

where P_j $(j = 0, 1)$ are the real Poisson kernels defined by

$$P_j(s+it,\tau) := \frac{e^{-\pi(\tau-t)}\sin \pi s}{\sin^2 \pi s + \left(\cos \pi s - e^{ij\pi - \pi(\tau-t)}\right)^2},$$

for $s \in (0,1)$, $t, \tau \in \mathbb{R}$. Note that $P_j(\theta,\cdot) \in L^2(\mathbb{R},\mathbb{R})$ and by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}} \|f(i\tau)\|_{X_0} P_0(\theta,\tau)d\tau &\leq \left(\int_{\mathbb{R}} \|f(i\tau)\|_{X_0}^2 d\tau\right)^{1/2} \left(\int_{\mathbb{R}} P_0^2(\theta,\tau)d\tau\right)^{1/2} \\ &\lesssim \|f\|_{\mathcal{F}^2} \lesssim \|a\|_{(X_0|X_1)_\theta}.\end{aligned}$$

In a similar way we get

$$\int_{\mathbb{R}} \|f(i\tau)\|_{X_0} P_0(\theta,\tau)d\tau \lesssim \|a\|_{(X_0|X_1)_\theta},$$

and combining with (32) one obtains

$$\|a\|_{(X_0,X_1)_\theta} \lesssim \|a\|_{(X_0|X_1)_\theta}. \tag{33}$$

By taking the closure we get $(X_0|X_1)_\theta \hookrightarrow (X_0, X_1)_\theta$.

Conversely, if $a \in (X_0, X_1)_\theta$, then there exists $g \in \mathcal{F}(X_0, X_1)$ (see [2, Chapter 4] for the standard notation $\mathcal{F}(X_0, X_1)$) such that $g(\theta) = a$ and

$$\max_{j=0,1} \sup_{t\in\mathbb{R}} \|g(j+it)\|_{X_j} \leq 2\|a\|_{(X_0,X_1)_\theta}. \tag{34}$$

Introduce the function $\widetilde{g} : S \to X_0 + X_1$ defined by $\widetilde{g}(z) := \exp(z^2 - \theta^2) g(z)$, for $z \in S$. We observe that, for any $j = 0, 1$,

$$\begin{aligned} \left(\int_{\mathbb{R}} \|\widetilde{g}(j+it)\|_{X_j}^2 \, dt \right)^{1/2} &\lesssim \left(\int_{\mathbb{R}} e^{-2t^2} \|g(j+it)\|_{X_j}^2 \, dt \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} e^{-2t^2} dt \right)^{1/2} \sup_{t \in \mathbb{R}} \|g(j+it)\|_{X_j} \\ &\sim \sup_{t \in \mathbb{R}} \|g(j+it)\|_{X_j} . \end{aligned} \tag{35}$$

Hence, $\widetilde{g} \in \mathcal{F}^2(X_0|X_1)$, $a = \widetilde{g}(\theta) \in \mathcal{C}_\theta(X_0|X_1)$ and by (34), (35),

$$\|a\|_{(X_0|X_1)_\theta} \lesssim \|a\|_{(X_0, X_1)_\theta} , \tag{36}$$

We have now $(X_0, X_1)_\theta \hookrightarrow (X_0|X_1)_\theta$ and Proposition 22 is proven. □

An immediate consequence of Proposition 22 is the following useful embedding result:

Corollary 24. *Suppose (X_0, X_1) is a compatible couple of Banach spaces. Then, for any Banach space A, we have the embedding*

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow A \cap (X_0, X_1)_\theta .$$

Proof. Consider the canonical inclusion $\iota : A \cap X_0 + A \cap X_1 \to X_0 + X_1$ as a linear bounded operator $\iota : A \cap X_j \to X_j$ and apply Proposition 21. We get

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow (X_0, X_0 \mid X_1, X_1)_\theta = (X_0|X_1)_\theta .$$

Since by Proposition 22 we have $(X_0|X_1)_\theta = (X_0, X_1)_\theta$, we now obtain the embedding

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow (X_0, X_1)_\theta . \tag{37}$$

Also, using the fact that $(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta$ is an intermediate space (see (29)), we have

$$(A \cap X_0, X_0 \mid A \cap X_1, X_1)_\theta \hookrightarrow A \cap X_0 + A \cap X_1 \hookrightarrow A,$$

which together with (37) proves Corollary 24. □

3.3 Solutions of linear equations

In this subsection we highlight the main strength of the $\mathcal{W}$-method. Namely, we show here how the $\mathcal{W}$-method can be used in order to "interpolate " underdetermined equations. Here we make an essential use of the fact that the Hilbert transform is bounded on spaces of the form $L^2(\mathbb{R}, Z)$, where Z is an UMD space. The UMD property plays here a key role.

Before stating the results in this subsection let us make some (common) notational conventions. The space $C_b^l(\mathbb{R}, Z)$, where $l \in \mathbb{N}$, is the space of all the functions $f : \mathbb{R} \to Z$ for which the k-th derivative $f^{(k)}$ is a continuous and bounded Z-valued function on $\mathbb{R}$ for all $k \in \mathbb{N}$ with $k \leq l$. We endow $C_b^l(\mathbb{R}, Z)$ with the norm

$$\|f\|_{C_b^l(\mathbb{R}, Z)} := \sum_{k=0}^{l} \left\| f^{(k)} \right\|_{L^\infty(\mathbb{R}, Z)} .$$

Given a measurable function $\omega : \mathbb{R} \to (0, \infty)$ we denote by $L^2(\omega, Z)$ the space of the strongly (Bochner-Lebesgue) measurable functions $f : \mathbb{R} \to Z$ for which the norm

$$\|f\|_{L^2(\omega,Z)} := \left(\int_{\mathbb{R}} \|f(t)\|_Z^2 \, \omega(t) dt \right)^{1/2},$$

is finite. When $\omega(t) = \exp(t^2)$ or $\omega(t) = \exp(-t^2)$ the space $L^2(\omega, Z)$ is denoted by $L^2(\exp(t^2), Z)$ or by $L^2(\exp(-t^2), Z)$ respectively. However, when $\omega \equiv 1$ we prefer to write $L^2(\mathbb{R}, Z)$ instead of $L^2(1, Z)$.

3.3.1 Boundary values of functions on the strip

Let us recall now some (at least implicitely) well-known facts related to the Hilbert transforms of vector-valued functions. Let Z be a Banach space and consider a function $f \in C_b^4(\mathbb{R}, Z) \cap L^2(\exp(t^2), Z)$. The Hilbert transform of f is defined by

$$Hf(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon < |t-s| < 1/\varepsilon} \frac{f(s)}{t-s} ds = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon < |s| < 1/\varepsilon} \frac{f(t-s)}{s} dy,$$

for $t \in \mathbb{R}$. As one can immediately check, for such f the above limit exists, for every $x \in \mathbb{R}$ (the convergence being in the norm of Z). Also, we get that

$$Hf \in C_b(\mathbb{R}, Z). \tag{38}$$

Indeed, for every $t \in \mathbb{R}$,

$$\begin{aligned}
\|Hf(t)\|_Z &\leq \frac{1}{\pi} \left\| \lim_{\varepsilon \to 0} \int_{\varepsilon < |t-s| < 1} \frac{f(s)}{t-s} ds \right\|_Z + \frac{1}{\pi} \left\| \lim_{\varepsilon \to 0} \int_{1 < |t-s| < 1/\varepsilon} \frac{f(s)}{t-s} ds \right\|_Z \\
&= \frac{1}{\pi} \left\| \lim_{\varepsilon \to 0} \int_{\varepsilon < |t-s| < 1} \frac{f(s) - f(t)}{t-s} ds \right\|_Z + \frac{1}{\pi} \left\| \lim_{\varepsilon \to 0} \int_{1 < |t-s| < 1/\varepsilon} \frac{f(s)}{t-s} ds \right\|_Z \\
&\leq \frac{2}{\pi} \|f\|_{Lip(\mathbb{R},Z)} + \frac{1}{\pi} \|f\|_{L^2(\exp(t^2),Z)} \\
&\lesssim \|f\|_{C_b^3(\mathbb{R},Z)} + \|f\|_{L^2(\exp(t^2),Z)}
\end{aligned}$$

and hence, $Hf(t)$ is uniformly bounded in Z. The continuity of Hf can be proved in a similar way, by estimating the expression $Hf(t_1) - Hf(t_2)$, when $t_1, t_2 \in \mathbb{R}$ are close to each other.

In what follows we need a vector-valued version of the Plemelj formula. The proof we give is completely similar to the one in the scalar valued case, however, we include it here for completeness (see for instance [19]).

Lemma 25. *Suppose Z is a Banach space and consider a function $f \in C_b^3(\mathbb{R}, Z) \cap L^2(\exp(t^2), Z)$. Then, when $\varepsilon \searrow 0$ we have*

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s - (t \pm i\varepsilon)} ds \to \frac{\pm f(t) + iHf(t)}{2},$$

in the norm of Z, uniformly in $t \in \mathbb{R}$.

Proof. We only consider the case of the sign "+", the other one being similar. We first show that

$$\left\| \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s - i\varepsilon} dy - \frac{f(0)}{2} - \frac{i}{2\pi} \int_{|s| > \varepsilon} \frac{f(s)}{-s} ds \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \, \varepsilon^{1/2}, \tag{39}$$

for any $\varepsilon \in (0,1)$, where the implicit constant does not depend on f or ε. Changing the variables ($s = \varepsilon\tau$), this is equivalent to

$$\left\| \int_{\mathbb{R}} f(\varepsilon\tau) \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau - \pi i f(0) \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon^{1/2}. \tag{40}$$

Notice that

$$\int_{\mathbb{R}} \left| \frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right| d\tau < \infty \quad \text{and} \quad \int_{\mathbb{R}} \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau = \pi i,$$

and hence, it remains to show that

$$\left\| \int_{\mathbb{R}} (f(\varepsilon\tau) - f(0)) \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau \right\|_Z \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon^{1/2}.$$

One can see this by a direct computation. Indeed, the quantity

$$\left\| \int_{\mathbb{R}} (f(\varepsilon\tau) - f(0)) \left(\frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right) d\tau \right\|_Z$$

is bounded by

$$\begin{aligned} \int_{\mathbb{R}} \|f(\varepsilon\tau) - f(0)\|_Z \left| \frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right| d\tau \;\lesssim\; & \int_{[-R,R]} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{1+|\tau|} d\tau \\ & + \int_{[-R,R]^c} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{\tau^2} d\tau, \end{aligned} \tag{41}$$

for any $R > 1$. Since,

$$\|f(\varepsilon\tau) - f(0)\|_Z \leq \varepsilon \|f\|_{C_b^1(\mathbb{R},Z)} |\tau|,$$

we get

$$\int_{[-R,R]} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{1+|\tau|} d\tau \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} \varepsilon R.$$

Also, since

$$\|f(\varepsilon\tau) - f(0)\|_Z \leq 2 \|f\|_{C_b^1(\mathbb{R},Z)},$$

we have

$$\int_{[-R,R]^c} \|f(\varepsilon\tau) - f(0)\|_Z \frac{1}{\tau^2} d\tau \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} / R.$$

From (41) we get now,

$$\int_{\mathbb{R}} \|f(\varepsilon\tau) - f(0)\|_Z \left| \frac{1}{\tau - i} - \mathbf{1}_{|\tau|>1}(\tau) \frac{1}{\tau} \right| d\tau \lesssim \|f\|_{C_b^1(\mathbb{R},Z)} (\varepsilon R + 1/R),$$

where the implicit constant does not depend on ε or R. Setting $R = \varepsilon^{-1/2}$ we obtain (40) and hence (39).

Now consider some function $f \in C_b^1(\mathbb{R}, Z)$ and for each $t \in \mathbb{R}$, define $f_t : \mathbb{R} \to Z$ by $f_t(x) := f(x+t)$ for all $x \in \mathbb{R}$. We can see that $f_t \in C_b^1(\mathbb{R}, Z)$ and

$$\|f_t\|_{C_b^1(\mathbb{R},Z)} = \|f\|_{C_b^1(\mathbb{R},Z)},$$

for any $t \in \mathbb{R}$. By a simple change of variables we can also observe that

$$\left\| \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f_t(s)}{s - i\varepsilon} ds - \frac{f_t(0)}{2} - \frac{i}{2\pi} \int_{|s|>\varepsilon} \frac{f_t(s)}{-s} ds \right\|_Z$$

equals

$$\left\| \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s-(t+i\varepsilon)} ds - \frac{f(t)}{2} - \frac{i}{2\pi} \int_{|t-s|>\varepsilon} \frac{f(s)}{t-s} ds \right\|_Z .$$

Hence, by applying (39) to f_t and letting $\varepsilon \to 0$ we obtain the conclusion. □

Let us introduce the some operators that quantify the boundary behaviour of analytic functions on the strip. For each $j = 0, 1$ let

$$\mathcal{H}_j : L^2\left(\exp\left(t^2\right), Z\right) \cap C_b^3\left(\mathbb{R}, Z\right) \to C_b\left(\mathbb{R}, Z\right),$$

be defined by

$$\mathcal{H}_j f(t) := \frac{-if(t) - (-1)^j Hf(t)}{2},$$

and $\mathcal{R}_j : L^2\left(\exp\left(t^2\right), Z\right) \cap C_b^3\left(\mathbb{R}, Z\right) \to C_b\left(\mathbb{R}, Z\right)$, be defined by

$$\mathcal{R}_j f(t) := \rho_j * f(t),$$

where $\rho_j : \mathbb{R} \to \mathbb{C}$ are the bounded functions

$$\rho_j(t) := \frac{1}{2\pi i} \frac{1}{1-(-1)^j it},$$

for all $t \in \mathbb{R}$. It is easy to see that, for $f \in L^2\left(\exp\left(t^2\right), Z\right)$, the quantity $\mathcal{R}_j f$ is indeed well-defined and $\mathcal{R}_j f \in C_b\left(\mathbb{R}, Z\right)$.

An easy consequence of (38) and Lemma 25 is the following fact:

Lemma 26. *Suppose (B_0, B_1) is a compatible couple of Banach spaces. Consider some functions $u_j \in C_b^3\left(\mathbb{R}, B_j\right) \cap L^2\left(\exp\left(t^2\right), B_j\right)$, $j = 0, 1$, and define $u : S \to B_0 + B_1$ by*

$$u(z) := -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{u_0(t)}{it - z} dt + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{u_1(t)}{1 + it - z} dt,$$

for all $z \in S_0$, and

$$u(j + it) := \mathcal{H}_j u_j(t) + \mathcal{R}_j u_{1-j}(t),$$

for all $t \in \mathbb{R}$. Then, $u \in C_b\left(S, B_0 + B_1\right)$.

