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# Critical 3-hypergraphs 

Abderrahim Boussaïri ${ }^{\star 8}$ Brahim Chergui ${ }^{* \ddagger}$ Pierre Ille ${ }^{\dagger}$<br>Mohamed Zaidi*\|

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#### Abstract

Given a 3-hypergraph $H$, a subset $M$ of $V(H)$ is a module of $H$ if for each $e \in E(H)$ such that $e \cap M \neq \varnothing$ and $e \backslash M \neq \varnothing$, there exists $m \in M$ such that $e \cap M=\{m\}$ and for every $n \in M$, we have $(e \backslash\{m\}) \cup\{n\} \in E(H)$. For example, $\varnothing, V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of $H$, called trivial modules. A 3-hypergraph with at least three vertices is prime if all its modules are trivial. Furthermore, a prime 3-hypergraph is critical if all its induced subhypergraphs, obtained by removing one vertex, are not prime. We characterize the critical 3-hypergraphs.


Mathematics Subject Classifications (2010): 05C65, 05C75, 05C76.
Key words: 3-hypergraph, module, prime, critical.

## 1 Introduction

Let $H$ be a 3-hypergraph. A tournament $T$, with the same vertex set as $H$, is a realization of $H$ if the edges of $H$ are exactly the 3 -element subsets of the vertex set of $T$ that induce 3-cycles. In [3], we characterized the 3-hypergraphs that admit realizations (see [5, Problem 1]). To obtain our characterization, we introduced a new notion of a module for hypergraphs. By using the modular decomposition tree, we demonstrated that a 3-hypergraph is realizable if and only if all its prime (in terms of modular decomposition) induced subhypergraphs are realizable (see [3, Theorem 13]). Moreover, given a realizable 3-hypergraph $H$, we proved that $H$ is prime if and only if all its realizations are prime (see [3, Theorem 12]). These results led us to study the prime and induced subhypergraphs of a prime 3 -hypergraph. Precisely, consider a prime 3-hypergraph $H$.

[^0]In [4, Corollary 18], we proved that $H$ admits a prime induced subhypergraph obtained by removing 1 or 2 vertices. Similar results were obtained for prime digraphs [10], for prime binary relational structures [11], or for prime 2-structures [7]. Our purpose is to characterize the critical 3-hypergraphs, that is, the prime 3-hypergraphs all the subhypergraphs of which, obtained by removing one vertex, are not prime. In particular, our characterization yields a simple manner to construct critical 3-hypergraphs that are not realizable.

At present, we formalize our presentation. We consider only finite structures. A hypergraph $H$ is defined by a vertex set $V(H)$ and an edge set $E(H)$, where $E(H) \subseteq 2^{V(H)} \backslash\{\varnothing\}$. Given a hypergraph $H, v(H)$ denotes the cardinality of $V(H)$. Given $k \geq 2$, a hypergraph $H$ is a $k$-hypergraph if

$$
E(H) \subseteq\binom{V(H)}{k}
$$

Furthermore, a hypergraph $H$ is a $\{k, k+1\}$-hypergraph if

$$
E(H) \subseteq\binom{V(H)}{k} \cup\binom{V(H)}{k+1} .
$$

Let $H$ be a hypergraph. With each $W \subseteq V(H)$, we associate the subhypergraph $H[W]$ of $H$ induced by $W$, which is defined on $V(H[W])=W$ by $E(H[W])=$ $\{e \in E(H): e \subseteq W\}$. Given $W \subseteq V(H), H[V(H) \backslash W]$ is also denoted by $H-W$, and by $H-w$ when $W=\{w\}$.

Definition 1. Let $H$ be a hypergraph. A subset $M$ of $V(H)$ is a module of $H$ if for each $e \in E(H)$ such that $e \cap M \neq \varnothing$ and $e \backslash M \neq \varnothing$, there exists $m \in M$ such that $e \cap M=\{m\}$, and for every $n \in M$, we have

$$
(e \backslash\{m\}) \cup\{n\} \in E(H)
$$

Given distinct $v, w \in V(H)$, we say that $v$ and $w$ are twins $[2,6]$ of $H$ if $\{v, w\}$ is a module of $H$.

Definition 2. Let $H$ be a hypergraph. Clearly, $\varnothing, V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of $H$, called trivial modules. A hypergraph $H$ is indecomposable if all its modules are trivial, otherwise it is decomposable. A hypergraph $H$ is prime if it is indecomposable, with $v(H) \geq 3$.

Remark 3. Recall that a hypergraph $H$ is connected if for distinct $v, w \in V(H)$, there exists a sequence $\left(e_{0}, \ldots, e_{n}\right)$ of edges of $H$, where $n \geq 0$, satisfying $v \in e_{0}$, $w \in e_{n}$, and (when $n \geq 1$ ) $e_{i} \cap e_{i+1} \neq \varnothing$ for every $0 \leq i \leq n-1$. Given a hypergraph $H$, a maximal connected subhypergraph of $H$ is called a component of $H$.

Let $H$ be a hypergraph. For each component $C$ of $H, V(C)$ and $V(H) \backslash V(C)$ are clearly modules of $H$. Therefore, a disconnected hypergraph with at least three vertices is decomposable.

In [4], we proved the following result (see [4, Theorem 17]). We need the following notation (see [4, Notation 13]).

Notation 4. Let $H$ be a 3 -hypergraph. Given $X \mp V(H)$ such that $H[X]$ is prime, we consider the following subsets of $V(H) \backslash X$.

- We denote by $\operatorname{Ext}_{(H, X)}$ the set of $v \in V(H) \backslash X$ such that $H[X \cup\{v\}]$ is prime. (Hence, given $v \in V(H) \backslash X, v \in \operatorname{Ext}_{(H, X)}$ if and only if $H[X \cup\{v\}]$ is a prime extension of $H[X]$.)
- We denote by $\operatorname{Dis}_{(H, X)}$ the set of $v \in V(H) \backslash X$ such that $X$ is a module of $H[X \cup\{v\}]$. (Let $v \in \operatorname{Dis}_{(H, X)}$. Since $X$ is a module of $H[X \cup\{v\}]$, we obtain $v \notin e$ for every $e \in E(H[X \cup\{v\}])$. Thus, $H[X \cup\{v\}]$ is disconnected (see Remark 3).)
- Let $y \in X$. We denote by $\operatorname{Twi}_{(H, X)}(y)$ the set of $v \in V(H) \backslash X$ such that $\{y, v\}$ is a module of $H[X \cup\{v\}]$. (Let $y \in X$. Given $v \in V(H) \backslash X$, $v \in \operatorname{Twi}_{(H, X)}(y)$ if and only if $v$ and $y$ are twins of $H[X \cup\{v\}]$.)

The set $\left\{\operatorname{Ext}_{(H, X)}, \operatorname{Dis}_{(H, X)}\right\} \cup\left\{\operatorname{Twi}_{(H, X)}(y): y \in X\right\}$ is denoted by $p_{(H, X)}$.
Theorem 5. Let $H$ be a 3-hypergraph. Consider $X \mp V(H)$ such that $H[X]$ is prime. Set

$$
\underline{X}=\left\{y \in X: \operatorname{Twi}_{(H, X)}(y) \neq \varnothing\right\} .
$$

If $H$ is prime, then there exist $v, w \in(V(H) \backslash X) \cup \underline{X}$ such that $H-\{v, w\}$ is prime.

The next result follows from Theorem 5 (see [4, Corollary 18])
Corollary 6. Let $H$ be a prime 3-hypergraph. If $v(H) \geq 4$, then there exist $v, w \in V(H)$ such that $H-\{v, w\}$ is prime.

Definition 7. A prime hypergraph $H$ is critical if $H-v$ is not prime for each $v \in V(H)$.

Our purpose is to characterize the critical 3-hypergraphs. An analogue of Corollary 6 for binary relations led Ille [9] to introduce the following auxiliary graph.

Definition 8. Let $H$ be a prime 3-hypergraph with $v(H) \geq 5$. The primality graph $\mathscr{P}(H)$ associated with $H$ is the graph defined on $V(H)$ as follows. Given distinct $v, w \in V(H), v w \in E(\mathscr{P}(H))$ if $H-\{v, w\}$ is prime. When $H$ is critical, it follows from Corollary 6 that $\mathscr{P}(H)$ is nonempty.

### 1.1 Critical and realizable 3-hypergraphs

Let $T$ be a tournament. A subset $M$ of $V(T)$ is a module [12] of $T$ provided that for any $x, y \in M$ and $v \in V(T)$, if $x v, v y \in A(T)$, then $v \in M$. Note that the notions of a module and of a convex subset [8] coincide for tournaments. Moreover, note that the notions of a module and of an interval coincide for linear orders. Given a tournament $T, \varnothing, V(T)$ and $\{v\}$, where $v \in V(T)$, are modules of $T$, called trivial modules. A tournament is indecomposable if all its
modules are trivial, otherwise it is decomposable. A tournament $T$ is prime if it is indecomposable, with $v(T) \geq 3$. Lastly, a prime tournament $T$ is critical if $T-v$ is not prime for each $v \in V(T)$. Schmerl and Trotter [11] characterized the critical tournaments. They obtained the tournaments $T_{2 n+1}, U_{2 n+1}$, and $W_{2 n+1}$ defined on $\{0, \ldots, 2 n\}$, where $n \geq 1$, as follows. We denote by $L_{2 n+1}$ the usual linear order on $\{0, \ldots, 2 n\}$.

- The tournament $T_{2 n+1}$ is obtained from $L_{2 n+1}$ by reversing all the arcs between even and odd vertices.
- The tournament $U_{2 n+1}$ is obtained from $L_{2 n+1}$ by reversing all the arcs between even vertices.
- The tournament $W_{2 n+1}$ is obtained from $L_{2 n+1}$ by reversing all the arcs between $2 n$ and the even elements of $\{0, \ldots, 2 n-1\}$.

Theorem 9 (Schmerl and Trotter [11]). Given a tournament $\tau$, with $v(\tau) \geq 5$, $\tau$ is critical if and only if $v(\tau)$ is odd and $\tau$ is isomorphic to $T_{v(\tau)}, U_{v(\tau)}$, or $W_{v(\tau)}$.

A realization of a 3-hypergraph is defined as follows (see [3, Definition 10]). To begin, we associate with each tournament a 3-hypergraph in the following way (see [3, Definition 9]).

Definition 10. The 3 -cycle is the tournament $C_{3}=(\{0,1,2\},\{01,12,20\})$. Given a tournament $T$, the $C_{3}$-structure of $T$ is the 3 -hypergraph $C_{3}(T)$ defined on $V\left(C_{3}(T)\right)=V(T)$ by

$$
E\left(C_{3}(T)\right)=\left\{X \subseteq V(T): T[X] \text { is isomorphic to } C_{3}\right\} .
$$

Definition 11. Given a 3-hypergraph $H$, a tournament $T$, with $V(T)=V(H)$, realizes $H$ if $H=C_{3}(T)$. We say also that $T$ is a realization of $H$.

In [3], we proved the following result (see [3, Theorem 12])
Theorem 12. Consider a realizable 3-hypergraph $H$. For a realization $T$ of $H$, $H$ is prime if and only if $T$ is prime.

The next result follows from Theorems 9 and 12 (see [3, Theorem 49]).
Corollary 13. Given a realizable 3-hypergraph $H$, with $v(H) \geq 5$, $H$ is critical if and only if $v(H)$ is odd and $H$ is isomorphic to $C_{3}\left(T_{v(H)}\right), C_{3}\left(U_{v(H)}\right)$, or $C_{3}\left(W_{v(H)}\right)$.

Definition 14. Given a 3-hypergraph $H$, we say that $H$ is circular if $v(H)$ is odd and $H$ is isomorphic to $C_{3}\left(T_{v(H)}\right)$.

### 1.2 A construction of non circular and critical 3-hypergraphs

The construction is provided from the set of the components of a graph. We use the following notation.

