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On a recolouring version of Hadwiger’s conjecture

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Abstract

We prove that for any $\varepsilon > 0$, for any large enough $t$, there is a graph $G$ that admits no $K_t$-minor but admits a $(\frac{3}{2} - \varepsilon)t$-colouring that is “frozen” with respect to Kempe changes, i.e. any two colour classes induce a connected component. This disproves three conjectures of Las Vergnas and Meyniel from 1981.

1 Introduction

In an attempt to prove the Four Colour Theorem in 1879, Kempe [Kem79] introduced an elementary operation on the colourings$^1$ of a graph that became known as a Kempe change. Given a $k$-colouring $\alpha$ of a graph $G$, a Kempe chain is a maximal bichromatic component$^2$. A Kempe change in $\alpha$ corresponds to swapping the two colours of a Kempe chain so as to obtain another $k$-colouring. Two $k$-colourings are Kempe equivalent if one can be obtained from the other through a series of Kempe changes.

The study of Kempe changes has a vast history, see e.g. [Moh06] for a comprehensive overview or [BBF19] for a recent result on general graphs. We refer the curious reader to the relevant chapter of a 2013 survey by Cereceda [vdH13]. Kempe equivalence falls within the wider setting of combinatorial recon/f_iguration, which [vdH13] is also an excellent introduction to. Perhaps surprisingly, Kempe equivalence has direct applications in approximate counting and applications in statistical physics (see e.g. [Sok00, MS09] for nice overviews). Closer to graph theory, Kempe equivalence can be studied with a goal of obtaining a random colouring by applying random walks and rapidly mixing Markov chains, see e.g. [Vig00].

Kempe changes were introduced as a mere tool, and are decisive in the proof of Vizing’s edge colouring theorem [Viz64]. However, the equivalence class they define on the set of $k$-colourings is itself highly interesting. In which cases is there a single equivalence class? In which cases does every equivalence class contain a colouring that uses the minimum number of colours? Vizing conjectured in 1965 [Viz68] that the second scenario should be true in every line graph, no matter the choice of $k$. Despite partial results [Asr09, AC16], this conjecture remains wildly open.

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$^2$Throughout this paper, all colourings are proper, i.e. no two vertices with the same colour are adjacent.
$^3$If a vertex of $G$ is coloured 1 and has no neighbour coloured 2 in $\alpha$, then it forms a Kempe chain of size 1.
In the setting of planar graphs, Meyniel proved in 1977 [Mey78] that all 5-colourings form a unique Kempe equivalence class. The result was then extended to all $K_5$-minor-free graphs in 1979 by Las Vergnas and Meyniel [LVM81]. They conjectured the following, which can be seen as a reconfiguration counterpoint to Hadwiger’s conjecture, though it neither implies it nor is implied by it.

**Conjecture 1.1** (Conjecture A in [LVM81]). For every $t$, all the $t$-colourings of a graph with no $K_t$-minor form a single equivalence class.

They also proposed a related conjecture that is weaker assuming Hadwiger’s conjecture holds.

**Conjecture 1.2** (Conjecture A’ in [LVM81]). For every $t$ and every graph with no $K_t$-minor, every equivalence class of $t$-colourings contains some $(t - 1)$-colouring.

Here, we disprove both Conjectures 1.1 and 1.2, as follows.

**Theorem 1.3.** For every $\varepsilon > 0$ and for any large enough $t$, there is a graph with no $K_t$-minor, whose $(\frac{3}{2} - \varepsilon)t$-colourings are not all Kempe equivalent.

In fact, we prove that for every $\varepsilon > 0$ and for any large enough $t$, there is a graph $G$ that does not admit a $K_t$-minor but admits a $(\frac{3}{2} - \varepsilon)t$-colouring that is frozen; Any pair of colours induce a connected component, so that no Kempe change can modify the colour partition. To obtain Theorem 1.3, we then argue that the graph admits a colouring with a different colour partition. The notion of frozen $k$-colouring is related to the notion of quasi-$K_p$-minor, introduced in [LVM81]. A graph $G$ admits a $K_p$-minor if it admits $p$ non-empty, pairwise disjoint and connected bags $B_1, \ldots, B_p \subset V(G)$ such that for any $i \neq j$, there is an edge between some vertex in $B_i$ and some vertex in $B_j$. For the notion of quasi-$K_p$-minor, we drop the restriction that each $B_i$ should induce a connected subgraph of $G$, and replace it with the condition that for any $i \neq j$, the set $B_i \cup B_j$ induces a connected subgraph of $G$. If the graph $G$ admits a frozen $p$-colouring, then it trivially admits a quasi-$K_p$-minor, while the converse may not be true. If all $p$-colourings of a graph form a single equivalence class, then either there is no frozen $p$-colouring or there is a unique $p$-colouring of the graph up to colour permutation. The latter situation in a graph with no $K_p$-minor would disprove Hadwiger’s conjecture, so Las Vergnas and Meyniel conjectured that there is no frozen $p$-colouring in that case. Namely, they conjectured the following.

