Critical 3-Hypergraphs

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Abstract

Given a 3-hypergraph H, a subset M of $V(H)$ is a module of H if for each $e \in E(H)$ such that $e \cap M \neq \emptyset$ and $e \setminus M \neq \emptyset$, there exists $m \in M$ such that $e \cap M = \{m\}$ and for every $n \in M$, we have $(e \setminus \{m\}) \cup \{n\} \in E(H)$. For example, $\emptyset$, $V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of H, called trivial modules. A 3-hypergraph with at least three vertices is prime if all its modules are trivial. Furthermore, a prime 3-hypergraph is critical if all its induced subhypergraphs, obtained by removing one vertex, are not prime. We characterize the critical 3-hypergraphs.

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1 Introduction

Let H be a 3-hypergraph. A tournament T, with the same vertex set as H, is a realization of H if the edges of H are exactly the 3-element subsets of the vertex set of T that induce 3-cycles. In [3], we characterized the 3-hypergraphs that admit realizations (see [5, Problem 1]). To obtain our characterization, we introduced a new notion of a module for hypergraphs. By using the modular decomposition tree, we demonstrated that a 3-hypergraph is realizable if and only if all its prime (in terms of modular decomposition) induced subhypergraphs are realizable (see [3, Theorem 13]). Moreover, given a realizable 3-hypergraph H, we proved that H is prime if and only if all its realizations are prime (see [3, Theorem 12]). These results led us to study the prime and induced subhypergraphs of a prime 3-hypergraph. Precisely, consider a prime 3-hypergraph H.

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In [4, Corollary 18], we proved that H admits a prime induced subhypergraph obtained by removing 1 or 2 vertices. Similar results were obtained for prime digraphs [10], for prime binary relational structures [11], or for prime 2-structures [7]. Our purpose is to characterize the critical 3-hypergraphs, that is, the prime 3-hypergraphs all the subhypergraphs of which, obtained by removing one vertex, are not prime. In particular, our characterization yields a simple manner to construct critical 3-hypergraphs that are not realizable.

At present, we formalize our presentation. We consider only finite structures. A hypergraph H is defined by a vertex set $V(H)$ and an edge set $E(H)$, where $E(H) \subseteq 2^{V(H)} \setminus \{\varnothing\}$. Given a hypergraph H, $v(H)$ denotes the cardinality of $V(H)$. Given $k \geq 2$, a hypergraph H is a *k-hypergraph* if

$$E(H) \subseteq \binom{V(H)}{k}.$$

Furthermore, a hypergraph H is a *$\{k, k+1\}$-hypergraph* if

$$E(H) \subseteq \binom{V(H)}{k} \cup \binom{V(H)}{k+1}.$$

Let H be a hypergraph. With each $W \subseteq V(H)$, we associate the *subhypergraph* $H[W]$ of H induced by W, which is defined on $V(H[W]) = W$ by $E(H[W]) = \{e \in E(H) : e \subseteq W\}$. Given $W \subseteq V(H)$, $H[V(H) \setminus W]$ is also denoted by $H - W$, and by $H - w$ when $W = \{w\}$.

Definition 1. Let H be a hypergraph. A subset M of $V(H)$ is a *module* of H if for each $e \in E(H)$ such that $e \cap M \neq \varnothing$ and $e \setminus M \neq \varnothing$, there exists $m \in M$ such that $e \cap M = \{m\}$, and for every $n \in M$, we have

$$(e \setminus \{m\}) \cup \{n\} \in E(H).$$

Given distinct $v, w \in V(H)$, we say that v and w are *twins* [2, 6] of H if $\{v, w\}$ is a module of H.

Definition 2. Let H be a hypergraph. Clearly, $\varnothing$, $V(H)$ and $\{v\}$, where $v \in V(H)$, are modules of H, called *trivial modules*. A hypergraph H is *indecomposable* if all its modules are trivial, otherwise it is *decomposable*. A hypergraph H is *prime* if it is indecomposable, with $v(H) \geq 3$.

Remark 3. Recall that a hypergraph H is *connected* if for distinct $v, w \in V(H)$, there exists a sequence $(e_0, \ldots, e_n)$ of edges of H, where $n \geq 0$, satisfying $v \in e_0$, $w \in e_n$, and (when $n \geq 1$) $e_i \cap e_{i+1} \neq \varnothing$ for every $0 \leq i \leq n-1$. Given a hypergraph H, a maximal connected subhypergraph of H is called a *component* of H.

Let H be a hypergraph. For each component C of H, $V(C)$ and $V(H) \setminus V(C)$ are clearly modules of H. Therefore, a disconnected hypergraph with at least three vertices is decomposable.

In [4], we proved the following result (see [4, Theorem 17]). We need the following notation (see [4, Notation 13]).

Notation 4. Let H be a 3-hypergraph. Given $X \subsetneq V(H)$ such that $H[X]$ is prime, we consider the following subsets of $V(H) \setminus X$.

- We denote by $\mathrm{Ext}_{(H,X)}$ the set of $v \in V(H) \setminus X$ such that $H[X \cup \{v\}]$ is prime. (Hence, given $v \in V(H) \setminus X$, $v \in \mathrm{Ext}_{(H,X)}$ if and only if $H[X \cup \{v\}]$ is a prime extension of $H[X]$.)
- We denote by $\mathrm{Dis}_{(H,X)}$ the set of $v \in V(H) \setminus X$ such that X is a module of $H[X \cup \{v\}]$. (Let $v \in \mathrm{Dis}_{(H,X)}$. Since X is a module of $H[X \cup \{v\}]$, we obtain $v \notin e$ for every $e \in E(H[X \cup \{v\}])$. Thus, $H[X \cup \{v\}]$ is disconnected (see Remark 3).)
- Let $y \in X$. We denote by $\mathrm{Twi}_{(H,X)}(y)$ the set of $v \in V(H) \setminus X$ such that $\{y, v\}$ is a module of $H[X \cup \{v\}]$. (Let $y \in X$. Given $v \in V(H) \setminus X$, $v \in \mathrm{Twi}_{(H,X)}(y)$ if and only if v and y are twins of $H[X \cup \{v\}]$.)

The set $\{\mathrm{Ext}_{(H,X)}, \mathrm{Dis}_{(H,X)}\} \cup \{\mathrm{Twi}_{(H,X)}(y) : y \in X\}$ is denoted by $p_{(H,X)}$.

Theorem 5. *Let H be a 3-hypergraph. Consider $X \subsetneq V(H)$ such that $H[X]$ is prime. Set*

$$\underline{X} = \{y \in X : \mathrm{Twi}_{(H,X)}(y) \neq \varnothing\}.$$

If H is prime, then there exist $v, w \in (V(H) \setminus X) \cup \underline{X}$ such that $H - \{v, w\}$ is prime.

The next result follows from Theorem 5 (see [4, Corollary 18])

Corollary 6. *Let H be a prime 3-hypergraph. If $v(H) \geq 4$, then there exist $v, w \in V(H)$ such that $H - \{v, w\}$ is prime.*

Definition 7. A prime hypergraph H is *critical* if $H - v$ is not prime for each $v \in V(H)$.

Our purpose is to characterize the critical 3-hypergraphs. An analogue of Corollary 6 for binary relations led Ille [9] to introduce the following auxiliary graph.

Definition 8. Let H be a prime 3-hypergraph with $v(H) \geq 5$. The *primality graph* $\mathscr{P}(H)$ associated with H is the graph defined on $V(H)$ as follows. Given distinct $v, w \in V(H)$, $vw \in E(\mathscr{P}(H))$ if $H - \{v, w\}$ is prime. When H is critical, it follows from Corollary 6 that $\mathscr{P}(H)$ is nonempty.

1.1 Critical and realizable 3-hypergraphs

Let T be a tournament. A subset M of $V(T)$ is a *module* [12] of T provided that for any $x, y \in M$ and $v \in V(T)$, if $xv, vy \in A(T)$, then $v \in M$. Note that the notions of a module and of a convex subset [8] coincide for tournaments. Moreover, note that the notions of a module and of an interval coincide for linear orders. Given a tournament T, $\varnothing$, $V(T)$ and $\{v\}$, where $v \in V(T)$, are modules of T, called *trivial modules*. A tournament is *indecomposable* if all its

modules are trivial, otherwise it is *decomposable*. A tournament T is *prime* if it is indecomposable, with $v(T) \geq 3$. Lastly, a prime tournament T is *critical* if $T - v$ is not prime for each $v \in V(T)$. Schmerl and Trotter [11] characterized the critical tournaments. They obtained the tournaments T_{2n+1}, U_{2n+1}, and W_{2n+1} defined on $\{0, \ldots, 2n\}$, where $n \geq 1$, as follows. We denote by L_{2n+1} the usual linear order on $\{0, \ldots, 2n\}$.

- The tournament T_{2n+1} is obtained from L_{2n+1} by reversing all the arcs between even and odd vertices.
- The tournament U_{2n+1} is obtained from L_{2n+1} by reversing all the arcs between even vertices.
- The tournament W_{2n+1} is obtained from L_{2n+1} by reversing all the arcs between $2n$ and the even elements of $\{0, \ldots, 2n-1\}$.

Theorem 9 (Schmerl and Trotter [11])**.** *Given a tournament τ, with $v(\tau) \geq 5$, τ is critical if and only if $v(\tau)$ is odd and τ is isomorphic to $T_{v(\tau)}$, $U_{v(\tau)}$, or $W_{v(\tau)}$.*

A realization of a 3-hypergraph is defined as follows (see [3, Definition 10]). To begin, we associate with each tournament a 3-hypergraph in the following way (see [3, Definition 9]).

Definition 10. The 3-*cycle* is the tournament $C_3 = (\{0,1,2\}, \{01, 12, 20\})$. Given a tournament T, the C_3-*structure* of T is the 3-hypergraph $C_3(T)$ defined on $V(C_3(T)) = V(T)$ by

$$E(C_3(T)) = \{X \subseteq V(T) : T[X] \text{ is isomorphic to } C_3\}.$$

Definition 11. Given a 3-hypergraph H, a tournament T, with $V(T) = V(H)$, *realizes* H if $H = C_3(T)$. We say also that T is a *realization* of H.

In [3], we proved the following result (see [3, Theorem 12])

Theorem 12. *Consider a realizable 3-hypergraph H. For a realization T of H, H is prime if and only if T is prime.*

The next result follows from Theorems 9 and 12 (see [3, Theorem 49]).

Corollary 13. *Given a realizable 3-hypergraph H, with $v(H) \geq 5$, H is critical if and only if $v(H)$ is odd and H is isomorphic to $C_3(T_{v(H)})$, $C_3(U_{v(H)})$, or $C_3(W_{v(H)})$.*

Definition 14. Given a 3-hypergraph H, we say that H is *circular* if $v(H)$ is odd and H is isomorphic to $C_3(T_{v(H)})$.

1.2 A construction of non circular and critical 3-hypergraphs

The construction is provided from the set of the components of a graph. We use the following notation.

