



**HAL**  
open science

# COMPACTNESS OF THE SET OF SOLUTIONS TO ELLIPTIC EQUATIONS IN 2 DIMENSIONS

Samy Skander Bahoura

► **To cite this version:**

Samy Skander Bahoura. COMPACTNESS OF THE SET OF SOLUTIONS TO ELLIPTIC EQUATIONS IN 2 DIMENSIONS. *Electronic Journal of Differential Equations*, 2022, 2022 (14), pp.1-10. hal-04096482

**HAL Id: hal-04096482**

**<https://hal.science/hal-04096482>**

Submitted on 12 May 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## COMPACTNESS OF THE SET OF SOLUTIONS TO ELLIPTIC EQUATIONS IN 2 DIMENSIONS

SAMY SKANDER BAHOURA

ABSTRACT. We study the behavior of solutions to elliptic equations in 2 dimensions. In particular, we show that the set of solutions is compact under a Lipschitz condition.

### 1. INTRODUCTION

Let us define the operator

$$e_\epsilon^L := \Delta + \epsilon(x_1\partial_1 + x_2\partial_2) = \frac{\operatorname{div}[a_\epsilon(x)\nabla]}{a_\epsilon(x)}, \quad \text{with } a_\epsilon(x) = e^{\epsilon|x|^2/2}.$$

We consider the equation

$$\begin{aligned} -\Delta u - \epsilon(x_1\partial_1 u + x_2\partial_2 u) &= -L_\epsilon u = V e^u \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\Omega$  is a starshaped set,  $u \in W_0^{1,1}(\Omega)$ ,  $e^u \in L^1(\Omega)$ ,  $0 \leq V \leq b$ ,  $1 \geq \epsilon \geq 0$ .

For  $\epsilon = 0$  equation (1.1) has been studied by many authors with and without the boundary condition. This equation also has been studied in Riemann surfaces; see [1]–[20], where one can find some existence and compactness results. Also we have a nice formulation in the sense of the distributions of this problem in [7]. Among the known results we find the following Theorem.

**Theorem 1.1** (Brezis-Merle [6]). *If  $(u_i)$  and  $(V_i)$  are two sequences of functions in problem (1.1) with  $\epsilon = 0$ , and*

$$0 < a \leq V_i \leq b < +\infty,$$

*then for all compact subset  $K$  of  $\Omega$  it holds*

$$\sup_K u_i \leq c,$$

*with  $c$  depending on  $a, b, K$  and  $\Omega$ .*

We can find an interior estimate if we assume  $a = 0$ , but we need an assumption on the integral of  $e^{u_i}$ .

---

2020 *Mathematics Subject Classification*. 35J60, 35B44, 35B45.

*Key words and phrases*. Blow-up; compactness; boundary; elliptic equation; Lipschitz condition; starshaped domain.

©2022. This work is licensed under a CC BY 4.0 license.

Submitted September 4, 2021. Published February 21, 2022.

**Theorem 1.2** (Brezis-Merle [6]). *Let  $(u_i)$  and  $(V_i)$  two sequences of functions in problem (1.1) with*

$$0 \leq V_i \leq b < +\infty \quad \text{and} \quad \int_{\Omega} e^{u_i} dy \leq C.$$

*Then, for all compact subset  $K$  of  $\Omega$  it holds*

$$\sup_K u_i \leq c,$$

*with  $c$  depending on  $b, C, K$  and  $\Omega$ .*

The condition  $\int_{\Omega} e^{u_i} dy \leq C$  is a necessary in Problem (1.1) as showed by the following statement for  $\epsilon = 0$ .

**Theorem 1.3** (Brezis-Merle [6]). *There are sequences  $(u_i)$  and  $(V_i)$  in problem (1.1) with*

$$0 \leq V_i \leq b < +\infty, \quad \int_{\Omega} e^{u_i} dy \leq C,$$

*such that  $\sup_{\Omega} u_i \rightarrow +\infty$ .*

To obtain Theorems 1.1 and 1.2 Brezis and Merle used an inequality [6, Theorem 1] obtained by an approximation argument, Fatou's lemma, and the maximum principle in  $W_0^{1,1}(\Omega)$ , which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer the reader to [5] for information about the weak form of the maximum principle.