Notation 15. Consider a graph $\Gamma$. The set of the components of $\Gamma$ is denoted by $\mathfrak{C}(\Gamma)$. Furthermore, set $\mathfrak{C}_{\text {even }}(\Gamma)=\{C \in \mathfrak{C}(\Gamma): v(C) \equiv 0(\bmod 2)\}$ and $\mathfrak{C}_{\text {odd }}(\Gamma)=\{C \in \mathfrak{C}(\Gamma): v(C) \equiv 1(\bmod 2)\}$. Lastly, set $\mathfrak{C}_{1}(\Gamma)=\{C \in \mathfrak{C}(\Gamma):$ $v(C)=1\}$ and $V_{1}(\Gamma)=\left\{v \in V(\Gamma):\{v\} \in \mathfrak{C}_{1}(\Gamma)\right\}$. For each $C \in \mathfrak{C}(\Gamma)$, set

$$
w(C)= \begin{cases}\frac{v(C)}{2} & \text { if } C \in \mathfrak{C}_{\text {even }}(\Gamma) \\ \frac{v(C)-1}{2} & \text { if } C \in \mathfrak{C}_{\text {odd }}(\Gamma)\end{cases}
$$

We consider a graph $\Gamma$,
all the components of which are paths.
For each $n \geq 1$, recall that the path $\mathcal{P}_{n}$ is defined on $V\left(\mathcal{P}_{n}\right)=\{0, \ldots, n-1\}$ by

$$
E\left(\mathcal{P}_{n}\right)= \begin{cases}\varnothing & \text { if } n=1 \\ \{k(k+1): 0 \leq k \leq n-2\} & \text { if } n \geq 2\end{cases}
$$

We suppose that
$\star \mathfrak{C}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma) \neq \varnothing$,
$\star$ for each $C \in \mathfrak{C}_{\text {odd }}(\Gamma)$, if $V(H) \backslash V(C) \neq \varnothing$, then

$$
\begin{equation*}
\left(\mathfrak{C}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)\right) \backslash\{C\} \neq \varnothing, \tag{2}
\end{equation*}
$$

$\star$ for each $C \in \mathfrak{C}_{\text {even }}(\Gamma)$, if $|V(H) \backslash V(C)| \geq 2$, then

$$
\left(\mathfrak{C}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)\right) \backslash\{C\} \neq \varnothing .
$$

Consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathfrak{C}(\Gamma)$ and satisfying

- for each $\varepsilon \in E(\mathbb{H})$, if $|\varepsilon|=2$, then $\varepsilon \cap \mathfrak{C}_{\text {even }}(\Gamma) \neq \varnothing$ and $\varepsilon \cap \mathfrak{C}_{\text {odd }}(\Gamma) \neq \varnothing$,
- for each $\varepsilon \in E(\mathbb{H})$, if $|\varepsilon|=3$, then $\varepsilon \subseteq \mathfrak{C}_{\text {odd }}(\Gamma)$.

With $\Gamma$ and $\mathbb{H}$, we associate the 3-hypergraph $\Gamma \bullet \mathbb{H}$ defined on $V(\Gamma)$ in the following manner. For each $C \in \mathfrak{C}(\Gamma)$, we consider an isomorphism $\varphi_{C}$ from the path $\mathcal{P}_{v(C)}$ onto $C$. With each $C \in \mathfrak{C}_{\text {odd }}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)$, we associate the set

$$
E_{C}=\varphi_{C}\left(E\left(C_{3}\left(U_{v(C)}\right)\right)\right) .
$$

Moreover, consider $\varepsilon \in E(\mathbb{H})$ such that $|\varepsilon|=2$. There exist $C \in \mathfrak{C}_{\text {even }}(\Gamma)$ and $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=C D$. Associate with $\varepsilon$ the set

$$
E_{\varepsilon}=\left\{\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k): 0 \leq i \leq j \leq w(C)-1,0 \leq k \leq w(D)\right\} .
$$

Lastly, consider $\varepsilon \in E(\mathbb{H})$ such that $|\varepsilon|=3$. There exist $I, J, K \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=I J K$. Associate with $\varepsilon$ the set

$$
E_{\varepsilon}=\left\{\varphi_{I}(2 i) \varphi_{J}(2 j) \varphi_{K}(2 k): 0 \leq i \leq w(I), 0 \leq j \leq w(J), 0 \leq k \leq w(K)\right\} .
$$

The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is defined on $V(\Gamma)$ by

$$
\begin{equation*}
E(\Gamma \bullet \mathbb{H})=\left(\bigcup_{C \in \mathfrak{C}_{\text {odd }}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)} E_{C}\right) \cup\left(\bigcup_{\varepsilon \in E(\mathbb{H})} E_{\varepsilon}\right) . \tag{4}
\end{equation*}
$$

### 1.3 The main results

To begin, we prove the following theorem.
Theorem 16. Let $H$ be a critical 3-hypergraph such that $v(H) \geq 5$. If $H$ is not circular, then $\mathscr{P}(\underset{\sim}{H})$ (see Definition 8) satisfies (1) and (2), and there exists a $\{2,3\}$-hypergraph $\widetilde{H}$ defined on $\mathfrak{C}(\mathscr{P}(H))$ satisfying (3) such that

$$
H=\mathscr{P}(H) \bullet \widetilde{H}
$$

To state the second main theorem, we consider a graph $\Gamma$ satisfying (1) and (2). We consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathfrak{C}(\Gamma)$ satisfying (3).

Theorem 17. Suppose that $v(\Gamma \bullet \mathbb{H}) \geq 5$. The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is critical if and only if the following three assertions hold
(C1) $\mathbb{H}$ is connected (see Remark 3);
(C2) for every nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \backslash \mathfrak{C}_{1}(\Gamma) \neq \varnothing$;
(C3) for each $v \in V_{1}(\Gamma)$, if $\mathbb{H}-\{v\}$ is connected, then $\mathbb{H}-\{v\}$ admits a nontrivial module $\mathbb{M}_{\{v\}}$ such that $\mathbb{M}_{\{v\}} \subseteq \mathfrak{C}_{1}(\Gamma) \backslash\{\{v\}\}$.

Before proving Theorem 17, we establish the next proposition.
Proposition 18. Suppose that $v(\Gamma \bullet \mathbb{H}) \geq 3$. The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is prime if and only if the following two assertions hold
(C1) $\mathbb{H}$ is connected;
(C2) for every nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \backslash \mathfrak{C}_{1}(\Gamma) \neq \varnothing$.
The next result yields an easy manner to construct critical 3-hypergraphs. It is a simple consequence of Corollary 13 and Theorem 17.

Corollary 19. Suppose that $v(\Gamma \bullet \mathbb{H}) \geq 5$. Suppose also that $\mathfrak{C}_{1}(\Gamma)=\varnothing$. If $\mathbb{H}$ is connected, then $\Gamma \bullet \mathbb{H}$ is critical. Moreover, if $\mathbb{H}$ is connected and $|\mathfrak{C}(\Gamma)| \geq 2$, then $\Gamma \bullet \mathbb{H}$ is critical and non realizable.

## 2 The critical tournaments

The primality graph associated with a prime tournament is defined in the same manner as that associated with a prime 3-hypergraph (see Definition 8).

Definition 20. Let $T$ be a prime tournament. The primality graph $\mathscr{P}(T)$ of $T$ is defined on $V(T)$ as follows. Given distinct $v, w \in V(T), v w \in E(\mathscr{P}(T))$ if $T-\{v, w\}$ is prime.

The basic properties of the primality graph follow. The next lemma is stated in [9] without a proof. For a proof, see [1, Lemma 10].

Lemma 21 (Ille [9]). Let $T$ be a prime tournament with $v(T) \geq 5$. Consider $v \in V(T)$ such that $T-v$ is decomposable. We have $d_{\mathscr{P}(T)}(v) \leq 2$. Moreover, the next two assertions hold.

1. If $d_{\mathscr{P}(T)}(v)=1$, then $V(T) \backslash\left(\{v\} \cup N_{\mathscr{P}(T)}(v)\right)$ is a module of $T-v$.
2. If $d_{\mathscr{P}(T)}(v)=2$, then $N_{\mathscr{P}(T)}(v)$ is a module of $T-v$.

For each $n \geq 3$, recall that the cycle $\mathcal{C}_{n}$ is defined on $V\left(\mathcal{C}_{n}\right)=\{0, \ldots, n-1\}$ by $E\left(\mathcal{C}_{n}\right)=E\left(\mathcal{P}_{n}\right) \cup\{0(n-1)\}$. The length of $\mathcal{C}_{n}$ is $n$. Given a critical tournament $T$, it follows from Lemma 21 that the connected components of $\mathscr{P}(T)$ are paths or cycles. Boudabbous and Ille [1] characterized the critical tournaments from their primality graphs. To begin, they examined the connected components of the primality graph associated with a critical tournament.

Lemma 22 (Corollary 17 [1]). If $T$ is a critical tournament, with $v(T) \geq 5$, then $\mathscr{P}(T)$ satisfies one of the following

1. $\mathscr{P}(T)$ is a cycle of odd length;
2. $\mathscr{P}(T)$ is a path;
3. $v(T)$ is odd and there is $v \in V(T)$ such that $\mathscr{P}(T)-v$ is a path and $N_{\mathscr{P}(T)}(v)=\varnothing$.

For each of the three shapes described in Lemma 22, Boudabbous and Ille [1] characterized the corresponding critical tournaments.

Proposition 23 (Proposition 18 [1]). Given a tournament such that $v(T) \geq 5$, $T$ is critical and $\mathscr{P}(T)=\mathcal{C}_{v(T)}$ if and only if $v(T)$ is odd and $T=T_{v(T)}$ or $\left(T_{v(T)}\right)^{\star}$.

Proposition 24 (Proposition 19 [1]). Given a tournament such that $v(T) \geq 5$, $T$ is critical and $\mathscr{P}(T)=\mathcal{P}_{v(T)}$ if and only if $v(T)$ is odd and $T=U_{v(T)}$ or $\left(U_{v(T)}\right)^{\star}$.
Proposition 25 (Proposition 21 [1]). Given a tournament defined on $\{0, \ldots, 2 n\}$ with $n \geq 2, T$ is critical, $\mathscr{P}(T)-2 n=\mathcal{P}_{2 n-1}$, and $N_{\mathscr{P}(T)}(2 n)=\varnothing$ if and only if $T=W_{2 n+1}$ or $\left(W_{2 n+1}\right)^{\star}$.

## 3 Proof of Theorem 16

The purpose of this section is to characterize the non circular and critical 3hypergraphs. We use the following notation.
Notation 26. Let $H$ be a 3 -hypergraph.

$$
\text { For } e, f \in\binom{V(H)}{3}, e \equiv_{H} f \text { means }\left\{\begin{array}{l}
e, f \in E(H) \\
\text { or } \\
e, f \notin E(H) .
\end{array}\right.
$$

An analogue of Lemma 21 for prime 3-hypergraphs follows. For a proof, see [1, Lemma 10].
Lemma 27. Let $H$ be a prime 3-hypergraph with $v(H) \geq 5$. Consider $v \in V(H)$ such that $H-v$ is decomposable. We have $d_{\mathscr{P}_{(H)}}(v) \leq 2$. Moreover, the next two assertions hold.

1. If $d_{\mathscr{P}(H)}(v)=1$, then $V(H) \backslash\left(\{v\} \cup N_{\mathscr{P}(H)}(v)\right)$ is a module of $H-v$.
2. If $d_{\mathscr{P}(H)}(v)=2$, then $N_{\mathscr{P}(H)}(v)$ is a module of $H-v$.

The next result is an immediate consequence of Lemma 27.
Corollary 28. Given a critical 3-hypergraph $H$ such that $v(H) \geq 5$, all the components of $\mathscr{P}(H)$ are cycles or paths.
Proposition 29. Let $H$ be a prime 3-hypergraph defined on $\{0, \ldots, p-1\}$, where $p \geq 5$. Suppose that there exists $k \in\{3, \ldots, p\}$ such that

$$
\mathscr{P}(H)[\{0, \ldots, k-1\}]=C_{k}
$$

If $H-i$ is decomposable for each $i \in\{0, \ldots, k-1\}$, then $k$ is odd, $k=p$, and $H=C_{3}\left(T_{p}\right)$.
Proof. Suppose that there exists $k \in\{3, \ldots, p\}$ such that $\mathscr{P}(H)[\{0, \ldots, k-1\}]=$ $\mathcal{C}_{k}$. Furthermore, suppose that $H-i$ is decomposable for each $i \in\{0, \ldots, k-1\}$. First, we show that

$$
\begin{equation*}
k \text { is odd. } \tag{5}
\end{equation*}
$$

Otherwise, there exists $l \geq 2$ such that $\mathscr{P}(H)[\{0, \ldots, 2 l-1\}]=\mathcal{C}_{2 l}$. We verify that $\{0,2\}$ is a module of $H$. Consider $e \in E(H)$ such that $e \cap\{0,2\} \neq \varnothing$ and $e \backslash\{0,2\} \neq \varnothing$. Suppose for a contradiction that $0,2 \in e$. There exists $i \in\{1\} \cup\{3, \ldots, 2 l-1\}$ such that $e=02 i$. Since $N_{\mathscr{P}_{(H)}}(1)=\{0,2\}$, it follows from Lemma 27 that $\{0,2\}$ is a module of $H-1$. It follows that $i=1$, that is, $e=012$. It follows from Lemma 27 that $01(2 l-2) \in E(H)$, which contradicts the fact that $\{0,2 l-2\}$ is a module of $H-(2 l-1)$. Consequently, $|e \cap\{0,2\}|=1$. For a contradiction, suppose that $1 \in e$. Since $|e \cap\{0,2\}|=1$, there exists $j \in\{3, \ldots, 2 l-1\}$ such that $e=01 j$ or $12 j$. Denote by $j^{\prime}$ the unique element of $\{2 l-2,2 l-1\}$ such that $j^{\prime} \equiv j(\bmod 2)$. It follows from Lemma 27 that $(e \backslash\{j\}) \cup\left\{j^{\prime}\right\} \in E(H)$. Similarly, by denoting by $j^{\prime \prime}$ the unique element of $\{3,4\}$ such that $j^{\prime \prime} \equiv j(\bmod 2)$, we obtain $(e \backslash\{j\}) \cup\left\{j^{\prime \prime}\right\} \in E(H)$. We distinguish the following two cases.