Notation 15. Consider a graph Γ. The set of the components of Γ is denoted by $\mathfrak{C}(\Gamma)$. Furthermore, set $\mathfrak{C}_{\text{even}}(\Gamma) = \{C \in \mathfrak{C}(\Gamma) : v(C) \equiv 0 \pmod 2\}$ and $\mathfrak{C}_{\text{odd}}(\Gamma) = \{C \in \mathfrak{C}(\Gamma) : v(C) \equiv 1 \pmod 2\}$. Lastly, set $\mathfrak{C}_1(\Gamma) = \{C \in \mathfrak{C}(\Gamma) : v(C) = 1\}$ and $V_1(\Gamma) = \{v \in V(\Gamma) : \{v\} \in \mathfrak{C}_1(\Gamma)\}$. For each $C \in \mathfrak{C}(\Gamma)$, set

$$w(C) = \begin{cases} \dfrac{v(C)}{2} & \text{if } C \in \mathfrak{C}_{\text{even}}(\Gamma) \\ \dfrac{v(C)-1}{2} & \text{if } C \in \mathfrak{C}_{\text{odd}}(\Gamma). \end{cases}$$

We consider a graph Γ,

$$\text{all the components of which are paths.} \tag{1}$$

For each $n \geq 1$, recall that the *path* $\mathcal{P}_n$ is defined on $V(\mathcal{P}_n) = \{0, \ldots, n-1\}$ by

$$E(\mathcal{P}_n) = \begin{cases} \varnothing & \text{if } n = 1 \\ \{k(k+1) : 0 \leq k \leq n-2\} & \text{if } n \geq 2. \end{cases}$$

We suppose that

$$\begin{aligned} &\star\ \mathfrak{C}(\Gamma) \setminus \mathfrak{C}_1(\Gamma) \neq \varnothing, \\ &\star\ \text{for each } C \in \mathfrak{C}_{\text{odd}}(\Gamma), \text{ if } V(H) \setminus V(C) \neq \varnothing, \text{ then} \\ &\qquad (\mathfrak{C}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)) \setminus \{C\} \neq \varnothing, \\ &\star\ \text{for each } C \in \mathfrak{C}_{\text{even}}(\Gamma), \text{ if } |V(H) \setminus V(C)| \geq 2, \text{ then} \\ &\qquad (\mathfrak{C}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)) \setminus \{C\} \neq \varnothing. \end{aligned} \tag{2}$$

Consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathfrak{C}(\Gamma)$ and satisfying

$$\begin{aligned} &\blacksquare\ \text{for each } \varepsilon \in E(\mathbb{H}), \text{ if } |\varepsilon| = 2, \text{ then } \varepsilon \cap \mathfrak{C}_{\text{even}}(\Gamma) \neq \varnothing \text{ and } \varepsilon \cap \mathfrak{C}_{\text{odd}}(\Gamma) \neq \varnothing, \\ &\blacksquare\ \text{for each } \varepsilon \in E(\mathbb{H}), \text{ if } |\varepsilon| = 3, \text{ then } \varepsilon \subseteq \mathfrak{C}_{\text{odd}}(\Gamma). \end{aligned} \tag{3}$$

With Γ and $\mathbb{H}$, we associate the 3-hypergraph $\Gamma \bullet \mathbb{H}$ defined on $V(\Gamma)$ in the following manner. For each $C \in \mathfrak{C}(\Gamma)$, we consider an isomorphism φ_C from the path $\mathcal{P}_{v(C)}$ onto C. With each $C \in \mathfrak{C}_{\text{odd}}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)$, we associate the set

$$E_C = \varphi_C(E(C_3(U_{v(C)}))).$$

Moreover, consider $\varepsilon \in E(\mathbb{H})$ such that $|\varepsilon| = 2$. There exist $C \in \mathfrak{C}_{\text{even}}(\Gamma)$ and $D \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = CD$. Associate with ε the set

$$E_\varepsilon = \{\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) : \ 0 \leq i \leq j \leq w(C) - 1, \ 0 \leq k \leq w(D)\}.$$

Lastly, consider $\varepsilon \in E(\mathbb{H})$ such that $|\varepsilon| = 3$. There exist $I, J, K \in \mathfrak{C}_{\mathrm{odd}}(\Gamma)$ such that $\varepsilon = IJK$. Associate with ε the set

$$E_\varepsilon = \{\varphi_I(2i)\varphi_J(2j)\varphi_K(2k) : 0 \le i \le w(I),\ 0 \le j \le w(J),\ 0 \le k \le w(K)\}.$$

The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is defined on $V(\Gamma)$ by

$$E(\Gamma \bullet \mathbb{H}) = \Big(\bigcup_{C \in \mathfrak{C}_{\mathrm{odd}}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)} E_C\Big) \cup \Big(\bigcup_{\varepsilon \in E(\mathbb{H})} E_\varepsilon\Big). \tag{4}$$

1.3 The main results

To begin, we prove the following theorem.

Theorem 16. *Let H be a critical 3-hypergraph such that $v(H) \ge 5$. If H is not circular, then $\mathscr{P}(H)$ (see Definition 8) satisfies* (1) *and* (2)*, and there exists a $\{2,3\}$-hypergraph $\widetilde{H}$ defined on $\mathfrak{C}(\mathscr{P}(H))$ satisfying* (3) *such that*

$$H = \mathscr{P}(H) \bullet \widetilde{H}.$$

To state the second main theorem, we consider a graph Γ satisfying (1) and (2). We consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathfrak{C}(\Gamma)$ satisfying (3).

Theorem 17. *Suppose that $v(\Gamma \bullet \mathbb{H}) \ge 5$. The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is critical if and only if the following three assertions hold*

(C1) $\mathbb{H}$ is connected (see Remark 3);

(C2) for every nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \setminus \mathfrak{C}_1(\Gamma) \ne \varnothing$;

(C3) for each $v \in V_1(\Gamma)$, if $\mathbb{H} - \{v\}$ is connected, then $\mathbb{H} - \{v\}$ admits a nontrivial module $\mathbb{M}_{\{v\}}$ such that $\mathbb{M}_{\{v\}} \subseteq \mathfrak{C}_1(\Gamma) \setminus \{\{v\}\}$.

Before proving Theorem 17, we establish the next proposition.

Proposition 18. *Suppose that $v(\Gamma \bullet \mathbb{H}) \ge 3$. The 3-hypergraph $\Gamma \bullet \mathbb{H}$ is prime if and only if the following two assertions hold*

(C1) $\mathbb{H}$ is connected;

(C2) for every nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \setminus \mathfrak{C}_1(\Gamma) \ne \varnothing$.

The next result yields an easy manner to construct critical 3-hypergraphs. It is a simple consequence of Corollary 13 and Theorem 17.

Corollary 19. *Suppose that $v(\Gamma \bullet \mathbb{H}) \ge 5$. Suppose also that $\mathfrak{C}_1(\Gamma) = \varnothing$. If $\mathbb{H}$ is connected, then $\Gamma \bullet \mathbb{H}$ is critical. Moreover, if $\mathbb{H}$ is connected and $|\mathfrak{C}(\Gamma)| \ge 2$, then $\Gamma \bullet \mathbb{H}$ is critical and non realizable.*

2 The critical tournaments

The primality graph associated with a prime tournament is defined in the same manner as that associated with a prime 3-hypergraph (see Definition 8).

Definition 20. Let T be a prime tournament. The *primality graph* $\mathscr{P}(T)$ of T is defined on $V(T)$ as follows. Given distinct $v, w \in V(T)$, $vw \in E(\mathscr{P}(T))$ if $T - \{v, w\}$ is prime.

The basic properties of the primality graph follow. The next lemma is stated in [9] without a proof. For a proof, see [1, Lemma 10].

Lemma 21 (Ille [9])**.** *Let T be a prime tournament with $v(T) \geq 5$. Consider $v \in V(T)$ such that $T - v$ is decomposable. We have $d_{\mathscr{P}(T)}(v) \leq 2$. Moreover, the next two assertions hold.*

1. *If $d_{\mathscr{P}(T)}(v) = 1$, then $V(T) \setminus (\{v\} \cup N_{\mathscr{P}(T)}(v))$ is a module of $T - v$.*
2. *If $d_{\mathscr{P}(T)}(v) = 2$, then $N_{\mathscr{P}(T)}(v)$ is a module of $T - v$.*

For each $n \geq 3$, recall that the *cycle* $\mathcal{C}_n$ is defined on $V(\mathcal{C}_n) = \{0, \ldots, n-1\}$ by $E(\mathcal{C}_n) = E(\mathcal{P}_n) \cup \{0(n-1)\}$. The *length* of $\mathcal{C}_n$ is n. Given a critical tournament T, it follows from Lemma 21 that the connected components of $\mathscr{P}(T)$ are paths or cycles. Boudabbous and Ille [1] characterized the critical tournaments from their primality graphs. To begin, they examined the connected components of the primality graph associated with a critical tournament.

Lemma 22 (Corollary 17 [1])**.** *If T is a critical tournament, with $v(T) \geq 5$, then $\mathscr{P}(T)$ satisfies one of the following*

1. *$\mathscr{P}(T)$ is a cycle of odd length;*
2. *$\mathscr{P}(T)$ is a path;*
3. *$v(T)$ is odd and there is $v \in V(T)$ such that $\mathscr{P}(T) - v$ is a path and $N_{\mathscr{P}(T)}(v) = \varnothing$.*

For each of the three shapes described in Lemma 22, Boudabbous and Ille [1] characterized the corresponding critical tournaments.

Proposition 23 (Proposition 18 [1])**.** *Given a tournament such that $v(T) \geq 5$, T is critical and $\mathscr{P}(T) = \mathcal{C}_{v(T)}$ if and only if $v(T)$ is odd and $T = T_{v(T)}$ or $(T_{v(T)})^\star$.*

Proposition 24 (Proposition 19 [1])**.** *Given a tournament such that $v(T) \geq 5$, T is critical and $\mathscr{P}(T) = \mathcal{P}_{v(T)}$ if and only if $v(T)$ is odd and $T = U_{v(T)}$ or $(U_{v(T)})^\star$.*

Proposition 25 (Proposition 21 [1])**.** *Given a tournament defined on $\{0, \ldots, 2n\}$ with $n \geq 2$, T is critical, $\mathscr{P}(T) - 2n = \mathcal{P}_{2n-1}$, and $N_{\mathscr{P}(T)}(2n) = \varnothing$ if and only if $T = W_{2n+1}$ or $(W_{2n+1})^\star$.*

3 Proof of Theorem 16

The purpose of this section is to characterize the non circular and critical 3-hypergraphs. We use the following notation.

Notation 26. Let H be a 3-hypergraph.

$$\text{For } e, f \in \binom{V(H)}{3},\ e \equiv_H f \text{ means } \begin{cases} e, f \in E(H) \\ \text{or} \\ e, f \notin E(H). \end{cases}$$

An analogue of Lemma 21 for prime 3-hypergraphs follows. For a proof, see [1, Lemma 10].

Lemma 27. *Let H be a prime 3-hypergraph with $v(H) \geq 5$. Consider $v \in V(H)$ such that $H - v$ is decomposable. We have $d_{\mathscr{P}(H)}(v) \leq 2$. Moreover, the next two assertions hold.*

1. *If $d_{\mathscr{P}(H)}(v) = 1$, then $V(H) \setminus (\{v\} \cup N_{\mathscr{P}(H)}(v))$ is a module of $H - v$.*
2. *If $d_{\mathscr{P}(H)}(v) = 2$, then $N_{\mathscr{P}(H)}(v)$ is a module of $H - v$.*

The next result is an immediate consequence of Lemma 27.