Note that for problem (1.1), by using the Pohozaev identity, we can prove that  $\int_{\Omega} e^{u_i}$  is uniformly bounded when  $0 < a \leq V_i \leq b < +\infty$ ,  $\|\nabla V_i\|_{L^\infty} \leq A$ , and  $\Omega$  starshaped. When  $a = 0$  and  $\nabla \log V_i$  is uniformly bounded, we can find a uniform bound for  $\int_{\Omega} V_i e^{u_i}$ .

Ma-Wei [17] proved that those results remain valid for all open sets not necessarily starshaped when  $a > 0$ . Chen-Li [9] proved that if  $a = 0$ ,  $\int_{\Omega} e^{u_i}$  is uniformly bounded, and  $\nabla \log V_i$  is uniformly bounded, then  $(u_i)$  is bounded near the boundary and we have directly the compactness result for the problem (1.1). Ma-Wei [17] extend this result in the case where  $a > 0$ .

When  $\epsilon = 0$  and if we assume  $V$  more regular we can have another type of estimates called sup + inf type inequalities. It was proved by Shafrir [19] that, if  $(u_i), (V_i)$  are two sequences of solutions to Problem (1.1), without assumption on the boundary and  $0 < a \leq V_i \leq b < +\infty$ , then it holds

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We find in [10] the explicit value  $C(a/b) = \sqrt{a/b}$ . In his proof, Shafrir [19] used the blow-up function, the Stokes formula and an isoperimetric inequality. Chen-Lin [10] used the blow-up analysis combined with some geometric type inequality for obtaining the integral curvature.

Now, if we assume  $(V_i)$  is uniformly Lipschitzian with constant  $A$ , then  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$  see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences  $(V_i)$  by Chen-Lin [10]. Also we have in [15], an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [17] an explicit form,  $(8\pi m, m \in \mathbb{N}^*$  exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here the notion of isolated

blow-up point is used. Also one can find in [11] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

Here we study the behavior of the blow-up points on the boundary, and give a compactness result with Lipschitz condition. Note that our problem is an extension of the Brezis-Merle Problem.

**Brezis-Merle Problem** [6]. Suppose that  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$  with  $0 \leq V_i$ , and consider a sequence of solutions  $(u_i)$  of (1.1) relative to  $(V_i)$  such that

$$\int_{\Omega} e^{u_i} dx \leq C.$$

Is it possible to have

$$\|u_i\|_{L^\infty} \leq C = C(b, C, V, \Omega)?$$

Here we give a blow-up analysis on the boundary when  $V_i$  are nonnegative and bounded (similar to the prescribed curvature when  $\epsilon = 0$ ). On the other hand, if we add the assumption that these functions (similar to the prescribed curvature) are uniformly Lipschitzian, we have a compactness of the solutions of problem (1.1) for  $\epsilon$  small enough. (In particular we can take a sequence of  $\epsilon_i$  tending to 0).

For the behavior of the blow-up points on the boundary, the following condition is sufficient,

$$0 \leq V_i \leq b,$$

The condition  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$  is not necessary. But for the compactness of the solutions we add the condition

$$\|\nabla V_i\|_{L^\infty} \leq A.$$

Our main results read as follows.

**Theorem 1.4.** *Assume that  $\max_{\Omega} u_i \rightarrow +\infty$ , where  $(u_i)$  are solutions of (1.1) with  $\epsilon = \epsilon_i$  and*

$$0 \leq V_i \leq b, \quad \int_{\Omega} e^{u_i} dx \leq C, \quad \epsilon_i \rightarrow 0.$$

*Then, after passing to a subsequence, there are a function  $u$ , a number  $N \in \mathbb{N}$ , and  $N$  points  $x_1, \dots, x_N \in \partial\Omega$ , such that*

$$\partial_\nu u_i \rightarrow \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi,$$

*in the sense of measures on  $\partial\Omega$ , and*

$$u_i \rightarrow u \quad \text{in } C_{\text{loc}}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

**Theorem 1.5.** *Assume that  $(u_i)$  are solutions of (1.1) with  $\epsilon = \epsilon_i$ , and*

$$0 \leq V_i \leq b, \quad \|\nabla V_i\|_{L^\infty} \leq A, \quad \int_{\Omega} e^{u_i} \leq C, \quad \epsilon_i \rightarrow 0.$$

*Then*

$$\|u_i\|_{L^\infty} \leq c(b, A, C, \Omega).$$

## 2. PROOFS OF MAIN RESULTS

**Proof of Theorem 1.4.** First we remark that

$$\begin{aligned} -\Delta u_i &= \epsilon_i(x_1 \partial_1 u_i + x_2 \partial_2 u_i) + V_i e^{u_i} \in L^1(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u_i &= 0 \quad \text{in } \partial\Omega. \end{aligned} \quad (2.1)$$

and  $u_i \in W_0^{1,1}(\Omega)$ .