1. Suppose that $e=01 j$. If $j$ is even, then $01(2 l-2) \in E(H)$, which contradicts the fact that $\{0,2 l-2\}$ is a module of $H-(2 l-1)$. If $j$ is odd, then $013 \in E(H)$, which contradicts the fact that $\{1,3\}$ is a module of $H-2$.
2. Suppose that $e=12 j$. If $j$ is even, then $124 \in E(H)$, which contradicts the fact that $\{2,4\}$ is a module of $H-3$. If $j$ is odd, then $12(2 l-1) \in E(H)$, which contradicts the fact that $\{1,2 l-1\}$ is a module of $H-0$.

It follows that $1 \notin e$. So, $e \in E(H-1)$. Since $N_{\mathscr{P}(H)}(1)=\{0,2\}$, it follows from Lemma 27 that $\{0,2\}$ is a module of $H-1$. Thus, there exists $i \in\{0,2\}$ such that $e \cap\{0,2\}=\{i\}$, and $(e \backslash\{i\}) \cup\left\{i^{\prime}\right\} \in E(H)$ for each $i^{\prime} \in\{0,2\}$. Consequently, $\{0,2\}$ is a module of $H$, which contradicts the fact that $H$ is prime. It follows that (5) holds. Set $k=2 n+1$, where $n \geq 1$.

Second, we prove that $\{0, \ldots, 2 n\}$ is a module of $H$. Consider $e \in E(H)$ such that $e \cap\{0, \ldots, 2 n\} \neq \varnothing$ and $e \backslash\{0, \ldots, 2 n\} \neq \varnothing$. We prove that

$$
\begin{equation*}
|e \cap\{0, \ldots, 2 n\}|=1 \tag{6}
\end{equation*}
$$

Otherwise, we have $|e \cap\{0, \ldots, 2 n\}|=2$. There exist $i, j \in\{0, \ldots, 2 n\}$, with $i<j$, such that $e \cap\{0, \ldots, 2 n\}=\{i, j\}$. Denote by $i^{\prime}$ the unique element of $\{0,1\}$ such that $i^{\prime} \equiv i(\bmod 2)$. As previously, we obtain $(e \backslash\{i\}) \cup\left\{i^{\prime}\right\} \in$ $E(H)$. Denote by $j^{\prime}$ the unique element of $\{2,3\}$ such that $j^{\prime} \equiv j(\bmod 2)$. We obtain $(e \backslash\{i, j\}) \cup\left\{i^{\prime}, j^{\prime}\right\} \in E(H)$. Set $e^{\prime}=(e \backslash\{i, j\}) \cup\left\{i^{\prime}, j^{\prime}\right\}$. Observe that $e^{\prime} \backslash\{0, \ldots, 2 n\}=e \backslash\{0, \ldots, 2 n\}$, and denote by $v$ the unique element of $e^{\prime} \backslash\{0, \ldots, 2 n\}$. If $i^{\prime}=0$ and $j^{\prime}=2$, then $02 v \in E(H)$, which contradicts the fact that $\{0,2\}$ is a module of $H-1$. If $i^{\prime}=1$ and $j^{\prime}=3$, then $13 v \in E(H)$, which contradicts the fact that $\{1,3\}$ is a module of $H-2$. Suppose that $i^{\prime}=0$ and $j^{\prime}=3$. We get $03 v \in E(H)$, and hence $0(2 n-1) v \in E(H)$, which contradicts the fact that $\{0,2 n-1\}$ is a module of $H-(2 n)$. Lastly, if $i^{\prime}=1$ and $j^{\prime}=4$, then $1(2 n) v \in E(H)$, which contradicts the fact that $\{1,2 n\}$ is a module of $H-0$. It follows that (6) holds. Denote by $i$ the unique element of $e \cap\{0, \ldots, 2 n\}$. For every $l \in\{1, \ldots, n\}$, we obtain

$$
\begin{equation*}
(e \backslash\{i\}) \cup\{i+2 l\} \in E(H) \tag{7}
\end{equation*}
$$

where $i+2 l$ is considered modulo $2 n+1$. In particular, $(e \backslash\{i\}) \cup\{i+2 n\}$, which is $(e \backslash\{i\}) \cup\{i-1\}$, belongs to $E(H)$. Since $\{i-1, i+1\}$ is a module of $H-i$, we get $(e \backslash\{i\}) \cup\{i+1\} \in E(H)$. For each $m \in\{0, \ldots, n-1\}$, we obtain

$$
\begin{equation*}
(e \backslash\{i\}) \cup\{i+2 m+1\} \in E(H) \tag{8}
\end{equation*}
$$

It follows from (7) and (8) that $(e \backslash\{i\}) \cup\left\{i^{\prime}\right\} \in E(H)$ for every $i^{\prime} \in\{0, \ldots, 2 n\}$. Consequently, $\{0, \ldots, 2 n\}$ is a module of $H$. Since $H$ is prime, we obtain $V(H)=$ $\{0, \ldots, 2 n\}$.

Third, we prove that $H=C_{3}\left(T_{2 n+1}\right)$. We have

$$
\begin{align*}
E\left(C_{3}\left(T_{2 n+1}\right)\right)= & \{(2 i)(2 l+1)(2 j): 0 \leq i \leq l<j \leq n\} \\
& \cup\{(2 i+1)(2 l)(2 j+1): 0 \leq i<l \leq j \leq n-1\} . \tag{9}
\end{align*}
$$

For a contradiction, suppose that there are $x, y, z \in\{0, \ldots, 2 n\}$ such that $x<y<$ $z, x \equiv y(\bmod 2)$, and $x y z \in E(H)$. It follows from Lemma 27 that $(y-2) y z \in$ $E(H)$, which contradicts the fact that $\{y-2, y\}$ is a module of $H-(y-1)$. Hence, $x y z \notin E(H)$ for $x, y, z \in\{0, \ldots, 2 n\}$ such that $x<y<z$ and $x \equiv y$ $(\bmod 2)$. Similarly, $x y z \notin E(H)$ for $x, y, z \in\{0, \ldots, 2 n\}$ such that $x<y<z$ and $y \equiv z(\bmod 2)$. It follows that

$$
\begin{equation*}
E(H) \subseteq E\left(C_{3}(T)\right) \tag{10}
\end{equation*}
$$

Now, consider $x, y, z \in\{0, \ldots, 2 n\}$ such that $x<y<z, x \not \equiv y(\bmod 2)$, and $y \not \equiv z$ $(\bmod 2)$. It follows from Lemma 27 that

$$
\begin{equation*}
x y z \equiv_{H} x(x+1)(x+2) \quad(\text { see Notation } 26) \tag{11}
\end{equation*}
$$

Now, we prove that the permutation

$$
\begin{aligned}
\theta:\{0, \ldots, 2 n\} & \longrightarrow\{0, \ldots, 2 n\} \\
x & \longmapsto x+1 \quad(\bmod 2 n+1)
\end{aligned}
$$

of $\{0, \ldots, 2 n\}$ is an automorphism of $H$. Given $x, y, z \in\{0, \ldots, 2 n\}$ such that $x<y<z$, we have to verify that $x y z \equiv_{H}(x+1)(y+1)(z+1)$. We have

$$
x y z \equiv_{H} x y(z+2) \equiv_{H} x(y+2)(z+2) \equiv_{H}(x+2)(y+2)(z+2) .
$$

Thus, $x y z \equiv_{H}(x+2 n+2)(y+2 n+2)(z+2 n+2)$, that is

$$
x y z \equiv_{H}(x+1)(y+1)(z+1) .
$$

Consequently, $\theta$ is an automorphism of $H$. Since $H$ is prime, there exist $u, v, w \in$ $\{0, \ldots, 2 n\}$ such that $u<v<w$ and $u v w \in E(H)$. By (10), uvw $\in E\left(C_{3}(T)\right)$. It follows from $(9)$ that $u \neq v(\bmod 2)$ and $v \not \equiv w(\bmod 2)$. It follows from (11) that $u(u+1)(u+2) \in E(H)$. Since $\theta$ is an automorphism of $H$, we obtain

$$
\begin{equation*}
012 \in E(H) \tag{12}
\end{equation*}
$$

Finally, consider $x, y, z \in\{0, \ldots, 2 n\}$ such that $x<y<z$ and $x y z \in E\left(C_{3}(T)\right)$. By (9), $x \not \equiv y(\bmod 2)$ and $y \not \equiv z(\bmod 2)$. It follows from (11) that $x y z \equiv_{H}$ $x(x+1)(x+2)$. Since $\theta$ is an automorphism of $H$, we obtain $x(x+1)(x+2) \equiv_{H}$ 012. Therefore, $x y z \equiv_{H} 012$. By (12), $x y z \in E(H)$. Consequently, we obtain $E\left(C_{3}(T)\right) \subseteq E(H)$. It follows from (10) that $H=C_{3}(T)$.

The next corollary is an easy consequence of Corollary 28 and Proposition 29.
Corollary 30. Given a critical 3-hypergraph $H$, with $v(H) \geq 5, H$ is not circular if and only if all the components of $\mathscr{P}(H)$ are paths.

Proof. Suppose that $H$ is circular. By Definition 14, $v(H)$ is odd and

$$
H \simeq C_{3}\left(T_{v(H)}\right)
$$

Hence, we obtain $\mathscr{P}(H) \simeq \mathscr{P}\left(C_{3}\left(T_{v(H)}\right)\right)$. It follows from Theorem 12 that

$$
\mathscr{P}\left(C_{3}\left(T_{v(H)}\right)\right)=\mathscr{P}\left(T_{v(H)}\right) .
$$

By Proposition 23, $\mathscr{P}\left(T_{v(H)}\right)=\mathcal{C}_{v(H)}$. It follows that $\mathscr{P}(H)$ contains a cycle among its components.

Conversely, suppose that $\mathscr{P}(H)$ admits a component $C$ which is not a path. By Corollary 28, $C$ is a cycle. Up to isomorphy, we can assume that $V(H)=\{0, \ldots, p-1\}$, where $p \geq 5$, and there exists $k \in\{3, \ldots, p\}$ satisfying $\mathscr{P}(H)[\{0, \ldots, k-1\}]=\mathcal{C}_{k}$. By Proposition $29, k$ is odd, $k=p$, and $H=C_{3}\left(T_{p}\right)$. Therefore, $H$ is circular.

Notation 31. Let $H$ be a non circular and critical 3-hypergraph. For each $C \in \mathscr{C}(\mathscr{P}(H)), C$ is a path. Thus, there exists an isomorphism denoted by $\varphi_{C}$ from $\mathcal{P}_{v(C)}$ onto $C$.

Lemma 32. Let $H$ be a critical 3-hypergraph with $v(H) \geq 5$. Suppose that $H$ is not circular.

1. Let $C \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$. For every $e \in E(H)$, if $e \cap V(C) \neq \varnothing$ and $e \backslash$ $V(C) \neq \varnothing$, then there exists $i \in\{0, \ldots, w(C)\}$ (see Notation 15) such that $e \cap V(C)=\left\{\varphi_{C}(2 i)\right\}$.
2. Let $C \in \mathfrak{C}_{\text {even }}(\mathscr{P}(H))$. For every $e \in E(H)$, if $e \cap V(C) \neq \varnothing$ and $e \backslash$ $V(C) \neq \varnothing$, then there exist $i, j \in\{0, \ldots, w(C)-1\}$ such that $i \leq j$ and $e \cap V(C)=\left\{\varphi_{C}(2 i), \varphi_{C}(2 j+1)\right\}$.

Proof. For the first assertion, consider $C \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$. Let $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \backslash V(C) \neq \varnothing$. We prove that

$$
\begin{equation*}
e \cap\left\{\varphi_{C}(2 i+1): i \in\{0, \ldots, w(C)-1\}\right\}=\varnothing . \tag{13}
\end{equation*}
$$

Otherwise, there exists $i \in\{0, \ldots, w(C)-1\}$ such that $\varphi_{C}(2 i+1) \in e$. We distinguish the following two cases.

- Suppose that there exists $j \in\{0, \ldots, i\}$ such that $\varphi_{C}(2 j) \in e$. Since $e \backslash$ $V(C) \neq \varnothing, e \cap V(C)=\left\{\varphi_{C}(2 j), \varphi_{C}(2 i+1)\right\}$. By denoting by $v$ the unique element of $e \backslash V(C)$, we obtain $e=\varphi_{C}(2 j) \varphi_{C}(2 i+1) v$. It follows from Lemma 27 that $\varphi_{C}(2 j) \varphi_{C}(2 w(C)-1) v \in E(H)$, which contradicts the fact that $V(H) \backslash\left\{\varphi_{C}(2 w(C)-1), \varphi_{C}(2 w(C))\right\}$ is a module of $H-\varphi_{C}(2 w(C))$.
- Suppose that $e \cap\left\{\varphi_{C}(2 j): j \in\{0, \ldots, i\}\right\}=\varnothing$. It follows from Lemma 27 that there exists $f \in E(H)$ such that $\varphi_{C}(1) \in f$ and $\varphi_{C}(0) \notin f$, which contradicts the fact that $V(H) \backslash\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $H-\varphi_{C}(0)$.