Proof. Clearly, by (38) (for $Z = B_0 + B_1$) we have $\mathcal{H}_j u_j \in C_b\left(\mathbb{R}, B_0 + B_1\right)$. Also, by Lemma 25 (for $Z = B_0 + B_1$) we have

$$\frac{(-1)^{j+1}}{2\pi i} \int_{\mathbb{R}} \frac{u_j(t)}{j + it - (j + i\tau + (-1)^j \varepsilon)} dt \to \mathcal{H}_j u_j(\tau),$$

in $B_0 + B_1$ uniformly in $\tau \in \mathbb{R}$, when $\varepsilon \searrow 0$. On the other hand, we have $\mathcal{R}_j u_{1-j} \in C_b\left(\mathbb{R}, B_0 + B_1\right)$ and, as one can easily see,

$$\frac{(-1)^{j+1}}{2\pi i} \int_{\mathbb{R}} \frac{u_{1-j}(t)}{j + it - (1 - j + i\tau + (-1)^j \varepsilon)} dt \to \mathcal{R}_j u_{1-j}(\tau),$$

in $B_0 + B_1$ uniformly in $\tau \in \mathbb{R}$, when $\varepsilon \searrow 0$.

Hence, u is approaching its boundary values uniformly. Since u is analytic on S_0 we obtain the conclusion. □

3.3.2 Interpolation of equations

Let us illustrate by some examples the fact that, in general, the surjectivity of operators is not preserved by interpolation:

Example 1. Consider the operator $T_s : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$, defined by the formula $T_s(f,g) := f+g$, for any $(f,g) \in L^2(\mathbb{T})$. One can easily see that $T_s : L^2(\mathbb{T}) \times L^4(\mathbb{T}) \to L^2(\mathbb{T})$ and $T_s : L^4(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$ are surjective operators however, the operator

$$T_s : (L^2(\mathbb{T}) \times L^4(\mathbb{T}), L^4(\mathbb{T}) \times L^2(\mathbb{T}))_{1/2} \to (L^2(\mathbb{T}), L^2(\mathbb{T}))_{1/2}, \tag{42}$$

is not surjective. Indeed, if T_s in (42) is surjective, then

$$T_s : L^3(\mathbb{T}) \times L^3(\mathbb{T}) \to L^2(\mathbb{T})$$

is surjective, and we get the false embedding $L^2(\mathbb{T}) \hookrightarrow L^3(\mathbb{T})$.

Example 2. Let us consider now another example which is more closely related to the equations we treat in this paper. For any $p \geq 1$ let $W_\sharp^{-1,p}(\mathbb{T}^2)$ be the spaces of those distributions f that are divergences of L^p-vector fields on $\mathbb{T}^2$ and with $\widehat{f}(0) = 0$ (in general, if Z is a function space on $\mathbb{T}^2$, we denote by $Z_\sharp$ the space of those $f \in Z$ with $\widehat{f}(0) = 0$). The norm on $W_\sharp^{-1,p}(\mathbb{T}^2)$ is given by

$$\|f\|_{W_\sharp^{-1,p}(\mathbb{T}^2)} = \inf\left\{ \|f_1\|_{L^p(\mathbb{T}^2)} + \|f_2\|_{L^p(\mathbb{T}^2)} \mid f = \partial_1 f_1 + \partial_2 f_2 \right\}.$$

We have that $\operatorname{div} : L^1(\mathbb{T}^2) \to W_\sharp^{-1,1}(\mathbb{T}^2)$ and $\operatorname{div} : L^3(\mathbb{T}^2) \to W_\sharp^{-1,3}(\mathbb{T}^2)$ are surjective operators. However, the operator

$$\operatorname{div} : L^2(\mathbb{T}^2) \to (W_\sharp^{-1,1}(\mathbb{T}^2), W_\sharp^{-1,3}(\mathbb{T}^2))_{1/2}, \tag{43}$$

cannot be surjective. Indeed, since

$$\operatorname{div} : L^2(\mathbb{T}^2) \to W_\sharp^{-1,2}(\mathbb{T}^2),$$

the surjectivity of the operator in (43) would imply that

$$(W_\sharp^{-1,1}(\mathbb{T}^2), W_\sharp^{-1,3}(\mathbb{T}^2))_{1/2} \hookrightarrow W_\sharp^{-1,2}(\mathbb{T}^2),$$

which, by duality is equivalent to

$$W_\sharp^{1,2}(\mathbb{T}^2) \hookrightarrow (W_\sharp^{1,\infty}(\mathbb{T}^2), W_\sharp^{1,3/2}(\mathbb{T}^2))_{1/2}. \tag{44}$$

Note that (44) is false (see cite [11, Section 4]).

In what follows we will work in a slightly different setting. Given two pairs of Banach spaces $A_j \hookrightarrow B_j$, $j = 0, 1$ and an operator T defined on $B_0 + B_1$ such that$T : A_j \to T((B_j)$ is surjective, we study the surjectivity of $T : (A_0, A_1)_\theta \to T((B_0, B_1)_\theta)$. Lemma 28 below gives some sufficient additional conditions under which surjectivity is preserved by the complex interpolation. To state and prove Lemma 28 we need the technical Lemma 27.

We introduce first some notation needed in the statement of Lemma 27. Let $\varphi \in C_c^\infty([-1,1], \mathbb{R})$ be a function of integral 1 and $\varepsilon > 0$. Define the function φ_ε by $\varphi_\varepsilon(t) := \varepsilon^{-1}\varphi(\varepsilon^{-1}t)$, for any $t \in \mathbb{R}$. For any $\varepsilon > 0$, and any (other) function $g : \mathbb{R} \to Z$ taking values in some Banach space Z, we define the function $g_\varepsilon := g * \varphi_\varepsilon$. With this notation we state the following:

Lemma 27. *Let A, X_0, X_1, B_0, B_1, E, F be Banach spaces such that $A, B_0, B_1 \hookrightarrow E$ and consider a bounded linear operator $T : E \to F$. Denote $A_0 := A \cap X_0$ and $A_1 := A \cap X_1$. Suppose moreover that the following conditions are satisfied:*

(i) $B_1 \hookrightarrow X_1 \hookrightarrow X_0$ and $B_0 \hookrightarrow A_0$;

(ii) A and X_1 have a separable preduals;

(iii) $T : A_j \to F$ and $T : B_j \to F$ are bounded for each $j = 0, 1$ and $T(B_1) \hookrightarrow T(A_1)$.

Then, for each $j = 0, 1$ we have the following:

For any $v_j \in L^2(\exp(t^2), B_j)$ and any $\varepsilon > 0$ there exists a function $u_j^\varepsilon \in L^2(\exp(t^2), A_j) \cap C_b^3(\mathbb{R}, A_j)$, such that

$$Tu_j^\varepsilon(t) = Tv_{j,\varepsilon}(t), \tag{45}$$

for any in $t \in \mathbb{R}$, and satisfying the estimates:

$$\left\|u_j^\varepsilon\right\|_{L^2(A_j)} \lesssim \|v_j\|_{L^2(B_j)}, \tag{46}$$

and

$$\left\|\mathcal{R}_{1-j} u_j^\varepsilon\right\|_{L^2(\exp(-t^2), A_{1-j})} \lesssim \|v_j\|_{L^2(\exp(t^2), B_j)} + \delta_{j0} \left\|\mathcal{R}_{1-j} v_j\right\|_{L^2(\exp(-t^2), B_{1-j})}. \tag{47}$$

where all the implicit constants do not depend on v_j and ε.

(Here, $v_{j,\varepsilon}(t) = v_j * \varphi_\varepsilon(t)$ and δ_{j0} is the Kronecker symbol, i.e., we have $\delta_{j0} = 1$ if $j = 0$ and $\delta_{j0} = 0$ if $j \neq 0$.)

Roughly speaking the conditions (45)–(47) are describing the fact that the equation $Tu = Tv$ can be solved efficiently on the boundary of the strip S. The role of Lemma 27 is to transform the easy to state conditions (i)–(iii) into the more technical conditions (45)–(47).

In order to prove Lemma 27 we need some simple facts that are easy consequences of classical inequalities.

Fact 1. *Suppose Z is a Banach space and consider some function $g \in L^2_{loc}(\mathbb{R}, Z)$. Then, we have:*

(i) $\|g_\varepsilon\|_{L^2(\exp(-t^2), Z)} \lesssim \|g\|_{L^2(\exp(-t^2), Z)}$, uniformly in $\varepsilon > 0$;

(ii) $\|g_\varepsilon\|_{L^2(\exp(t^2), Z)} \lesssim \|g\|_{L^2(\exp(t^2), Z)}$, uniformly in $\varepsilon > 0$.

Proof of Fact 1. We prove only item (i), item (ii) being similar. By Minkowski's inequality we have

$$\begin{aligned}
\|g_\varepsilon\|_{L^2(\exp(-t^2), Z)} &= \left(\int_{\mathbb{R}} e^{-t^2} \left\|\int_{B(0,1)} g(t - \varepsilon s)\varphi(s) ds\right\|_Z^{} dt\right)^{1/2} \\
&\leq \int_{B_{\mathbb{R}}(0,1)} \left(\int_{\mathbb{R}} e^{-t^2} \|g(t - \varepsilon s)\|_Z^2 \varphi(s) dt\right)^{1/2} ds \\
&= \int_{B_{\mathbb{R}}(0,1)} \left(\int_{\mathbb{R}} e^{-(t+\varepsilon s)^2} \|g(t)\|_Z^2 dt\right)^{1/2} \varphi(s) ds \\
&\lesssim \left(\int_{\mathbb{R}} e^{-(t+\varepsilon s)^2} \|g(t)\|_Z^2 dt\right)^{1/2} \\
&\lesssim \|g_\varepsilon\|_{L^2(\exp(-t^2), Z)},
\end{aligned}$$

where we have used the fact that $e^{-(t+\varepsilon s)^2} \sim e^{-t^2}$, when $s \in B_{\mathbb{R}}(0, 1)$. □

Fact 2. *Suppose Z is a Banach space and consider some function $g \in L^2(\exp(-t^2), Z)$. Then, we have $\|\mathcal{R}_j g\|_{L^2(\exp(-t^2),Z)} \lesssim \|g\|_{L^2(\exp(t^2),Z)}$, for any $j = 0, 1$.*

Proof of Fact 2. Fix $j \in \{0, 1\}$. Note first that if $g \in L^2(\exp(-t^2), Z)$, then $g \in L^1(\mathbb{R}, Z)$. Using the boundedness of the function ρ_j on $\mathbb{R}$, one writes

$$\begin{aligned}
\|\mathcal{R}_j g\|_{L^2(\exp(-t^2),Z)} &= \left(\int_{\mathbb{R}} e^{-t^2} \left\| \int_{\mathbb{R}} \rho_j(t-s) g(s) ds \right\|_Z^2 dt \right)^{1/2} \\
&\lesssim \left(\int_{\mathbb{R}} e^{-t^2} \left(\int_{\mathbb{R}} \|g(s)\|_Z \, ds \right)^2 dt \right)^{1/2} \\
&\lesssim \int_{\mathbb{R}} \|g(s)\|_Z \, ds.
\end{aligned}$$

By the Cauchy-Schwarz inequality we get

$$\int_{\mathbb{R}} \|g(s)\|_Z \, ds \leq \|g\|_{L^2(\exp(t^2),Z)},$$

and Fact 2 is proved. □

Let us introduce some more notation. Let Z be a Banach space and fix some $N \in \mathbb{N}^*$. For any function $g \in L^1_{loc}(\mathbb{R}, Z)$ we denote by $\mathbb{E}_N g$ the conditional expectation of g with the respect to the σ-algebra generated by the intervals $I_N^k := [k/N, (k+1)/N)$, were $k \in \mathbb{Z}$. In other words, if $(g)_I$ is the mean of g on one of these intervals I, i.e.,

$$(g)_I := \frac{1}{|I|} \int_I g(t) dt,$$

we define the corresponding conditional expectation of g by

$$\mathbb{E}_N g := \sum_{k \in \mathbb{Z}} (g)_{I_N^k} \mathbf{1}_{I_N^k}.$$

See [12, Chapter 5] for some fundamental properties of the conditional expectation operator $\mathbb{E}_N$.

Now we can pass to the proof of Lemma 27.

Proof of Lemma 27. For each $j = 0, 1$ consider some functions $v_j \in L^2(\exp(t^2), B_j)$. In the case where $j = 0$ one can simply set $u_0^\varepsilon := v_{0,\varepsilon}$. Clearly, $u_0^\varepsilon \in L^2(\exp(t^2), A_0) \cap C_b^3(\mathbb{R}, A_0)$. It is also clear that the conditions (45), (46) are satisfied thanks to the fact that $B_0 \hookrightarrow A_0$. Let us verify (47). We have

$$\|\mathcal{R}_1 u_0^\varepsilon\|_{L^2(\exp(-t^2),A_1)} \sim \|\mathcal{R}_1 v_{0,\varepsilon}\|_{L^2(\exp(-t^2),A)} + \|\mathcal{R}_1 v_{0,\varepsilon}\|_{L^2(\exp(-t^2),X_1)}, \tag{48}$$

and it remains to bound each term in the right hand side of (48). Since $B_0 \hookrightarrow A \cap X_0$, we have in particular that $B_0 \hookrightarrow A$ and hence,

$$\|\mathcal{R}_1 v_{0,\varepsilon}\|_{L^2(\exp(-t^2),A)} \lesssim \|\mathcal{R}_1 v_{0,\varepsilon}\|_{L^2(\exp(-t^2),B_0)} \lesssim \|v_{0,\varepsilon}\|_{L^2(\exp(t^2),B_0)} \lesssim \|v_0\|_{L^2(\exp(t^2),B_0)}, \tag{49}$$

where for the second "$\lesssim$" we have used Fact 2 and for the third "$\lesssim$" we have used Fact 1 (ii). Since $B_1 \hookrightarrow X_1$,

$$\begin{aligned}\left\|\mathcal{R}_1 v_{0,\varepsilon}\right\|_{L^2(\exp(-t^2),X_1)} &\lesssim \left\|\mathcal{R}_1 v_{0,\varepsilon}\right\|_{L^2(\exp(-t^2),B_1)} = \left\|(\mathcal{R}_1 v_0)_\varepsilon\right\|_{L^2(\exp(-t^2),B_1)} \\ &\lesssim \left\|\mathcal{R}_1 v_0\right\|_{L^2(\exp(-t^2),B_1)}, \end{aligned} \tag{50}$$

where for "=" we have used the relation $\rho * \varphi_\varepsilon = \varphi_\varepsilon * \rho$ and for the last "$\lesssim$" we have used Fact 1 (i). From (48), (49) and (50) we obtain (47) in the case $j = 0$.

