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On a recolouring version of Hadwiger’s conjecture

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Abstract

We prove that for any $\varepsilon > 0$, for any large enough $t$, there is a graph $G$ that admits no $K_t$-minor but admits a $(3^2 - \varepsilon)t$-colouring that is “frozen” with respect to Kempe changes, i.e. any two colour classes induce a connected component. This disproves three conjectures of Las Vergnas and Meyniel from 1981.

1 Introduction

In an attempt to prove the Four Colour Theorem in 1879, Kempe [Kem79] introduced an elementary operation on the colourings of a graph that became known as a Kempe change. Given a $k$-colouring $\alpha$ of a graph $G$, a Kempe chain is a maximal bichromatic component. A Kempe change in $\alpha$ corresponds to swapping the two colours of a Kempe chain so as to obtain another $k$-colouring. Two $k$-colourings are Kempe equivalent if one can be obtained from the other through a series of Kempe changes.

The study of Kempe changes has a vast history, see e.g. [Moh06] for a comprehensive overview or [BBF19] for a recent result on general graphs. We refer the curious reader to the relevant chapter of a 2013 survey by Cereceda [vdH13]. Kempe equivalence falls within the wider setting of combinatorial reconconfiguration, which [vdH13] is also an excellent introduction to. Perhaps surprisingly, Kempe equivalence has direct applications in approximate counting and applications in statistical physics (see e.g. [Sok00, MS09] for nice overviews). Closer to graph theory, Kempe equivalence can be studied with a goal of obtaining a random colouring by applying random walks and rapidly mixing Markov chains, see e.g. [Vig00].

Kempe changes were introduced as a mere tool, and are decisive in the proof of Vizing’s edge colouring theorem [Viz64]. However, the equivalence class they define on the set of $k$-colourings is itself highly interesting. In which cases is there a single equivalence class? In which cases does every equivalence class contain a colouring that uses the minimum number of colours? Vizing conjectured in 1965 [Viz68] that the second scenario should be true in every line graph, no matter the choice of $k$. Despite partial results [Asr09, AC16], this conjecture remains wildly open.

1 Throughout this paper, all colourings are proper, i.e. no two vertices with the same colour are adjacent.
2 If a vertex of $G$ is coloured 1 and has no neighbour coloured 2 in $\alpha$, then it forms a Kempe chain of size 1.
In the setting of planar graphs, Meyniel proved in 1977 [Mey78] that all 5-colourings form a unique Kempe equivalence class. The result was then extended to all $K_5$-minor-free graphs in 1979 by Las Vergnas and Meyniel [LVM81]. They conjectured the following, which can be seen as a reconfiguration counterpoint to Hadwiger’s conjecture, though it neither implies it nor is implied by it.

**Conjecture 1.1** (Conjecture A in [LVM81]). For every $t$, all the $t$-colourings of a graph with no $K_t$-minor form a single equivalence class.

They also proposed a related conjecture that is weaker assuming Hadwiger’s conjecture holds.

**Conjecture 1.2** (Conjecture A’ in [LVM81]). For every $t$ and every graph with no $K_t$-minor, every equivalence class of $t$-colourings contains some $(t-1)$-colouring.

Here, we disprove both Conjectures 1.1 and 1.2, as follows.

**Theorem 1.3.** For every $\varepsilon > 0$ and for any large enough $t$, there is a graph with no $K_t$-minor, whose $(\frac{3}{2} - \varepsilon)t$-colourings are not all Kempe equivalent.

In fact, we prove that for every $\varepsilon > 0$ and for any large enough $t$, there is a graph $G$ that does not admit a $K_t$-minor but admits a $(\frac{3}{2} - \varepsilon)t$-colouring that is frozen; any pair of colours induce a connected component, so that no Kempe change can modify the colour partition. To obtain Theorem 1.3, we then argue that the graph admits a colouring with a different colour partition. The notion of frozen $k$-colouring is related to the notion of quasi-$K_p$-minor, introduced in [LVM81]. A graph $G$ admits a $K_p$-minor if it admits $p$ non-empty, pairwise disjoint and connected bags $B_1, \ldots, B_p \subset V(G)$ such that for any $i \neq j$, there is an edge between some vertex in $B_i$ and some vertex in $B_j$. For the notion of quasi-$K_p$-minor, we drop the restriction that each $B_i$ should induce a connected subgraph of $G$, and replace it with the condition that for any $i \neq j$, the set $B_i \cup B_j$ induces a connected subgraph of $G$. If the graph $G$ admits a frozen $p$-colouring, then it trivially admits a quasi-$K_p$-minor, while the converse may not be true. If all $p$-colourings of a graph form a single equivalence class, then either there is no frozen $p$-colouring or there is a unique $p$-colouring of the graph up to colour permutation. The latter situation in a graph with no $K_p$-minor would disprove Hadwiger’s conjecture, so Las Vergnas and Meyniel conjectured that there is no frozen $p$-colouring in that case. Namely, they conjectured the following.