By [6, Corollary 1] we have  $e^{u_i} \in L^k(\Omega)$  for all  $k > 2$  and the elliptic estimates of Agmon and the Sobolev embedding see [1] imply that

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Also we remark that for two positive constants  $C_q = C(q, \Omega)$  and  $C_1 = C_1(\Omega)$ , we have

$$\|\nabla u_i\|_{L^q} \leq C_q \|\Delta u_i\|_{L^1} \leq (C'_q + \epsilon C_1 \|\nabla u_i\|_{L^1}), \quad \forall i \text{ and } 1 < q < 2.$$

(see [7]). Thus, if  $\epsilon > 0$  is small enough and by Holder's inequality,

$$\|\nabla u_i\|_{L^q} \leq C''_q, \quad \forall i \text{ and } 1 < q < 2.$$

**Step 1: Interior estimate.** First we consider the equation

$$\begin{aligned} -\Delta w_i &= \epsilon_i(x_1 \partial_1 u_i + x_2 \partial_2 u_i) \in L^q, \quad 1 < q < 2 \quad \text{in } \Omega \subset \mathbb{R}^2, \\ w_i &= 0 \quad \text{in } \partial\Omega. \end{aligned} \quad (2.2)$$

If we consider  $v_i$  as the Newtonian potential of  $\epsilon_i(x_1 \partial_1 u_i + x_2 \partial_2 u_i)$ , we have

$$v_i \in C^0(\bar{\Omega}), \quad \Delta(w_i - v_i) = 0.$$

By the maximum principle  $w_i - v_i \in C^0(\bar{\Omega})$  and thus  $w_i \in C^0(\bar{\Omega})$ .

Also we have by elliptic estimates that  $w_i \in W^{2,1+\epsilon} \subset L^\infty$ , and we can write the equation of the Problem as

$$\begin{aligned} -\Delta(u_i - w_i) &= \tilde{V}_i e^{u_i - w_i} \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u_i - w_i &= 0 \quad \text{in } \partial\Omega, \end{aligned} \quad (2.3)$$

with

$$0 \leq \tilde{V}_i = V_i e^{w_i} \leq \tilde{b}, \quad \int_{\Omega} e^{u_i - w_i} \leq \tilde{C}.$$

We apply the Brezis-Merle theorem to  $u_i - w_i$  to have  $u_i - w_i \in L^\infty_{\text{loc}}(\Omega)$ , and, thus  $u_i \in L^\infty_{\text{loc}}(\Omega)$ .

**Step2: Boundary estimate.** Let  $\partial_\nu u_i$  be the inner derivative of  $u_i$ . By the maximum principle  $\partial_\nu u_i \geq 0$ . Then we have

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C.$$

We have the existence of a nonnegative Radon measure  $\mu$  such that

$$\int_{\partial\Omega} \partial_\nu u_i \phi d\sigma \rightarrow \mu(\phi), \quad \forall \phi \in C^0(\partial\Omega).$$

We take an  $x_0 \in \partial\Omega$  such that  $\mu(x_0) < 4\pi$ . Set  $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$ . We choose a function  $\eta_\epsilon$  such that

$$\begin{aligned} \eta_\epsilon &\equiv 1, \quad \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon &\equiv 0, \quad \text{outside } I_{2\epsilon}, \end{aligned}$$

$$0 \leq \eta_\epsilon \leq 1,$$

$$\|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.$$

We take a  $\tilde{\eta}_\epsilon$  such that

$$-\Delta \tilde{\eta}_\epsilon = 0 \quad \text{in } \Omega \subset \mathbb{R}^2,$$

$$\tilde{\eta}_\epsilon = \eta_\epsilon \quad \text{in } \partial\Omega.$$

**Remark 2.1.** We use the following steps in the construction of  $\tilde{\eta}_\epsilon$ , taking a cutoff function  $\eta_0$  in  $B(0, 2)$  or in  $B(x_0, 2)$ :