Consequently, (13) holds.
Now, we prove that there exists $i \in\{0, \ldots, w(C)\}$ such that

$$
\begin{equation*}
e \cap V(C)=\left\{\varphi_{C}(2 i)\right\} \tag{14}
\end{equation*}
$$

Otherwise, it follows from (13) that there exist distinct $i, j \in\{0, \ldots, w(C)\}$ such that $0 \leq i<j \leq w(C)$ such that $e \cap V(C)=\left\{\varphi_{C}(2 i), \varphi_{C}(2 j)\right\}$. It follows from Lemma 27 that there exists $f \in E(H)$ such that $f \cap V(C)=\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ and $f \backslash V(C) \neq \varnothing$, which contradicts the fact that $\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $H-\varphi_{C}(2 i+1)$. Consequently, (14) holds.

For the second assertion, consider $C \in \mathfrak{C}_{\text {even }}(\mathscr{P}(H))$. Let $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \backslash V(C) \neq \varnothing$. Set

$$
p=\min \left(\left\{i \in\{0, \ldots, 2 w(C)-1\}: \varphi_{C}(i) \in e\right\}\right)
$$

For a contradiction, suppose that $p$ is odd. It follows from Lemma 27 that $\left(e \backslash\left\{\varphi_{C}(p)\right\}\right) \cup\left\{\varphi_{C}(1)\right\} \in E(H)$, which contradicts the fact that $V(C)$ \ $\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $H-\varphi_{C}(0)$. It follows that $p$ is even. Thus, there exists $i \in\{0, \ldots, w(C)-1\}$ such that $p=\varphi_{C}(2 i)$. Similarly, $\max (\{i \in$ $\left.\left.\{0, \ldots, 2 w(C)-1\}: \varphi_{C}(i) \in e\right\}\right)$ is odd. Thus there exists $j \in\{0, \ldots, w(C)-1\}$ such that $\max \left(\left\{i \in\{0, \ldots, 2 w(C)-1\}: \varphi_{C}(i) \in e\right\}\right)=\varphi_{C}(2 j+1)$. We obtain that $i \leq j$ and $e \cap V(C)=\left\{\varphi_{C}(2 i), \varphi_{C}(2 j+1)\right\}$.

Proposition 33. Let $H$ be a critical 3-hypergraph with $v(H) \geq 5$. Suppose that $H$ is not circular. Consider $C \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H)) \backslash \mathfrak{C}_{1}(\mathscr{P}(H))$.

1. Suppose that $V(C) \mp V(H)$. There exists $e \in E(H)$, such that $e \cap V(C) \neq \varnothing$ and $e \backslash V(C) \neq \varnothing$. Moreover, one of the following two assertions holds

- there exist distinct $D, D^{\prime} \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H)) \backslash\{C\}$ such that $e \subseteq V(C) \cup$ $V(D) \cup V\left(D^{\prime}\right)$, $e \cap V(D) \neq \varnothing$, and $e \cap V\left(D^{\prime}\right) \neq \varnothing$;
- there exists $D \in \mathfrak{C}_{\text {even }}(\mathscr{P}(H))$ such that $e \subseteq V(C) \cup V(D)$ and $e \cap$ $V(D) \neq \varnothing$.

Furthermore, there exists $i \in\{0, \ldots, w(C)\}$ such that $e \cap V(C)=\left\{\varphi_{C}(2 i)\right\}$. Lastly, for each $j \in\{0, \ldots, w(C)\},\left(e \backslash\left\{\varphi_{C}(2 i)\right\}\right) \cup\left\{\varphi_{C}(2 j)\right\} \in E(H)$.
2. The function $\varphi_{C}$ is an isomorphism from $C_{3}\left(U_{v(C)}\right)$ onto $H[V(C)]$.
3. If $V(C) \mp V(H)$, then there exists $C^{\prime} \in \mathfrak{C}(\mathscr{P}(H)) \backslash \mathfrak{C}_{1}(\mathscr{P}(H))$ such that $C^{\prime} \neq C$.

Proof. For the first assertion, suppose that $V(C) \mp V(H)$. Since $V(C)$ is not a module of $H$, there exists $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \backslash V(C) \neq \varnothing$. It follows from Lemma 32 that there exists $i \in\{0, \ldots, w(C)\}$ such that $e \cap V(C)=$ $\left\{\varphi_{C}(2 i)\right\}$. It follows from Lemma 27 that $\left(e \backslash\left\{\varphi_{C}(2 i)\right\}\right) \cup\left\{\varphi_{C}(2 j)\right\} \in E(H)$ for each $j \in\{0, \ldots, w(C)\}$. Moreover, since $|e \cap V(C)|=1$, there exists $D \in$ $\mathfrak{C}(\mathscr{P}(H)) \backslash\{C\}$ such that $e \cap V(D) \neq \varnothing$. We distinguish the following two cases.

- Suppose that $D \in \mathfrak{C}_{\text {even }}(\mathscr{P}(H))$. By Lemma 32, $|e \cap V(D)|=2$. Thus, $e \subseteq V(C) \cup V(D)$.
- Suppose that $D \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$. By Lemma $32,|e \cap V(D)|=1$. Therefore, there exists $D^{\prime} \in \mathfrak{C}(\mathscr{P}(H)) \backslash\{C, D\}$ such that $e \cap V\left(D^{\prime}\right) \neq \varnothing$. Thus, $e \subseteq$ $V(C) \cup V(D) \cup V\left(D^{\prime}\right)$. It follows from Lemma 32 that $D^{\prime} \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$.

For the second assertion, suppose for a contradiction that for every $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$, we have $e \backslash V(C) \neq \varnothing$. We verify that $\left\{\varphi_{C}(2 i): i \in\right.$ $\{0, \ldots, w(C)\}\}$ is a module of $H$. Let $e \in E(H)$ such that $e \cap\left\{\varphi_{C}(2 i): i \in\right.$ $\{0, \ldots, w(C)\}\} \neq \varnothing$ and $e \backslash\left\{\varphi_{C}(2 i): i \in\{0, \ldots, w(C)\}\right\} \neq \varnothing$. Since $e \cap\left\{\varphi_{C}(2 i)\right.$ : $i \in\{0, \ldots, w(C)\}\} \neq \varnothing$, we have $e \cap V(C) \neq \varnothing$, and hence $e \backslash V(C) \neq \varnothing$. By Lemma 32, there exists $i \in\{0, \ldots, w(C)\}$ such that $e \cap V(C)=\left\{\varphi_{C}(2 i)\right\}$. Given $j \in\{0, \ldots, w(C)\}$, it follows from Lemma 27 that $\left(e \backslash\left\{\varphi_{C}(2 i)\right\}\right) \cup\left\{\varphi_{C}(2 j)\right\} \in$ $E(H)$. Therefore, $\left\{\varphi_{C}(2 i): i \in\{0, \ldots, w(C)\}\right\}$ is a module of $H$, which contradicts the fact that $H$ is prime. Consequently, there exists $e \in V(H)$ such that $e \subseteq V(C)$. Set

$$
p=\min \left(\left\{i \in\{0, \ldots, 2 w(C)\}: \varphi_{C}(i) \in e\right\}\right)
$$

For a contradiction, suppose that $p$ is odd. It follows from Lemma 27 that $\left(e \backslash\left\{\varphi_{C}(p)\right\}\right) \cup\left\{\varphi_{C}(1)\right\} \in E(H)$, which contradicts the fact that $V(C)$ \} $\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $H-\varphi_{C}(0)$. Thus, there exists $i \in\{0, \ldots, w(C)\}$ such that $p=\varphi_{C}(2 i)$. Similarly, there exists $k \in\{0, \ldots, w(C)\}$ such that $\max \left(\left\{i \in\{0, \ldots, 2 w(C)\}: \varphi_{C}(i) \in e\right\}\right)=\varphi_{C}(2 k)$. Since $e \subseteq V(C)$, we have $i<k$. Consider $q \in\{2 i+1, \ldots, 2 k-1\}$ such that $e=\varphi_{C}(2 i) \varphi_{C}(q) \varphi_{C}(2 k)$. For a contradiction, suppose that $q$ is even. It follows from Lemma 27 that $\varphi_{C}(2 i) \varphi_{C}(2 i+$ 2) $\varphi_{C}(2 k) \in E(H)$, which contradicts the fact that $\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $H-\varphi_{C}(2 i+1)$. It follows that $p$ is odd. Hence, there exists $j \in\{i, \ldots, k-1\}$ such that $q=\varphi_{C}(2 j+1)$. We obtain

$$
\begin{equation*}
e=\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{C}(2 k) \tag{15}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
E(H[V(C)]) \subseteq\left\{\varphi_{C}\left(2 i^{\prime}\right) \varphi_{C}\left(2 j^{\prime}+1\right) \varphi_{C}\left(2 k^{\prime}\right): 0 \leq i^{\prime} \leq j^{\prime}<k^{\prime} \leq w(C)\right\} . \tag{16}
\end{equation*}
$$

It follows from Lemma 27 that

$$
\begin{equation*}
\varphi_{C}\left(2 i^{\prime}\right) \varphi_{C}\left(2 j^{\prime}+1\right) \varphi_{C}\left(2 k^{\prime}\right) \equiv_{H} \varphi_{C}(0) \varphi_{C}(1) \varphi_{C}(2) \tag{17}
\end{equation*}
$$

for $0 \leq i^{\prime} \leq j^{\prime}<k^{\prime} \leq w(C)$. Since $\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{C}(2 k) \in E(H[V(C)])$ by (15), we obtain $\varphi_{C}(0) \varphi_{C}(1) \varphi_{C}(2) \in E(H[V(C)])$. It follows from (16) and (17) that

$$
E(H[V(C)])=\left\{\varphi_{C}\left(2 i^{\prime}\right) \varphi_{C}\left(2 j^{\prime}+1\right) \varphi_{C}\left(2 k^{\prime}\right): 0 \leq i^{\prime} \leq j^{\prime}<k^{\prime} \leq w(C)\right\} .
$$

In other words, $\varphi_{C}$ is an isomorphism from $C_{3}\left(U_{v(C)}\right)$ onto $H[V(C)]$.
For the third assertion, suppose that $V(C) \mp V(H)$. Set

$$
Y=V(C)
$$

By Theorem $9, U_{v(C)}$ is prime. Hence, $C_{3}\left(U_{v(C)}\right)$ is prime by Theorem 12. It follows from the second assertion above that $H[Y]$ is prime. By Theorem 5 ,
there exist $v, w \in(V(H) \backslash Y) \cup \underline{Y}$ such that $H-\{v, w\}$ is prime. Since $H$ is critical, we have $v \neq w$. Therefore, there exists $C^{\prime} \in \mathfrak{C}(\mathscr{P}(H)) \backslash \mathfrak{C}_{1}(\mathscr{P}(H))$ such that $v, w \in C^{\prime}$. Lastly, suppose for a contradiction that $\underline{Y} \neq \varnothing$. Therefore, there exist $y \in Y$ and $u \in V(H) \backslash Y$ such that $\{y, u\}$ is a module of $H[Y \cup\{u\}]$. Since $H[Y]$ is isomorphic to $C_{3}\left(U_{v(C)}\right)$ by the second assertion above, there exists $e \in E(H[Y])$ such that $y \in e$. Since $\{y, u\}$ is a module of $H[Y \cup\{u\}]$, we get $(e \backslash\{y\}) \cup\{u\} \in E(H)$, which contradicts Lemma 32. It follows that $\underline{Y}=\varnothing$. Thus, we obtain $v, w \notin Y$, that is, $v, w \notin V(C)$. Hence, we have $C^{\prime} \neq C$.

Proposition 34. Let $H$ be a critical 3-hypergraph such that $v(H) \geq 5$. Suppose that $H$ is not circular. Consider $C \in \mathfrak{C}_{\text {even }}(\mathscr{P}(H))$.

1. There exists $D \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$ such that $\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k) \in E(H)$, where $i, j \in\{0, \ldots, w(C)-1\}$, with $i \leq j$, and $k \in\{0, \ldots, w(D)\}$;
2. For each $k \in\{0, \ldots, w(D)\}$, the extension $\psi_{C}^{2 k}:\{0, \ldots, v(C)\} \longrightarrow V(C) \cup$ $\left\{\varphi_{D}(2 k)\right\}$ of $\varphi_{C}$ defined by

$$
\left\{\begin{array}{l}
\left(\psi_{C}^{2 k}\right)_{\uparrow\{0, \ldots, v(C)-1\}}=\varphi_{C} \\
\text { and } \\
\psi_{C}^{2 k}(v(C))=\varphi_{D}(2 k),
\end{array}\right.
$$

is an isomorphism from $C_{3}\left(W_{v(C)+1}\right)$ onto $H\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]$.
3. If $|V(H) \backslash V(C)| \geq 2$, then there exists $C^{\prime} \in \mathfrak{C}(\mathscr{P}(H)) \backslash \mathfrak{C}_{1}(\mathscr{P}(H))$ such that $C^{\prime} \neq C$.