Now, we deal with the case $j = 1$. By using the open mapping theorem one gets that, if $b \in B_1$, then there exists $a \in A_1$ such that $Ta = Tb$ and $\|a\|_{A_1} \leq C \|b\|_{B_1}$ for some constant $C > 0$. As a consequence, for each $k \in \mathbb{Z}$ we can find some elements $a_N^k \in A_1$ with

$$Ta_N^k = T\left(v_1\right)_{I_N^k}$$

and such that

$$\left\|a_N^k\right\|_{A_1} \leq C \left\|\left(v_1\right)_{I_N^k}\right\|_{B_1}.$$

Hence, defining $u_N : \mathbb{R} \to A_1$ by

$$u_{1,N} := \sum_{k\in\mathbb{Z}} a_N^k \mathbf{1}_{I_N^k},$$

we have

$$Tu_{1,N}(t) = T\mathbb{E}_N v_1(t), \tag{51}$$

for any $t \in \mathbb{R}$, and

$$\left\|u_{1,N}(t)\right\|_{A_1} \lesssim \left\|\mathbb{E}_N v_1(t)\right\|_{B_1}, \tag{52}$$

uniformly in $t \in \mathbb{R}$.

Define now the function $u_{1,N}^\varepsilon := u_{1,N} * \varphi_\varepsilon = (u_{1,N})_\varepsilon$, the convolution being in the t variable. Thanks to (51) we have

$$Tu_{1,N}^\varepsilon(t) = T(\mathbb{E}_N v_{1,})_\varepsilon(t), \tag{53}$$

for any $t \in \mathbb{R}$.

Let us observe that, when $N \to \infty$,

$$\left\|(\mathbb{E}_N v_1)_\varepsilon(t) - v_{1,\varepsilon}(t)\right\|_{B_1} \to 0, \tag{54}$$

uniformly in $t \in \mathbb{R}$.

Indeed, by Jensen's inequality and [12, Corollary 2, p. 126] we have

$$\begin{aligned}\left\|(\mathbb{E}_N v_1)_\varepsilon(t) - v_{1,\varepsilon}(t)\right\|_{B_1} &\leq \int_{B_{\mathbb{R}}(t,1/\varepsilon)} \left\|(\mathbb{E}_N v_1)_\varepsilon(s) - v_{1,\varepsilon}(s)\right\|_{B_1} \varphi_\varepsilon(t-s)ds \\ &\leq \left(\int_{B_{\mathbb{R}}(t,1/\varepsilon)} \left\|(\mathbb{E}_N v_1)_\varepsilon(s) - v_{1,\varepsilon}(s)\right\|_{B_1}^2 \varphi_\varepsilon(t-s)ds\right)^{1/2} \\ &\lesssim \varepsilon \left\|\mathbb{E}_N v_1 - v_1\right\|_{L^2(\mathbb{R},B_1)} \to 0. \end{aligned}$$

Also one easily observe that the sequence of functions $(u_{1,N}^\varepsilon)_{N\geq 1}$ is equi-continuous and uniformly bounded. Indeed, using the Cauchy-Schwaz inequality,

$$\begin{aligned}\left\|u_{1,N}^\varepsilon(t_1) - u_{1,N}^\varepsilon(t_2)\right\|_{A_1} &\leq \int_{\mathbb{R}} \left\|u_{1,N}(s)\right\|_{A_1} \left|\varphi_\varepsilon(t_1-s) - \varphi_\varepsilon(t_2-s)\right| ds \\ &\leq \left\|u_{1,N}\right\|_{L^2(\mathbb{R},A_1)} \left(\int_{\mathbb{R}} \left|\varphi_\varepsilon(t_1-s) - \varphi_\varepsilon(t_2-s)\right|^2 ds\right)^2, \end{aligned} \tag{55}$$

for any $t_1, t_2 \in \mathbb{R}$ and it remains to notice that, by (52), we have

$$\|u_{1,N}\|_{L^2(\mathbb{R},A_1)} \leq \|\mathbb{E}_N v_1\|_{L^2(\mathbb{R},B_1)} \lesssim \|v_1\|_{L^2(\mathbb{R},B_1)} < \infty.$$

Since $(u^\varepsilon_{1,N})_{N\geq 1}$ is equi-continuous and uniformly bounded sequence, and A, X_1 have a separable preduals, there exists some $u^\varepsilon_1 \in L^2(\mathbb{R}, A_1)$ such that $u^\varepsilon_{1,N}(t) \to u^\varepsilon_1(t)$ in the w^*-topology on A and in the w^*-topology on X_1, up to a subsequence, for all $t \in \mathbb{R}$. By an argument similar to the one used in (55), it is easy to see that one can choose $u^\varepsilon_1 \in C^3_b(\mathbb{R}, A_1)$. Thanks to (53) and (54) one can write

$$Tu^\varepsilon_1(t) = Tv_{1,\varepsilon}(t),$$

for all $t \in \mathbb{R}$, which proves (45). In order to verify (46) one uses the Young inequality and (52):

$$\begin{aligned} \|u^\varepsilon_1\|_{L^2(\mathbb{R},A_1)} &\leq \liminf_{N\to\infty} \left\|u^\varepsilon_{1,N}\right\|_{L^2(\mathbb{R},A_1)} \leq \liminf_{N\to\infty} \|u_{1,N}\|_{L^2(\mathbb{R},A_1)} \\ &\lesssim \liminf_{N\to\infty} \|\mathbb{E}_N v_1\|_{L^2(\mathbb{R},B_1)} \lesssim \|v_1\|_{L^2(\mathbb{R},B_1)}. \end{aligned} \tag{56}$$

Observe now that, as in (56), we get

$$\|u^\varepsilon_1\|_{L^2(\exp(t^2),A_1)} \lesssim \|v_{1,\varepsilon}\|_{L^2(\exp(t^2),B_1)} \lesssim \|v_1\|_{L^2(\exp(t^2),B_1)}, \tag{57}$$

where for the second "$\lesssim$" we have used Fact 1 (ii). In particular, we have that $u^\varepsilon_1 \in L^2(\exp(t^2), A_1) \cap C^3_b(\mathbb{R}, A_1)$.

Let us verify now that u^ε_1 also satisfies (47). We can write

$$\|\mathcal{R}_0 u^\varepsilon_1\|_{L^2(\exp(-t^2),A_0)} \lesssim \|u^\varepsilon_1\|_{L^2(\exp(t^2),A_0)} \lesssim \|u^\varepsilon_1\|_{L^2(\exp(t^2),A_1)}, \tag{58}$$

where for the first "$\lesssim$" we have used Fact 2 and for the second "$\lesssim$" we have used the embedding $X_1 \hookrightarrow X_0$ (that implies $A_1 \hookrightarrow A_0$). Combining (57) with (58) we get

$$\|\mathcal{R}_0 u^\varepsilon_1\|_{L^2(\exp(-t^2),A_0)} \lesssim \|v_1\|_{L^2(\exp(t^2),B_1)},$$

and (47) is proved in the case $j = 1$. Lemma 27 is proved. □

We are now able to state and prove the main result of subsection 3.3:

Lemma 28. *Fix some number $\theta \in (0,1)$. Let the Banch spaces A, X_0, X_1, A_0, A_1, B_0, B_1, E, F and the operator T be as in Lemma 27. Moreover, we assume that X_0, X_1 and B_1 are UMD spaces and that $(X_0, X_1)_\theta$ has a separable predual. Then, for any $b \in (B_0, B_1)_\theta$ there exists some $a \in A \cap (X_0, X_1)_\theta$ such that*

$$Ta = Tb,$$

and

$$\|a\|_{A\cap(X_0,X_1)_\theta} \lesssim \|b\|_{(B_0,B_1)_\theta}. \tag{59}$$

Proof. Fix some $b \in \mathcal{C}_\theta(B_0|B_1)$. Consider $v \in \mathcal{F}^2(B_0|B_1)$ such that $v(\theta) = b$ and

$$\|v\|_{\mathcal{F}^2(B_0|B_1)} \leq 2\|b\|_{(B_0|B_1)_\theta}. \tag{60}$$

Since we can replace (if necessary) v by $\exp(z^2 - \theta^2)v$, we can assume without loss of generality that $v_j \in L^2(\exp(t^2), B_j)$, where $v_j(t) := v(j + it)$, for all $t \in \mathbb{R}$. Define, for each $\varepsilon \in (0,1)$, the function v_ε on S by $v_\varepsilon := v * \varphi_\varepsilon$, as in the statement of Lemma 27.

Note that, thanks to Lemma 26,

$$v_{j,\varepsilon}(t) := \mathcal{H}_j v_{j,\varepsilon}(t) + \mathcal{R}_j v_{1-j,\varepsilon}(t),$$

for all $t \in \mathbb{R}$. From this identity, since B_1 is an UMD space, we can write

$$\begin{aligned} \|\mathcal{R}_1 v_{0,\varepsilon}\|_{L^2(\mathbb{R},B_1)} &\leq \|v_{1,\varepsilon}\|_{L^2(\mathbb{R},B_1)} + \|\mathcal{H} v_{1,\varepsilon}\|_{L^2(\mathbb{R},B_1)} \\ &\lesssim \|v_{1,\varepsilon}\|_{L^2(\mathbb{R},B_1)} \lesssim \|v_1\|_{L^2(\mathbb{R},B_1)} \\ &\lesssim \|b\|_{(B_0|B_1)_\theta}, \end{aligned}$$

where for the second "$\lesssim$" we have used Young's inequality and for the last "$\lesssim$" we have used (60). In particular, we get

$$\|\mathcal{R}_1 v_{0,\varepsilon}\|_{L^2(\exp(-t^2),B_1)} \lesssim \|b\|_{(B_0|B_1)_\theta}. \tag{61}$$

By Lemma 27 there exist some functions $u_j^\varepsilon \in L^2(\exp(t^2), A_j) \cap C_b^2(\mathbb{R}, A_j)$ satisfying (45), (46), (47). Define $\tilde{u}^\varepsilon : S \to A_0 + A_1$ by

$$\tilde{u}^\varepsilon(z) := -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{u_0^\varepsilon(t)}{it - z} dt + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{u_1^\varepsilon(t)}{1 + it - z} dt, \tag{62}$$

for all $z \in S_0$, and

$$\tilde{u}_j^\varepsilon(t) = \tilde{u}^\varepsilon(j + it) := \mathcal{H}_j u_j^\varepsilon(t) + \mathcal{R}_j u_{1-j}^\varepsilon(t), \tag{63}$$

for all $t \in \mathbb{R}$. Notice that, since $u_j^\varepsilon \in L^2(\exp(t^2), A_j) \cap C_b^3(\mathbb{R}, A_j)$, thanks to Lemma 26, $\tilde{u}^\varepsilon$ is well-defined and $\tilde{u}^\varepsilon \in C_b(S, A_0 + A_1)$. Let us verify that $\exp(z^2 - \theta^2)\tilde{u}^\varepsilon \in \mathcal{F}^2(A_0, X_0 \mid A_1, X_1)$. We show that, for any $j = 0, 1$, we have the estimate:

$$\left(\int_{\mathbb{R}} e^{-t^2} \|\Lambda_j \tilde{u}^\varepsilon(j + it)\|_{X_j}^2 dt\right)^{1/2} \lesssim \|b\|_{(B_0|B_1)_\theta}, \tag{64}$$

for any bounded linear operator $\Lambda_j : A_j \to X_j$ with $\|\Lambda_j\| \leq 1$, the implicit constant not depending on Λ_j.

Using (63) we write:

$$\begin{aligned} \left(\int_{\mathbb{R}} e^{-t^2} \|\Lambda_j \tilde{u}^\varepsilon(j + it)\|_{X_j}^2 dt\right)^{1/2} &\leq \left(\int_{\mathbb{R}} e^{-t^2} \left\|\mathcal{H}_j \Lambda_j u_j^\varepsilon(t)\right\|_{X_j}^2 dt\right)^{1/2} \\ &\quad + \left(\int_{\mathbb{R}} e^{-t^2} \left\|\Lambda_j \mathcal{R}_j u_{1-j}^\varepsilon(t)\right\|_{X_j}^2 dt\right)^{1/2}. \end{aligned} \tag{65}$$

Since the spaces X_j have the UMD property, we get

$$\begin{aligned} \left(\int_{\mathbb{R}} e^{-t^2} \left\|\mathcal{H}_j \Lambda_j u_j^\varepsilon(t)\right\|_{X_j}^2 dt\right)^{1/2} &\leq \left\|\mathcal{H}_j \Lambda_j u_j^\varepsilon\right\|_{L^2(\mathbb{R},X_j)} \lesssim \left\|\Lambda_j u_j^\varepsilon\right\|_{L^2(\mathbb{R},X_j)} \\ &\leq \left\|u_j^\varepsilon\right\|_{L^2(\mathbb{R},A_j)} \leq \|v_j\|_{L^2(\mathbb{R},B_j)} \\ &\lesssim \|b\|_{(B_0|B_1)_\theta}, \end{aligned} \tag{66}$$

where for the third "$\leq$" we have used (46) and for the last "$\lesssim$" we have used (60). It remains to estimate the second term in the right hand side of (65):

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-t^2}\left\|\Lambda_j\mathcal{R}_j u_{1-j}^{\varepsilon}(t)\right\|_{X_j}^2 dt\right)^{1/2} &\leq \left\|\mathcal{R}_j u_{1-j}^{\varepsilon}\right\|_{L^2(\exp(-t^2)A_j)}\\ &\lesssim \|v_{1-j}\|_{L^2(\exp(t^2),B_{1-j})} + \delta_{j1}\left\|\mathcal{R}_j v_{1-j}\right\|_{L^2(\exp(-t^2),B_j)}\\ &\lesssim \|v_{1-j}\|_{L^2(\mathbb{R},B_{1-j})} + \delta_{j1}\left\|\mathcal{R}_j v_{1-j}\right\|_{L^2(\exp(-t^2),B_j)}\\ &\lesssim \|b\|_{(B_0|B_1)_\theta}, \end{aligned} \tag{67}$$

where for the first "$\lesssim$" we have used (47) and for the last "$\lesssim$" we have used (60) and (61). By (65), (66), (67) we have proved (64). Hence, we have obtained

$$\left\|\exp(z^2-\theta^2)\tilde{u}^{\varepsilon}\right\|_{\mathcal{F}^2(A_0,X_0\,|\,A_1,X_1)} \lesssim \|b\|_{(B_0|B_1)_\theta}.$$