- (1) We set  $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$  in the case of the unit disk it is sufficient.
- (2) Or, in the general case: we use a chart  $(f, \tilde{\Omega})$  with  $f(0) = x_0$  and we take  $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$  to have connected sets  $I_\epsilon$  and we take  $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$ . Because  $f, f^{-1}$  are Lipschitz,  $|f(x) - x_0| \leq k_2|x| \leq 1$  for  $|x| \leq 1/k_2$  and  $|f(x) - x_0| \geq k_1|x| \geq 2$  for  $|x| \geq 2/k_1 > 1/k_2$ , the support of  $\eta$  is in  $I_{(2/k_1)\epsilon}$ .

$$\eta_\epsilon \equiv 1, \quad \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2,$$

$$\eta_\epsilon \equiv 0, \quad \text{outside } f(I_{(2/k_1)\epsilon}),$$

$$0 \leq \eta_\epsilon \leq 1,$$

$$\|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.$$

- (3) Also, we can take:  $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$  and  $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$ , we extend it by 0 outside  $f(B_1(0))$ . We have  $f(B_1(0)) = D_1(x_0)$ ,  $f(B_\epsilon(0)) = D_\epsilon(x_0)$  and  $f(B_\epsilon^+) = D_\epsilon^+(x_0)$  with  $f$  and  $f^{-1}$  smooth diffeomorphism.

$$\eta_\epsilon \equiv 1, \quad \text{on the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2,$$

$$\eta_\epsilon \equiv 0, \quad \text{outside } J'_\epsilon = f(I_{2\epsilon}),$$

$$0 \leq \eta_\epsilon \leq 1,$$

$$\|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.$$

And  $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$ , because  $f$  is Lipschitz. Here  $H_1$  is the Hausdorff measure. We solve the Dirichlet Problem

$$\Delta \bar{\eta}_\epsilon = \Delta \eta_\epsilon \quad \text{in } \Omega \subset \mathbb{R}^2,$$

$$\bar{\eta}_\epsilon = 0 \quad \text{in } \partial\Omega.$$

and finally we set  $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$ . Also, by the maximum principle and the elliptic estimates we have

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with  $C_1$  depending on  $\Omega$ .

As we said in the beginning, see also [3, 7, 13, 20], we have

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \forall i, \quad 1 < q < 2.$$

We deduce from the above estimate that,  $(u_i)$  converge weakly in  $W_0^{1,q}(\Omega)$ , almost everywhere to a function  $u \geq 0$  and  $\int_\Omega e^u < +\infty$  (by Fatou lemma). Also,  $V_i$

weakly converge to a nonnegative function  $V$  in  $L^\infty$ . The function  $u$  is in  $W_0^{1,q}(\Omega)$  solution of

$$\begin{aligned} -\Delta u &= Ve^u \in L^1(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

According to [6, Corollary 1], we have  $e^{ku} \in L^1(\Omega)$ ,  $k > 1$ . By the elliptic estimates, we have  $u \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega})$ .

We denote by  $f \cdot g$  the inner product of any two vectors  $f$  and  $g$  of  $\mathbb{R}^2$ . Then we can write

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon + \epsilon_i(\nabla u_i \cdot x)\tilde{\eta}_\epsilon. \quad (2.4)$$

We use the interior estimate in Brezis-Merle [6].

**Step 1:** Estimate of the integral of the first term of the right-hand side of (2.4). We use Green's formula between  $\tilde{\eta}_\epsilon$  and  $u$ , to obtain

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon \, dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C\epsilon = O(\epsilon) \quad (2.5)$$

then we have

$$\begin{aligned} -\Delta u_i - \epsilon_i \nabla u_i \cdot x &= V_i e^{u_i} \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

We use Green's formula between  $u_i$  and  $\tilde{\eta}_\epsilon$  to have

$$\begin{aligned} \int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon \, dx &= \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon \, d\sigma - \epsilon_i \int_{\Omega} (\nabla u_i \cdot x) \tilde{\eta}_\epsilon \\ &= \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon \, d\sigma + o(1) \\ &\rightarrow \mu(\eta_\epsilon) \leq \mu(J'_\epsilon) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \end{aligned} \quad (2.6)$$

From (2.5) and (2.6) we have that for all  $\epsilon > 0$  there is  $i_0$  such that, for  $i \geq i_0$ ,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| \, dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (2.7)$$

**Step 2.1:** Estimate of integral of the second term of the right hand side of (2.4). Let  $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$  and  $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$ ,  $\epsilon > 0$ . Then, for  $\epsilon$  small enough,  $\Sigma_\epsilon$  is an hypersurface.