Corollary 28. *Given a critical 3-hypergraph H such that $v(H) \geq 5$, all the components of $\mathscr{P}(H)$ are cycles or paths.*

Proposition 29. *Let H be a prime 3-hypergraph defined on $\{0, \ldots, p-1\}$, where $p \geq 5$. Suppose that there exists $k \in \{3, \ldots, p\}$ such that*

$$\mathscr{P}(H)[\{0, \ldots, k-1\}] = C_k.$$

If $H - i$ is decomposable for each $i \in \{0, \ldots, k-1\}$, then k is odd, $k = p$, and $H = C_3(T_p)$.

Proof. Suppose that there exists $k \in \{3, \ldots, p\}$ such that $\mathscr{P}(H)[\{0, \ldots, k-1\}] = \mathcal{C}_k$. Furthermore, suppose that $H - i$ is decomposable for each $i \in \{0, \ldots, k-1\}$. First, we show that

$$k \text{ is odd.} \tag{5}$$

Otherwise, there exists $l \geq 2$ such that $\mathscr{P}(H)[\{0, \ldots, 2l-1\}] = \mathcal{C}_{2l}$. We verify that $\{0, 2\}$ is a module of H. Consider $e \in E(H)$ such that $e \cap \{0, 2\} \neq \varnothing$ and $e \setminus \{0, 2\} \neq \varnothing$. Suppose for a contradiction that $0, 2 \in e$. There exists $i \in \{1\} \cup \{3, \ldots, 2l-1\}$ such that $e = 02i$. Since $N_{\mathscr{P}(H)}(1) = \{0, 2\}$, it follows from Lemma 27 that $\{0, 2\}$ is a module of $H - 1$. It follows that $i = 1$, that is, $e = 012$. It follows from Lemma 27 that $01(2l-2) \in E(H)$, which contradicts the fact that $\{0, 2l-2\}$ is a module of $H - (2l-1)$. Consequently, $|e \cap \{0, 2\}| = 1$. For a contradiction, suppose that $1 \in e$. Since $|e \cap \{0, 2\}| = 1$, there exists $j \in \{3, \ldots, 2l-1\}$ such that $e = 01j$ or $12j$. Denote by j' the unique element of $\{2l-2, 2l-1\}$ such that $j' \equiv j \pmod 2$. It follows from Lemma 27 that $(e \setminus \{j\}) \cup \{j'\} \in E(H)$. Similarly, by denoting by j'' the unique element of $\{3, 4\}$ such that $j'' \equiv j \pmod 2$, we obtain $(e \setminus \{j\}) \cup \{j''\} \in E(H)$. We distinguish the following two cases.

1. Suppose that $e = 01j$. If j is even, then $01(2l-2) \in E(H)$, which contradicts the fact that $\{0, 2l-2\}$ is a module of $H-(2l-1)$. If j is odd, then $013 \in E(H)$, which contradicts the fact that $\{1,3\}$ is a module of $H-2$.

2. Suppose that $e = 12j$. If j is even, then $124 \in E(H)$, which contradicts the fact that $\{2,4\}$ is a module of $H-3$. If j is odd, then $12(2l-1) \in E(H)$, which contradicts the fact that $\{1, 2l-1\}$ is a module of $H-0$.

It follows that $1 \notin e$. So, $e \in E(H-1)$. Since $N_{\mathscr{P}(H)}(1) = \{0,2\}$, it follows from Lemma 27 that $\{0,2\}$ is a module of $H-1$. Thus, there exists $i \in \{0,2\}$ such that $e \cap \{0,2\} = \{i\}$, and $(e \setminus \{i\}) \cup \{i'\} \in E(H)$ for each $i' \in \{0,2\}$. Consequently, $\{0,2\}$ is a module of H, which contradicts the fact that H is prime. It follows that (5) holds. Set $k = 2n+1$, where $n \geq 1$.

Second, we prove that $\{0, \ldots, 2n\}$ is a module of H. Consider $e \in E(H)$ such that $e \cap \{0, \ldots, 2n\} \neq \varnothing$ and $e \setminus \{0, \ldots, 2n\} \neq \varnothing$. We prove that

$$|e \cap \{0, \ldots, 2n\}| = 1. \tag{6}$$

Otherwise, we have $|e \cap \{0, \ldots, 2n\}| = 2$. There exist $i, j \in \{0, \ldots, 2n\}$, with $i < j$, such that $e \cap \{0, \ldots, 2n\} = \{i, j\}$. Denote by i' the unique element of $\{0,1\}$ such that $i' \equiv i \pmod 2$. As previously, we obtain $(e \setminus \{i\}) \cup \{i'\} \in E(H)$. Denote by j' the unique element of $\{2,3\}$ such that $j' \equiv j \pmod 2$. We obtain $(e \setminus \{i,j\}) \cup \{i', j'\} \in E(H)$. Set $e' = (e \setminus \{i,j\}) \cup \{i', j'\}$. Observe that $e' \setminus \{0, \ldots, 2n\} = e \setminus \{0, \ldots, 2n\}$, and denote by v the unique element of $e' \setminus \{0, \ldots, 2n\}$. If $i' = 0$ and $j' = 2$, then $02v \in E(H)$, which contradicts the fact that $\{0,2\}$ is a module of $H-1$. If $i' = 1$ and $j' = 3$, then $13v \in E(H)$, which contradicts the fact that $\{1,3\}$ is a module of $H-2$. Suppose that $i' = 0$ and $j' = 3$. We get $03v \in E(H)$, and hence $0(2n-1)v \in E(H)$, which contradicts the fact that $\{0, 2n-1\}$ is a module of $H-(2n)$. Lastly, if $i' = 1$ and $j' = 4$, then $1(2n)v \in E(H)$, which contradicts the fact that $\{1, 2n\}$ is a module of $H-0$. It follows that (6) holds. Denote by i the unique element of $e \cap \{0, \ldots, 2n\}$. For every $l \in \{1, \ldots, n\}$, we obtain

$$(e \setminus \{i\}) \cup \{i + 2l\} \in E(H), \tag{7}$$

where $i+2l$ is considered modulo $2n+1$. In particular, $(e \setminus \{i\}) \cup \{i+2n\}$, which is $(e \setminus \{i\}) \cup \{i-1\}$, belongs to $E(H)$. Since $\{i-1, i+1\}$ is a module of $H-i$, we get $(e \setminus \{i\}) \cup \{i+1\} \in E(H)$. For each $m \in \{0, \ldots, n-1\}$, we obtain

$$(e \setminus \{i\}) \cup \{i + 2m + 1\} \in E(H). \tag{8}$$

It follows from (7) and (8) that $(e \setminus \{i\}) \cup \{i'\} \in E(H)$ for every $i' \in \{0, \ldots, 2n\}$. Consequently, $\{0, \ldots, 2n\}$ is a module of H. Since H is prime, we obtain $V(H) = \{0, \ldots, 2n\}$.

Third, we prove that $H = C_3(T_{2n+1})$. We have

$$\begin{aligned} E(C_3(T_{2n+1})) = &\{(2i)(2l+1)(2j) : 0 \leq i \leq l < j \leq n\} \\ &\cup \{(2i+1)(2l)(2j+1) : 0 \leq i < l \leq j \leq n-1\}. \end{aligned} \tag{9}$$

For a contradiction, suppose that there are $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$, $x \equiv y \pmod 2$, and $xyz \in E(H)$. It follows from Lemma 27 that $(y-2)yz \in E(H)$, which contradicts the fact that $\{y-2, y\}$ is a module of $H - (y-1)$. Hence, $xyz \notin E(H)$ for $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$ and $x \equiv y \pmod 2$. Similarly, $xyz \notin E(H)$ for $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$ and $y \equiv z \pmod 2$. It follows that

$$E(H) \subseteq E(C_3(T)). \tag{10}$$

Now, consider $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$, $x \not\equiv y \pmod 2$, and $y \not\equiv z \pmod 2$. It follows from Lemma 27 that

$$xyz \equiv_H x(x+1)(x+2) \quad \text{(see Notation 26).} \tag{11}$$

Now, we prove that the permutation

$$\begin{array}{rccl} \theta: & \{0, \ldots, 2n\} & \longrightarrow & \{0, \ldots, 2n\} \\ & x & \longmapsto & x+1 \pmod{2n+1} \end{array}$$

of $\{0, \ldots, 2n\}$ is an automorphism of H. Given $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$, we have to verify that $xyz \equiv_H (x+1)(y+1)(z+1)$. We have

$$xyz \equiv_H xy(z+2) \equiv_H x(y+2)(z+2) \equiv_H (x+2)(y+2)(z+2).$$

Thus, $xyz \equiv_H (x+2n+2)(y+2n+2)(z+2n+2)$, that is

$$xyz \equiv_H (x+1)(y+1)(z+1).$$

Consequently, θ is an automorphism of H. Since H is prime, there exist $u, v, w \in \{0, \ldots, 2n\}$ such that $u < v < w$ and $uvw \in E(H)$. By (10), $uvw \in E(C_3(T))$. It follows from (9) that $u \not\equiv v \pmod 2$ and $v \not\equiv w \pmod 2$. It follows from (11) that $u(u+1)(u+2) \in E(H)$. Since θ is an automorphism of H, we obtain

$$012 \in E(H). \tag{12}$$

Finally, consider $x, y, z \in \{0, \ldots, 2n\}$ such that $x < y < z$ and $xyz \in E(C_3(T))$. By (9), $x \not\equiv y \pmod 2$ and $y \not\equiv z \pmod 2$. It follows from (11) that $xyz \equiv_H x(x+1)(x+2)$. Since θ is an automorphism of H, we obtain $x(x+1)(x+2) \equiv_H 012$. Therefore, $xyz \equiv_H 012$. By (12), $xyz \in E(H)$. Consequently, we obtain $E(C_3(T)) \subseteq E(H)$. It follows from (10) that $H = C_3(T)$. □

The next corollary is an easy consequence of Corollary 28 and Proposition 29.

Corollary 30. *Given a critical 3-hypergraph H, with $v(H) \geq 5$, H is not circular if and only if all the components of $\mathscr{P}(H)$ are paths.*

Proof. Suppose that H is circular. By Definition 14, $v(H)$ is odd and

$$H \simeq C_3(T_{v(H)}).$$

Hence, we obtain $\mathscr{P}(H) \simeq \mathscr{P}(C_3(T_{v(H)}))$. It follows from Theorem 12 that

$$\mathscr{P}(C_3(T_{v(H)})) = \mathscr{P}(T_{v(H)}).$$

By Proposition 23, $\mathscr{P}(T_{v(H)}) = \mathcal{C}_{v(H)}$. It follows that $\mathscr{P}(H)$ contains a cycle among its components.

Conversely, suppose that $\mathscr{P}(H)$ admits a component C which is not a path. By Corollary 28, C is a cycle. Up to isomorphy, we can assume that $V(H) = \{0, \ldots, p-1\}$, where $p \geq 5$, and there exists $k \in \{3, \ldots, p\}$ satisfying $\mathscr{P}(H)[\{0, \ldots, k-1\}] = \mathcal{C}_k$. By Proposition 29, k is odd, $k = p$, and $H = C_3(T_p)$. Therefore, H is circular. □

Notation 31. Let H be a non circular and critical 3-hypergraph. For each $C \in \mathfrak{C}(\mathscr{P}(H))$, C is a path. Thus, there exists an isomorphism denoted by φ_C from $\mathcal{P}_{v(C)}$ onto C.