This implies that for $a^{\varepsilon} := \tilde{u}^{\varepsilon}(\theta)$ we have

$$\|a^{\varepsilon}\|_{(A_0,X_0\,|\,A_1,X_1)_\theta} \lesssim \|b\|_{(B_0|B_1)_\theta}. \tag{68}$$

Note that, by Proposition 22, $(B_0|B_1)_\theta = (B_0,B_1)_\theta$, and by Corollary 24, $(A_0,X_0\mid A_1,X_1)_\theta \hookrightarrow A\cap(X_0,X_1)_\theta$. From this and (68) we get

$$\|a^{\varepsilon}\|_{A\cap(X_0,X_1)_\theta} \lesssim \|b\|_{(B_0,B_1)_\theta}. \tag{69}$$

We observe that for $b_\varepsilon := v_\varepsilon(\theta)$ we have

$$Ta^{\varepsilon} = Tb_{\varepsilon}. \tag{70}$$

Indeed, by applying Lemma 20, (62), the continuity of $T: E\to F$ and (45), one gets

$$T\tilde{u}^{\varepsilon}(\theta) - Lv_\varepsilon(\theta) = -\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{Tu_0^{\varepsilon}(t)-Tv_{0,\varepsilon}(t)}{it-\theta}dt + \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{Tu_1^{\varepsilon}(t)-Tv_{1,\varepsilon}(t)}{1+it-\theta}dt = 0.$$

We let $\varepsilon\to 0$. Since $v_{j,\varepsilon}\to v_j$ in $L^2(\mathbb{R},B_j)$, for each $j=0,1$ we get that $b_\varepsilon\to b$ in $(B_0|B_1)_\theta = (B_0,B_1)_\theta$. Also, thanks to (69), since A and$(X_0,X_1)_\theta$ have separable preduals, by the sequential Banach-Aloglu theorem, there exists some $a\in A\cap(X_0,X_1)_\theta$ such that $a^{1/n}\to a$ ($n\in\mathbb{N}^*$) in the w^*-topology on A and in the w^*-topology on $(X_0,X_1)_\theta$, up to a subsequence. Also, by (69) we get

$$\|a\|_{A\cap(X_0,X_1)_\theta} \lesssim \|b\|_{(B_0,B_1)_\theta}.$$

It follows that $Tb_{1/n}\to Tb$ and $Ta^{1/n}\to Ta$ in the w^*-topology of F, up to a subsequence. Consequently, by (70) we have

$$Ta = Tb.$$

Since $\mathcal{C}_\theta(B_0|B_1)$ is dense in $(B_0|B_1)_\theta = (B_0,B_1)_\theta$ we can use the above compactness argument in order to obtain a solution for any $b\in(B_0|B_1)_\theta$. Lemma 28 is proved. □

Remark 29. *One can easily adapt Lemma 27 and Lemma 28 to the more general case of the equations $Tu = Lv$, where $T, L: E\to F$ are possibly different operators. For this we have to change the conditions (i) and (iii) in Lemma 27 by*

(i') $B_1\hookrightarrow X_1\hookrightarrow X_0$ and there exists an operator $L_T: E\to F$ such that $L_T: B_j\to A_j$ is bounded for each $j=0,1$ and $T\circ L_T = L$ on B_0;

(iii') $T: A_j\to F$ and $L: B_j\to F$ are bounded for each $j=0,1$ and $L(B_1)\hookrightarrow T(A_1)$.

The modifications needed for the corresponding proofs are minor. However, for the sake of simplicity, we preferred to present the proofs only in the case $T=L$.

4 Spectral analysis

In this section we study the solutions of divergence-like equations via L^2-based Fourier analysis methods. This is done by a slight modification of the ideas of Bourgain and Brézis used in the proof of [5, Lemma 2]. While in this case the techniques we use are essentially those of Bourgain and Brézis, we give more general existence results that take into account the shape of the Fourier spectrum of solutions. The final results of this section will represent the "1-endpoint" when we apply the $\mathcal{W}$-method (see Lemma 35 and Lemma 36 below).

4.1 Symbols with bounded Fourier transform

Let $1 \leq \ell \leq d$ be some integers and let $m : \mathbb{R}^d \to \mathbb{C}$ be a function. We say that m is an ℓ-BB symbol[5] if the following conditions are satisfied:

(i) there exists a constant $C > 0$ such that, in the case $\ell < d$,

$$\int_{\mathbb{R}^{d-\ell}} |\partial_1^{\alpha_1} ... \partial_\ell^{\alpha_l} m(\xi', \xi'')| \, d\xi'' \leq \frac{C}{|\xi'|^{\ell+|\alpha|}}, \tag{71}$$

for all $\alpha = (\alpha_1, ..., \alpha_\ell) \in \{0,1\}^\ell$ and all $\xi' \in (0,\infty)^\ell$, and, in the case $\ell = d$,

$$|\partial_1^{\alpha_1} ... \partial_d^{\alpha_d} m(\xi)| \leq \frac{C}{|\xi|^{\ell+|\alpha|}}, \tag{72}$$

for all $\alpha = (\alpha_1, ..., \alpha_d) \in \{0,1\}^d$ and all $\xi \in (0,\infty)^d$;

(ii) m is an odd function in each of the components ξ_1, ξ_2,....,ξ_ℓ, i.e.,

$$m\left(\xi_1, ..., \xi_{j-1}, -\xi_j, \xi_{j+1}...., \xi_d\right) = -m\left(\xi_1, ..., \xi_{j-1}, \xi_j, \xi_{j+1}...., \xi_d\right), \tag{73}$$

for all $1 \leq j \leq \ell$, and all $\xi_1, \xi_2, ..., \xi_d \in \mathbb{R}$.

For any integer ν we denote by I_ν the interval $[2^{\nu-1}, 2^\nu]$. For every $k' = (k_1, ..., k_\ell) \in \mathbb{Z}^\ell$ we consider the positive dyadic box $I_{k'} := I_{k_1} \times ... \times I_{k_\ell}$ and we associate to it the ℓ-symmetric set:

$$s_\ell(I_{k'}) := \bigcup_{\alpha_1,...,\alpha_\ell \in \{0,1\}} ((-1)^{\alpha_1} I_{k_1}) \times ... \times ((-1)^{\alpha_\ell} I_{k_\ell}) \subset \mathbb{R}^\ell.$$

The next technical Lemma is the basis for all of our results in this section. Its proof consists in slightly adapting some arguments of Bourgain and Br´ezis (see [5, Lemma 3, p. 404]):

Lemma 30. *Let $d \geq 1$ be an integer and let $m : \mathbb{R}^d \to \mathbb{C}$ be an ℓ-BB symbol for some $1 \leq \ell \leq d$ and some constant C.*

Then,

$$\sum_{k' \in \mathbb{Z}^\ell} \left| \int_{s_\ell(I_{k'}) \times M} m(\xi) e^{i\langle \xi, x \rangle} d\xi \right| \lesssim C,$$

for any measurable subset $M \subseteq \mathbb{R}^{d-\ell}$ of finite measure, uniformly in $x \in \mathbb{R}^d$ and in M. (By convention, if $\ell = d$, then $s_\ell(I_{k'}) \times M$ is replaced by $s_d(I_{k'})$.)

[5]Here, BB stands for "Bourgain-Brézis".

Proof. We first prove Lemma 30 in the case where m is a d-BB symbol. In this case we have to prove that

$$\sum_{k\in\mathbb{Z}^d}\left|\int_{s_d(I_k)} m(\xi)e^{i\langle\xi,x\rangle}d\xi\right|\lesssim C, \tag{74}$$

uniformly in $x=(x_1,...,x_d)\in\mathbb{R}^d$.

Since m is odd in the variables $\xi_1,...,\xi_d$, for any $k\in\mathbb{Z}^d$ one can write,

$$\int_{s_d(I_k)} m(\xi)e^{i\langle\xi,x\rangle}d\xi=\int_{s_d(I_k)} m(\xi)\prod_{j=1}^{d}e^{i\xi_j x_j}d\xi=(2i)^d\int_{I_k} m(\xi)\prod_{j=1}^{d}\sin\left(\xi_j x_j\right)d\xi. \tag{75}$$

Now, let us notice that whenever $a:\mathbb{R}^d\to\mathbb{C}$, $b_1,...,b_d:\mathbb{R}\to\mathbb{C}$ are sufficiently smooth functions and $J_j=[q_j,r_j]$, $(q_j<r_j)$, $j=1,...,d$ are d intervals, we have

$$\int_{J_1\times...\times J_d} a\left(\xi\right)\prod_{j=1}^{d}b_j\left(\xi_j\right)d\xi=\sum_{\alpha\in\{0,1\}^d}\int_{J_1\times...\times J_d}\left(-\nabla\right)^{\alpha}a\left(\xi\right)\prod_{j=1}^{d}\left[b_j\left(\xi_j\right)\right]_{J_j}^{\alpha_j}d\xi, \tag{76}$$

where, for each $1\leq j\leq d$, the quantity $\left[b_j\left(\xi_j\right)\right]_{J_j}^{\alpha_j}$ is defined[6] as follows

$$\left[b_j\left(\xi_j\right)\right]_{J_j}^{0}:=\left(\int_{q_j}^{r_j}b_j\left(t\right)dt\right)\delta_{r_j}\left(\xi_j\right)\quad\text{and}\quad\left[b_j\left(\xi_j\right)\right]_{J_j}^{1}:=\int_{q_j}^{\xi_j}b_j\left(t\right)dt,$$

where δ_{r_j} is the Dirac measure on $\mathbb{R}$ concentrated in r_j. The formula (76) easily follows by induction on d and integration by parts.

Fix some integers $k_1,...,k_d$ and consider the intervals $I_{k_j}=\left[2^{k_j-1},2^{k_j}\right]$, $j=1,...,d$. By applying (76) to the functions $a=m$ and $b_j\left(\xi_j\right)=\sin\left(\xi_j x_j\right)$, we obtain

$$\int_{I_{k_1}\times...\times I_{k_d}} m(\xi)\prod_{j=1}^{d}\sin\left(\xi_j x_j\right)d\xi=\sum_{\alpha\in\{0,1\}^d}\int_{I_{k_1}\times...\times I_{k_d}}\left(-\nabla\right)^{\alpha}m\left(\xi\right)\prod_{j=1}^{d}\left[\sin\left(\xi_j x_j\right)\right]_{I_{k_j}}^{\alpha_j}d\xi. \tag{77}$$

By a direct computation,

$$\left|\int_J\sin\left(t\cdot x_j\right)dt\right|\lesssim\min\left(4^{k_j}\left|x_j\right|,\frac{1}{\left|x_j\right|}\right),$$

for any subinterval $J\subset I_{k_j}$, and consequently,

$$\left|\left[\sin\left(\xi_j x_j\right)\right]_{I_{k_j}}^{\alpha_j}\right|\lesssim\min\left(4^{k_j}\left|x_j\right|,\frac{1}{\left|x_j\right|}\right), \tag{78}$$

for any $j=1,...,d$, any $\alpha_j\in\{0,1\}$ and any $\xi_j\in I_{k_j}$.

Let us fix some integer $0\leq l\leq d$ and consider the case of $\alpha=(1,...,1,0,...,0)\in\{0,1\}^d$ (l values equal to 1). By (78) we get that the quantity

$$\left|\int_{I_{k_1}\times...\times I_{k_d}}\left(-\nabla\right)^{\alpha}m\left(\xi\right)\prod_{j=1}^{d}\left[\sin\left(\xi_j x_j\right)\right]_{I_{k_j}}^{\alpha_j}d\xi\right|$$

[6]Strictly speaking we should write $\left[b_j\left(\cdot\right)\right]_{J_j}^{\alpha_j}\left(\xi_j\right)$ instead of $\left[b_j\left(\xi_j\right)\right]_{J_j}^{\alpha_j}$. However, for simplicity we prefer here to use the last notation.

is bounded by

$$\left(\int_{I_{k_1}\times...\times I_{k_l}}\left|\partial_1...\partial_l m\left(\xi_1,...,\xi_l,2^{k_{l+1}},...,2^{k_d}\right)\right|d\xi_1...d\xi_l\right)\prod_{j=1}^{d}\min\left(4^{k_j}\left|x_j\right|,\frac{1}{\left|x_j\right|}\right). \tag{79}$$

Fix some index $\mu\in\{1,...,d\}$. Using the fact that m is a d-BB symbol one can write (see (72))

$$\begin{aligned}\int_{I_{k_1}\times...\times I_{k_l}}\left|\partial_1...\partial_l m\left(\xi_1,...,\xi_l,2^{k_{l+1}},...,2^{k_d}\right)\right|d\xi_1...d\xi_l &\leq C\int_{I_{k_1}\times...\times I_{k_l}}\frac{1}{2^{k_\mu(d+l)}}d\xi_1...d\xi_l\\ &\sim C\frac{2^{k_1+...+k_l}}{2^{k_\mu(d+l)}}.\end{aligned}$$

Combining this with (79) we get that

$$\left|\int_{I_{k_1}\times...\times I_{k_d}}(-\nabla)^\alpha m\left(\xi\right)\prod_{j=1}^{d}\left[\sin\left(\xi_j x_j\right)\right]_{I_{k_j}}^{\alpha_j}d\xi\right|\lesssim C\frac{2^{k_1+...+k_l}}{2^{k_\mu(d+l)}}\prod_{j=1}^{d}\min\left(4^{k_j}\left|x_j\right|,\frac{1}{\left|x_j\right|}\right),$$

and hence, for $\Gamma_\mu:=\left\{k=(k_1,...,k_d)\in\mathbb{Z}^d\mid k_\mu=\max_{1\leq j\leq d}k_j\right\}$,

$$\sum_{k\in\Gamma_\mu}\left|\int_{I_{k_1}\times...\times I_{k_d}}(-\nabla)^\alpha m\left(\xi\right)\prod_{j=1}^{d}\left[\sin\left(\xi_j x_j\right)\right]_{I_{k_j}}^{\alpha_j}d\xi\right|$$

is bounded by

$$C\sum_{k\in\Gamma_\mu}\frac{2^{k_1+...+k_l}}{2^{k_\mu(d+l)}}\prod_{j=1}^{d}\min\left(4^{k_j}\left|x_j\right|,\frac{1}{\left|x_j\right|}\right)=C\sum_{k\in\Gamma_\mu}\frac{2^{k_1+...+k_l}}{2^{k_\mu(d+l)}}2^{k_1+...+k_d}\prod_{j=1}^{d}\min\left(2^{k_j}\left|x_j\right|,\frac{1}{2^{k_j}\left|x_j\right|}\right). \tag{80}$$