The measure of  $\Omega - \Omega_{\epsilon^3}$  is  $k_2\epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1\epsilon^3$ .

**Remark 2.2.** For the unit ball  $\bar{B}(0, 1)$ , our new manifold is  $\bar{B}(0, 1 - \epsilon^3)$ . To prove this fact, we consider consider  $d(x, \partial\Omega) = d(x, z_0)$ ,  $z_0 \in \partial\Omega$ , which implies that  $(d(x, z_0))^2 \leq (d(x, z))^2$  for all  $z \in \partial\Omega$ . This is equivalent to  $(z - z_0) \cdot (2x - z - z_0) \leq 0$  for all  $z \in \partial\Omega$ . Let us consider a chart around  $z_0$  and  $\gamma(t)$  a curve in  $\partial\Omega$ , we have  $(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$  if we divide by  $(t - t_0)$  (with the sign and tend  $t$  to  $t_0$ ), we have  $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$ . This implies that  $x = z_0 - s\nu_0$  where  $\nu_0$  is the outward normal of  $\partial\Omega$  at  $z_0$

From the above remark, we can say that

$$S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}.$$

It is sufficient to work on  $\partial\Omega$ . Let us consider charts  $(z, D = B(z, 4\epsilon_z), \gamma_z)$  with  $z \in \partial\Omega$  such that  $\cup_z B(z, \epsilon_z)$  is cover of  $\partial\Omega$ . One can extract a finite cover

$(B(z_k, \epsilon_k)), k = 1, \dots, m$ , by the area formula the measure of  $S \cap B(z_k, \epsilon_k)$  is less than a  $k\epsilon$  (a  $\epsilon$ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary). We write

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx. \tag{2.8}$$

**Step 2.1.1:** Estimate of  $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$ . First, we know from elliptic estimates that  $\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$ ,  $C_1$  depends on  $\Omega$ .

We know that  $(|\nabla u_i|)_i$  is bounded in  $L^q$ ,  $1 < q < 2$ , we can extract from this sequence a subsequence which converge weakly to  $h \in L^q$ . But, we know that we have locally the uniform convergence to  $|\nabla u|$  (by Brezis-Merle's theorem), then,  $h = |\nabla u|$  a.e. Let  $q'$  be the conjugate of  $q$ .

We have that for all  $f \in L^{q'}(\Omega)$ ,

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take  $f = 1_{\Omega - \Omega_{\epsilon^3}}$ , for each  $\epsilon > 0$  there exists  $i_1 = i_1(\epsilon) \in \mathbb{N}$ , such that  $i \geq i_1$  implies

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for  $i \geq i_1(\epsilon)$ ,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3 (k_1 \|\nabla u\|_{L^\infty} + 1) = O(\epsilon^3).$$

Thus, we obtain

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 1) = O(\epsilon) \tag{2.9}$$

The constant  $C_1$  does not depend on  $\epsilon$  but on  $\Omega$ .

**Step 2.1.2:** Estimate of  $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$ . We know that,  $\Omega_\epsilon \subset\subset \Omega$ , and (because of Brezis-Merle's interior estimates)  $u_i \rightarrow u$  in  $C^1(\Omega_{\epsilon^3})$ . We have

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3.$$

We write

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} = C_1 \epsilon = O(\epsilon)$$

for  $i \geq i_3$ . For  $\epsilon > 0$ , and  $i \in \mathbb{N}$ , with  $i \geq i'$ , we have

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) = O(\epsilon) \tag{2.10}$$

From (2.7) and (2.10), for  $\epsilon > 0$ , there is  $i''$  such that  $i \geq i''$ , we have

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) = 4\pi - \epsilon_0 + O(\epsilon) \tag{2.11}$$

Now we choose  $\epsilon > 0$  small enough to have a good estimate of (2.4). Indeed, we have

$$\begin{aligned} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] &= g_{i,\epsilon} \quad \text{text in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$



with  $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$ .