Proof. To begin, we verify that

$$
\begin{equation*}
e \backslash V(C) \neq \varnothing \tag{18}
\end{equation*}
$$

for every $e \in E(H)$. Otherwise, there exists $e \in E(H)$ such that $e \subseteq V(C)$. Set

$$
p=\min \left(\left\{i \in\{0, \ldots, 2 w(C)-1\}: \varphi_{C}(i) \in e\right\}\right)
$$

For a contradiction, suppose that $p$ is odd. It follows from Lemma 27 that $\left(e \backslash\left\{\varphi_{C}(p)\right\}\right) \cup\left\{\varphi_{C}(1)\right\} \in E(H)$, which contradicts the fact that $V(C)$ \} $\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $H-\varphi_{C}(0)$. Thus, there exists $i \in\{0, \ldots, w(C)-1\}$ such that $p=\varphi_{C}(2 i)$. Similarly, there exists $k \in\{i, \ldots, w(C)-1\}$ such that $\max \left(\left\{i \in\{0, \ldots, 2 w(C)\}: \varphi_{C}(i) \in e\right\}\right)=\varphi_{C}(2 k+1)$. Since $e \subseteq V(C)$, we have $i<k$. Consider $q \in\{2 i+1, \ldots, 2 k\}$ such that $e=\varphi_{C}(2 i) \varphi_{C}(q) \varphi_{C}(2 k+1)$. We distinguish the following two cases. In each of them, we obtain a contradiction.

- Suppose that $q$ is even. It follows from Lemma 27 that $\varphi_{C}(2 i) \varphi_{C}(2 i+$ 2) $\varphi_{C}(2 k+1) \in E(H)$, which contradicts the fact that $\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $H-\varphi_{C}(2 i+1)$.
- Suppose that $q$ is odd. It follows from Lemma 27 that $\varphi_{C}(2 i) \varphi_{C}(2 k-$ 1) $\varphi_{C}(2 k+1) \in E(H)$, which contradicts the fact that $\left\{\varphi_{C}(2 k-1), \varphi_{C}(2 k+\right.$ $1)\}$ is a module of $H-\varphi_{C}(2 k)$.

It follows that (18) holds. Since $H$ is prime, we have $E(H) \neq \varnothing$. It follows from (18) that

$$
V(C) \mp V(H) .
$$

For the first assertion, since $V(C) \mp V(H), V(C)$ is not a module of $H$. Therefore, there exists $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \backslash V(C) \neq \varnothing$. By Lemma 32, there exist $i, j \in\{0, \ldots, w(C)-1\}$ such that $i \leq j$ and $e \cap V(C)=$ $\left\{\varphi_{C}(2 i), \varphi_{C}(2 j+1)\right\}$. Since $|e \cap V(C)|=2$, there exists $D \in \mathfrak{C}(\mathscr{P}(H)) \backslash\{C\}$ such that $e \cap V(D) \neq \varnothing$. It follows from Lemma 32 that $D \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$ and $e \cap V(D)=\left\{\varphi_{D}(2 k)\right\}$, where $k \in\{0, \ldots, w(D)\}$.

For the second assertion, it follows from the first assertion above that there exists $D \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$ such that

$$
\begin{equation*}
\varphi_{C}\left(2 i_{0}\right) \varphi_{C}\left(2 j_{0}+1\right) \varphi_{D}\left(2 k_{0}\right) \in E(H) \tag{19}
\end{equation*}
$$

where $i_{0}, j_{0} \in\{0, \ldots, w(C)-1\}$, with $i_{0} \leq j_{0}$, and $k_{0} \in\{0, \ldots, w(D)\}$. It follows from Lemma 27 that

$$
\begin{equation*}
\varphi_{C}\left(2 i_{0}\right) \varphi_{C}\left(2 j_{0}+1\right) \varphi_{D}(2 k) \in E(H) \tag{20}
\end{equation*}
$$

for each $k \in\{0, \ldots, w(D)\}$. Let $k \in\{0, \ldots, w(D)\}$. Consider $i, j \in\{0, \ldots, w(C)-$ $1\}$ such that $i \leq j$. It follows from Lemma 27 that

$$
\begin{equation*}
\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k) \equiv_{H} \varphi_{C}(0) \varphi_{C}(1) \varphi_{D}(2 k) \tag{21}
\end{equation*}
$$

By (20) and (21), we have

$$
\begin{equation*}
\varphi_{C}(0) \varphi_{C}(1) \varphi_{D}(2 k) \in E(H) \tag{22}
\end{equation*}
$$

We prove that

$$
\begin{align*}
E\left(H\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]\right)= & \left\{\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k):\right.  \tag{23}\\
& 0 \leq i \leq j \leq w(C)-1\} .
\end{align*}
$$

Consider $i, j \in\{0, \ldots, w(C)-1\}$ such that $i \leq j$. By (21) and (22), we have $\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k) \in E(H)$. Therefore, we obtain

$$
\begin{align*}
& E\left(H\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]\right) \supseteq\left\{\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k):\right.  \tag{24}\\
&0 \leq i \leq j \leq w(C)-1\} .
\end{align*}
$$

Conversely, consider $e \in E\left(H\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]\right)$. By (18), we have $\varphi_{D}(2 k) \in e$. It follows from Lemma 32 that there exist $i, j \in\{0, \ldots, w(C)-1\}$ such that $i \leq j$ and $e \cap V(C)=\left\{\varphi_{C}(2 i), \varphi_{C}(2 j+1)\right\}$. Thus,

$$
\begin{align*}
& E\left(H\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]\right) \subseteq\left\{\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k):\right.  \tag{25}\\
&0 \leq i \leq j \leq w(C)-1\} .
\end{align*}
$$

It follows from (24) and (25) that (23) holds. Therefore, the extension $\psi_{C}^{2 k}$ of $\varphi_{C}$ is an isomorphism from $C_{3}\left(W_{v(C)+1}\right)$ onto $H\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]$.

For the third assertion, suppose that $|V(H) \backslash V(C)| \geq 2$. Let $k \in\{0, \ldots, w(D)\}$.
Set

$$
Y=V(C) \cup\left\{\varphi_{D}(2 k)\right\} .
$$

We have $Y \nsubseteq V(H)$. By Theorem 9, $W_{v(C)+1}$ is prime. Hence, $C_{3}\left(W_{v(C)+1}\right)$ is prime by Theorem 12. It follows from the second assertion above that $H[Y]$ is prime. By Theorem 5, there exist $v, w \in(V(H) \backslash Y) \cup \underline{Y}$ such that $H-\{v, w\}$ is prime. Since $H$ is critical, we have $v \neq w$. Therefore, there exists $C^{\prime} \in$ $\mathfrak{C}(\mathscr{P}(H)) \backslash \mathfrak{C}_{1}(\mathscr{P}(H))$ such that $v, w \in C^{\prime}$. Lastly, suppose for a contradiction that $V(C) \cap \underline{Y} \neq \varnothing$. There exist $c \in V(C)$ and $u \in V(H) \backslash Y$ such that $\{c, u\}$ is a module of $H[Y \cup\{u\}]$. By the second assertion above, there exists $d \in V(C) \backslash\{c\}$ such that $c d \varphi_{D}(2 k) \in E(H[Y])$. Since $\{c, u\}$ is a module of $H[Y \cup\{u\}]$, we get $u d \varphi_{D}(2 k) \in E(H[Y])$, which contradicts Lemma 32. Consequently, we have $V(C) \cap \underline{Y}=\varnothing$. It follows that $v, w \notin V(C)$, so $C^{\prime} \neq C$.

Proof of Theorem 16. Let $H$ be a critical 3-hypergraph such that $v(H) \geq 5$. Suppose that $H$ is not circular. By Corollary 30, all the components of $\mathscr{P}(H)$ are paths. Hence, $\mathscr{P}(H)$ satisfies (1).

Now, we verify that $\mathscr{P}(H)$ satisfies (2). By Corollary 6, there exist $v, w \in$ $V(H)$ such that $H-\{v, w\}$ is prime. Since $H$ is critical, we have $v \neq w$. Thus, $v w \in E(\mathscr{P}(H))$. Denote by $C$ the component of $\mathscr{P}(H)$ containing $v$ and $w$. We obtain $C \in\left(\mathscr{C}(\mathscr{P}(H)) \backslash \mathfrak{C}_{1}(\mathscr{P}(H))\right)$. Therefore, it follows from Propositions 33 and 34 that $\mathscr{P}(H)$ satisfies (2).

Lastly, we associate with $H$ the hypergraph $\widetilde{H}$ defined on $\mathfrak{C}(\mathscr{P}(H))$ as follows

1. given distinct $C, D \in \mathfrak{C}(\mathscr{P}(H)), C D \in E(\widetilde{H})$ if $C \in \mathfrak{C}_{\text {even }}(\mathscr{P}(H)), D \in$ $\mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$, and there exists $e \in E(H)$ such that $|e \cap V(C)|=2$ and $|e \cap V(D)|=1 ;$
2. given distinct $I, J, K \in \mathfrak{C}(\mathscr{P}(H)), I J K \in E(\widetilde{H})$ if $I, J, K \in \mathfrak{C}_{\text {odd }}(\mathscr{P}(H))$ and there exists $e \in E(H)$ such that $|e \cap V(I)|=|e \cap V(J)|=|e \cap V(K)|=1$.

Clearly, $\widetilde{H}$ satisfies (3). Using Lemma 27, Lemma 32, Proposition 33, and Proposition 34, it is not difficult to verify that $H=\mathscr{P}(H) \bullet \widetilde{H}$.

## 4 Proof of Proposition 18

We use the following preliminary result (see [4, Fact 29]).
Lemma 35. Let $H$ be a 3-hypergraph. Consider $X \mp V(H)$ such that $H[X]$ is prime. Let $M$ be a module of $H$. We have $M \cap X=\varnothing, M \supseteq X$, or $M \cap X=\{y\}$, where $y \in X$. Moreover, the following assertions hold (see Notation 4).

1. If $M \cap X=\varnothing$, then all the elements of $M$ belong to the same block of $p_{(H, X)}$.
2. If $M \supseteq X$, then all the elements of $V(H) \backslash M$ belong to $\operatorname{Dis}_{(H, X)}$.
3. If $M \cap X=\{y\}$, where $y \in X$, then all the elements of $M \backslash\{y\}$ belong to $\operatorname{Twi}_{(H, X)}(y)$.

In this section and the next one, we consider a graph $\Gamma$ satisfying (1) and (2). We consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathfrak{C}(\Gamma)$ satisfying (3). We use the following notation.

Notation 36. For $\mathbb{W} \subseteq V(\mathbb{H})$, set

$$
\overline{\mathbb{W}}=\bigcup_{C \in \mathbb{W}} V(C)
$$

Conversely, for $W \subseteq V(\Gamma \bullet \mathbb{H})$, set

$$
W / \mathfrak{C}(\Gamma)=\{C \in \mathfrak{C}(\Gamma): V(C) \cap W \neq \varnothing\}
$$

We utilize the next two remarks.
Remark 37. Let $C \in \mathfrak{C}_{\text {odd }}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)$. First, suppose that $w(C) \geq 2$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $\varphi_{C}$ is an isomorphism from $C_{3}\left(U_{v(C)}\right)$ onto $(\Gamma \bullet \mathbb{H})[V(C)]$. By Theorem $9, U_{v(C)}$ is critical. By Theorem 12, $C_{3}\left(U_{v(C)}\right)$ is critical and $\mathscr{P}\left(C_{3}\left(U_{v(C)}\right)\right)=\mathscr{P}\left(U_{v(C)}\right)$. Furthermore, we have $\mathscr{P}\left(U_{v(C)}\right)=$ $\mathcal{P}_{v(C)}$ by Proposition 24. Since $\varphi_{C}$ is an isomorphism from $C_{3}\left(U_{v(C)}\right)$ onto $(\Gamma \bullet \mathbb{H})[V(C)],(\Gamma \bullet \mathbb{H})[V(C)]$ is critical and

$$
\mathscr{P}((\Gamma \bullet \mathbb{H})[V(C)])=\left(V(C),\left\{\varphi_{C}(m) \varphi_{C}(m+1): 0 \leq m<2 w(C)\right\}\right) .
$$

It follows from Lemma 27 that for each $1 \leq m \leq 2 w(C)-1$,

$$
\begin{equation*}
\left\{\varphi_{C}(m-1), \varphi_{C}(m+1)\right\} \text { is a module of }((\Gamma \bullet \mathbb{H})[V(C)])-\varphi_{C}(m) \tag{26}
\end{equation*}
$$

Second, suppose that $w(C)=1$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $E((\Gamma \bullet \mathbb{H})[V(C)])=\{V(C)\}$. Therefore, $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime. Moreover, since $v(C)=3$, observe that (26) holds.

Remark 38. Let $C \in \mathfrak{C}_{\text {even }}(\Gamma)$. We have $w(C) \geq 1$. Suppose that there exists $\varepsilon \in E(\mathbb{H})$ such that $C \in \varepsilon$. We have $\varepsilon=C D$, where $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$. Let $k \in$ $\{0, \ldots, w(D)\}$. We verify that $(\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]$ is prime

First, suppose that $w(C)=1$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $E\left((\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]\right)=\left\{V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right\}$. Therefore, $(\Gamma \bullet \mathbb{H})[V(C) \cup$ $\left.\left\{\varphi_{D}(2 k)\right\}\right]$ is prime.