For any $k\in\Gamma_\mu$ we have $k_\mu=\max_{1\leq j\leq d}k_j$. Hence, we can write

$$\frac{2^{k_1+...+k_l}}{2^{k_\mu(d+l)}}2^{k_1+...+k_d}=\prod_{j=1}^{l}2^{k_j-k_\mu}\prod_{j=1}^{d}2^{k_j-k_\mu}\leq 1,$$

and therefore, the right hand side of (80) is at most

$$\begin{aligned}C\sum_{k\in\Gamma_\mu}\prod_{j=1}^{d}\min\left(2^{k_j}\left|x_j\right|,\frac{1}{2^{k_j}\left|x_j\right|}\right) &\leq C\sum_{k_1,k_2,...,k_d\in\mathbb{Z}}\prod_{j=1}^{d}\min\left(2^{k_j}\left|x_j\right|,\frac{1}{2^{k_j}\left|x_j\right|}\right)\\ &= C\prod_{j=1}^{d}\sum_{k_j\in\mathbb{Z}}\min\left(2^{k_j}\left|x_j\right|,\frac{1}{2^{k_j}\left|x_j\right|}\right)\lesssim C.\end{aligned}$$

In other words, we have seen that for any $\mu\in\{1,...,d\}$ and any multiindex $\alpha\in\{0,1\}^d$ of the form $\alpha=(1,...1,0,...,0)$,

$$\sum_{k\in\Gamma_\mu}\left|\int_{I_{k_1}\times...\times I_{k_d}}(-\nabla)^\alpha m\left(\xi\right)\prod_{j=1}^{d}\left[\sin\left(\xi_j x_j\right)\right]_{I_{k_j}}^{\alpha_j}d\xi\right|\lesssim C. \tag{81}$$

Clearly, (81) remains true for any $\alpha \in \{0,1\}^d$. Now, observing that $\mathbb{Z}^d$ is covered by the union of the sets $\Gamma_1, ...\Gamma_d$, one can write

$$\sum_{k\in\mathbb{Z}^d}\left|\int_{I_{k_1}\times...\times I_{k_d}}(-\nabla)^{\alpha} m(\xi)\prod_{j=1}^{d}\left[\sin(\xi_j x_j)\right]_{I_{k_j}}^{\alpha_j} d\xi\right| \leq \sum_{\mu=1}^{d}\sum_{k\in\Gamma_\mu}... \lesssim C.$$

This, together with (75) and (77) proves Lemma 30 in the case $\ell = d$ (i.e., (74)). The case where $1 \leq \ell < d$, can be obtained from the case $\ell = d$ as follows.

Suppose that $1 \leq \ell < d$ and m is an ℓ-BB symbol. Let M be a measurable subset of $\mathbb{R}^{d-\ell}$ of finite measure. For each $x'' \in \mathbb{R}^{d-\ell}$ define the function $\widetilde{m}_{x''} : \mathbb{R}^{\ell} \to \mathbb{C}$ by

$$\widetilde{m}_{x''}(\xi') := \int_M m(\xi', \xi'') e^{i\langle \xi'', x''\rangle} d\xi'',$$

for all $\xi' \in \mathbb{R}^{\ell}$. Thanks to the fact that m satisfies the condition (71), $\widetilde{m}_{x''}$ is well-defined and it satisfies uniformly in θ'' the condition (72) as a symbol on $\mathbb{R}^{\ell}$. Indeed,

$$\begin{aligned}
|\partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell}\widetilde{m}_{x''}(\xi')| &= \left|\int_M \partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell} m(\xi', \xi'') e^{i\langle \xi'', x''\rangle} d\xi''\right| \\
&\leq \int_M |\partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell} m(\xi', \xi'')|\, d\xi'' \\
&\leq \int_{\mathbb{R}^{d-\ell}} |\partial_1^{\alpha_1}...\partial_\ell^{\alpha_\ell} m(\xi', \xi'')|\, d\xi'' \\
&\leq \frac{C}{|\xi'|^{\ell+|\alpha|}}, \qquad (82)
\end{aligned}$$

for all $\alpha = (\alpha_1, ..., \alpha_\ell) \in \{0,1\}^{\ell}$ and all $\xi' \in (0,\infty)^{\ell}$. (Note that the final estimate in (82) does not depend on the set M.)

Also, $\widetilde{m}_{\theta''}$ is odd in each variable and hence, we have (73). Now, using (72) for $\widetilde{m}_{\theta''}$, we have

$$\begin{aligned}
\sum_{k'\in\mathbb{Z}^{\ell}}\left|\int_{s_\ell(I_{k'})\times M} m(\xi) e^{i\langle \xi, x\rangle} d\xi\right| &= \sum_{k'\in\mathbb{N}^{\ell}}\left|\int_{s_\ell(I_{k'})}\left(\int_M m(\xi', \xi'') e^{i\langle \xi'', x''\rangle} d\xi''\right) e^{i\langle \xi', x'\rangle} d\xi'\right| \\
&= \sum_{k'\in\mathbb{N}^{\ell}}\left|\int_{s_\ell(I_{k'})} \widetilde{m}_{x''}(\xi')\, e^{i\langle \xi', x'\rangle} d\xi'\right| \lesssim C,
\end{aligned}$$

uniformly in $x \in \mathbb{R}^d$, which proves Lemma 30. □

By applying Lemma 30 we can deduce the following useful fact:

Lemma 31. *Let $d \geq 1$ and $1 \leq \ell \leq d$ be some integers, and let $m : \mathbb{R}^d \to \mathbb{C}$ be an ℓ-BB symbol satisfying condition (71) or (71) (when $\ell = d$) for some constant C. Then, there exists some kernel $K \in L^{\infty}\left(\mathbb{R}^d\right)$ such that $\widehat{K}(\xi) = m(\xi)$ on $\mathbb{R}^d$ and $\|K\|_{L^{\infty}} \lesssim C$, the implicit constant not depending on C.*

Remark 32. *The meaning of Lemma 31 is that there exists a unique function $K \in L^{\infty}$, such that $\|K\|_{L^{\infty}} \lesssim C$ and*

$$\langle K, \psi\rangle = \left\langle m, \widehat{\psi}\right\rangle,$$

for any function $\psi \in \mathcal{S}_{\sharp}$.

Proof. For each $n \in \mathbb{N}$, we consider the functions K_n defined by

$$\widehat{K_n}(\xi) = m(\xi)\mathbf{1}_{J_n^d}(\xi)$$

for all $\xi \in \mathbb{R}^d$, where $J_n := [-2^n, -2^{-n}] \cup [2^{-n}, 2^n]$ and $J_n^d = J_n \times ... \times J_n$ (d times). It is easy to see that K_n are well defined continuous functions. One can also see that K^n are uniformly bounded. Indeed, applying Lemma 30 for $M = J_n^{d-\ell}$ (we suppose that $\ell < d$; the case $\ell = d$ is similar) we can write (using the triangle inequality),

$$\begin{aligned} |K_n(x)| &\sim \left| \int_{J_n^\ell \times J_n^{d-\ell}} m(\xi) e^{i\langle \xi, x\rangle} d\xi \right| \\ &= \left| \sum_{\substack{k' \in \mathbb{Z}^\ell \\ s_\ell(I_{k'}) \subseteq J_n^\ell}} \int_{s_\ell(I_{k'}) \times J_n^{d-\ell}} m(\xi) e^{i\langle \xi, x\rangle} d\xi \right| \\ &\leq \sum_{k' \in \mathbb{Z}^\ell} \left| \int_{s_\ell(I_{k'}) \times J_n^{d-\ell}} m(\xi) e^{i\langle \xi, x\rangle} d\xi \right| \lesssim C, \end{aligned}$$

uniformly in $x \in \mathbb{R}^d$, and in $n \in \mathbb{N}$. (Here, for "=", we have used the fact that the sets $s_\ell(I_{k'})$ with $s_\ell(I_{k'}) \subseteq J_n^\ell$ are pairwise almost disjoint and they cover J_n^ℓ.) Hence, $\|K_n\|_{L^\infty} \lesssim C$ uniformly in $n \in \mathbb{N}$. Using the sequential Banach-Alaoglu theorem, we can find some $K \in L^\infty$, with $\|K\|_{L^\infty} \lesssim C$ and such that $K_n \to K$ in the w^*-topology on L^∞, up to a subsequence.

Consider now some $\psi \in \mathcal{S}_\sharp$. Clearly, for any n,

$$\langle K_n, \psi \rangle = \left\langle m\mathbf{1}_{J_n^d}, \widehat{\psi} \right\rangle. \tag{83}$$

Since $\psi \in L^1$ we have $\langle K_n, \psi \rangle \to \langle K, \psi \rangle$ up to a subsequence when $n \to \infty$. Also, we have

$$\left\langle m\mathbf{1}_{J_n^d}, \widehat{\psi} \right\rangle \to \left\langle m, \widehat{\psi} \right\rangle,$$

and by (83) we get

$$\langle K, \psi \rangle = \left\langle m, \widehat{\psi} \right\rangle. \tag{84}$$

The uniqueness of K immediately follows from (84). Lemma 31 is proved. □

4.2 Divergence-like equations in $\dot{W}^{d/2,2}$

We can now prove our existence result for divergence-like equations in a particular type of critical spectral spaces. We start by an analogue of [5, Lemma 2]. The proof we give below rests on some elaborations of the main ideas used by Bourgain and Brézis in the proof of [5, Lemma 2]. Lemma 31 from the previous subsection will play here an important role.

In what follows let us denote by $\nabla_2^\sigma u$ the first two components of the "σ-gradient" of u, namely,

$$\nabla_2^\sigma u := (\partial_1^\sigma u, \partial_2^\sigma u),$$

where ∂_j^σ is the Fourier multiplier $\partial_j^\sigma := \sigma_j(\nabla)$.

Lemma 33. *Let $d \geq 3$ be an integer. Consider the set $U := \mathbb{R}^{d-1} \times (0, \infty)$. If $\sigma : \mathbb{R}^d \to \mathbb{R}$ satisfies (P1),(P2) then,*

$$\begin{aligned} \left\| |\nabla|^{-1} |\nabla_2^\sigma|^2 u \right\|_{Y^*/U} &\lesssim \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U} \\ &\quad + \|\nabla_2^\sigma u\|_{Y^*/U}^{1/2} \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}^{1/2}, \end{aligned} \tag{85}$$

for any $u \in \mathcal{S}_\sharp$, where $Y := \dot{W}^{d/2,2}$.

(For the meaning of the notation Y^*/U and $(L^1 + Y^*)/U$ see subsection 2.3.)

Proof. Clearly, $Y^* = \dot{W}^{-d/2,2}$. First we show that

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 \, d\xi \lesssim \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}^2 + \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}, \tag{86}$$

for any function $u \in \mathcal{S}_\sharp$.

Consider some functions $F_1, F_2, h_1, h_2, F_1^-, F_2^- \in \mathcal{S}_\sharp$ with $spec(F_1^-), spec(F_2^-) \subseteq U^c$ such that

$$\nabla_2^\sigma u = (F_1, F_2) + (h_1, h_2) + (F_1^-, F_2^-), \tag{87}$$

and

$$\|F_1\|_{L^1} + \|F_2\|_{L^1} + \|h_1\|_{Y^*} + \|h_2\|_{Y^*} \leq 2 \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}. \tag{88}$$

We have

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 \, d\xi = c \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) \widehat{\partial_1^\sigma u}(\xi) \overline{\widehat{\partial_2^\sigma u}(\xi)} d\xi,$$

Using this, (87) and the fact that $spec(F_j^-) \subseteq U^c$ (and hence $\mathbf{1}_U(\xi)\widehat{F_j^-}(\xi) = 0$, for $j = 1, 2$ and for all $\xi \in \mathbb{R}^d$), we can write

$$\int_{\mathbb{R}^d} \frac{\sigma_1^2(\xi)\sigma_2^2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) |\widehat{u}(\xi)|^2 \, d\xi = I + II,$$

where

$$I := c \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) \widehat{F_1}(\xi) \overline{\widehat{F_2}(\xi)} d\xi,$$

and II is the sum of a finite number of terms of the form

$$c \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) g_1(\xi) g_2(\xi) d\xi, \tag{89}$$

where each $g_k : \mathbb{R}^d \to \mathbb{C}$ is one of the functions

$$\widehat{h}_1, \ \widehat{h}_2, \ \overline{\widehat{h}}_1, \ \overline{\widehat{h}}_2, \ \widehat{F}_1, \ \widehat{F}_2, \ \overline{\widehat{F}}_1, \ \overline{\widehat{F}}_2$$

and at least one g_k is $\widehat{h}_j$ or $\overline{\widehat{h}}_j$ for some $j \in \{1, 2\}$.

One can verify immediately that the symbol m defined by

$$m(\xi) := \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi), \ \xi \in \mathbb{R}^d \setminus \{0\},$$

satisfies the conditions in Lemma 31. Hence, by a applying Lemma 31 and (88),

$$\begin{aligned} |I| &\sim |\langle K * F_1, F_2\rangle| \leq \|K * F_1\|_{L^\infty} \|F_2\|_{L^1} \\ &\leq \|K\|_{L^\infty} \|F_1\|_{L^1} \|F_2\|_{L^1} \lesssim \|\nabla_2^\sigma u\|^2_{(L^1+Y^*)/U}. \end{aligned} \tag{90}$$

(Here, we have used the notation from Lemma 31: $\widehat{K} = m$.)

In order to estimate II we estimate each of its terms of the form (89). By the Cauchy-Schwarz inequality, we get

$$\left| \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) g_1(\xi) g_2(\xi) d\xi \right| \leq \prod_{k=1}^{2} \left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) \frac{|g_k(\xi)|^2}{|\xi|^d} d\xi \right)^{1/2}, \tag{91}$$

where we have used the inequality

$$\left| \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^2} \right| \lesssim 1,$$

which follows directly from (P1).