Lemma 32. *Let H be a critical 3-hypergraph with $v(H) \geq 5$. Suppose that H is not circular.*

1. *Let $C \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$. For every $e \in E(H)$, if $e \cap V(C) \neq \varnothing$ and $e \setminus V(C) \neq \varnothing$, then there exists $i \in \{0, \ldots, w(C)\}$ (see Notation 15) such that $e \cap V(C) = \{\varphi_C(2i)\}$.*
2. *Let $C \in \mathfrak{C}_{\text{even}}(\mathscr{P}(H))$. For every $e \in E(H)$, if $e \cap V(C) \neq \varnothing$ and $e \setminus V(C) \neq \varnothing$, then there exist $i, j \in \{0, \ldots, w(C)-1\}$ such that $i \leq j$ and $e \cap V(C) = \{\varphi_C(2i), \varphi_C(2j+1)\}$.*

Proof. For the first assertion, consider $C \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$. Let $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \setminus V(C) \neq \varnothing$. We prove that

$$e \cap \{\varphi_C(2i+1) : i \in \{0, \ldots, w(C)-1\}\} = \varnothing. \tag{13}$$

Otherwise, there exists $i \in \{0, \ldots, w(C)-1\}$ such that $\varphi_C(2i+1) \in e$. We distinguish the following two cases.

- Suppose that there exists $j \in \{0, \ldots, i\}$ such that $\varphi_C(2j) \in e$. Since $e \setminus V(C) \neq \varnothing$, $e \cap V(C) = \{\varphi_C(2j), \varphi_C(2i+1)\}$. By denoting by v the unique element of $e \setminus V(C)$, we obtain $e = \varphi_C(2j)\varphi_C(2i+1)v$. It follows from Lemma 27 that $\varphi_C(2j)\varphi_C(2w(C)-1)v \in E(H)$, which contradicts the fact that $V(H) \setminus \{\varphi_C(2w(C)-1), \varphi_C(2w(C))\}$ is a module of $H - \varphi_C(2w(C))$.
- Suppose that $e \cap \{\varphi_C(2j) : j \in \{0, \ldots, i\}\} = \varnothing$. It follows from Lemma 27 that there exists $f \in E(H)$ such that $\varphi_C(1) \in f$ and $\varphi_C(0) \notin f$, which contradicts the fact that $V(H) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $H - \varphi_C(0)$.

Consequently, (13) holds.

Now, we prove that there exists $i \in \{0, \ldots, w(C)\}$ such that

$$e \cap V(C) = \{\varphi_C(2i)\}. \tag{14}$$

Otherwise, it follows from (13) that there exist distinct $i, j \in \{0, \ldots, w(C)\}$ such that $0 \leq i < j \leq w(C)$ such that $e \cap V(C) = \{\varphi_C(2i), \varphi_C(2j)\}$. It follows from Lemma 27 that there exists $f \in E(H)$ such that $f \cap V(C) = \{\varphi_C(2i), \varphi_C(2i+2)\}$ and $f \smallsetminus V(C) \neq \varnothing$, which contradicts the fact that $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $H - \varphi_C(2i+1)$. Consequently, (14) holds.

For the second assertion, consider $C \in \mathfrak{C}_{\text{even}}(\mathscr{P}(H))$. Let $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \smallsetminus V(C) \neq \varnothing$. Set

$$p = \min(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\}).$$

For a contradiction, suppose that p is odd. It follows from Lemma 27 that $(e \smallsetminus \{\varphi_C(p)\}) \cup \{\varphi_C(1)\} \in E(H)$, which contradicts the fact that $V(C) \smallsetminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $H - \varphi_C(0)$. It follows that p is even. Thus, there exists $i \in \{0, \ldots, w(C) - 1\}$ such that $p = \varphi_C(2i)$. Similarly, $\max(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\})$ is odd. Thus there exists $j \in \{0, \ldots, w(C) - 1\}$ such that $\max(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\}) = \varphi_C(2j + 1)$. We obtain that $i \leq j$ and $e \cap V(C) = \{\varphi_C(2i), \varphi_C(2j + 1)\}$. □

Proposition 33. *Let H be a critical 3-hypergraph with $v(H) \geq 5$. Suppose that H is not circular. Consider $C \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H)) \smallsetminus \mathfrak{C}_1(\mathscr{P}(H))$.*

1. *Suppose that $V(C) \subsetneq V(H)$. There exists $e \in E(H)$, such that $e \cap V(C) \neq \varnothing$ and $e \smallsetminus V(C) \neq \varnothing$. Moreover, one of the following two assertions holds*

   - *there exist distinct $D, D' \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H)) \smallsetminus \{C\}$ such that $e \subseteq V(C) \cup V(D) \cup V(D')$, $e \cap V(D) \neq \varnothing$, and $e \cap V(D') \neq \varnothing$;*
   - *there exists $D \in \mathfrak{C}_{\text{even}}(\mathscr{P}(H))$ such that $e \subseteq V(C) \cup V(D)$ and $e \cap V(D) \neq \varnothing$.*

   Furthermore, there exists $i \in \{0, \ldots, w(C)\}$ such that $e \cap V(C) = \{\varphi_C(2i)\}$. Lastly, for each $j \in \{0, \ldots, w(C)\}$, $(e \smallsetminus \{\varphi_C(2i)\}) \cup \{\varphi_C(2j)\} \in E(H)$.

2. *The function φ_C is an isomorphism from $C_3(U_{v(C)})$ onto $H[V(C)]$.*

3. *If $V(C) \subsetneq V(H)$, then there exists $C' \in \mathfrak{C}(\mathscr{P}(H)) \smallsetminus \mathfrak{C}_1(\mathscr{P}(H))$ such that $C' \neq C$.*

Proof. For the first assertion, suppose that $V(C) \subsetneq V(H)$. Since $V(C)$ is not a module of H, there exists $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \smallsetminus V(C) \neq \varnothing$. It follows from Lemma 32 that there exists $i \in \{0, \ldots, w(C)\}$ such that $e \cap V(C) = \{\varphi_C(2i)\}$. It follows from Lemma 27 that $(e \smallsetminus \{\varphi_C(2i)\}) \cup \{\varphi_C(2j)\} \in E(H)$ for each $j \in \{0, \ldots, w(C)\}$. Moreover, since $|e \cap V(C)| = 1$, there exists $D \in \mathfrak{C}(\mathscr{P}(H)) \smallsetminus \{C\}$ such that $e \cap V(D) \neq \varnothing$. We distinguish the following two cases.

- Suppose that $D \in \mathfrak{C}_{\text{even}}(\mathscr{P}(H))$. By Lemma 32, $|e \cap V(D)| = 2$. Thus, $e \subseteq V(C) \cup V(D)$.

- Suppose that $D \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$. By Lemma 32, $|e \cap V(D)| = 1$. Therefore, there exists $D' \in \mathfrak{C}(\mathscr{P}(H)) \setminus \{C, D\}$ such that $e \cap V(D') \neq \varnothing$. Thus, $e \subseteq V(C) \cup V(D) \cup V(D')$. It follows from Lemma 32 that $D' \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$.

For the second assertion, suppose for a contradiction that for every $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$, we have $e \setminus V(C) \neq \varnothing$. We verify that $\{\varphi_C(2i) : i \in \{0, \ldots, w(C)\}\}$ is a module of H. Let $e \in E(H)$ such that $e \cap \{\varphi_C(2i) : i \in \{0, \ldots, w(C)\}\} \neq \varnothing$ and $e \setminus \{\varphi_C(2i) : i \in \{0, \ldots, w(C)\}\} \neq \varnothing$. Since $e \cap \{\varphi_C(2i) : i \in \{0, \ldots, w(C)\}\} \neq \varnothing$, we have $e \cap V(C) \neq \varnothing$, and hence $e \setminus V(C) \neq \varnothing$. By Lemma 32, there exists $i \in \{0, \ldots, w(C)\}$ such that $e \cap V(C) = \{\varphi_C(2i)\}$. Given $j \in \{0, \ldots, w(C)\}$, it follows from Lemma 27 that $(e \setminus \{\varphi_C(2i)\}) \cup \{\varphi_C(2j)\} \in E(H)$. Therefore, $\{\varphi_C(2i) : i \in \{0, \ldots, w(C)\}\}$ is a module of H, which contradicts the fact that H is prime. Consequently, there exists $e \in V(H)$ such that $e \subseteq V(C)$. Set

$$p = \min(\{i \in \{0, \ldots, 2w(C)\} : \varphi_C(i) \in e\}).$$

For a contradiction, suppose that p is odd. It follows from Lemma 27 that $(e \setminus \{\varphi_C(p)\}) \cup \{\varphi_C(1)\} \in E(H)$, which contradicts the fact that $V(C) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $H - \varphi_C(0)$. Thus, there exists $i \in \{0, \ldots, w(C)\}$ such that $p = \varphi_C(2i)$. Similarly, there exists $k \in \{0, \ldots, w(C)\}$ such that $\max(\{i \in \{0, \ldots, 2w(C)\} : \varphi_C(i) \in e\}) = \varphi_C(2k)$. Since $e \subseteq V(C)$, we have $i < k$. Consider $q \in \{2i+1, \ldots, 2k-1\}$ such that $e = \varphi_C(2i)\varphi_C(q)\varphi_C(2k)$. For a contradiction, suppose that q is even. It follows from Lemma 27 that $\varphi_C(2i)\varphi_C(2i+2)\varphi_C(2k) \in E(H)$, which contradicts the fact that $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $H - \varphi_C(2i+1)$. It follows that p is odd. Hence, there exists $j \in \{i, \ldots, k-1\}$ such that $q = \varphi_C(2j+1)$. We obtain

$$e = \varphi_C(2i)\varphi_C(2j+1)\varphi_C(2k). \tag{15}$$

It follows that

$$E(H[V(C)]) \subseteq \{\varphi_C(2i')\varphi_C(2j'+1)\varphi_C(2k') : 0 \leq i' \leq j' < k' \leq w(C)\}. \tag{16}$$

It follows from Lemma 27 that

$$\varphi_C(2i')\varphi_C(2j'+1)\varphi_C(2k') \equiv_H \varphi_C(0)\varphi_C(1)\varphi_C(2) \tag{17}$$

for $0 \leq i' \leq j' < k' \leq w(C)$. Since $\varphi_C(2i)\varphi_C(2j+1)\varphi_C(2k) \in E(H[V(C)])$ by (15), we obtain $\varphi_C(0)\varphi_C(1)\varphi_C(2) \in E(H[V(C)])$. It follows from (16) and (17) that

$$E(H[V(C)]) = \{\varphi_C(2i')\varphi_C(2j'+1)\varphi_C(2k') : 0 \leq i' \leq j' < k' \leq w(C)\}.$$

In other words, φ_C is an isomorphism from $C_3(U_{v(C)})$ onto $H[V(C)]$.