Second, suppose that $w(C) \geq 2$. Let $\psi_{C}^{2 k}:\{0, \ldots, v(C)\} \longrightarrow V(C) \cup\left\{\varphi_{D}(2 k)\right\}$ satisfying $\left(\psi_{C}^{2 k}\right)_{\uparrow\{0, \ldots, v(C)-1\}}=\varphi_{C}$ and $\psi_{C}^{2 k}(v(C))=\varphi_{D}(2 k)$. By $(4), \psi_{C}^{2 k}$ is an isomorphism from $C_{3}\left(W_{v(C)+1}\right)$ onto $(\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]$. By Theorem 9 , $W_{v(C)+1}$ is critical, so $W_{v(C)+1}$ is prime. By Theorem $12, C_{3}\left(W_{v(C)+1}\right)$ is prime as well. Since $\psi_{C}^{2 k}$ is an isomorphism from $C_{3}\left(W_{v(C)+1}\right)$ onto $(\Gamma \bullet \mathbb{H})[V(C) \cup$ $\left.\left\{\varphi_{D}(2 k)\right\}\right],(\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(2 k)\right\}\right]$ is prime too.

If $\mathbb{H}$ is disconnected, then $\Gamma \bullet \mathbb{H}$ is decomposable, whence the necessity of Assertion (C1) in Proposition 18. Indeed, we have

Lemma 39. If $\mathbb{C}$ is a component of $\mathbb{H}$, then $\overline{V(\mathbb{C})}$ and $V(\Gamma \bullet \mathbb{H}) \backslash \overline{V(\mathbb{C})}$ are modules of $\Gamma \bullet \mathbb{H}$.

Proof. For a contradiction, suppose that there exists $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \overline{V(\mathbb{C})} \neq \varnothing$ and $e \cap(V(\Gamma \bullet \mathbb{H}) \backslash \overline{V(\mathbb{C})}) \neq \varnothing$. There exist $C \in V(\mathbb{C})$ and $D \notin V(\mathbb{C})$ such that $e \cap V(C) \neq \varnothing$ and $e \cap V(D) \neq \varnothing$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that there exists $\varepsilon \in E(\mathbb{H})$ such that $C, D \in \varepsilon$, which contradicts the fact that $\mathbb{C}$ is a component of $\mathbb{H}$. Consequently, for each $e \in E(\Gamma \bullet \mathbb{H})$, we have $e \subseteq \overline{V(\mathbb{C})}$ or $e \subseteq(V(\Gamma \bullet \mathbb{H}) \backslash \overline{V(\mathbb{C})})$. It follows that $\overline{V(\mathbb{C})}$ and $V(\Gamma \bullet \mathbb{H}) \backslash \overline{V(\mathbb{C})}$ are modules of $\Gamma \bullet \mathbb{H}$.

Each edge of $\mathbb{H}$ induces a prime subhypergraph of $\Gamma \bullet \mathbb{H}$. Precisely, we have
Lemma 40. For every $\varepsilon \in E(\mathbb{H})$, $(\Gamma \bullet \mathbb{H})[\bar{\varepsilon}]$ is prime.
Proof. Let $\varepsilon \in E(\mathbb{H})$. First, suppose that $|\varepsilon|=2$. There exist $C \in \mathfrak{C}_{\text {even }}(\Gamma)$ and $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=C D$. Hence, we have to show that $(\Gamma \bullet \mathbb{H})[V(C) \cup$ $V(D)]$ is prime. If $w(D)=0$, then $(\Gamma \bullet \mathbb{H})[V(C) \cup V(D)]$ is prime by Remark 38 . Suppose that $w(D) \geq 1$. Set

$$
X=V(D)
$$

By Remark 37, $(\Gamma \bullet \mathbb{H})[X]$ is prime. We have $V(C) \subseteq \operatorname{Dis}_{(\Gamma \bullet \mathbb{H}, X)}$ (see Notation 4). Let $M$ be a module of $(\Gamma \bullet \mathbb{H})[X \cup V(C)]$ such that $|M| \geq 2$. We have to show that $M=X \cup V(C)$. By Lemma 35, we have $M \cap X=\varnothing$, $M \supseteq X$, or $M \cap X=\{y\}$, where $y \in X$. It follows from the third assertion of Lemma 35 that $|M \cap X| \neq 1$. Now, suppose that $M \cap X=\varnothing$. Hence, we have $M \subseteq V(C)$. By Remark $38,(\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(0)\right\}\right]$ is prime. Since $|M| \geq 2$ and $M \subseteq V(C)$, we obtain $M=V(C) \cup\left\{\varphi_{D}(0)\right\}$, which contradicts $M \cap X=\varnothing$. Therefore, we have $M \supseteq X$. Let $i, j \in\{0, \ldots, w(C)-1\}$ with $i \leq j$. We have $\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(0) \in E(H)$. Since $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(C)]$ and $\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(1) \notin E(H)$, we obtain $\varphi_{C}(2 i), \varphi_{C}(2 j+1) \in M$. It follows that $M=X \cup V(C)$.

Second, suppose that $|\varepsilon|=3$. There exist $I, J, K \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=$ $I J K$. Hence, we have to show that $(\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)]$ is prime. If $w(I)=w(J)=w(K)=0$, then $|V(I) \cup V(J) \cup V(K)|=3$, and hence $E((\Gamma \bullet$ $\mathbb{H})[V(I) \cup V(J) \cup V(K)])=\left\{\varphi_{I}(0) \varphi_{J}(0) \varphi_{K}(0)\right\}$. It follows that $(\Gamma \bullet \mathbb{H})[V(I) \cup$ $V(J) \cup V(K)]$ is prime. Now, suppose that $w(I)>0, w(J)>0$, or $w(K)>0$. For instance, assume that $w(I)>0$. By Remark $37,(\Gamma \bullet \mathbb{H})[V(I)]$ is prime. Set

$$
X=V(I)
$$

We have $V(J) \cup V(K) \subseteq \operatorname{Dis}_{(\Gamma \bullet \mathbb{H}, X)}$. Let $M$ be a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup$ $V(K)]$ such that $|M| \geq 2$. We have to show that $M=X \cup V(J) \cup V(K)$. It follows from the third assertion of Lemma 35 that $M \cap X=\varnothing$ or $X \subseteq M$.

For a contradiction, suppose that $M \cap X=\varnothing$. For $0 \leq j \leq w(J)$ and $0 \leq k \leq$ $w(K)$, we have $\varphi_{I}(0) \varphi_{J}(2 j) \varphi_{K}(2 k) \in E(\Gamma \bullet \mathbb{H})$. It follows that $M \cap\left\{\varphi_{J}(2 j)\right.$ : $0 \leq j \leq w(J)\}=\varnothing$ or $M \cap\left\{\varphi_{K}(2 k): 0 \leq k \leq w(K)\right\}=\varnothing$. For instance, assume that

$$
\begin{equation*}
M \cap\left\{\varphi_{J}(2 j): 0 \leq j \leq w(J)\right\}=\varnothing \tag{27}
\end{equation*}
$$

We distinguish the following two cases.

1. Suppose that $M \cap\left\{\varphi_{J}(2 j+1): 0 \leq j \leq w(J)-1\right\} \neq \varnothing$. Hence, there exists $j \in\{0, \ldots, w(J)-1\}$ such that $\varphi_{J}(2 j+1) \in M$. We have $\varphi_{J}(2 j) \varphi_{J}(2 j+$ 1) $\varphi_{J}(2 j+2) \in E(\Gamma \bullet \mathbb{H})$. By $(27), \varphi_{J}(2 j), \varphi_{J}(2 j+2) \notin M$. Since $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$, we obtain $\varphi_{J}(2 j) x \varphi_{J}(2 j+2) \in E(\Gamma \bullet$ $\mathbb{H})$ for every $x \in M$. But, for each $v \in V(K)$, we have $\varphi_{J}(2 j) v \varphi_{J}(2 j+2) \notin$ $E(\Gamma \bullet \mathbb{H})$. It follows that $M \cap V(K)=\varnothing$. Therefore, $M \subseteq V(J)$, and hence $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J)]$. Since $|M| \geq 2$, we have $w(J) \geq 1$. By Remark 37, $(\Gamma \bullet \mathbb{H})[V(J)]$ is prime. Therefore, $M=V(J)$, which contradicts (27).
2. Suppose that $M \cap\left\{\varphi_{J}(2 j+1): 0 \leq j \leq w(J)-1\right\}=\varnothing$. By (27), we have $M \cap V(J)=\varnothing$, and hence $M \subseteq V(K)$. As previously for $J$, we obtain $w(k) \geq 1$ and $M=V(K)$, which is impossible because $\varphi_{I}(0) \varphi_{J}(0) \varphi_{K}(0) \epsilon$ $E(\Gamma \bullet \mathbb{H})$ and $\varphi_{I}(0) \varphi_{J}(0) \varphi_{K}(1) \notin E(\Gamma \bullet \mathbb{H})$.

It follows that $X \subseteq M$. Recall that $\varphi_{I}(0) \varphi_{J}(2 j) \varphi_{K}(2 k) \in E(\Gamma \bullet \mathbb{H})$ for $0 \leq j \leq w(J)$ and $0 \leq k \leq w(K)$. Let $j \in\{0, \ldots, w(J)\}$ and $k \in\{0, \ldots, w(J)\}$. We have $\varphi_{I}(0) \in M$ because $X \subseteq M$. Since $\varphi_{I}(1) \varphi_{J}(2 j) \varphi_{K}(2 k) \notin E(\Gamma \bullet \mathbb{H})$ and $M$ is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$, we obtain $\varphi_{J}(2 j), \varphi_{K}(2 k) \in M$. It follows that $X \cup\left\{\varphi_{J}(2 j): 0 \leq j \leq w(J)\right\} \cup\left\{\varphi_{K}(2 k): 0 \leq k \leq w(K)\right\} \subseteq M$. Clearly, $V(J) \subseteq M$ when $w(J)=0$. Suppose that $w(J) \geq 1$. By Remark 37, $(\Gamma \bullet \mathbb{H})[V(J)]$ is prime. Since $M \cap V(J)$ is a module of $(\Gamma \bullet \mathbb{H})[V(J)]$ such that $\left\{\varphi_{J}(2 j): 0 \leq j \leq w(J)\right\} \subseteq(M \cap V(J))$, we obtain $V(J) \subseteq M$. Similarly, $V(K) \subseteq M$. Consequently, $M=X \cup V(J) \cup V(K)$.

In the next four results, we compare the modules of $\mathbb{H}$ with those of $\Gamma \bullet \mathbb{H}$.
Lemma 41. Suppose that $\mathbb{H}$ is connected. For each nontrivial module $M$ of $\Gamma \bullet \mathbb{H}$, we have $M=\overline{M / \mathfrak{C}(\Gamma)},|M / \mathfrak{C}(\Gamma)| \geq 2$, and $M / \mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$.

Proof. Let $M$ be a nontrivial module of $\Gamma \bullet \mathbb{H}$. First, to prove that $M=\overline{M / \mathfrak{C}(\Gamma)}$, it suffices to show that $V(C) \subseteq M$ for every $C \in M / \mathfrak{C}(\Gamma)$. To begin, consider $C \in \mathfrak{C}_{\text {odd }}(\Gamma) \cap(M / \mathfrak{C}(\Gamma))$. If $w(C)=0$, then we clearly have $V(C) \subseteq M$. Hence, suppose that $w(C) \geq 1$. By Remark $37,(\Gamma \bullet \mathbb{H})[V(C)]$ is prime. For a contradiction, suppose that $|V(C) \cap M|=1$. Denote by $c$ the unique element of $V(C) \cap M$. Since $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime, there exist distinct $d, d^{\prime} \in V(C) \backslash M$ such that $c d d^{\prime} \in E(\Gamma \bullet \mathbb{H})$. Since $|M| \geq 2$, there exists $v \in M \backslash V(C)$. Since $M$ is a module of $\Gamma \bullet \mathbb{H}$, we obtain $v d d^{\prime} \in E(\Gamma \bullet \mathbb{H})$, which contradicts the definition of $\Gamma \bullet \mathbb{H}$. It follows that $|V(C) \cap M| \geq 2$. Since $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime, $V(C) \subseteq M$. Now, consider $C \in \mathfrak{C}_{\text {even }}(\Gamma) \cap(M / \mathfrak{C}(\Gamma))$. Since $\mathbb{H}$ is connected, there exists $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $C D \in E(\mathbb{H})$. For a contradiction, suppose that $|V(C) \cap M|=1$. Denote by $c$ the unique element of $V(C) \cap M$. There exists $c^{\prime} \in V(C) \backslash M$ such that $c c^{\prime} \varphi_{D}(0) \in E(\Gamma \bullet \mathbb{H})$. By Remark 38, $(\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(0)\right\}\right]$ is prime. Hence, if $\left|\left(V(C) \cup\left\{\varphi_{D}(0)\right\}\right) \cap M\right| \geq 2$, then $V(C) \cup\left\{\varphi_{D}(0)\right\} \subseteq M$, which contradicts $|V(C) \cap M|=1$. Therefore, we have $\left|\left(V(C) \cup\left\{\varphi_{D}(0)\right\}\right) \cap M\right|=1$. Since $|M| \geq 2$, there exists $v \in M \backslash\left(V(C) \cup\left\{\varphi_{D}(0)\right\}\right)$. Since $M$ is a module of $\Gamma \bullet \mathbb{H}$,
we obtain $v c^{\prime} \varphi_{D}(0) \in E(\Gamma \bullet \mathbb{H})$, which contradicts the definition of $\Gamma \bullet \mathbb{H}$. It follows that $|V(C) \cap M| \geq 2$. Since $(\Gamma \bullet \mathbb{H})\left[V(C) \cup\left\{\varphi_{D}(0)\right\}\right]$ is prime, we obtain $V(C) \cup\left\{\varphi_{D}(0)\right\} \subseteq M$. Consequently, we have $M=\overline{M / \mathfrak{C}(\Gamma)}$.