**Conjecture 1.4** (Conjecture C in [LVM81]). For any $t$, any graph that admits a quasi-$K_t$-minor admits a $K_t$-minor.

Conjecture 1.4 is known to hold for $t \leq 8$ [Jor94]. As discussed above, we strongly disprove Conjecture 1.4 for large $t$. It is unclear how large $t$ needs to be for a counter-example.

**Theorem 1.5.** For every $\varepsilon > 0$ and for any large enough $t$, there is a graph $G$ that admits a quasi-$K_t$-minor but does not admit a $K_{(\frac{3}{2} + \varepsilon)t}$-minor.

Trivially, every graph that admits a quasi-$K_{2t}$-minor admits a $K_t$-minor. We leave the following two open questions, noting that $\frac{2}{3} \geq c \geq \frac{1}{2}$ and $c' \geq \frac{3}{2}$.

**Question 1.6.** What is the infimum $c$ such that for any large enough $t$, there is a graph $G$ that admits a quasi-$K_t$-minor but no $K_{ct}$-minor?

**Question 1.7.** Is there a constant $c'$ such that for every $t$, all the $c' \cdot t$- colourings of a graph with no $K_t$-minor form a single equivalence class?

3One bag for each colour class.
2 Construction

Let \( n \in \mathbb{N} \) and let \( \eta > 0 \). We build a random graph \( G_n \) on vertex set \( \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \): for every \( i \neq j \) independently, we select one pair uniformly at random among \( \{(a_i, a_j), (a_i, b_j), (b_i, a_j), (b_i, b_j)\} \) and add the three other pairs as edges to the graph \( G_n \).

Note that the sets \( \{a_i, b_i\}_{1 \leq i \leq n} \) form a quasi-\( K_n \)-minor, as for every \( i \neq j \), the set \( \{a_i, b_i, a_j, b_j\} \) induces a path on four vertices in \( G_n \), hence is connected.

Our goal is to argue that if \( n \) is sufficiently large then with high probability the graph \( G_n \) does not admit any \( K\left(\frac{2}{3} + \eta\right)n \)-minor. This will yield Theorem 1.5. To additionally obtain Theorem 1.3, we need to argue that with high probability, \( G_n \) admits an \( n \)-colouring with a different colour partition than the natural one, where the colour classes are of the form \( \{a_i, b_i\} \). Informally, we can observe that each of \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \) induces a graph behaving like a graph in \( G_{n, \frac{3}{4}} \) (i.e. each edge exists with probability \( \frac{3}{4} \)) though the two processes are not independent. This argument indicates that \( \chi(G_n) = O\left(\frac{n}{\log n}\right) \), but we prefer a simpler, more pedestrian approach.

Assume that for some \( i, j, k, \ell \), none of the edges \( a_ib_j, a_jb_k, a_kb_\ell \) and \( a_\ell b_i \) exist. Then the graph \( G_n \) admits an \( n \)-colouring \( \alpha \) where \( \alpha(a_p) = \alpha(b_p) = p \) for every \( p \not\in \{i, j, k, \ell\} \) and \( \alpha(a_i) = \alpha(b_j) = i, \alpha(a_j) = \alpha(b_k) = j, \alpha(a_k) = \alpha(b_\ell) = k \) and \( \alpha(a_\ell) = \alpha(b_i) = \ell \) (see Figure 1). Since every quadruple \( (i, j, k, \ell) \) has a positive and constant probability of satisfying this property, \( G_n \) contains such a quadruple with overwhelmingly high probability when \( n \) is large.

We are now ready to prove that the probability that \( G_n \) admits a \( K\left(\frac{2}{3} + \eta\right)n \)-minor tends to 0 as \( n \) grows to infinity. We consider three types of \( K_p \)-minors in \( G \), depending on the size of the bags involved. If every bag is of size 1, we say that it is a simple \( K_p \)-minor – in fact, it is a subgraph. If every bag is of size 2, we say it is a double \( K_p \)-minor. If every bag is of size at least 3, we say it is a triple \( K_p \)-minor. We prove three claims, as follows.

**Claim 2.1.** For any \( \varepsilon > 0 \), \( \mathbb{P}(G_n \text{ contains a simple } K_{\varepsilon n}-\text{minor}) \to 0 \) as \( n \to \infty \).

**Claim 2.2.** For any \( \varepsilon > 0 \), \( \mathbb{P}(G_n \text{ contains a double } K_{\varepsilon n}-\text{minor}) \to 0 \) as \( n \to \infty \).