For the third assertion, suppose that $V(C) \subsetneq V(H)$. Set

$$Y = V(C).$$

By Theorem 9, $U_{v(C)}$ is prime. Hence, $C_3(U_{v(C)})$ is prime by Theorem 12. It follows from the second assertion above that $H[Y]$ is prime. By Theorem 5,

there exist $v, w \in (V(H) \setminus Y) \cup \underline{Y}$ such that $H - \{v, w\}$ is prime. Since H is critical, we have $v \neq w$. Therefore, there exists $C' \in \mathfrak{C}(\mathscr{P}(H)) \setminus \mathfrak{C}_1(\mathscr{P}(H))$ such that $v, w \in C'$. Lastly, suppose for a contradiction that $\underline{Y} \neq \varnothing$. Therefore, there exist $y \in Y$ and $u \in V(H) \setminus Y$ such that $\{y, u\}$ is a module of $H[Y \cup \{u\}]$. Since $H[Y]$ is isomorphic to $C_3(U_{v(C)})$ by the second assertion above, there exists $e \in E(H[Y])$ such that $y \in e$. Since $\{y, u\}$ is a module of $H[Y \cup \{u\}]$, we get $(e \setminus \{y\}) \cup \{u\} \in E(H)$, which contradicts Lemma 32. It follows that $\underline{Y} = \varnothing$. Thus, we obtain $v, w \notin Y$, that is, $v, w \notin V(C)$. Hence, we have $C' \neq C$. □

Proposition 34. *Let H be a critical 3-hypergraph such that $v(H) \geq 5$. Suppose that H is not circular. Consider $C \in \mathfrak{C}_{\text{even}}(\mathscr{P}(H))$.*

1. *There exists $D \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$ such that $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) \in E(H)$, where $i, j \in \{0, \ldots, w(C) - 1\}$, with $i \leq j$, and $k \in \{0, \ldots, w(D)\}$;*
2. *For each $k \in \{0, \ldots, w(D)\}$, the extension $\psi_C^{2k} : \{0, \ldots, v(C)\} \longrightarrow V(C) \cup \{\varphi_D(2k)\}$ of φ_C defined by*

$$\begin{cases} (\psi_C^{2k})_{\restriction \{0,\ldots,v(C)-1\}} = \varphi_C \\ \text{and} \\ \psi_C^{2k}(v(C)) = \varphi_D(2k), \end{cases}$$

   is an isomorphism from $C_3(W_{v(C)+1})$ onto $H[V(C) \cup \{\varphi_D(2k)\}]$.
3. *If $|V(H) \setminus V(C)| \geq 2$, then there exists $C' \in \mathfrak{C}(\mathscr{P}(H)) \setminus \mathfrak{C}_1(\mathscr{P}(H))$ such that $C' \neq C$.*

Proof. To begin, we verify that

$$e \setminus V(C) \neq \varnothing \tag{18}$$

for every $e \in E(H)$. Otherwise, there exists $e \in E(H)$ such that $e \subseteq V(C)$. Set

$$p = \min(\{i \in \{0, \ldots, 2w(C) - 1\} : \varphi_C(i) \in e\}).$$

For a contradiction, suppose that p is odd. It follows from Lemma 27 that $(e \setminus \{\varphi_C(p)\}) \cup \{\varphi_C(1)\} \in E(H)$, which contradicts the fact that $V(C) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $H - \varphi_C(0)$. Thus, there exists $i \in \{0, \ldots, w(C) - 1\}$ such that $p = \varphi_C(2i)$. Similarly, there exists $k \in \{i, \ldots, w(C) - 1\}$ such that $\max(\{i \in \{0, \ldots, 2w(C)\} : \varphi_C(i) \in e\}) = \varphi_C(2k + 1)$. Since $e \subseteq V(C)$, we have $i < k$. Consider $q \in \{2i + 1, \ldots, 2k\}$ such that $e = \varphi_C(2i)\varphi_C(q)\varphi_C(2k + 1)$. We distinguish the following two cases. In each of them, we obtain a contradiction.

- Suppose that q is even. It follows from Lemma 27 that $\varphi_C(2i)\varphi_C(2i + 2)\varphi_C(2k+1) \in E(H)$, which contradicts the fact that $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $H - \varphi_C(2i + 1)$.
- Suppose that q is odd. It follows from Lemma 27 that $\varphi_C(2i)\varphi_C(2k - 1)\varphi_C(2k+1) \in E(H)$, which contradicts the fact that $\{\varphi_C(2k-1), \varphi_C(2k+1)\}$ is a module of $H - \varphi_C(2k)$.

It follows that (18) holds. Since H is prime, we have $E(H) \neq \varnothing$. It follows from (18) that

$$V(C) \subsetneq V(H).$$

For the first assertion, since $V(C) \subsetneq V(H)$, $V(C)$ is not a module of H. Therefore, there exists $e \in E(H)$ such that $e \cap V(C) \neq \varnothing$ and $e \setminus V(C) \neq \varnothing$. By Lemma 32, there exist $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$ and $e \cap V(C) = \{\varphi_C(2i), \varphi_C(2j+1)\}$. Since $|e \cap V(C)| = 2$, there exists $D \in \mathfrak{C}(\mathscr{P}(H)) \setminus \{C\}$ such that $e \cap V(D) \neq \varnothing$. It follows from Lemma 32 that $D \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$ and $e \cap V(D) = \{\varphi_D(2k)\}$, where $k \in \{0, \ldots, w(D)\}$.

For the second assertion, it follows from the first assertion above that there exists $D \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$ such that

$$\varphi_C(2i_0)\varphi_C(2j_0+1)\varphi_D(2k_0) \in E(H), \tag{19}$$

where $i_0, j_0 \in \{0, \ldots, w(C) - 1\}$, with $i_0 \leq j_0$, and $k_0 \in \{0, \ldots, w(D)\}$. It follows from Lemma 27 that

$$\varphi_C(2i_0)\varphi_C(2j_0+1)\varphi_D(2k) \in E(H) \tag{20}$$

for each $k \in \{0, \ldots, w(D)\}$. Let $k \in \{0, \ldots, w(D)\}$. Consider $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$. It follows from Lemma 27 that

$$\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) \equiv_H \varphi_C(0)\varphi_C(1)\varphi_D(2k). \tag{21}$$

By (20) and (21), we have

$$\varphi_C(0)\varphi_C(1)\varphi_D(2k) \in E(H). \tag{22}$$

We prove that

$$\begin{aligned} E(H[V(C) \cup \{\varphi_D(2k)\}]) = \{&\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) : \\ &0 \leq i \leq j \leq w(C) - 1\}. \end{aligned} \tag{23}$$

Consider $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$. By (21) and (22), we have $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) \in E(H)$. Therefore, we obtain

$$\begin{aligned} E(H[V(C) \cup \{\varphi_D(2k)\}]) \supseteq \{&\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) : \\ &0 \leq i \leq j \leq w(C) - 1\}. \end{aligned} \tag{24}$$

Conversely, consider $e \in E(H[V(C) \cup \{\varphi_D(2k)\}])$. By (18), we have $\varphi_D(2k) \in e$. It follows from Lemma 32 that there exist $i, j \in \{0, \ldots, w(C) - 1\}$ such that $i \leq j$ and $e \cap V(C) = \{\varphi_C(2i), \varphi_C(2j+1)\}$. Thus,

$$\begin{aligned} E(H[V(C) \cup \{\varphi_D(2k)\}]) \subseteq \{&\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k) : \\ &0 \leq i \leq j \leq w(C) - 1\}. \end{aligned} \tag{25}$$

It follows from (24) and (25) that (23) holds. Therefore, the extension ψ_C^{2k} of φ_C is an isomorphism from $C_3(W_{v(C)+1})$ onto $H[V(C) \cup \{\varphi_D(2k)\}]$.

For the third assertion, suppose that $|V(H) \setminus V(C)| \geq 2$. Let $k \in \{0, \ldots, w(D)\}$. Set

$$Y = V(C) \cup \{\varphi_D(2k)\}.$$

We have $Y \subsetneq V(H)$. By Theorem 9, $W_{v(C)+1}$ is prime. Hence, $C_3(W_{v(C)+1})$ is prime by Theorem 12. It follows from the second assertion above that $H[Y]$ is prime. By Theorem 5, there exist $v, w \in (V(H) \setminus Y) \cup \underline{Y}$ such that $H - \{v, w\}$ is prime. Since H is critical, we have $v \neq w$. Therefore, there exists $C' \in \mathfrak{C}(\mathscr{P}(H)) \setminus \mathfrak{C}_1(\mathscr{P}(H))$ such that $v, w \in C'$. Lastly, suppose for a contradiction that $V(C) \cap \underline{Y} \neq \varnothing$. There exist $c \in V(C)$ and $u \in V(H) \setminus Y$ such that $\{c, u\}$ is a module of $H[Y \cup \{u\}]$. By the second assertion above, there exists $d \in V(C) \setminus \{c\}$ such that $cd\varphi_D(2k) \in E(H[Y])$. Since $\{c, u\}$ is a module of $H[Y \cup \{u\}]$, we get $ud\varphi_D(2k) \in E(H[Y])$, which contradicts Lemma 32. Consequently, we have $V(C) \cap \underline{Y} = \varnothing$. It follows that $v, w \notin V(C)$, so $C' \neq C$. □

Proof of Theorem 16. Let H be a critical 3-hypergraph such that $v(H) \geq 5$. Suppose that H is not circular. By Corollary 30, all the components of $\mathscr{P}(H)$ are paths. Hence, $\mathscr{P}(H)$ satisfies (1).

Now, we verify that $\mathscr{P}(H)$ satisfies (2). By Corollary 6, there exist $v, w \in V(H)$ such that $H - \{v, w\}$ is prime. Since H is critical, we have $v \neq w$. Thus, $vw \in E(\mathscr{P}(H))$. Denote by C the component of $\mathscr{P}(H)$ containing v and w. We obtain $C \in (\mathfrak{C}(\mathscr{P}(H)) \setminus \mathfrak{C}_1(\mathscr{P}(H)))$. Therefore, it follows from Propositions 33 and 34 that $\mathscr{P}(H)$ satisfies (2).

Lastly, we associate with H the hypergraph $\widetilde{H}$ defined on $\mathfrak{C}(\mathscr{P}(H))$ as follows

1. given distinct $C, D \in \mathfrak{C}(\mathscr{P}(H))$, $CD \in E(\widetilde{H})$ if $C \in \mathfrak{C}_{\text{even}}(\mathscr{P}(H))$, $D \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$, and there exists $e \in E(H)$ such that $|e \cap V(C)| = 2$ and $|e \cap V(D)| = 1$;

2. given distinct $I, J, K \in \mathfrak{C}(\mathscr{P}(H))$, $IJK \in E(\widetilde{H})$ if $I, J, K \in \mathfrak{C}_{\text{odd}}(\mathscr{P}(H))$ and there exists $e \in E(H)$ such that $|e \cap V(I)| = |e \cap V(J)| = |e \cap V(K)| = 1$.

Clearly, $\widetilde{H}$ satisfies (3). Using Lemma 27, Lemma 32, Proposition 33, and Proposition 34, it is not difficult to verify that $H = \mathscr{P}(H) \bullet \widetilde{H}$. □

4 Proof of Proposition 18

We use the following preliminary result (see [4, Fact 29]).

Lemma 35. *Let H be a 3-hypergraph. Consider $X \subsetneq V(H)$ such that $H[X]$ is prime. Let M be a module of H. We have $M \cap X = \varnothing$, $M \supseteq X$, or $M \cap X = \{y\}$, where $y \in X$. Moreover, the following assertions hold (see Notation 4).*

1. *If $M \cap X = \varnothing$, then all the elements of M belong to the same block of $p_{(H,X)}$.*

2. *If $M \supseteq X$, then all the elements of $V(H) \setminus M$ belong to $\mathrm{Dis}_{(H,X)}$.*

3. *If $M \cap X = \{y\}$, where $y \in X$, then all the elements of $M \setminus \{y\}$ belong to* $\mathrm{Twi}_{(H,X)}(y)$.