Second, we prove that $|M / \mathfrak{C}(\Gamma)| \geq 2$. If $v(C)=1$ for each $C \in M / \mathfrak{C}(\Gamma)$, then $|M / \mathfrak{C}(\Gamma)| \geq 2$ because $|M| \geq 2$. Hence, consider $C \in M / \mathfrak{C}(\Gamma)$ such that $v(C) \geq 2$. Since $\mathbb{H}$ is connected, there exists $\varepsilon \in E(\mathbb{H})$ such that $C \in \varepsilon$. By Lemma 40, $(\Gamma \bullet \mathbb{H})[\bar{\varepsilon}]$ is prime. We have $V(C) \subseteq M$ because $M=\overline{M / \mathfrak{C}(\Gamma)}$. Since $v(C) \geq 2$, we get $\bar{\varepsilon} \subseteq M$. Thus, we have $\varepsilon \subseteq(M / \mathfrak{C}(\Gamma))$, so $|M / \mathfrak{C}(\Gamma)| \geq 2$.

Third, we prove that $M / \mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$. Let $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap(M / \mathfrak{C}(\Gamma)) \neq \varnothing$ and $\varepsilon \backslash(M / \mathfrak{C}(\Gamma)) \neq \varnothing$. We distinguish the following two cases.

- Suppose that $|\varepsilon|=2$. There exist $C \in \mathfrak{C}_{\text {even }}(\Gamma)$ and $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=C D$. Thus, there exist distinct $c, c^{\prime} \in V(C)$ and $d \in V(D)$ such that $c c^{\prime} d \in E(\Gamma \bullet \mathbb{H})$. For a contradiction, suppose that $C \in M / \mathfrak{C}(\Gamma)$. Since $\varepsilon \backslash(M / \mathfrak{C}(\Gamma)) \neq \varnothing$, we have $D \notin M / \mathfrak{C}(\Gamma)$. Since $M=\overline{M / \mathfrak{C}(\Gamma)}$, we have $V(C) \subseteq M$ and $V(D) \cap M=\varnothing$. We obtain $c c^{\prime} d \in E(\Gamma \bullet \mathbb{H})$, with $c, c^{\prime} \in V(C) \cap M$ and $d \in V(D) \backslash M$, which contradicts the fact that $M$ is a module of $\Gamma \bullet \mathbb{H}$. It follows that $D \in M / \mathbb{C}(\Gamma)$ and $C \notin M / \mathfrak{C}(\Gamma)$. Since $M=\overline{M / \mathfrak{C}(\Gamma)}$, we have $V(D) \subseteq M$ and $V(C) \cap M=\varnothing$. We obtain $c c^{\prime} d \in E(\Gamma \bullet \mathbb{H})$, with $c, c^{\prime} \in V(C) \backslash M$ and $d \in V(D) \cap M$. Since $M$ is a module of $\Gamma \bullet \mathbb{H}$, we obtain

$$
\begin{equation*}
c c^{\prime} x \in E(\Gamma \bullet \mathbb{H}) \text { for every } x \in M \tag{28}
\end{equation*}
$$

Consider $D^{\prime} \in M / \mathfrak{C}(\Gamma)$. Since $M=\overline{M / \mathfrak{C}(\Gamma)}$, we have $V\left(D^{\prime}\right) \subseteq M$. Let $d^{\prime} \in V\left(D^{\prime}\right)$. By $(28), c c^{\prime} d^{\prime} \in E(\Gamma \bullet \mathbb{H})$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $C D^{\prime} \in E(\mathbb{H})$.

- Suppose that $|\varepsilon|=3$. There exist distinct $I, J, K \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=I J K$. Moreover, we can assume that $I \in M / \mathfrak{C}(\Gamma)$ and $K \notin M / \mathfrak{C}(\Gamma)$. Since $M=\overline{M / \mathfrak{C}(\Gamma)}$, we have $V(I) \subseteq M$ and $V(K) \cap M=\varnothing$. By the definition of $\Gamma \bullet \mathbb{H}$, we obtain $\varphi_{I}(0) \varphi_{J}(0) \varphi_{K}(0) \in E(\Gamma \bullet \mathbb{H})$. Since $M$ is a module of $\Gamma \bullet \mathbb{H}$, we have $\varphi_{J}(0) \notin M$. Since $M=\overline{M / \mathfrak{C}(\Gamma)}$, we obtain $V(J) \cap M=\varnothing$. Thus, $\varphi_{I}(0) \varphi_{J}(0) \varphi_{K}(0) \in E(\Gamma \bullet \mathbb{H})$, with $\varphi_{I}(0) \in M$ and $\varphi_{J}(0), \varphi_{K}(0) \notin M$. Since $M$ is a module of $\Gamma \bullet \mathbb{H}$, we obtain

$$
\begin{equation*}
x \varphi_{J}(0) \varphi_{K}(0) \in E(\Gamma \bullet \mathbb{H}) \text { for every } x \in M \tag{29}
\end{equation*}
$$

Consider any $L \in M / \mathfrak{C}(\Gamma)$. Since $M=\overline{M / \mathfrak{C}(\Gamma)}$, we have $V(L) \subseteq M$. Let $d \in V(L)$. By $(29), d \varphi_{J}(0) \varphi_{K}(0) \in E(\Gamma \bullet \mathbb{H})$. Since $J, K, L$ are distinct element of $\mathfrak{C}(\Gamma)$, it follows from the definition of $\Gamma \bullet \mathbb{H}$ that $J K L \in E(\mathbb{H})$.

Consequently, $M / \mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$.
Lemma 42. Suppose that $\mathbb{H}$ is connected. For a nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \subseteq \mathfrak{C}_{\text {even }}(\Gamma)$ or $\mathbb{M} \subseteq \mathfrak{C}_{\text {odd }}(\Gamma)$.

Proof. Let $\mathbb{M}$ be a nontrivial module of $\mathbb{H}$. Suppose that $\mathbb{M} \backslash \mathfrak{C}_{\text {odd }}(\Gamma) \neq \varnothing$. We have to show that $\mathbb{M} \subseteq \mathfrak{C}_{\text {even }}(\Gamma)$. Since $\mathbb{M} \backslash \mathfrak{C}_{\text {odd }}(\Gamma) \neq \varnothing$, there exists $C \in \mathbb{M} \cap \mathfrak{C}_{\text {even }}(\Gamma)$. Since $\mathbb{H}$ is connected and $\mathbb{M} \mp V(\mathbb{H})$, there exists $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap \mathbb{M} \neq \varnothing$ and $\varepsilon \backslash \mathbb{M} \neq \varnothing$. Furthermore, since $\mathbb{M}$ is a module of $\mathbb{H}$, we have $|\varepsilon \cap \mathbb{M}|=1$. It follows that $(\varepsilon \backslash \mathbb{M}) \cup\{C\} \in E(\mathbb{H})$. Since $C \in \mathbb{M} \cap \mathfrak{C}_{\text {even }}(\Gamma)$, we obtain $|\varepsilon \backslash \mathbb{M}|=1$ and $(\varepsilon \backslash \mathbb{M}) \subseteq \mathfrak{C}_{\text {odd }}(\Gamma)$. Lastly, consider $D \in \mathbb{M}$. Since $\mathbb{M}$ is a module of $\mathbb{H}$, we have $(\varepsilon \backslash \mathbb{M}) \cup\{D\} \in E(\mathbb{H})$. Since $|(\varepsilon \backslash \mathbb{M}) \cup\{D\}|=2$ and $(\varepsilon \backslash \mathbb{M}) \subseteq \mathfrak{C}_{\text {odd }}(\Gamma)$, we obtain $D \in\left(\mathbb{M} \cap \mathfrak{C}_{\text {even }}(\Gamma)\right)$. Consequently, we have $\mathbb{M} \subseteq \mathfrak{C}_{\text {even }}(\Gamma)$.

The next result follows from Lemmas 41 and 42.
Corollary 43. Suppose that $\mathbb{H}$ is connected. For each nontrivial module $M$ of $\Gamma \bullet \mathbb{H}$, we have $M \subseteq V_{1}(\Gamma)$.

Proof. Let $M$ be a nontrivial module of $\Gamma \bullet \mathbb{H}$. It follows from Lemma 41 that $M / \mathfrak{C}(\Gamma)$ is a nontrivial module of $\mathbb{H}$. By Lemma $42, M / \mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text {even }}(\Gamma)$ or $M / \mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text {odd }}(\Gamma)$.

For a contradiction, suppose that $M / \mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text {even }}(\Gamma)$. Since $\mathbb{H}$ is connected, there exists $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap(M / \mathfrak{C}(\Gamma)) \neq \varnothing$ and $\varepsilon \backslash(M / \mathfrak{C}(\Gamma)) \neq \varnothing$. Since $M / \mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text {even }}(\Gamma)$, there exist $C \in M / \mathfrak{C}(\Gamma)$ and $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $\varepsilon=C D$. By Lemma 41, we have $M=\overline{M / \mathfrak{C}(\Gamma)}$. Therefore, we obtain $V(C) \subseteq M$ and $V(D) \cap M=\varnothing$. It follows that $V(C)$ is a nontrivial module of $(\Gamma \bullet \mathbb{H})[V(C) \cup V(D)]$, which contradicts Lemma 40. Consequently, we have $M / \mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text {odd }}(\Gamma)$.

For a contradiction, suppose that $M \backslash V_{1}(\Gamma) \neq \varnothing$. Hence, there exists $D \in$ $(M / \mathfrak{C}(\Gamma)) \cap\left(\mathfrak{C}_{\text {odd }}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)\right)$. Since $\mathbb{H}$ is connected, there exists $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap(M / \mathfrak{C}(\Gamma)) \neq \varnothing$ and $\varepsilon \backslash(M / \mathfrak{C}(\Gamma)) \neq \varnothing$. Since $M / \mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$, there exists $C \in M / \mathfrak{C}(\Gamma)$ such that $\varepsilon \cap(M / \mathfrak{C}(\Gamma))=\{C\}$ and $(\varepsilon \backslash\{C\}) \cup\{D\} \in$ $E(\mathbb{H})$. By Lemma 41, we have $V(D) \subseteq M$. We obtain that $V(D)$ is a nontrivial module of $(\Gamma \bullet \mathbb{H})[(\varepsilon \backslash\{C\}) \cup\{D\}]$, which contradicts Lemma 40. Consequently, $M \subseteq V_{1}(\Gamma)$.

Lemma 44. Given a module $\mathbb{M}$ of $\mathbb{H}$, if $\mathbb{M} \subseteq \mathfrak{C}_{1}(\Gamma)$, then $\overline{\mathbb{M}}$ is a module of $\Gamma \bullet \mathbb{H}$.

Proof. Let $\mathbb{M}$ be a module of $\mathbb{H}$. Suppose that $\mathbb{M} \subseteq \mathfrak{C}_{1}(\Gamma)$. Let $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \overline{\mathbb{M}} \neq \varnothing$ and $e \backslash \overline{\mathbb{M}} \neq \varnothing$. There exist $c \in e \cap \overline{\mathbb{M}}$ and $x \in e \backslash \overline{\mathbb{M}}$. We have $\{c\} \in \mathfrak{C}(\Gamma)$. Consider $C_{x} \in \mathfrak{C}(\Gamma)$ such that $x \in C_{x}$. For a contradiction, suppose that $e=c d x$, where $d \in \overline{\mathbb{M}} \backslash\{c\}$. We have $\{d\} \in \mathfrak{C}_{1}(\Gamma) \backslash\{\{c\}\}$. Since $c d x \in E(\Gamma \bullet \mathbb{H})$, we obtain $\{c\}\{d\} C_{x} \in E(\mathbb{H})$, which contradicts the fact that $\mathbb{M}$ is a module of $\mathbb{H}$. It follows that $e=c x y$, where $y \in V(\Gamma \bullet \mathbb{H}) \backslash \overline{\mathbb{M}}$. Consider $C_{y} \in \mathfrak{C}(\Gamma)$ such that $y \in C_{y}$. Given $d \in \overline{\mathbb{M}}$, we have to verify that $d x y \in E(\Gamma \bullet \mathbb{H})$. We have $\{d\} \in \mathfrak{C}(\Gamma)$. Note that we can have $C_{x}=C_{y}$. We distinguish the following two cases.