Note that, if $|g_k| = |\widehat{h}_j|$, for some $j \in \{1, 2\}$, then, by (88)

$$\left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) \frac{|g_k(\xi)|^2}{|\xi|^d} d\xi \right)^{1/2} \leq \|h_j\|_{Y^*} \leq 2 \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}. \tag{92}$$

If $|g_k| = |\widehat{F}_j|$, for some $j \in \{1, 2\}$, then, since $\widehat{F}_j \mathbf{1}_U = (\widehat{\partial_j^\sigma u} - \widehat{h}_j)\mathbf{1}_U$, the triangle inequality together with (88) gives

$$\begin{aligned} \left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) \frac{|g_k(\xi)|^2}{|\xi|^d} d\xi \right)^{1/2} &= \left(\int_{\mathbb{R}^d} \mathbf{1}_U(\xi) |\xi|^{-d} \left| \widehat{\partial_j^\sigma u}(\xi) - \widehat{h}_j(\xi) \right|^2 d\xi \right)^{1/2} \\ &\leq \left\| \partial_j^\sigma u \right\|_{Y^*/U} + \|h_j\|_{Y^*} \\ &\leq \|\nabla_2^\sigma u\|_{Y^*/U} + 2 \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U} \\ &\leq 3 \|\nabla_2^\sigma u\|_{Y^*/U}. \end{aligned} \tag{93}$$

Since $|g_k| = |\widehat{h}_j|$ (for some j) for at least one k, we get from (91), (92) and (93) that

$$\left| \int_{\mathbb{R}^d} \frac{\sigma_1(\xi)\sigma_2(\xi)}{|\xi|^{d+2}} \mathbf{1}_U(\xi) g_1(\xi) g_2(\xi) d\xi \right| \lesssim \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}.$$

Hence,

$$|II| \lesssim \|\nabla_2^\sigma u\|_{Y^*/U} \|\nabla_2^\sigma u\|_{(L^1+Y^*)/U}. \tag{94}$$

By (93) and (94) we get (86).

Consider the rotation

$$R(\xi) = (\xi_1 - \xi_2, \xi_1 + \xi_2, \xi_3, ..., \xi_d),$$

for any $\xi \in \mathbb{R}^d$. Consider now the functions

$$\sigma_1' := \sigma_1 \circ R - \sigma_2 \circ R, \quad \text{and} \quad \sigma_2' := \sigma_1 \circ R + \sigma_2 \circ R.$$

One can immediately check that the function

$$\sigma_1'\sigma_2' := (\sigma_1 \circ R)^2 - (\sigma_2 \circ R)^2$$

is odd in each of the variables ξ_1, ξ_2. Using this, we easily observe that the symbol m' defined by

$$m'(\xi) := \frac{\sigma_1'(\xi)\sigma_2'(\xi)}{|\xi|^{d+2}}\mathbf{1}_U(\xi),\ \xi \in \mathbb{R}^d \setminus \{0\},$$

satisfies the conditions in Lemma 31. Hence, as in (86), we obtain that

$$\int_{\mathbb{R}^d} \frac{\sigma_1'(\xi)^2\sigma_2'(\xi)^2}{|\xi|^{d+2}}\mathbf{1}_U(\xi)\,|\widehat{v}(\xi)|^2\,d\xi \lesssim \left\|\nabla_2^{\sigma'}v\right\|^2_{(L^1+Y^*)/U} + \left\|\nabla_2^{\sigma'}v\right\|_{Y^*/U}\left\|\nabla_2^{\sigma'}v\right\|_{(L^1+Y^*)/U}, \tag{95}$$

for any function $v \in \mathcal{S}_\sharp$, where

$$\nabla_2^{\sigma'}v := (\partial_1^{\sigma'}v, \partial_2^{\sigma'}v),$$

and $\partial_1^{\sigma'}$ is the Fourier multiplier of symbol σ_j', for any $j = 0, 1$.

Since the spaces Y^*/U and $(L^1 + Y^*)/U$ are invariant under the rotation R, by applying (95) to the function $v = u \circ R^t$ we obtain (by changing the variables) that

$$\int_{\mathbb{R}^d} \frac{(\sigma_1(\xi) - \sigma_2(\xi))^2\,(\sigma_1(\xi) + \sigma_2(\xi))^2}{|\xi|^{d+2}}\mathbf{1}_U(\xi)\,|\widehat{u}(\xi)|^2\,d\xi$$

is bounded by

$$\|\nabla_2^{\sigma}u\|^2_{(L^1+Y^*)/U} + \|\nabla_2^{\sigma}u\|_{Y^*/U}\,\|\nabla_2^{\sigma}u\|_{(L^1+Y^*)/U}, \tag{96}$$

for any function $u \in \mathcal{S}_\sharp$. By adding up, we get from (86) and (96) that

$$\int_{\mathbb{R}^d} \frac{\widetilde{\sigma}(\xi)}{|\xi|^{d+2}}\mathbf{1}_U(\xi)\,|\widehat{u}(\xi)|^2\,d\xi \lesssim \|\nabla_2^{\sigma}u\|^2_{(L^1+Y^*)/U} + \|\nabla_2^{\sigma}u\|_{Y^*/U}\,\|\nabla_2^{\sigma}u\|_{(L^1+Y^*)/U}, \tag{97}$$

where

$$\widetilde{\sigma}(\xi) := \sigma_1^2(\xi)\sigma_2^2(\xi) + (\sigma_1(\xi) - \sigma_2(\xi))^2\,(\sigma_1(\xi) + \sigma_2(\xi))^2,$$

for any $\xi \in \mathbb{R}^d$. Since for any real numbers a, b we have

$$a^2b^2 + (a-b)^2\,(a+b)^2 \sim a^4 + b^4,$$

we obtain

$$\widetilde{\sigma}(\xi) \sim \sigma_1^4(\xi) + \sigma_2^4(\xi),$$

for all $\xi \in \mathbb{R}^d$, and now (97) gives us

$$\int_{\mathbb{R}^d} \frac{\sigma_1^4(\xi) + \sigma_2^4(\xi)}{|\xi|^{d+2}}\mathbf{1}_U(\xi)\,|\widehat{u}(\xi)|^2\,d\xi \lesssim \|\nabla_2^{\sigma}u\|^2_{(L^1+Y^*)/U} + \|\nabla_2^{\sigma}u\|_{Y^*/U}\,\|\nabla_2^{\sigma}u\|_{(L^1+Y^*)/U}. \tag{98}$$

Note that by (20) (with $D = U$), we have

$$\left\||\nabla|^{-1}\,|\nabla_2^{\sigma}|^2u\right\|^N_{Y^*/U} \sim \int_{\mathbb{R}^d} \frac{\sigma_1^4(\xi) + \sigma_2^4(\xi)}{|\xi|^{d+2}}\mathbf{1}_U(\xi)\,|\widehat{u}(\xi)|^2\,d\xi,$$

and together with (98) this concludes the proof of Lemma 33. □

By composition with rotations and by adding up inequalities of the form (85), Lemma 33 easily implies the following:

Lemma 34. *Let $d \geq 3$ be an integer. Suppose that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is adapted to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$. For any $j \in \{1, ..., d\}$ we have*

$$\begin{aligned}\left\| |\nabla|^{-1} |G_j(\nabla)|^2 u \right\|_{Y^*/U} &\lesssim \|G_j(\nabla)u\|_{(L^1+Y^*)/D_j} \\ &\quad + \|G_j(\nabla)u\|^{1/2}_{Y^*/D_j} \|G_j(\nabla)u\|^{1/2}_{(L^1+Y^*)/D_j},\end{aligned}$$

for any $u \in \mathcal{S}_\sharp$, where $Y := \dot{W}^{d/2,2}$.

(For the properties of the functions $G_1, ..., G_d$ and their relation with the half-spaces $D_1, ..., D_d$ see the subsection 1.3 in the introduction of this paper.)

We can now state and prove the existence results of this subsection:

Lemma 35. *Let $d \geq 3$ be an integer. Suppose that the family of functions $G_1, ..., G_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is adapted to the family of half-spaces $D_1, ..., D_d \subset \mathbb{R}^d$. Then, for any system of $(d-1)$-vector fields $(v_j)_{j=1,..,d}$ with $v_j \in \dot{W}^{d/2,2}(\mathbb{R}^d)$ and $spec(v_j) \subseteq D_j$, there exists a system of $(d-1)$-vector fields $(u_j)_{j=1,..,d}$, with $u_j \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{d/2,2}(\mathbb{R}^d)$ and $spec(u_j) \subseteq D_j$, such that*

$$\sum_{j=1}^d G_j(\nabla) \cdot u_j = \sum_{j=1}^d G_j(\nabla) \cdot v_j,$$

and

$$\sum_{j=1}^d \|u_j\|_{L^\infty \cap \dot{W}^{d/2,2}} \lesssim \sum_{j=1}^d \|v_j\|_{\dot{W}^{d/2,2}}.$$

Proof. As before, let Y be the space $\dot{W}^{d/2,2}$. According to Lemma 34, for any $\varphi \in \mathcal{S}_\sharp$, we have

$$\begin{aligned}\left\| |\nabla|^{-1} |G_j(\nabla)| G_j(\nabla)\varphi \right\|_{Y^*/D_j} &\lesssim \|G_j(\nabla)\varphi\|_{(L^1+Y^*)/D_j} \\ &\quad + \|G_j(\nabla)\varphi\|^{1/2}_{Y^*/D_j} \|G_j(\nabla)\varphi\|^{1/2}_{(L^1+Y^*)/D_j} \\ &= \|G_j(\nabla)\varphi\|_{(L^1+Y^*)/D_j} \\ &\quad + (\varepsilon^{1/2} \|G_j(\nabla)\varphi\|^{1/2}_{Y^*/D_j})(\varepsilon^{-1/2} \|G_j(\nabla)\varphi\|^{1/2}_{(L^1+Y^*)/D_j}) \\ &\leq \|G_j(\nabla)\varphi\|_{(L^1+Y^*)/D_j} \\ &\quad + \varepsilon \|G_j(\nabla)\varphi\|_{Y^*/D_j} + \varepsilon^{-1} \|G_j(\nabla)\varphi\|_{(L^1+Y^*)/D_j},\end{aligned}$$

for any $\varepsilon \in (0,1)$ and any $j \in \{1, ..., d\}$. By adding up these inequalities we get

$$\begin{aligned}\sum_{j=1}^d \left\| |\nabla|^{-1} |G_j(\nabla)| G_j(\nabla)\varphi \right\|_{Y^*/D_j} &\lesssim \varepsilon \sum_{j=1}^d \|G_j(\nabla)\varphi\|_{Y^*/D_j} \\ &\quad + \varepsilon^{1-N} \sum_{j=1}^d \|G_j(\nabla)\varphi\|_{(L^1+Y^*)/D_j}. \qquad (99)\end{aligned}$$

Since the family G_1,...,G_d is adapted to D_1,...,D_d, (see (10)) we easily get

$$\sum_{j=1}^d |G_j(\xi)|^\beta \mathbf{1}_{D_j}(\xi) \sim_\beta |\xi|^\beta \mathbf{1}_D(\xi), \qquad (100)$$

on $\mathbb{R}^d$, for any $\beta > 0$. Using now (100), with $\beta = 2$, we can write

$$\begin{aligned}
\sum_{j=1}^{d} \left\| |\nabla|^{-1} |G_j(\nabla)| G_j(\nabla) \varphi \right\|_{Y^*/D_j} &\sim \sum_{j=1}^{d} \left\| |\xi|^{-1} |G_j(\xi)|^2 \mathbf{1}_{D_j}(\xi) |\widehat{\varphi}(\xi)| |\xi|^{-d/2} \right\|_{L^2_\xi} \\
&\sim \left\| |\xi|^{-1} \left(\sum_{j=1}^{d} |G_j(\xi)|^2 \mathbf{1}_{D_j}(\xi) \right) |\widehat{\varphi}(\xi)| |\xi|^{-d/2} \right\|_{L^2_\xi} \\
&\sim \left\| |\xi|^{-1} |\xi|^2 \mathbf{1}_D(\xi) |\widehat{\varphi}(\xi)| |\xi|^{-d/2} \right\|_{L^2_\xi} \\
&= \left\| |\xi| \mathbf{1}_D(\xi) |\widehat{\varphi}(\xi)| |\xi|^{-d/2} \right\|_{L^2_\xi}. \qquad (101)
\end{aligned}$$

In a similar way (by (10)) we can write

$$\begin{aligned}
\sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{Y^*/D_j} &\sim \sum_{j=1}^{d} \left\| |G_j(\xi)| \mathbf{1}_{D_j}(\xi) \widehat{\varphi}(\xi) |\xi|^{-d/2} \right\|_{L^2_\xi} \\
&\sim \left\| \left(\sum_{j=1}^{d} |G_j(\xi)| \mathbf{1}_{D_j}(\xi) \right) \widehat{\varphi}(\xi) |\xi|^{-d/2} \right\|_{L^2_\xi} \\
&\sim \left\| |\xi| \mathbf{1}_D(\xi) |\widehat{\varphi}(\xi)| |\xi|^{-d/2} \right\|_{L^2_\xi}. \qquad (102)
\end{aligned}$$

Note that (101) and (102) gives us

$$\sum_{j=1}^{d} \left\| |\nabla|^{-1} |G_j(\nabla)| G_j(\nabla) \varphi \right\|_{Y^*/D_j} \sim \sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{Y^*/D_j},$$

and together with (99) yields

$$\sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{Y^*/D_j} \lesssim \varepsilon \sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{Y^*/D_j} + \varepsilon^{-1} \sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{(L^1+Y^*)/D_j}.$$

Choosing ε sufficiently small one can write

$$\sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{Y^*/D_j} \lesssim \sum_{j=1}^{d} \left\| G_j(\nabla) \varphi \right\|_{(L^1+Y^*)/D_j}.$$

By duality (using the closed range theorem) we get Lemma 35. □

With the same methods one can prove an analogue of Theorem 8 when $d = 2$ and the source space is $\dot{W}^{1,2}(\mathbb{R}^2)$:

Lemma 36. *Consider the numbers $\delta \in (0, \pi/8)$ and $\varepsilon \in (0, 2]$. Then, for any vector field $v \in \dot{W}^{1,2}(\mathbb{R}^2)$, with spec$(v) \subseteq C_\delta$, there exist a vector field $u \in L^\infty(\mathbb{R}^2) \cap \dot{W}^{1,2}(\mathbb{R}^2)$, with spec$(u_j) \subseteq C_{(1+\varepsilon)\delta}$, such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{W}^{1,2}} \lesssim \|v\|_{\dot{W}^{1,2}}.$$

(For the meaning of C_δ see subsection 1.3.)