**Conjecture 1.4** (Conjecture C in [LVM81]). For any $t$, any graph that admits a quasi-$K_t$-minor admits a $K_t$-minor.

Conjecture 1.4 is known to hold for $t \leq 8$ [Jor94]. As discussed above, we strongly disprove Conjecture 1.4 for large $t$. It is unclear how large $t$ needs to be for a counter-example.

**Theorem 1.5.** For every $\varepsilon > 0$ and for any large enough $t$, there is a graph $G$ that admits a quasi-$K_t$-minor but does not admit a $K_{(\frac{3}{2} + \varepsilon)t}$-minor.

Trivially, every graph that admits a quasi-$K_{2t}$-minor admits a $K_t$-minor. We leave the following two open questions, noting that $\frac{2}{3} \geq c \geq \frac{1}{2}$ and $c' \geq \frac{3}{2}$.

**Question 1.6.** What is the infimum $c$ such that for any large enough $t$, there is a graph $G$ that admits a quasi-$K_t$-minor but no $K_{ct}$-minor?

**Question 1.7.** Is there a constant $c'$ such that for every $t$, all the $c' \cdot t$-colourings of a graph with no $K_t$-minor form a single equivalence class?

\[3\] One bag for each colour class.
2 Construction

Let $n \in \mathbb{N}$ and let $\eta > 0$. We build a random graph $G_n$ on vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$: for every $i \neq j$ independently, we select one pair uniformly at random among $\{(a_i, a_j), (a_i, b_j), (b_i, a_j), (b_i, b_j)\}$ and add the three other pairs as edges to the graph $G_n$.

Note that the sets $\{a_i, b_i\}_{1 \leq i \leq n}$ form a quasi-$K_n$-minor, as for every $i \neq j$, the set $\{a_i, a_j, b_i, b_j\}$ induces a path on four vertices in $G_n$, hence is connected.

Our goal is to argue that if $n$ is sufficiently large then with high probability the graph $G_n$ does not admit any $K_{(2^3 + \eta)n}$-minor. This will yield Theorem 1.5. To additionally obtain Theorem 1.3, we need to argue that with high probability, $G_n$ admits an $n$-colouring with a different colour partition than the natural one, where the colour classes are of the form $\{a_i, b_i\}$. Informally, we can observe that each of $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ induces a graph behaving like a graph in $G_{n,4}$ (i.e. each edge exists with probability $\frac{3}{4}$) though the two processes are not independent. This argument indicates that $\chi(G_n) = O(\frac{n}{\log n})$, but we prefer a simpler, more pedestrian approach.

Assume that for some $i, j, k, \ell$, none of the edges $a_ib_j, a_jb_k, a_kb_\ell$ and $a_\ell b_i$ exist. Then the graph $G_n$ admits an $n$-colouring $\alpha$ where $\alpha(a_p) = \alpha(b_p) = p$ for every $p \notin \{i, j, k, \ell\}$ and $\alpha(a_i) = \alpha(b_j) = i, \alpha(a_j) = \alpha(b_k) = j, \alpha(a_k) = \alpha(b_\ell) = k$ and $\alpha(a_\ell) = \alpha(b_i) = \ell$ (see Figure 1). Since every quadruple $(i, j, k, \ell)$ has a positive and constant probability of satisfying this property, $G_n$ contains such a quadruple with overwhelmingly high probability when $n$ is large.

![Figure 1: A different n-colouring given an appropriate quadruple.](image)

We are now ready to prove that the probability that $G_n$ admits a $K_{(2^3 + \eta)n}$-minor tends to 0 as $n$ grows to infinity. We consider three types of $K_p$-minors in $G$, depending on the size of the bags involved. If every bag is of size 1, we say that it is a simple $K_p$-minor – in fact, it is a subgraph. If every bag is of size 2, we say it is a double $K_p$-minor. If every bag is of size at least 3, we say it is a triple $K_p$-minor. We prove three claims, as follows.

Claim 2.1. For any $\varepsilon > 0$, $\mathbb{P}(G_n \text{ contains a simple } K_{\varepsilon n}\text{-minor}) \rightarrow 0$ as $n \rightarrow \infty$.

Claim 2.2. For any $\varepsilon > 0$, $\mathbb{P}(G_n \text{ contains a double } K_{\varepsilon n}\text{-minor}) \rightarrow 0$ as $n \rightarrow \infty$. 