In this section and the next one, we consider a graph Γ satisfying (1) and (2). We consider also a $\{2,3\}$-hypergraph $\mathbb{H}$ defined on $\mathfrak{C}(\Gamma)$ satisfying (3). We use the following notation.

Notation 36. For $\mathbb{W} \subseteq V(\mathbb{H})$, set

$$\overline{\mathbb{W}} = \bigcup_{C \in \mathbb{W}} V(C).$$

Conversely, for $W \subseteq V(\Gamma \bullet \mathbb{H})$, set

$$W/\mathfrak{C}(\Gamma) = \{C \in \mathfrak{C}(\Gamma) : V(C) \cap W \neq \varnothing\}.$$

We utilize the next two remarks.

Remark 37. Let $C \in \mathfrak{C}_{\mathrm{odd}}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)$. First, suppose that $w(C) \geq 2$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that φ_C is an isomorphism from $C_3(U_{v(C)})$ onto $(\Gamma \bullet \mathbb{H})[V(C)]$. By Theorem 9, $U_{v(C)}$ is critical. By Theorem 12, $C_3(U_{v(C)})$ is critical and $\mathscr{P}(C_3(U_{v(C)})) = \mathscr{P}(U_{v(C)})$. Furthermore, we have $\mathscr{P}(U_{v(C)}) = \mathcal{P}_{v(C)}$ by Proposition 24. Since φ_C is an isomorphism from $C_3(U_{v(C)})$ onto $(\Gamma \bullet \mathbb{H})[V(C)]$, $(\Gamma \bullet \mathbb{H})[V(C)]$ is critical and

$$\mathscr{P}((\Gamma \bullet \mathbb{H})[V(C)]) = (V(C), \{\varphi_C(m)\varphi_C(m+1) : 0 \leq m < 2w(C)\}).$$

It follows from Lemma 27 that for each $1 \leq m \leq 2w(C) - 1$,

$$\{\varphi_C(m-1),\ \varphi_C(m+1)\} \text{ is a module of } ((\Gamma \bullet \mathbb{H})[V(C)]) - \varphi_C(m). \tag{26}$$

Second, suppose that $w(C) = 1$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $E((\Gamma \bullet \mathbb{H})[V(C)]) = \{V(C)\}$. Therefore, $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime. Moreover, since $v(C) = 3$, observe that (26) holds.

Remark 38. Let $C \in \mathfrak{C}_{\mathrm{even}}(\Gamma)$. We have $w(C) \geq 1$. Suppose that there exists $\varepsilon \in E(\mathbb{H})$ such that $C \in \varepsilon$. We have $\varepsilon = CD$, where $D \in \mathfrak{C}_{\mathrm{odd}}(\Gamma)$. Let $k \in \{0, \ldots, w(D)\}$. We verify that $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$ is prime

First, suppose that $w(C) = 1$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $E((\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]) = \{V(C) \cup \{\varphi_D(2k)\}\}$. Therefore, $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$ is prime.

Second, suppose that $w(C) \geq 2$. Let $\psi_C^{2k} : \{0, \ldots, v(C)\} \longrightarrow V(C) \cup \{\varphi_D(2k)\}$ satisfying $(\psi_C^{2k})_{\restriction \{0,\ldots,v(C)-1\}} = \varphi_C$ and $\psi_C^{2k}(v(C)) = \varphi_D(2k)$. By (4), ψ_C^{2k} is an isomorphism from $C_3(W_{v(C)+1})$ onto $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$. By Theorem 9, $W_{v(C)+1}$ is critical, so $W_{v(C)+1}$ is prime. By Theorem 12, $C_3(W_{v(C)+1})$ is prime as well. Since ψ_C^{2k} is an isomorphism from $C_3(W_{v(C)+1})$ onto $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$, $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(2k)\}]$ is prime too.

If $\mathbb{H}$ is disconnected, then $\Gamma \bullet \mathbb{H}$ is decomposable, whence the necessity of Assertion (C1) in Proposition 18. Indeed, we have

Lemma 39. *If $\mathbb{C}$ is a component of $\mathbb{H}$, then $\overline{V(\mathbb{C})}$ and $V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(\mathbb{C})}$ are modules of $\Gamma \bullet \mathbb{H}$.*

Proof. For a contradiction, suppose that there exists $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \overline{V(\mathbb{C})} \neq \varnothing$ and $e \cap (V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(\mathbb{C})}) \neq \varnothing$. There exist $C \in V(\mathbb{C})$ and $D \notin V(\mathbb{C})$ such that $e \cap V(C) \neq \varnothing$ and $e \cap V(D) \neq \varnothing$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that there exists $\varepsilon \in E(\mathbb{H})$ such that $C, D \in \varepsilon$, which contradicts the fact that $\mathbb{C}$ is a component of $\mathbb{H}$. Consequently, for each $e \in E(\Gamma \bullet \mathbb{H})$, we have $e \subseteq \overline{V(\mathbb{C})}$ or $e \subseteq (V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(\mathbb{C})})$. It follows that $\overline{V(\mathbb{C})}$ and $V(\Gamma \bullet \mathbb{H}) \setminus \overline{V(\mathbb{C})}$ are modules of $\Gamma \bullet \mathbb{H}$. □

Each edge of $\mathbb{H}$ induces a prime subhypergraph of $\Gamma \bullet \mathbb{H}$. Precisely, we have

Lemma 40. *For every $\varepsilon \in E(\mathbb{H})$, $(\Gamma \bullet \mathbb{H})[\overline{\varepsilon}]$ is prime.*

Proof. Let $\varepsilon \in E(\mathbb{H})$. First, suppose that $|\varepsilon| = 2$. There exist $C \in \mathfrak{C}_{\text{even}}(\Gamma)$ and $D \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = CD$. Hence, we have to show that $(\Gamma \bullet \mathbb{H})[V(C) \cup V(D)]$ is prime. If $w(D) = 0$, then $(\Gamma \bullet \mathbb{H})[V(C) \cup V(D)]$ is prime by Remark 38. Suppose that $w(D) \geq 1$. Set

$$X = V(D).$$

By Remark 37, $(\Gamma \bullet \mathbb{H})[X]$ is prime. We have $V(C) \subseteq \text{Dis}_{(\Gamma \bullet \mathbb{H}, X)}$ (see Notation 4). Let M be a module of $(\Gamma \bullet \mathbb{H})[X \cup V(C)]$ such that $|M| \geq 2$. We have to show that $M = X \cup V(C)$. By Lemma 35, we have $M \cap X = \varnothing$, $M \supseteq X$, or $M \cap X = \{y\}$, where $y \in X$. It follows from the third assertion of Lemma 35 that $|M \cap X| \neq 1$. Now, suppose that $M \cap X = \varnothing$. Hence, we have $M \subseteq V(C)$. By Remark 38, $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(0)\}]$ is prime. Since $|M| \geq 2$ and $M \subseteq V(C)$, we obtain $M = V(C) \cup \{\varphi_D(0)\}$, which contradicts $M \cap X = \varnothing$. Therefore, we have $M \supseteq X$. Let $i, j \in \{0, \ldots, w(C) - 1\}$ with $i \leq j$. We have $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(0) \in E(H)$. Since M is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(C)]$ and $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(1) \notin E(H)$, we obtain $\varphi_C(2i), \varphi_C(2j+1) \in M$. It follows that $M = X \cup V(C)$.

Second, suppose that $|\varepsilon| = 3$. There exist $I, J, K \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = IJK$. Hence, we have to show that $(\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)]$ is prime. If $w(I) = w(J) = w(K) = 0$, then $|V(I) \cup V(J) \cup V(K)| = 3$, and hence $E((\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)]) = \{\varphi_I(0)\varphi_J(0)\varphi_K(0)\}$. It follows that $(\Gamma \bullet \mathbb{H})[V(I) \cup V(J) \cup V(K)]$ is prime. Now, suppose that $w(I) > 0$, $w(J) > 0$, or $w(K) > 0$. For instance, assume that $w(I) > 0$. By Remark 37, $(\Gamma \bullet \mathbb{H})[V(I)]$ is prime. Set

$$X = V(I).$$

We have $V(J) \cup V(K) \subseteq \text{Dis}_{(\Gamma \bullet \mathbb{H}, X)}$. Let M be a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$ such that $|M| \geq 2$. We have to show that $M = X \cup V(J) \cup V(K)$. It follows from the third assertion of Lemma 35 that $M \cap X = \varnothing$ or $X \subseteq M$.

For a contradiction, suppose that $M \cap X = \varnothing$. For $0 \leq j \leq w(J)$ and $0 \leq k \leq w(K)$, we have $\varphi_I(0)\varphi_J(2j)\varphi_K(2k) \in E(\Gamma \bullet \mathbb{H})$. It follows that $M \cap \{\varphi_J(2j) : 0 \leq j \leq w(J)\} = \varnothing$ or $M \cap \{\varphi_K(2k) : 0 \leq k \leq w(K)\} = \varnothing$. For instance, assume that

$$M \cap \{\varphi_J(2j) : 0 \leq j \leq w(J)\} = \varnothing. \tag{27}$$

We distinguish the following two cases.

1. Suppose that $M \cap \{\varphi_J(2j+1) : 0 \leq j \leq w(J)-1\} \neq \varnothing$. Hence, there exists $j \in \{0, \ldots, w(J)-1\}$ such that $\varphi_J(2j+1) \in M$. We have $\varphi_J(2j)\varphi_J(2j+1)\varphi_J(2j+2) \in E(\Gamma \bullet \mathbb{H})$. By (27), $\varphi_J(2j), \varphi_J(2j+2) \notin M$. Since M is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$, we obtain $\varphi_J(2j)x\varphi_J(2j+2) \in E(\Gamma \bullet \mathbb{H})$ for every $x \in M$. But, for each $v \in V(K)$, we have $\varphi_J(2j)v\varphi_J(2j+2) \notin E(\Gamma \bullet \mathbb{H})$. It follows that $M \cap V(K) = \varnothing$. Therefore, $M \subseteq V(J)$, and hence M is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J)]$. Since $|M| \geq 2$, we have $w(J) \geq 1$. By Remark 37, $(\Gamma \bullet \mathbb{H})[V(J)]$ is prime. Therefore, $M = V(J)$, which contradicts (27).

2. Suppose that $M \cap \{\varphi_J(2j+1) : 0 \leq j \leq w(J)-1\} = \varnothing$. By (27), we have $M \cap V(J) = \varnothing$, and hence $M \subseteq V(K)$. As previously for J, we obtain $w(k) \geq 1$ and $M = V(K)$, which is impossible because $\varphi_I(0)\varphi_J(0)\varphi_K(0) \in E(\Gamma \bullet \mathbb{H})$ and $\varphi_I(0)\varphi_J(0)\varphi_K(1) \notin E(\Gamma \bullet \mathbb{H})$.