- Suppose that $C_{x}=C_{y}$. Since cxy $\in E(\Gamma \bullet \mathbb{H})$, we obtain $C_{x} \in \mathfrak{C}_{\text {even }}(\Gamma)$, $\{x, y\}=\left\{\varphi_{C_{x}}(2 i), \varphi_{C_{x}}(2 j+1)\right\}$, where $0 \leq i \leq j \leq w\left(C_{x}\right)-1$, and $\{c\} C_{x} \in$
$E(\mathbb{H})$. Since $\mathbb{M}$ is a module of $\mathbb{H},\{d\} C_{x} \in E(\mathbb{H})$. Since $C_{x} \in \mathfrak{C}_{\text {even }}(\Gamma)$ and $\{x, y\}=\left\{\varphi_{C_{x}}(2 i), \varphi_{C_{x}}(2 j+1)\right\}$, where $0 \leq i \leq j \leq w\left(C_{x}\right)-1$, we obtain $d x y \in E(\Gamma \bullet \mathbb{H})$.
- Suppose that $C_{x} \neq C_{y}$. Since $c x y \in E(\Gamma \bullet \mathbb{H})$, we obtain $C_{x}, C_{y} \in \mathfrak{C}_{\text {odd }}(\Gamma)$, $\{x, y\}=\left\{\varphi_{C_{x}}(2 i), \varphi_{C_{y}}(2 j)\right\}$, where $0 \leq i \leq w\left(C_{x}\right)$ and $0 \leq j \leq w\left(C_{y}\right)$, and $\{c\} C_{x} C_{y} \in E(\mathbb{H})$. Since $\mathbb{M}$ is a module of $\mathbb{H},\{d\} C_{x} C_{y} \in E(\mathbb{H})$. Since $C_{x}, C_{y} \in \mathfrak{C}_{\text {odd }}(\Gamma)$ and $\{x, y\}=\left\{\varphi_{C_{x}}(2 i), \varphi_{C_{y}}(2 j)\right\}$, where $0 \leq i \leq w\left(C_{x}\right)$ and $0 \leq j \leq w\left(C_{y}\right)$, we obtain $d x y \in E(\Gamma \bullet \mathbb{H})$.

Proof of Proposition 18. To begin, suppose that $\Gamma \bullet \mathbb{H}$ is decomposable. If $\mathbb{H}$ is disconnected, then Assertion (C1) does not hold. Hence, suppose that $\mathbb{H}$ is connected. Consider a nontrivial module $M$ of $\Gamma \bullet \mathbb{H}$. It follows from Lemma 41 that $M / \mathfrak{C}(\Gamma)$ is a nontrivial module of $\mathbb{H}$. Furthermore, we have $M \subseteq V_{1}(\Gamma)$ by Corollary 43. Thus, we have $M / \mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{1}(\Gamma)$. Consequently, Assertion (C2) does not hold.

Conversely, if $\mathbb{H}$ is disconnected, then it follows from Lemma 39 that $\Gamma \bullet \mathbb{H}$ is decomposable. Hence, suppose that $\mathbb{H}$ is connected. Moreover, suppose that there exists a nontrivial module $\mathbb{M}$ of $\mathbb{H}$ such that $\mathbb{M} \subseteq \mathfrak{C}_{1}(\Gamma)$. It follows from Lemma 44 that $\overline{\mathbb{M}}$ is a nontrivial module of $\Gamma \bullet \mathbb{H}$. Hence, $\Gamma \bullet \mathbb{H}$ is decomposable.

## 5 Proof of Theorem 17

In the next remark, we describe a simple way to obtain automorphisms of $\Gamma \bullet \mathbb{H}$.
Remark 45. For every $C \in \mathfrak{C}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)$, consider the function

$$
\begin{array}{rll}
F_{C}: & V(\Gamma \bullet \mathbb{H}) & \\
& \longrightarrow V(\Gamma \bullet \mathbb{H}) \\
& \varphi_{C}(m)(0 \leq m \leq v(C)-1) & \longmapsto \varphi_{C}(v(C)-1-m) \\
& v \notin V(C) & \longmapsto v .
\end{array}
$$

Clearly, $F_{C}$ is an automorphism of $\Gamma$. It is easy to verify that $F_{C}$ is an automorphism of $\Gamma \bullet \mathbb{H}$. Moreover, consider a permutation $F$ of $V(\Gamma \bullet \mathbb{H})$ such that $F(v)=v$ for each $v \in V_{1}(\Gamma)$. If for every $C \in \mathfrak{C}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma), F_{\mid V(C)}=\left(F_{C}\right)_{\mid V(C)}$ or $\operatorname{Id}_{V(C)}$, then $F$ is an automorphism of $\Gamma \bullet \mathbb{H}$.

The next two results are useful to study the criticality of $\Gamma \bullet \mathbb{H}$. They allow us to prove Theorem 17.

Lemma 46. Given $C \in \mathfrak{C}_{\text {odd }}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)$, the following assertions hold

1. $V(\Gamma \bullet \mathbb{H}) \backslash\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(0)$;
2. $V(\Gamma \bullet \mathbb{H}) \backslash\left\{\varphi_{C}(2 w(C)-1), \varphi_{C}(2 w(C))\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 w(C))$;
3. if $w(C) \geq 1$, then for $m \in\{1, \ldots, 2 w(C)-1\}$, $\left\{\varphi_{C}(m-1), \varphi_{C}(m+1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(m)$.

Proof. We begin with the first assertion. For every $e \in E(\Gamma \bullet \mathbb{H})$, if $\varphi_{C}(1) \in e$, then $e \subseteq V(C)$ and $\varphi_{C}(0) \in e$. Consequently, there does not exist $e \in E((\Gamma \bullet$ $\left.\mathbb{H})-\varphi_{C}(0)\right)$ such that $\varphi_{C}(1) \in e$. Therefore, $V(\Gamma \bullet \mathbb{H}) \backslash\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(0)$.

For the second assertion, recall that the function $F_{C}$ is an automorphism of $\Gamma \bullet \mathbb{H}$ (see Remark 45). It follows from the first assertion above that $F_{C}(V(\Gamma \bullet$ $\left.\mathbb{H}) \backslash\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}\right)$ is a module of $(\Gamma \bullet \mathbb{H})-F_{C}\left(\varphi_{C}(0)\right)$, that is, $V(C)$ \} $\left\{\varphi_{C}(2 w(C)-1), \varphi_{C}(2 w(C))\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 w(C))$.

For the third assertion, suppose that $w(C) \geq 1$. Let $i \in\{0, \ldots, w(C)-1\}$. By $(26),\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $((\Gamma \bullet \mathbb{H})[V(C)])-\varphi_{C}(2 i+1)$. Let $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\} \neq \varnothing$ and $e \backslash V(C) \neq \varnothing$. By the definition of $\Gamma \bullet \mathbb{H}$, $\left|e \cap\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}\right|=1,|e \backslash V(C)|=2$, and $(e \backslash V(C)) \cup$ $\left\{\varphi_{C}(2 i)\right\},(e \backslash V(C)) \cup\left\{\varphi_{C}(2 i+2)\right\} \in E(\Gamma \bullet \mathbb{H})$. Thus, $\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i+1)$. Lastly, suppose that $w(C) \geq 2$, and consider $i \in\{1, \ldots, w(C)-1\}$. By (26), $\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\}$ is a module of $((\Gamma \bullet$ $\mathbb{H})[V(C)])-\varphi_{C}(2 i)$. For every $e \in E(\Gamma \bullet \mathbb{H})$, if $e \cap\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\} \neq \varnothing$, then $e \subseteq V(C)$. Therefore, $\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-$ $\varphi_{C}(2 i)$.

Lemma 47. Given $C \in \mathfrak{C}_{\text {even }}(\Gamma)$, the following assertions hold

1. $V(\Gamma \bullet \mathbb{H}) \backslash\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(0)$;
2. $V(\Gamma \bullet \mathbb{H}) \backslash\left\{\varphi_{C}(2 w(C)-2), \varphi_{C}(2 w(C)-1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-$ $\varphi_{C}(2 w(C)-1)$;
3. if $w(C) \geq 2$, then for $m \in\{1, \ldots, 2 w(C)-2\}$, $\left\{\varphi_{C}(m-1), \varphi_{C}(m+1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(m)$.

Proof. For the first assertion, consider $e \in E(\Gamma \bullet \mathbb{H})$ such that $\varphi_{C}(1) \in e$. By definition of $\Gamma \bullet \mathbb{H}$, there exists $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ such that $e=\varphi_{C}(0) \varphi_{C}(1) \varphi_{D}(2 k)$, where $0 \leq k \leq w(D)$. Consequently, there does not exist $e \in E\left((\Gamma \bullet \mathbb{H})-\varphi_{C}(0)\right)$ such that $\varphi_{C}(1) \in e$. It follows that $V(\Gamma \bullet \mathbb{H}) \backslash\left\{\varphi_{C}(0), \varphi_{C}(1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(0)$.

As in the proof of the second assertion of Lemma 46, the second assertion is deduced from the first one by using Remark 45.

For the third assertion, suppose that $w(C) \geq 2$. Consider $i \in\{1, \ldots, w(C)-$ $1\}$. We prove that $\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i)$. Let $e \in E\left((\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i)\right)$ such that $e \cap\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\} \neq \varnothing$ and $e \backslash\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\} \neq \varnothing$. By definition of $\Gamma \bullet \mathbb{H}$, there exist $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ and $k \in\{0, \ldots, w(D)\}$ such that

$$
\left\{\begin{array}{l}
e=\varphi_{C}(2 j) \varphi_{C}(2 i-1) \varphi_{D}(2 k), \text { where } j \in\{0, \ldots, i-1\} \\
\text { or } \\
e=\varphi_{C}(2 j) \varphi_{C}(2 i+1) \varphi_{D}(2 k), \text { where } j \in\{0, \ldots, i\}
\end{array}\right.
$$

In the second instance, we have $j \in\{0, \ldots, i-1\}$ because $\varphi_{C}(2 i) \notin e$. By the definition of $\Gamma \bullet \mathbb{H}$, we have $\varphi_{C}(2 j) \varphi_{C}(2 i-1) \varphi_{D}(2 k), \varphi_{C}(2 j) \varphi_{C}(2 i+1) \varphi_{D}(2 k) \epsilon$
$E\left((\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i)\right)$. Thus, $\left\{\varphi_{C}(2 i-1), \varphi_{C}(2 i+1)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i)$. Lastly, consider $i \in\{0, \ldots, w(C)-2\}$. We prove that $\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i+1)$. Let $e \in E\left((\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i+1)\right)$ such that $e \cap\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\} \neq \varnothing$ and $e \backslash\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\} \neq \varnothing$. By the definition of $\Gamma \bullet \mathbb{H}$, there exist $D \in \mathfrak{C}_{\text {odd }}(\Gamma)$ and $k \in\{0, \ldots, w(D)\}$ such that

$$
\left\{\begin{array}{l}
e=\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k), \text { where } j \in\{i, \ldots, w(C)-1\} \\
\text { or } \\
e=\varphi_{C}(2 i+2) \varphi_{C}(2 j+1) \varphi_{D}(2 k), \text { where } j \in\{i+1, \ldots, w(C)-1\} .
\end{array}\right.
$$

In the first instance, we have $j \in\{i+1, \ldots, w(C)-1\}$ because $\varphi_{C}(2 i+1) \notin e$. By the definition of $\Gamma \bullet \mathbb{H}$, we have $\varphi_{C}(2 i) \varphi_{C}(2 j+1) \varphi_{D}(2 k), \varphi_{C}(2 i+2) \varphi_{C}(2 j+$ 1) $\varphi_{D}(2 k) \in E\left((\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i+1)\right)$. Thus, $\left\{\varphi_{C}(2 i), \varphi_{C}(2 i+2)\right\}$ is a module of $(\Gamma \bullet \mathbb{H})-\varphi_{C}(2 i+1)$.

Proof of Theorem 17. To begin, suppose that $\Gamma \bullet \mathbb{H}$ is critical. In particular, $\Gamma \bullet \mathbb{H}$ is prime. By Proposition 18, Assertions (C1) and (C2) hold. For Assertion (C3), consider $v \in V_{1}(\Gamma)$. We have $\{v\} \in \mathfrak{C}_{1}(\Gamma)$. Suppose that $\mathbb{H}-\{v\}$ is connected. Since $\Gamma \bullet \mathbb{H}$ is critical, $(\Gamma \bullet \mathbb{H})-v$ is decomposable. We have

$$
(\Gamma \bullet \mathbb{H})-v=(\Gamma-v) \bullet(\mathbb{H}-\{v\}) .
$$

Since $\mathbb{H}-\{v\}$ is connected, it follows from Proposition 18 applied to $(\Gamma \bullet \mathbb{H})-v$ that there exists a nontrivial module $\mathbb{M}_{\{v\}}$ of $\mathbb{H}-\{v\}$ such that $\mathbb{M}_{\{v\}} \subseteq \mathfrak{C}_{1}(\Gamma-v)$. Since $\mathfrak{C}_{1}(\Gamma-v)=\mathfrak{C}_{1}(\Gamma) \backslash\{\{v\}\}$, we have $\mathbb{M}_{\{v\}} \subseteq \mathfrak{C}_{1}(\Gamma) \backslash\{\{v\}\}$.

Conversely, suppose that Assertions (C1), (C2), and (C3) hold. Since Assertions ( C 1 ) and ( C 2 ) hold, it follows from Proposition 18 that $\Gamma \bullet \mathbb{H}$ is prime. Furthermore, it follows from Assertion (C3) and Proposition 18 that $(\Gamma \bullet \mathbb{H})-v$ is decomposable for each $v \in V_{1}(\Gamma)$. Lastly, consider $v \in V(\Gamma) \backslash V_{1}(\Gamma)$. There exists $C \in \mathfrak{C}(\Gamma) \backslash \mathfrak{C}_{1}(\Gamma)$ such that $v \in V(C)$. It follows from Lemmas 46 and 47 that $(\Gamma \bullet \mathbb{H})-v$ is decomposable. Consequently, $\Gamma \bullet \mathbb{H}$ is critical.

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