Sketch of the proof. First we establish the result for $\varepsilon = 2$. Let us denote by $\mathcal{D}(C_\delta)$ the set defined by

$$\mathcal{D}(C_\delta) := \bigcup_{I \subset C_\delta} I_\nu,$$

the union being taken after all the dyadic boxes $I_\nu = [2^{\nu_1 - 1}, 2^{\nu_1}] \times [2^{\nu_2 - 1}, 2^{\nu_2}]$ (where $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$) that are included in C_δ and are maximal (with respect to the inclusion relation) with this property. One can find a finite number of rotations $R_1, ..., R_n : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$C_\delta \backslash B_{\mathbb{R}^2}(0, r) \subset \widetilde{C}_\delta := \bigcup_{j=1}^{n} R_j(\mathcal{D}(C_\delta)) \subset C_{2\delta}, \tag{103}$$

for some sufficiently large $r > 0$. As in the proof of (86) we get

$$\int_{\mathbb{R}^d} \frac{(\xi_1 \xi_2)^2}{|\xi|^{d+2}} \mathbf{1}_{\mathcal{D}(C_\delta)}(\xi) \left|\widehat{u}(\xi)\right|^2 d\xi \lesssim \|\nabla u\|^2_{(L^1+Y^*)/\mathcal{D}(C_\delta)} + \|\nabla u\|_{Y^*/\mathcal{D}(C_\delta)} \|\nabla u\|_{(L^1+Y^*)/\mathcal{D}(C_\delta)}, \tag{104}$$

for any $u \in \mathcal{S}_\sharp$, where $Y = \dot{W}^{d/2,2}$. As in the proof of Lemma 31, by Lemma 30 (applied in the case $\ell = d = 2$) one can see that there exists $K \in L^\infty$ such that

$$\widehat{K}(\xi) = \frac{\xi_1 \xi_2}{|\xi|^{d+2}} \mathbf{1}_{\mathcal{D}(C_\delta)}(\xi),$$

with the same meaning as in Lemma 31. We use then the same method as in (90). The rest of the argument remains essentially the same as the one used in the proof of (86).

Note that, by (104) we get (by composition with rotations) that

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{(R_j^1(\xi) R_j^2(\xi))^2}{|\xi|^{d+2}} \mathbf{1}_{R_j(\mathcal{D}(C_\delta))}(\xi) \left|\widehat{u}(\xi)\right|^2 d\xi \quad &\lesssim \quad \|\nabla u\|^2_{(L^1+Y^*)/R_j(\mathcal{D}(C_\delta))} \\ &\quad + \|\nabla u\|_{Y^*/R_j(\mathcal{D}(C_\delta))} \|\nabla u\|_{(L^1+Y^*)/R_j(\mathcal{D}(C_\delta))}, \end{aligned}$$

for all $j \in \{1, ..., n\}$, where $R_j^l(\xi)$ is the l-th coordinate of the vector $R_j(\xi)$. Using the (103) we get

$$\|\nabla u\|_{Y^*/R_j(\mathcal{D}(C_\delta))} \leq \|\nabla u\|_{Y^*/\widetilde{C}_\delta} \quad \text{and} \quad \|\nabla u\|_{(L^1+Y^*)/R_j(\mathcal{D}(C_\delta))} \leq \|\nabla u\|_{(L^1+Y^*)/\widetilde{C}_\delta},$$

for all $j \in \{1, ..., n\}$, and one can write

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{(R_j^1(\xi) R_j^2(\xi))^2}{|\xi|^{d+2}} \mathbf{1}_{R_j(\mathcal{D}(C_\delta))}(\xi) \left|\widehat{u}(\xi)\right|^2 d\xi \quad &\lesssim \quad \|\nabla u\|^2_{(L^1+Y^*)/\widetilde{C}_\delta} \\ &\quad + \|\nabla u\|_{Y^*/\widetilde{C}_\delta} \|\nabla u\|_{(L^1+Y^*)/\widetilde{C}_\delta}, \end{aligned} \tag{105}$$

for all $j \in \{1, ..., n\}$. One can easily check that

$$\sum_{j=1}^{n} (R_j^1(\xi) R_j^2(\xi))^2 \mathbf{1}_{R_j(\mathcal{D}(C_\delta))}(\xi) \sim |\xi|^4 \mathbf{1}_{\widetilde{C}_\delta}(\xi),$$

for all $\xi \in \mathbb{R}^2$. Hence, by adding up the inequalities (105) we get

$$\int_{\mathbb{R}^d} \frac{|\xi|^2}{|\xi|^d} \mathbf{1}_{\widetilde{C}_\delta}(\xi) \left|\widehat{u}(\xi)\right|^2 d\xi \lesssim \|\nabla u\|^2_{(L^1+Y^*)/\widetilde{C}_\delta} + \|\nabla u\|_{Y^*/\widetilde{C}_\delta} \|\nabla u\|_{(L^1+Y^*)/\widetilde{C}_\delta},$$

which can be rewritten as

$$\|\nabla u\|_{Y^*/\widetilde{C}_\delta} \lesssim \|\nabla u\|_{(L^1+Y^*)/\widetilde{C}_\delta} + \|\nabla u\|^{1/2}_{Y^*/\widetilde{C}_\delta} \|\nabla u\|^{1/2}_{(L^1+Y^*)/\widetilde{C}_\delta}. \tag{106}$$

By duality (as in the proof of Lemma 35) we obtain that for any vector field $v \in \dot{W}^{1,2}(\mathbb{R}^2)$, with $spec(v) \subseteq \widetilde{C}_\delta$, there exist a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,2}(\mathbb{R}^2)$, with $spec(u_j) \subseteq \widetilde{C}_\delta$, such that

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{W}^{1,2}} \lesssim \|v\|_{\dot{W}^{1,2}}.$$

Thanks to (103) this immediately implies Lemma 36 in the case $\varepsilon = 2$. To obtain the result for any $\varepsilon \in (0,1)$ simply cover the symmetric cone C_δ with a small union of rotated copies of the symmetric cones $C_{\delta/n}$ for some large integer $n > 0$. It suffices now to apply the result corresponding to the case $\varepsilon = 2$ to each rotated copy of $C_{\delta/n}$ and then add the obtained solutions. □

5 Solutions in interpolation spaces

5.1 Proof of the main results

We now discuss some immediate applications of the $\mathcal{W}$-method to the divergence-like equation. First we formulate a general result:

Theorem 37. *Let X, $\tilde{X}$, Y, $\tilde{Y}$, F be Banach function spaces on $\mathbb{R}^d$ satisfying the embeddings $X \hookrightarrow \tilde{X}$, $Y \hookrightarrow \tilde{Y} \hookrightarrow \tilde{X}$ and consider a bounded linear operator $T : \tilde{X} \to F$. Suppose moreover that the following conditions are satisfied:*

(i) $\tilde{X}$, Y, $\tilde{Y}$ are UMD separable spaces;

(ii) T is bounded from X to F and from Y to F, and $T(Y) \hookrightarrow T(L^\infty \cap \tilde{Y})$.

Fix some $\theta \in (0,1)$. Then, for any vector field $v \in (L^\infty \cap X, Y)_\theta$, there exists a vector field $u \in L^\infty \cap (\tilde{X}, \tilde{Y})_\theta$ such that

$$Tu = Tv,$$

and

$$\|u\|_{L^\infty \cap (\tilde{X},\tilde{Y})_\theta} \lesssim \|v\|_{(L^\infty \cap X, Y)_\theta}.$$

Proof. We apply Lemma 28 for the Banach spaces $X_0 = \tilde{X}$, $X_1 = \tilde{Y}$, $A = L^\infty$, $A_0 = L^\infty \cap \tilde{X}$, $B_0 = L^\infty \cap X$, $A_1 = L^\infty \cap \tilde{Y}$, $B_1 = Y$ and the operator T. One can easily observe that in this setting the conditions of Lemma 28 (part of them are explicitly stated in Lemma 27) are satisfied. Indeed, in order to verify the condition (i) in Lemma 27 it suffices to see that $X, Y \hookrightarrow \tilde{X}$ and hence,

$$B_1 \hookrightarrow X_1 = \tilde{Y} \hookrightarrow X_0 = \tilde{X},$$

and $B_0 \hookrightarrow A_0$.

The space $\tilde{Y}$ is reflexive (since it has the UMD property) and hence, it has a separable predual[7]. Also, $A = (L^1)^*$ has a separable predual. Thus, the condition (ii) in Lemma 27 is verified. Condition (iii) in Lemma 27 is ensured by condition (ii) in Theorem 37.

[7]Here we use the fact that if the dual X^* of a Banach space X is separable, then, X is separable (see for instance [14, Theorem 4.6-8, 245]).

Notice that $\tilde{X} \cap \tilde{Y}$ is a separable space that is dense in $(\tilde{X}, \tilde{Y})_\theta$ (see [2, Theorem 4.2.2 (a), p. 91]). It follows that $(\tilde{X}, \tilde{Y})_\theta$ is a reflexive and separable space and consequently it has a separable predual. We also have by (i) that the spaces $\tilde{X}$, Y, $\tilde{Y}$ have the UMD property. We can apply now Lemma 28 and we get Theorem 37. □

Let us see now that Theorem 37 above implies Theorem 10, Theorem 11, Theorem 9 and Theorem 8. In what follows we will ignore the space F since it is easy for the operators T we use to find a space F sufficiently large such that $T : \tilde{X} \to F$ (one can simply set F of the form $F = B_\infty^{-a,b}$ for some $a, b \in (1, \infty)$, with a sufficiently large).

Proof of Theorem 10. Let us consider some parameter $r \in [2, \infty)$ such that $1/p = (1-\theta)/r + \theta/2$. We apply Theorem 37 for the Banach spaces $X = \dot{B}_1^{d/r,r}$, $\tilde{X} = \dot{B}_2^{d/r,r}$ and $Y = \tilde{Y} = \dot{B}_2^{d/2,2}$. Since $r \geq 2$, we have $\dot{B}_2^{d/2,2}, B_1^{d/r,r} \hookrightarrow \dot{B}_2^{d/r,r}$. Hence, $X \hookrightarrow \tilde{X}$ and $Y \hookrightarrow \tilde{Y} \hookrightarrow \tilde{X}$. Also, by Mazya's theorem (Theorem 3 in the case $p = q = 2$) we have $T(Y) \hookrightarrow T(L^\infty \cap \tilde{Y})$, for the operator $T = \mathrm{div}$. Now the hypotheses of Theorem 37 are satisfyed.

Observe that, since $X = \dot{B}_1^{d/r,r} \hookrightarrow L^\infty$, we have $L^\infty \cap X = \dot{B}_1^{d/r,r}$ and we can write

$$(L^\infty \cap X, \dot{B}_2^{d/2,2})_\theta = (B_1^{d/r,r}, \dot{B}_2^{d/2,2})_\theta = B_q^{d/p,p}.$$

Since we also have

$$(\dot{B}_2^{d/r,r}, \dot{B}_2^{d/2,2})_\theta = \dot{B}_2^{d/p,p},$$

it remains to apply Theorem 37 and Theorem 10 is proved. □

Proof of Theorem 11. As in the proof of Theorem 10 let us consider $r \in [2, \infty)$ such that $1/p = (1-\theta)/r + \theta/2$. We apply Theorem 37 for the Banach spaces $X = \dot{W}^{d/r} L^{r,1}$, $\tilde{X} = \dot{W}^{d/r} L^{r,2}$ and $Y = \tilde{Y} = \dot{W}^{d/2,2}$. It remains to verify that the hypotheses of Theorem 37 are satisfyed. Indeed, by the monotonicity properties of the Lorentz spaces we also have $\dot{W}^{d/r} L^{r,1} \hookrightarrow \dot{W}^{d/r} L^{r,2}$, i.e., $X \hookrightarrow \tilde{X}$. By Lemma 16 we get $\dot{W}^{d/2,2} \hookrightarrow \tilde{X}$, i.e., $Y = \tilde{Y} \hookrightarrow \tilde{X}$. Also, by Mazya's theorem (Theorem 3 in the case $p = q = 2$) we have $T(Y) \hookrightarrow T(L^\infty \cap \tilde{Y})$ and now the hypotheses of Theorem 37 are satisfyed.

By Lemma 17 we have $\dot{W}^{d/2,2} \hookrightarrow \tilde{X}$ and $X = \dot{W}^{d/r} L^{r,1} \hookrightarrow L^\infty$ and hence $L^\infty \cap X = \dot{W}^{d/r} L^{r,1}$. From this and Lemma 18 we get

$$(L^\infty \cap X, \dot{W}^{d/2,2})_\theta = (\dot{W}^{d/r} L^{r,1}, \dot{W}^{d/2,2})_\theta = \dot{W}^{d/p} L^{p,q}. \tag{107}$$

Lemma 18 also gives

$$(\dot{W}^{d/r} L^{r,2}, \dot{W}^{d/2,2})_\theta = \dot{W}^{d/p} L^{p,2}, \tag{108}$$

and now one can easily conclude the proof of Theorem 11 by a direct application of Theorem 37. □

Proof of Theorem 9. The proof is very similar to the one of Theorem 10. Suppose p, r, θ are as in the proof of Theorem 10. We put

$$X = \prod_{j=1}^d (\dot{B}_1^{d/r,r})_{D_j}, \quad \widetilde{X} = \prod_{j=1}^d (\dot{B}_2^{d/r,r})_{D_j} \text{ and } Y = \tilde{Y} = \prod_{j=1}^d (\dot{B}_2^{d/2,2})_{D_j}.$$

Now, the operator T is the operator formaly defined for the systems of $(d-1)$-vector fields by the formula

$$T(v_1, ..., v_d) = \sum_{j=1}^d G_j(\nabla) \cdot v_j,$$

where $v_1, ..., v_d \in \mathcal{S}'$ are $(d-1)$-vector fields. In order to verify the item (ii) in Theorem 37 we use Lemma 35 instead of Mazya's theorem. It remains to apply Theorem 37 and to observe that, by the retraction method,

$$((\dot{B}_1^{d/r,r})_{D_j}, (\dot{B}_2^{d/2,2})_{D_j})_\theta = (\dot{B}_q^{d/p,p})_{D_j}$$

and

$$((\dot{B}_2^{d/r,r})_{D_j}, (\dot{B}_2^{d/2,2})_{D_j})_\theta = (\dot{B}_2^{d/p,p})_{D_j},$$

for any $j \in \{1, ..., d\}$. □

Proof of Theorem 8. Again, suppose p, r, θ are as in the proof of Theorem 10. We put $X = (\dot{B}_1^{d/r,r})_{C_\delta}$, $\widetilde{X} = (\dot{B}_2^{d/r,r})_{C_{(1+\varepsilon)\delta}}$ and $Y = (\dot{B}_2^{d/2,2})_{C_\delta}$, $\widetilde{Y} = (\dot{B}_2^{d/2,2})_{C_{(1+\varepsilon)\delta}}$. The operator T is the usual divergence operator $T = \mathrm{div}$. It remains to apply Theorem 37 and to observe that, by the retraction method,

$$((\dot{B}_1^{d/r,r})_{C_\delta}, (\dot{B}_2^{d/2,2})_{C_\delta})_\theta = (\dot{B}_q^{d/p,p})_{C_\delta}$$

and

$$((\dot{B}_2^{d/r,r})_{C_{(1+\varepsilon)\delta}}, (\dot{B}_2^{d/2,2})_{C_{(1+\varepsilon)\delta}})_\theta = (\dot{B}_2^{d/p,p})_{C_{(1+\varepsilon)\delta}}.$$

Both of these equalities rest on the fact that the Fourier projections P_{C_δ} on the sets C_δ are a sum of two rotated and dilated Riesz projections. Hence, P_{C_δ} is bounded on each of the spaces $\dot{B}_1^{d/r,r}$, $\dot{B}_2^{d/r,r}$ and $\dot{B}_2^{d/2,2}$. □

Let us see now that Theorem 9 implies Theorem 7. For this we need only some elementary geometry. Suppose $d \geq 3$ and consider the unit vectors $\nu_j := (1, ..., 1, 2, 1, ..., 1)/\sqrt{d+1}$ in $\mathbb{R}^d$ (with value 2 on the j-th position), $j \in \{1, ..., d\}$. For each $j \in \{1, ..., d\}$ define the half-spaces

$$D_j := \left\{\xi \in \mathbb{R}^d \mid \langle \xi, \nu_j \rangle > 0\right\},$$

and let D be the set $D := D_1 \cup ... \cup D_d$. By $p_{D_j}(\xi)$ we denote the orthogonal projection of the point ξ on the support hyperplane Π_j of D_j:

$$\Pi_j := \left\{\xi \in \mathbb{R}^d \mid \langle \xi, \nu_j \rangle = 0\right\}.$$

Note that

$$\left|p_{D_j}(\xi)\right| = \left(|\xi|^2 - |\langle \xi, \nu_j \rangle|^2\right)^{1/2},$$

for any $j \in \{1, ..., d\}$.