We can use [6, Theorem 1] to conclude that there are  $q \geq \tilde{q} > 1$  such that

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|} dx \leq \int_{\Omega} e^{q|u_i-u|^{\tilde{q}\epsilon}} dx \leq C(\epsilon, \Omega),$$

where,  $V_\epsilon(x_0)$  is a neighborhood of  $x_0$  in  $\bar{\Omega}$ . Here we have used that in a neighborhood of  $x_0$  by the elliptic estimates,  $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$ .

Thus, for each  $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$  there is  $\epsilon_0 > 0, q_0 > 1$  such that

$$\int_{B(x_0, \epsilon_0)} e^{q_0 u_i} dx \leq C, \quad \forall i.$$

By elliptic estimates see [14], we have

$$\|u_i\|_{C^{1,\theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

We have proved that there is a finite number of points  $\bar{x}_1, \dots, \bar{x}_m$  such that the sequence  $(u_i)$  is locally uniformly bounded in  $C^{1,\theta}$ , ( $\theta > 0$ ) on  $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$ .

**Proof of theorem 1.5.** The Pohozaev identity gives

$$\int_{\partial\Omega} \frac{1}{2}(x \cdot \nu)(\partial_\nu u_i)^2 d\sigma + \epsilon \int_{\Omega} (x \cdot \nabla u_i)^2 dx + \int_{\partial\Omega} (x \cdot \nu) V_i e^{u_i} d\sigma = \int_{\Omega} (x \cdot \nabla V_i + 2V_i) e^{u_i} dx.$$

We use the boundary condition, that  $\Omega$  is starshaped, and that  $\epsilon > 0$  to have

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \quad (2.12)$$

Thus we can use the weak convergence in  $L^2(\partial\Omega)$  to have a subsequence  $\partial_\nu u_i$ , such that

$$\int_{\partial\Omega} \partial_\nu u_i \phi dx \rightarrow \int_{\partial\Omega} \partial_\nu u \phi dx, \quad \forall \phi \in L^2(\partial\Omega),$$

Thus,  $\alpha_j = 0, j = 1, \dots, N$  and  $(u_i)$  is uniformly bounded.

**Remark 2.3.** If we assume the open set bounded starshaped and  $V_i$  uniformly Lipschitzian and between two positive constants we can bound, by using the inner normal derivative  $\int_{\Omega} e^{u_i}$ .

If we assume the open set bounded starshaped and  $\nabla \log V_i$  uniformly bounded, by the previous Pohozaev identity (we consider the inner normal derivative) one can bound  $\int_{\Omega} V_i e^{u_i}$  uniformly.

One can consider the problem on the unit ball and an ellipse. These two problems are different, because:

- (1) if we use a linear transformation,  $(y_1, y_2) = (x_1/a, x_2/b)$ , the Laplacian is not invariant under this map.
- (2) If we use a conformal transformation, by a Riemann theorem, the quantity  $x \cdot \nabla u$  is not invariant under this map.

We can not use, after using those transformations, the Pohozaev identity.

## 3. A COUNTEREXAMPLE

We start with the notation of the counterexample of Brezis and Merle. The domain  $\Omega$  is the unit ball centered in  $x_0 = (1, 0)$ . Consider  $z_i$  (obtained by the variational method), such that

$$-\Delta z_i - \epsilon_i(x - x_0) \cdot \nabla z_i = -\tilde{L}_{\epsilon_i}(z_i) = f_{\epsilon_i},$$

with Dirichlet condition. By the regularity theorem,  $z_i \in C^1(\bar{\Omega})$ . Then we have

$$\|f_{\epsilon_i}\|_1 = 4\pi A.$$

Thus by the duality theorem of Stampacchia or Brezis-Strauss, we have

$$\|\nabla z_i\|_q \leq C_q, \quad 1 \leq q < 2.$$

We solve

$$-\Delta w_i = \epsilon_i(x - x_0) \cdot \nabla z_i,$$

with Dirichlet boundary condition.