![Figure 1: A different n-colouring given an appropriate quadruple.](image-url)
Claim 2.3. $G_n$ does not contain a triple $K_{3n+1}$-minor.

Claims 2.1, 2.2 and 2.3 are proved in Sections 2.1, 2.2 and 2.3, respectively. If a graph admits a $K_p$-minor, then in particular it admits a simple $K_n$-minor, a double $K_r$-minor and a triple $K_c$-minor such that $a + b + c \geq p$. Combining Claims 2.1, 2.2 and 2.3, we derive the desired conclusion.

2.1 No large simple minor

Proof of Claim 2.1. Let $S$ be a subset of $k$ vertices of $G_n$. The probability that $S$ induces a clique in $G_n$ is at most $\left(\frac{3}{4}\right)^{\binom{n}{2}}$. Indeed, if $\{a_i, b_i\} \subseteq S$ for some $i$, then the probability is 0. Otherwise, $|S \cap \{a_i, b_i\}| \leq 1$ for every $i$, so we have $G[S] \in G_{k,\frac{3}{4}}$, i.e. edges exist independently with probability $\frac{3}{4}$. Therefore, the probability that $S$ induces a clique is $\left(\frac{3}{4}\right)^{\binom{n}{2}}$.

By union-bound, the probability that some subset on $k$ vertices induces a clique is at most $(2^n/k) \cdot \left(\frac{3}{4}\right)^{\binom{n}{2}}$. For any $\varepsilon > 0$, we note that $(2^n/\varepsilon n) \leq 2^n$. Therefore, the probability that $G_n$ contains a simple $K_{\varepsilon n}$-minor is at most $2^n \cdot \left(\frac{3}{4}\right)^{\binom{n}{2}}$, which tends to 0 as $n$ grows to infinity.

2.2 No large double minor

Proof of Claim 2.2. Let $S'$ be a subset of $k$ pairwise disjoint pairs of vertices in $G_n$, such that for every $i$, at most one of $\{a_i, b_i\}$ is involved in $S'$.

We consider the probability that $G_n/S'$ induces a clique, where $G_n/S'$ is defined as the graph obtained from $G_n$ by considering only vertices involved in some pair of $S'$ and identifying the vertices in each pair.

We consider two distinct pairs $(x_1, y_1), (x_2, y_2)$ of $S'$. Without loss of generality, $\{x_1, x_2, y_1, y_2\} = \{a_i, a_j, a_k, a_l\}$ for some $i, j, k, \ell$. The probability that there is an edge between $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is $1 - \left(\frac{1}{4}\right)^4$. In other words, $\mathbb{P}(E((x_1, y_1), (x_2, y_2)) = \emptyset) = \left(\frac{1}{4}\right)^4$ and since at most one of $\{a_i, b_i\}$ is involved in $S'$ for all $i$, all such events are mutually independent. Therefore, the probability that $S'$ yields a quasi-$K_{|S'|}$-minor is $\left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{|S'|}{2}}$.

For any $\varepsilon' > 0$, the number of candidates for $S'$ is at most $(2^n/2\varepsilon' n)$ (the number of choices for a ground set of $2\varepsilon'n$ vertices) times $(2\varepsilon'n)!$ (a rough upper bound on the number of ways to pair them). Note that $(2^n/2\varepsilon' n) \cdot (2\varepsilon' n)! \leq (2n)^{2\varepsilon'n}$. We derive that the probability that there is a set $S'$ of size $\varepsilon'n$ such that $G_n/S' = K_{|S'|}$ is at most $(2n)^{2\varepsilon'n} \cdot \left(1 - \left(\frac{1}{4}\right)^4\right)^{\binom{n}{2}}$, which tends to 0 as $n$ grows large.

Consider a double $K_k$-minor $S$ of $G_n$. Note that no pair in $S$ is equal to $\{a_i, b_i\}$ (for any $i$), as every bag induces a connected subgraph in $G_n$. We build greedily a maximal subset $S' \subseteq S$ such that $S'$ involves at most one vertex out of every set of type $\{a_i, b_i\}$. Note that $|S'| \geq \frac{|S|}{3}$. By taking $\varepsilon' = \frac{\varepsilon}{2}$ in the above analysis, we obtain that the probability that there is a set $S$ of $\varepsilon n$ pairs that induces a quasi-$K_{\varepsilon n}$-minor tends to 0 as $n$ grows large.

2.3 No large triple minor

Proof of Claim 2.3. The graph $G_n$ has $2n$ vertices, and a triple $K_k$-minor involves at least $3k$ vertices. It follows that if $G_n$ contains a triple $K_k$-minor then $k \leq \frac{2n}{3}$.
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References