Consider the function $\sigma : \mathbb{R}^d \to \mathbb{R}$ defined by $\sigma(\xi) = \xi_1$. We can immediately see that this σ satisfies the conditions (P1), (P2). We have now $\sigma_j(\xi) = \xi_j$, for all $j \in \{1, ..., d\}$. Consider $G_0 := (\sigma_1, ..., \sigma_{d-1})$ and let G_j be obtained by composing G_0 with a rotation that transforms $\mathbb{R}^{d-1} \times (0, \infty)$ in D_j. One can see that $|G_j(\xi)| \sim |p_{D_j}(\xi)|$, for all $j \in \{1, ..., d\}$. In order to see that the family of functions $G_1, ..., G_d$ is adapted to the family $D_1, ..., D_d$ of half-spaces it remains to prove the following equivalence that corresponds to (10):

Lemma 38. *With the above notation we have*

$$\sum_{j=1}^{d} \left|p_{D_j}(\xi)\right| \mathbf{1}_{D_j}(\xi) \sim |\xi| \, \mathbf{1}_D(\xi),$$

for any $\xi \in \mathbb{R}^d$.

Proof. Observe that $|\nu_i - \nu_j| < 1$ and $|\langle \nu_i, \nu_j \rangle| < 1$ for any $i, j \in \{1, ..., d\}$, $i \neq j$. It follows from this that we can find some sufficiently small number $\alpha \in (0, 1)$ such that

$$|\nu_i - \nu_j| < \sqrt{1-\alpha}, \tag{109}$$

for any $i, j \in \{1, ..., d\}$, and

$$\sqrt{\frac{\alpha}{1-\alpha}} + |\langle \nu_1, \nu_2 \rangle| < 1. \tag{110}$$

Let $c \in (0, 1)$ such that $\sqrt{1-c}$ equals the left hand side of (110). In order to prove Lemma 38 it suffices to see that, for any $\xi \in \mathbb{R}^d$,

$$\sum_{j=1}^{d} \left| p_{D_j}(\xi) \right| \mathbf{1}_{D_j}(\xi) \geq c_1 \, |\xi| \, \mathbf{1}_D(\xi), \tag{111}$$

with $c_1 = \min(\sqrt{\alpha}, \sqrt{c})$.

Pick some $\xi \in D_1 \cap ... \cap D_d$. If $|\langle \xi, \nu_1 \rangle| \leq \sqrt{1-\alpha} \, |\xi|$, then the left hand side of (111) is at least

$$|p_{D_1}(\xi)| = \left(|\xi|^2 - |\langle \xi, \nu_1 \rangle|^2 \right)^{1/2} \geq \sqrt{\alpha} \, |\xi| \, ,$$

and we are done. Else, we have $|\langle \xi, \nu_1 \rangle| > \sqrt{1-\alpha} \, |\xi|$ and decomposing ξ as $\xi = \beta \nu_1 + w$, for some $\beta \in \mathbb{R}$ and $w \in \mathbb{R}^d$ with $w \perp \nu_1$, we can rewrite this inequality as

$$|\beta| > \sqrt{1-\alpha} \left(|\beta|^2 + |w|^2 \right)^{1/2},$$

or, equivalently,

$$|w| < |\beta| \sqrt{\frac{\alpha}{1-\alpha}}. \tag{112}$$

Now, note that, using (110)

$$\begin{aligned} |\langle \xi, \nu_2 \rangle| &\leq |\beta| \, |\langle \nu_1, \nu_2 \rangle| + |\langle w, \nu_2 \rangle| \\ &\leq |\beta| \, |\langle \nu_1, \nu_2 \rangle| + |w| \\ &< |\beta| \left(|\langle \nu_1, \nu_2 \rangle| + \sqrt{\frac{\alpha}{1-\alpha}} \right) \\ &\leq |\xi| \sqrt{1-c}. \end{aligned}$$

As above we get $|p_{D_2}(\xi)| = \left(|\xi|^2 - |\langle \xi, \nu_1 \rangle|^2 \right)^{1/2} \geq \sqrt{c} \, |\xi|$ and we are done. It remains to treat the case $\xi \in D \backslash (D_1 \cap ... \cap D_d)$.

Suppose $\xi \in D_k \backslash (D_1 \cap ... \cap D_d)$ for some $k \in \{1, ..., d\}$. Then, we cannot have $\langle \xi, \nu_k \rangle > \sqrt{1-\alpha} \, |\xi|$. Otherwise, using (109),

$$\begin{aligned} \langle \xi, \nu_j \rangle &= \langle \xi, \nu_k \rangle - \langle \xi, \nu_k - \nu_j \rangle \\ &> \sqrt{1-\alpha} \, |\xi| - |\nu_i - \nu_j| \, |\xi| > 0, \end{aligned}$$

for any $j \in \{1, ..., d\}$, and we obtain that $\xi \in D_1 \cap ... \cap D_d$ contradicting the choice of ξ. Hence, we must have $\langle \xi, \nu_k \rangle \leq \sqrt{1-\alpha} \, |\xi|$ and since $\langle \xi, \nu_k \rangle > 0$ (thanks to the fact that $\xi \in D_k$) we get $|\langle \xi, \nu_k \rangle| \leq \sqrt{1-\alpha} \, |\xi|$. Now we can conclude as above. □

By applying Theorem 9 we get a version of Theorem 7 in which the Fourier support of the solutions lie in the set D instead of $\Delta = \mathbb{R}^d \backslash (-\infty, 0)^d$. One can easily deduce from this the original version of Theorem 7 by composing the involved functions (the source term and the solution) by rotations and dilations.

Remark 39. *It would be interesting if one could replace the set $\Delta = \mathbb{R}^d \backslash (-\infty, 0)^d$ in Theorem 7 with the set $(-\infty, 0)^d$. This will give a stronger version of Theorem 7. It is not known whether this stronger version is true or not. The methods used in this paper seem to not apply in the case of the set $(-\infty, 0)^d$.*

5.2 Remark concerning the "third" parameter

Let us consider here the problem related to the nonoptimality of the third parameter. For the sake of simplicity we are concerned here only with the divergence equation. Similar observations can be made for the case of the divergence-like equations.

Recall that, in Theorem 10 in contrast to Theorem 3, we lose some control of the parameter q of the Besov spaces involved: we start with a source term in $\dot{B}^{d/p,p}_q$ and we end up with a solution in $\dot{B}^{d/p,p}_2$ which, despite the fact that it has the "right" differential regularity (the exponents p and $s = d/p$ are the right ones), it is a space strictly larger than $\dot{B}^{d/p,p}_q$. This is due to the fact that in order to easily compute the source space we have chosen X such that $X \hookrightarrow L^\infty$. Consequently, we have to take $\tilde{X}$ strictly larger than X. Indeed, choosing $\tilde{X} = X$ the hypotheses of Theorem 37 imply that $\dot{W}^{d/2,2} \hookrightarrow X \hookrightarrow L^\infty$, however, $\dot{W}^{d/2,2}$ is not embedded in L^∞. By the method we used to prove Theorem 10 it is unlikely to improve the solution space to $L^\infty \cap \dot{B}^{d/p,p}_q$. A similar remark can be made for Theorem 11.

When we use Theorem 37, in order to not lose any regularity, we would like to have that $\tilde{X} = X$ and

$$(L^\infty \cap X, \dot{W}^{d/2,2})_\theta = (X, \dot{W}^{d/2,2})_\theta. \tag{113}$$

Since $\dot{W}^{d/2,2} \hookrightarrow X$ we cannot impose the condition $X \hookrightarrow L^\infty$. Apart from this situation, there are other natural candidates for the space X that one may expect to satisfy (113). However, this condition (113) is too restrictive. For instance we cannot pick $X = \dot{B}^{d/r,r}_r$ for some $r \in (2, \infty)$. Indeed, in this case, we have the following negative result:

Proposition 40. *Let $r \in (2, \infty)$ and $\theta \in (0, 1)$ be some fixed parameters. Then,*

$$(L^\infty(\mathbb{R}^d) \cap \dot{B}^{d/r,r}_r(\mathbb{R}^d), \dot{W}^{d/2,2}(\mathbb{R}^d))_\theta \neq (\dot{B}^{d/r,r}_r(\mathbb{R}^d), \dot{W}^{d/2,2}(\mathbb{R}^d))_\theta.$$

(The corresponding norms on the two interpolation spaces are not equivalent.)

Proof. Suppose by contradiction that we have

$$(L^\infty \cap \dot{B}^{d/r,r}_r, \dot{W}^{d/2,2})_\theta = (\dot{B}^{d/r,r}_r, \dot{W}^{d/2,2})_\theta.$$

This implies that

$$\begin{aligned} (L^\infty \cap \dot{B}^{d/r,r}_r, \dot{W}^{d/2,2})_\theta &= (\dot{B}^{d/r,r}_r, \dot{B}^{d/2,2}_2)_\theta \\ &= \dot{B}^{d/p,p}_p, \end{aligned} \tag{114}$$

where $1/p = (1-\theta)/r + \theta/2$. On the other hand, since $p > 2$, there exists some $\eta \in (0, 1)$ such that

$$(\dot{B}^{d/p,p}_p, \dot{B}^{d,1}_1)_\theta = \dot{B}^{d/2,2}_2.$$

This, together with (114) and T. Wolff's interpolation theorem (see [25, Theorem 2]) implies that, there exists some $\theta_1 \in (0, 1)$ such that

$$(L^\infty \cap \dot{B}^{d/r,r}_r, \dot{B}^{d,1}_1)_{\theta_1} = \dot{B}^{d/2,2}_2. \tag{115}$$

Consider some function $\psi \in C_c^\infty(B(0,1))$ such that $\psi \equiv 1$ on $B(0,1/2)$ and define the operator T_ψ by

$$T_\psi f = f - f * \widehat{\psi},$$

for any Schwartz function f. We extend T_ψ by continuity to the spaces $L^\infty \cap \dot{B}_r^{d/r,r}$, $\dot{B}_1^{d,1}$, $\dot{B}_2^{d/2,2}$ and we easily observe that we have the embeddings $T_\psi(L^\infty \cap \dot{B}_r^{d/r,r}) \hookrightarrow L^\infty$ and $T_\psi(\dot{B}_1^{d,1}) \hookrightarrow B_1^{d,1} \hookrightarrow L^\infty$.

Thanks to this and (115), $T_\psi(\dot{B}_2^{d/2,2})$ must be embedded in L^∞. In other words,

$$\left\| f - f * \widehat{\psi} \right\|_{L^\infty} \lesssim \|f\|_{\dot{B}_2^{d/2,2}} = \|f\|_{\dot{W}^{d/2,2}} \lesssim \|f\|_{W^{d/2,2}}, \tag{116}$$

for any Schwartz function f. Young's inequality and the fact that $\widehat{\psi}$ is Schwartz gives us that

$$||f * \widehat{\psi}||_{L^\infty} \lesssim \left\| \widehat{\psi} \right\|_{L^2} \|f\|_{L^2} \lesssim \|f\|_{W^{d/2,2}},$$

which together with (115) yields

$$\|f\|_{L^\infty} \lesssim \left\| f - f * \widehat{\psi} \right\|_{L^\infty} + \left\| f * \widehat{\psi} \right\|_{L^\infty} \lesssim \|f\|_{W^{d/2,2}}.$$

In other words we have obtained the embedding $W^{d/2,2} \hookrightarrow L^\infty$, which is false. □

Open problem. *Suppose $D = \mathbb{R}^{d-1} \times (0,\infty)$ and X is a function space on $\mathbb{R}^d$ such that*

$$\dot{W}^{d/2,2} \hookrightarrow X \hookrightarrow BMO.$$

Is it true that

$$\operatorname{div}(L^\infty \cap X_D, \dot{W}_D^{d/2,2})_\theta = \operatorname{div}(X_D, \dot{W}_D^{d/2,2})_\theta$$

(in the sense that any divergence of a $(X_D, \dot{W}_D^{d/2,2})_\theta$ vector field is a divergence of a $(L^\infty \cap X_D, \dot{W}_D^{d/2,2})_\theta$ vector field)?

If the answer to this question is yes, then, by using Theorem 37 we would be able to provide a version of Theorem 7 with no loss of regularity in the third parameter:

Conjecture. *Let $d \geq 3$ be an integer and consider the set $\Delta := \mathbb{R}^d \backslash (-\infty, 0)^d$. Consider some parameters $p \in [2,\infty)$ and $q \in (1,\infty)$. Then, for any vector field $v \in \mathcal{S}'(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$, with $spec(v) \subseteq \Delta$ there exists a vector field $u \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{d/p,p}(\mathbb{R}^d)$, with $spec(u) \subseteq \Delta$ such that*

$$\operatorname{div} u = \operatorname{div} v,$$

and

$$\|u\|_{L^\infty \cap \dot{F}_q^{d/p,p}} \lesssim \|v\|_{\dot{F}_q^{d/p,p}}.$$

(And a similar statement with $\dot{B}_q^{d/p,p}$ in place of $\dot{F}_q^{d/p,p}$.)

One can formulate similar conjectures corresponding to the statements of Theorem 9 and Theorem 8.

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