It follows that $X \subseteq M$. Recall that $\varphi_I(0)\varphi_J(2j)\varphi_K(2k) \in E(\Gamma \bullet \mathbb{H})$ for $0 \leq j \leq w(J)$ and $0 \leq k \leq w(K)$. Let $j \in \{0, \ldots, w(J)\}$ and $k \in \{0, \ldots, w(J)\}$. We have $\varphi_I(0) \in M$ because $X \subseteq M$. Since $\varphi_I(1)\varphi_J(2j)\varphi_K(2k) \notin E(\Gamma \bullet \mathbb{H})$ and M is a module of $(\Gamma \bullet \mathbb{H})[X \cup V(J) \cup V(K)]$, we obtain $\varphi_J(2j), \varphi_K(2k) \in M$. It follows that $X \cup \{\varphi_J(2j) : 0 \leq j \leq w(J)\} \cup \{\varphi_K(2k) : 0 \leq k \leq w(K)\} \subseteq M$. Clearly, $V(J) \subseteq M$ when $w(J) = 0$. Suppose that $w(J) \geq 1$. By Remark 37, $(\Gamma \bullet \mathbb{H})[V(J)]$ is prime. Since $M \cap V(J)$ is a module of $(\Gamma \bullet \mathbb{H})[V(J)]$ such that $\{\varphi_J(2j) : 0 \leq j \leq w(J)\} \subseteq (M \cap V(J))$, we obtain $V(J) \subseteq M$. Similarly, $V(K) \subseteq M$. Consequently, $M = X \cup V(J) \cup V(K)$. □

In the next four results, we compare the modules of $\mathbb{H}$ with those of $\Gamma \bullet \mathbb{H}$.

Lemma 41. *Suppose that $\mathbb{H}$ is connected. For each nontrivial module M of $\Gamma \bullet \mathbb{H}$, we have $M = \overline{M/\mathfrak{C}(\Gamma)}$, $|M/\mathfrak{C}(\Gamma)| \geq 2$, and $M/\mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$.*

Proof. Let M be a nontrivial module of $\Gamma \bullet \mathbb{H}$. First, to prove that $M = \overline{M/\mathfrak{C}(\Gamma)}$, it suffices to show that $V(C) \subseteq M$ for every $C \in M/\mathfrak{C}(\Gamma)$. To begin, consider $C \in \mathfrak{C}_{\mathrm{odd}}(\Gamma) \cap (M/\mathfrak{C}(\Gamma))$. If $w(C) = 0$, then we clearly have $V(C) \subseteq M$. Hence, suppose that $w(C) \geq 1$. By Remark 37, $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime. For a contradiction, suppose that $|V(C) \cap M| = 1$. Denote by c the unique element of $V(C) \cap M$. Since $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime, there exist distinct $d, d' \in V(C) \setminus M$ such that $cdd' \in E(\Gamma \bullet \mathbb{H})$. Since $|M| \geq 2$, there exists $v \in M \setminus V(C)$. Since M is a module of $\Gamma \bullet \mathbb{H}$, we obtain $vdd' \in E(\Gamma \bullet \mathbb{H})$, which contradicts the definition of $\Gamma \bullet \mathbb{H}$. It follows that $|V(C) \cap M| \geq 2$. Since $(\Gamma \bullet \mathbb{H})[V(C)]$ is prime, $V(C) \subseteq M$. Now, consider $C \in \mathfrak{C}_{\mathrm{even}}(\Gamma) \cap (M/\mathfrak{C}(\Gamma))$. Since $\mathbb{H}$ is connected, there exists $D \in \mathfrak{C}_{\mathrm{odd}}(\Gamma)$ such that $CD \in E(\mathbb{H})$. For a contradiction, suppose that $|V(C) \cap M| = 1$. Denote by c the unique element of $V(C) \cap M$. There exists $c' \in V(C) \setminus M$ such that $cc'\varphi_D(0) \in E(\Gamma \bullet \mathbb{H})$. By Remark 38, $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(0)\}]$ is prime. Hence, if $|(V(C) \cup \{\varphi_D(0)\}) \cap M| \geq 2$, then $V(C) \cup \{\varphi_D(0)\} \subseteq M$, which contradicts $|V(C) \cap M| = 1$. Therefore, we have $|(V(C) \cup \{\varphi_D(0)\}) \cap M| = 1$. Since $|M| \geq 2$, there exists $v \in M \setminus (V(C) \cup \{\varphi_D(0)\})$. Since M is a module of $\Gamma \bullet \mathbb{H}$,

we obtain $vc'\varphi_D(0) \in E(\Gamma \bullet \mathbb{H})$, which contradicts the definition of $\Gamma \bullet \mathbb{H}$. It follows that $|V(C) \cap M| \geq 2$. Since $(\Gamma \bullet \mathbb{H})[V(C) \cup \{\varphi_D(0)\}]$ is prime, we obtain $V(C) \cup \{\varphi_D(0)\} \subseteq M$. Consequently, we have $M = \overline{M/\mathfrak{C}(\Gamma)}$.

Second, we prove that $|M/\mathfrak{C}(\Gamma)| \geq 2$. If $v(C) = 1$ for each $C \in M/\mathfrak{C}(\Gamma)$, then $|M/\mathfrak{C}(\Gamma)| \geq 2$ because $|M| \geq 2$. Hence, consider $C \in M/\mathfrak{C}(\Gamma)$ such that $v(C) \geq 2$. Since $\mathbb{H}$ is connected, there exists $\varepsilon \in E(\mathbb{H})$ such that $C \in \varepsilon$. By Lemma 40, $(\Gamma \bullet \mathbb{H})[\overline{\varepsilon}]$ is prime. We have $V(C) \subseteq M$ because $M = \overline{M/\mathfrak{C}(\Gamma)}$. Since $v(C) \geq 2$, we get $\overline{\varepsilon} \subseteq M$. Thus, we have $\varepsilon \subseteq (M/\mathfrak{C}(\Gamma))$, so $|M/\mathfrak{C}(\Gamma)| \geq 2$.

Third, we prove that $M/\mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$. Let $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap (M/\mathfrak{C}(\Gamma)) \neq \varnothing$ and $\varepsilon \setminus (M/\mathfrak{C}(\Gamma)) \neq \varnothing$. We distinguish the following two cases.

- Suppose that $|\varepsilon| = 2$. There exist $C \in \mathfrak{C}_{\text{even}}(\Gamma)$ and $D \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = CD$. Thus, there exist distinct $c, c' \in V(C)$ and $d \in V(D)$ such that $cc'd \in E(\Gamma \bullet \mathbb{H})$. For a contradiction, suppose that $C \in M/\mathfrak{C}(\Gamma)$. Since $\varepsilon \setminus (M/\mathfrak{C}(\Gamma)) \neq \varnothing$, we have $D \notin M/\mathfrak{C}(\Gamma)$. Since $M = \overline{M/\mathfrak{C}(\Gamma)}$, we have $V(C) \subseteq M$ and $V(D) \cap M = \varnothing$. We obtain $cc'd \in E(\Gamma \bullet \mathbb{H})$, with $c, c' \in V(C) \cap M$ and $d \in V(D) \setminus M$, which contradicts the fact that M is a module of $\Gamma \bullet \mathbb{H}$. It follows that $D \in M/\mathfrak{C}(\Gamma)$ and $C \notin M/\mathfrak{C}(\Gamma)$. Since $M = \overline{M/\mathfrak{C}(\Gamma)}$, we have $V(D) \subseteq M$ and $V(C) \cap M = \varnothing$. We obtain $cc'd \in E(\Gamma \bullet \mathbb{H})$, with $c, c' \in V(C) \setminus M$ and $d \in V(D) \cap M$. Since M is a module of $\Gamma \bullet \mathbb{H}$, we obtain

$$cc'x \in E(\Gamma \bullet \mathbb{H}) \text{ for every } x \in M. \tag{28}$$

  Consider $D' \in M/\mathfrak{C}(\Gamma)$. Since $M = \overline{M/\mathfrak{C}(\Gamma)}$, we have $V(D') \subseteq M$. Let $d' \in V(D')$. By (28), $cc'd' \in E(\Gamma \bullet \mathbb{H})$. It follows from the definition of $\Gamma \bullet \mathbb{H}$ that $CD' \in E(\mathbb{H})$.

- Suppose that $|\varepsilon| = 3$. There exist distinct $I, J, K \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = IJK$. Moreover, we can assume that $I \in M/\mathfrak{C}(\Gamma)$ and $K \notin M/\mathfrak{C}(\Gamma)$. Since $M = \overline{M/\mathfrak{C}(\Gamma)}$, we have $V(I) \subseteq M$ and $V(K) \cap M = \varnothing$. By the definition of $\Gamma \bullet \mathbb{H}$, we obtain $\varphi_I(0)\varphi_J(0)\varphi_K(0) \in E(\Gamma \bullet \mathbb{H})$. Since M is a module of $\Gamma \bullet \mathbb{H}$, we have $\varphi_J(0) \notin M$. Since $M = \overline{M/\mathfrak{C}(\Gamma)}$, we obtain $V(J) \cap M = \varnothing$. Thus, $\varphi_I(0)\varphi_J(0)\varphi_K(0) \in E(\Gamma \bullet \mathbb{H})$, with $\varphi_I(0) \in M$ and $\varphi_J(0), \varphi_K(0) \notin M$. Since M is a module of $\Gamma \bullet \mathbb{H}$, we obtain

$$x\varphi_J(0)\varphi_K(0) \in E(\Gamma \bullet \mathbb{H}) \text{ for every } x \in M. \tag{29}$$

  Consider any $L \in M/\mathfrak{C}(\Gamma)$. Since $M = \overline{M/\mathfrak{C}(\Gamma)}$, we have $V(L) \subseteq M$. Let $d \in V(L)$. By (29), $d\varphi_J(0)\varphi_K(0) \in E(\Gamma \bullet \mathbb{H})$. Since J, K, L are distinct element of $\mathfrak{C}(\Gamma)$, it follows from the definition of $\Gamma \bullet \mathbb{H}$ that $JKL \in E(\mathbb{H})$.

Consequently, $M/\mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$. □

Lemma 42. *Suppose that $\mathbb{H}$ is connected. For a nontrivial module $\mathbb{M}$ of $\mathbb{H}$, we have $\mathbb{M} \subseteq \mathfrak{C}_{\text{even}}(\Gamma)$ or $\mathbb{M} \subseteq \mathfrak{C}_{\text{odd}}(\Gamma)$.*

Proof. Let $\mathbb{M}$ be a nontrivial module of $\mathbb{H}$. Suppose that $\mathbb{M} \setminus \mathfrak{C}_{\text{odd}}(\Gamma) \neq \varnothing$. We have to show that $\mathbb{M} \subseteq \mathfrak{C}_{\text{even}}(\Gamma)$. Since $\mathbb{M} \setminus \mathfrak{C}_{\text{odd}}(\Gamma) \neq \varnothing$, there exists $C \in \mathbb{M} \cap \mathfrak{C}_{\text{even}}(\Gamma)$. Since $\mathbb{H}$ is connected and $\mathbb{M} \subsetneq V(\mathbb{H})$, there exists $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap \mathbb{M} \neq \varnothing$ and $\varepsilon \setminus \mathbb{M} \neq \varnothing$. Furthermore, since $\mathbb{M}$ is a module of $\mathbb{H}$, we have $|\varepsilon \cap \mathbb{M}| = 1$. It follows that $(\varepsilon \setminus \mathbb{M}) \cup \{C\} \in E(\mathbb{H})$. Since $C \in \mathbb{M} \cap \mathfrak{C}_{\text{even}}(\Gamma)$, we obtain $|\varepsilon \setminus \mathbb{M}| = 1$ and $(\varepsilon \setminus \mathbb{M}) \subseteq \mathfrak{C}_{\text{odd}}(\Gamma)$. Lastly, consider $D \in \mathbb{M}$. Since $\mathbb{M}$ is a module of $\mathbb{H}$, we have $(\varepsilon \setminus \mathbb{M}) \cup \{D\} \in E(\mathbb{H})$. Since $|(\varepsilon \setminus \mathbb{M}) \cup \{D\}| = 2$ and $(\varepsilon \setminus \mathbb{M}) \subseteq \mathfrak{C}_{\text{odd}}(\Gamma)$, we obtain $D \in (\mathbb{M} \cap \mathfrak{C}_{\text{even}}(\Gamma))$. Consequently, we have $\mathbb{M} \subseteq \mathfrak{C}_{\text{even}}(\Gamma)$. □

The next result follows from Lemmas 41 and 42.