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Claim 2.3. $G_n$ does not contain a triple $K_{2n+1}$-minor.

Claims 2.1, 2.2 and 2.3 are proved in Sections 2.1, 2.2 and 2.3, respectively. If a graph admits a $K_p$-minor, then in particular it admits a simple $K_n$-minor, a double $K_b$-minor and a triple $K_c$-minor such that $a + b + c \geq p$. Combining Claims 2.1, 2.2 and 2.3, we derive the desired conclusion.

2.1 No large simple minor

Proof of Claim 2.1. Let $S$ be a subset of $k$ vertices of $G_n$. The probability that $S$ induces a clique in $G_n$ is at most $\binom{\frac{n}{3}}{\frac{k}{3}}$. Indeed, if $\{a_i, b_i\} \subseteq S$ for some $i$, then the probability is 0. Otherwise, $|S \cap \{a_i, b_i\}| \leq 1$ for every $i$, so we have $G[S] \in G_{k, \frac{n}{3}}$, i.e. edges exist independently with probability $\frac{3}{4}$. Therefore, the probability that $S$ induces a clique is $\binom{\frac{n}{3}}{\frac{k}{3}}$.

By union-bound, the probability that some subset on $k$ vertices induces a clique is at most $\binom{2n}{k} \cdot \frac{3}{4}^k$. For any $\varepsilon > 0$, we note that $\binom{2n}{\varepsilon n} \leq 2^{\varepsilon n}$. Therefore, the probability that $G_n$ contains a simple $K_{\varepsilon n}$-minor is at most $2^{\varepsilon n} \cdot \left(\frac{3}{4}\right)^{\varepsilon n}$, which tends to 0 as $n$ grows to infinity.

2.2 No large double minor

Proof of Claim 2.2. Let $S'$ be a subset of $k$ pairwise disjoint pairs of vertices in $G_n$ such that for every $i$, at most one of $\{a_i, b_i\}$ is involved in $S'$.

We consider the probability that $G_n/_{S'}$ induces a clique, where $G_n/_{S'}$ is defined as the graph obtained from $G_n$ by considering only vertices involved in some pair of $S'$ and identifying the vertices in each pair.

We consider two distinct pairs $(x_1, y_1), (x_2, y_2)$ of $S'$. Without loss of generality, $\{x_1, x_2, y_1, y_2\} = \{a_i, a_j, a_k, a_\ell\}$ for some $i, j, k, \ell$. The probability that there is an edge between $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is $1 - \left(\frac{1}{4}\right)^4$. In other words, $\mathbb{P}(E((x_1, y_1), (x_2, y_2)) = \emptyset) = \left(\frac{1}{4}\right)^4$ and since at most one of $\{a_i, b_i\}$ is involved in $S'$ for all $i$, all such events are mutually independent. Therefore, the probability that $S'$ yields a quasi-$K_{|S'|}$-minor is $\left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{|S'|}{2}}$.

For any $\varepsilon' > 0$, the number of candidates for $S'$ is at most $\binom{2n}{2\varepsilon'n}$ (the number of choices for a ground set of $2\varepsilon'n$ vertices) times $(2\varepsilon'n)!$ (a rough upper bound on the number of ways to pair them). Note that $\binom{2n}{2\varepsilon'n} \cdot (2\varepsilon'n)! \leq (2n)^{2\varepsilon'n}$. We derive that the probability that there is a set $S'$ of size $\varepsilon'n$ such that $G_n/_{S'} = K_{|S'|}$ is at most $(2n)^{2\varepsilon'n} \cdot \left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{|S'|}{2}}$, which tends to 0 as $n$ grows large.

Consider a double $K_k$-minor $S$ of $G_n$. Note that no pair in $S$ is equal to $\{a_i, b_i\}$ (for any $i$), as every bag induces a connected subgraph in $G_n$. We build greedily a maximal subset $S' \subseteq S$ such that $S'$ involves at most one vertex out of every set of type $\{a_i, b_i\}$. Note that $|S'| \geq \frac{|S|}{3}$. By taking $\varepsilon' = \frac{2}{3}$ in the above analysis, we obtain that the probability that there is a set $S$ of $\varepsilon n$ pairs that induces a quasi-$K_{|S'|}$-minor tends to 0 as $n$ grows large.

2.3 No large triple minor

Proof of Claim 2.3. The graph $G_n$ has $2n$ vertices, and a triple $K_k$-minor involves at least $3k$ vertices. It follows that if $G_n$ contains a triple $K_k$-minor then $k \leq \frac{2n}{3}$.
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