By elliptic estimates,  $w_i \in C^1(\bar{\Omega})$  and  $w_i \in C^0(\bar{\Omega})$  uniformly. By the maximum principle we have

$$z_i - w_i \equiv u_i.$$

Where  $u_i$  is the function of the counterexample of Brezis Merle. Then we write

$$-\Delta z_i - \epsilon_i(x - x_0) \cdot \nabla z_i = f_{\epsilon_i} = V_i e^{z_i}.$$

Thus, we have

$$\int_{\Omega} e^{z_i} \leq C_1, \quad 0 \leq V_i \leq C_2,$$

$$z_i(a_i) \geq u_i(a_i) - C_3 \rightarrow +\infty, \quad a_i \rightarrow O.$$

To have a counterexample on the unit disk, we do a translation  $x \rightarrow x - x_0$  in the previous counterexample.

**Acknowledgments.** The author would like to thank the anonymous reviewers.

## REFERENCES

- [1] T. Aubin; *Some Nonlinear Problems in Riemannian Geometry*. Springer-Verlag 1998.
- [2] C. Bandle; *Isoperimetric Inequalities and Applications*. Pitman, 1980.
- [3] L. Boccardo, T. Gallouet; *Nonlinear elliptic and parabolic equations involving measure data*. J. Funct. Anal. 87 no. 1, (1989), 149–169.
- [4] H. Brezis, Y.Y. Li, I. Shafirir; *A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities*. J. Funct. Anal. 115 (1993), 344–358.
- [5] H. Brezis, M. Marcus, A. C. Ponce; *Nonlinear elliptic equations with measures revisited. Mathematical aspects of nonlinear dispersive equations*, 55–109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.
- [6] H. Brezis, F. Merle; *Uniform estimates and Blow-up behavior for solutions of  $-\Delta u = V(x)e^u$  in two dimension*. Comm. Part. Diff. Equations, 16 (8 and 9) (1991), 1223–1253.
- [7] H. Brezis, W. A. Strauss; *Semi-linear second-order elliptic equations in  $L^1$* . J. Math. Soc. Japan 25 (1973), 565–590.
- [8] A. Sun-Yung Chang, Matthew J. Gursky, Paul C. Yang; *Scalar curvature equation on 2- and 3-spheres*. Calc. Var. Partial Differential Equations 1 (1993), no. 2, 205–229.
- [9] W. Chen, C. Li; *A priori estimates for solutions to nonlinear elliptic equations*. Arch. Rational. Mech. Anal., 122 (1993) 145–157.
- [10] C.-C. Chen, C-S. Lin; *A sharp sup+inf inequality for a nonlinear elliptic equation in  $\mathbb{R}^2$* . Comm. Anal. Geom. 6, no.1 (1998), 1–19.
- [11] C.-C.Chen, C-S. Lin; *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*. Comm. Pure Appl. Math. 55 (2002), no. 6, 728–771.

- [12] D. G. De Figueiredo, P. L. Lions, R. D. Nussbaum; *A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations*, J. Math. Pures et Appl., vol 61, 1982, pp.41–63.
- [13] W. Ding, J. Jost, J. Li, G. Wang; *The differential equation  $\Delta u = 8\pi - 8\pi h e^u$  on a compact Riemann surface*. Asian J. Math. 1 (1997), no. 2, 230–248.
- [14] D. Gilbarg, N. S. Trudinger; *Elliptic Partial Differential Equations of Second order*, Berlin Springer-Verlag.
- [15] Y. Y. Li; *Harnack Type Inequality: the method of moving planes*. Commun. Math. Phys. 200 (1999), 421–444.
- [16] Y. Y. Li, I. Shafrir; *Blow-up analysis for solutions of  $-\Delta u = V e^u$  in dimension two*. Indiana Math. J. Vol 3, no. 4 (1994). 1255–1270.
- [17] L. Ma, J.-C. Wei; *Convergence for a Liouville equation*. Comment. Math. Helv. 76 (2001), 506–514.
- [18] K. Nagasaki, T. Suzuki; *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities*. Asymptotic Anal. 3 (1990), no. 2, 173–188.
- [19] I. Shafrir; *A sup+inf inequality for the equation  $-\Delta u = V e^u$* . C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159–164.
- [20] G. Tarantello; *Multiple condensate solutions for the Chern-Simons-Higgs theory*. J. Math. Phys. 37 (1996), no. 8, 3769–3796.

SAMY SKANDER BAHOURA

DEPARTMENT OF MATHEMATICS, PIERRE ET MARIE CURIE UNIVERSITY, 4 PLACE JUSSIEU, 75005, PARIS, FRANCE

*Email address:* samybahoura@gmail.com