Corollary 43. *Suppose that $\mathbb{H}$ is connected. For each nontrivial module M of $\Gamma \bullet \mathbb{H}$, we have $M \subseteq V_1(\Gamma)$.*

Proof. Let M be a nontrivial module of $\Gamma \bullet \mathbb{H}$. It follows from Lemma 41 that $M/\mathfrak{C}(\Gamma)$ is a nontrivial module of $\mathbb{H}$. By Lemma 42, $M/\mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text{even}}(\Gamma)$ or $M/\mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text{odd}}(\Gamma)$.

For a contradiction, suppose that $M/\mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text{even}}(\Gamma)$. Since $\mathbb{H}$ is connected, there exists $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap (M/\mathfrak{C}(\Gamma)) \neq \varnothing$ and $\varepsilon \setminus (M/\mathfrak{C}(\Gamma)) \neq \varnothing$. Since $M/\mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text{even}}(\Gamma)$, there exist $C \in \overline{M/\mathfrak{C}(\Gamma)}$ and $D \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $\varepsilon = CD$. By Lemma 41, we have $M = \overline{M/\mathfrak{C}(\Gamma)}$. Therefore, we obtain $V(C) \subseteq M$ and $V(D) \cap M = \varnothing$. It follows that $V(C)$ is a nontrivial module of $(\Gamma \bullet \mathbb{H})[V(C) \cup V(D)]$, which contradicts Lemma 40. Consequently, we have $M/\mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_{\text{odd}}(\Gamma)$.

For a contradiction, suppose that $M \setminus V_1(\Gamma) \neq \varnothing$. Hence, there exists $D \in (M/\mathfrak{C}(\Gamma)) \cap (\mathfrak{C}_{\text{odd}}(\Gamma) \setminus \mathfrak{C}_1(\Gamma))$. Since $\mathbb{H}$ is connected, there exists $\varepsilon \in E(\mathbb{H})$ such that $\varepsilon \cap (M/\mathfrak{C}(\Gamma)) \neq \varnothing$ and $\varepsilon \setminus (M/\mathfrak{C}(\Gamma)) \neq \varnothing$. Since $M/\mathfrak{C}(\Gamma)$ is a module of $\mathbb{H}$, there exists $C \in M/\mathfrak{C}(\Gamma)$ such that $\varepsilon \cap (M/\mathfrak{C}(\Gamma)) = \{C\}$ and $(\varepsilon \setminus \{C\}) \cup \{D\} \in E(\mathbb{H})$. By Lemma 41, we have $V(D) \subseteq M$. We obtain that $V(D)$ is a nontrivial module of $(\Gamma \bullet \mathbb{H})[\overline{(\varepsilon \setminus \{C\}) \cup \{D\}}]$, which contradicts Lemma 40. Consequently, $M \subseteq V_1(\Gamma)$. □

Lemma 44. *Given a module $\mathbb{M}$ of $\mathbb{H}$, if $\mathbb{M} \subseteq \mathfrak{C}_1(\Gamma)$, then $\overline{\mathbb{M}}$ is a module of $\Gamma \bullet \mathbb{H}$.*

Proof. Let $\mathbb{M}$ be a module of $\mathbb{H}$. Suppose that $\mathbb{M} \subseteq \mathfrak{C}_1(\Gamma)$. Let $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \overline{\mathbb{M}} \neq \varnothing$ and $e \setminus \overline{\mathbb{M}} \neq \varnothing$. There exist $c \in e \cap \overline{\mathbb{M}}$ and $x \in e \setminus \overline{\mathbb{M}}$. We have $\{c\} \in \mathfrak{C}(\Gamma)$. Consider $C_x \in \mathfrak{C}(\Gamma)$ such that $x \in C_x$. For a contradiction, suppose that $e = cdx$, where $d \in \overline{\mathbb{M}} \setminus \{c\}$. We have $\{d\} \in \mathfrak{C}_1(\Gamma) \setminus \{\{c\}\}$. Since $cdx \in E(\Gamma \bullet \mathbb{H})$, we obtain $\{c\}\{d\}C_x \in E(\mathbb{H})$, which contradicts the fact that $\mathbb{M}$ is a module of $\mathbb{H}$. It follows that $e = cxy$, where $y \in V(\Gamma \bullet \mathbb{H}) \setminus \overline{\mathbb{M}}$. Consider $C_y \in \mathfrak{C}(\Gamma)$ such that $y \in C_y$. Given $d \in \overline{\mathbb{M}}$, we have to verify that $dxy \in E(\Gamma \bullet \mathbb{H})$. We have $\{d\} \in \mathfrak{C}(\Gamma)$. Note that we can have $C_x = C_y$. We distinguish the following two cases.

- Suppose that $C_x = C_y$. Since $cxy \in E(\Gamma \bullet \mathbb{H})$, we obtain $C_x \in \mathfrak{C}_{\text{even}}(\Gamma)$, $\{x, y\} = \{\varphi_{C_x}(2i), \varphi_{C_x}(2j+1)\}$, where $0 \leq i \leq j \leq w(C_x) - 1$, and $\{c\}C_x \in$

$E(\mathbb{H})$. Since $\mathbb{M}$ is a module of $\mathbb{H}$, $\{d\}C_x \in E(\mathbb{H})$. Since $C_x \in \mathfrak{C}_{\text{even}}(\Gamma)$ and $\{x,y\} = \{\varphi_{C_x}(2i), \varphi_{C_x}(2j+1)\}$, where $0 \leq i \leq j \leq w(C_x) - 1$, we obtain $dxy \in E(\Gamma \bullet \mathbb{H})$.

- Suppose that $C_x \neq C_y$. Since $cxy \in E(\Gamma \bullet \mathbb{H})$, we obtain $C_x, C_y \in \mathfrak{C}_{\text{odd}}(\Gamma)$, $\{x,y\} = \{\varphi_{C_x}(2i), \varphi_{C_y}(2j)\}$, where $0 \leq i \leq w(C_x)$ and $0 \leq j \leq w(C_y)$, and $\{c\}C_xC_y \in E(\mathbb{H})$. Since $\mathbb{M}$ is a module of $\mathbb{H}$, $\{d\}C_xC_y \in E(\mathbb{H})$. Since $C_x, C_y \in \mathfrak{C}_{\text{odd}}(\Gamma)$ and $\{x,y\} = \{\varphi_{C_x}(2i), \varphi_{C_y}(2j)\}$, where $0 \leq i \leq w(C_x)$ and $0 \leq j \leq w(C_y)$, we obtain $dxy \in E(\Gamma \bullet \mathbb{H})$. □

Proof of Proposition 18. To begin, suppose that $\Gamma \bullet \mathbb{H}$ is decomposable. If $\mathbb{H}$ is disconnected, then Assertion (C1) does not hold. Hence, suppose that $\mathbb{H}$ is connected. Consider a nontrivial module M of $\Gamma \bullet \mathbb{H}$. It follows from Lemma 41 that $M/\mathfrak{C}(\Gamma)$ is a nontrivial module of $\mathbb{H}$. Furthermore, we have $M \subseteq V_1(\Gamma)$ by Corollary 43. Thus, we have $M/\mathfrak{C}(\Gamma) \subseteq \mathfrak{C}_1(\Gamma)$. Consequently, Assertion (C2) does not hold.

Conversely, if $\mathbb{H}$ is disconnected, then it follows from Lemma 39 that $\Gamma \bullet \mathbb{H}$ is decomposable. Hence, suppose that $\mathbb{H}$ is connected. Moreover, suppose that there exists a nontrivial module $\mathbb{M}$ of $\mathbb{H}$ such that $\mathbb{M} \subseteq \mathfrak{C}_1(\Gamma)$. It follows from Lemma 44 that $\overline{\mathbb{M}}$ is a nontrivial module of $\Gamma \bullet \mathbb{H}$. Hence, $\Gamma \bullet \mathbb{H}$ is decomposable. □

5 Proof of Theorem 17

In the next remark, we describe a simple way to obtain automorphisms of $\Gamma \bullet \mathbb{H}$.

Remark 45. For every $C \in \mathfrak{C}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)$, consider the function

$$\begin{array}{rccc} F_C: & V(\Gamma \bullet \mathbb{H}) & \longrightarrow & V(\Gamma \bullet \mathbb{H}) \\ \varphi_C(m) & (0 \leq m \leq v(C) - 1) & \longmapsto & \varphi_C(v(C) - 1 - m) \\ & v \notin V(C) & \longmapsto & v. \end{array}$$

Clearly, F_C is an automorphism of Γ. It is easy to verify that F_C is an automorphism of $\Gamma \bullet \mathbb{H}$. Moreover, consider a permutation F of $V(\Gamma \bullet \mathbb{H})$ such that $F(v) = v$ for each $v \in V_1(\Gamma)$. If for every $C \in \mathfrak{C}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)$, $F_{\restriction V(C)} = (F_C)_{\restriction V(C)}$ or $\mathrm{Id}_{V(C)}$, then F is an automorphism of $\Gamma \bullet \mathbb{H}$.

The next two results are useful to study the criticality of $\Gamma \bullet \mathbb{H}$. They allow us to prove Theorem 17.

Lemma 46. *Given $C \in \mathfrak{C}_{\text{odd}}(\Gamma) \setminus \mathfrak{C}_1(\Gamma)$, the following assertions hold*

1. *$V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(0)$;*
2. *$V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(2w(C)-1), \varphi_C(2w(C))\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(2w(C))$;*
3. *if $w(C) \geq 1$, then for $m \in \{1, \ldots, 2w(C) - 1\}$, $\{\varphi_C(m-1), \varphi_C(m+1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(m)$.*

Proof. We begin with the first assertion. For every $e \in E(\Gamma \bullet \mathbb{H})$, if $\varphi_C(1) \in e$, then $e \subseteq V(C)$ and $\varphi_C(0) \in e$. Consequently, there does not exist $e \in E((\Gamma \bullet \mathbb{H}) - \varphi_C(0))$ such that $\varphi_C(1) \in e$. Therefore, $V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(0)$.

For the second assertion, recall that the function F_C is an automorphism of $\Gamma \bullet \mathbb{H}$ (see Remark 45). It follows from the first assertion above that $F_C(V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\})$ is a module of $(\Gamma \bullet \mathbb{H}) - F_C(\varphi_C(0))$, that is, $V(C) \setminus \{\varphi_C(2w(C)-1), \varphi_C(2w(C))\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(2w(C))$.

For the third assertion, suppose that $w(C) \geq 1$. Let $i \in \{0, \ldots, w(C)-1\}$. By (26), $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $((\Gamma \bullet \mathbb{H})[V(C)]) - \varphi_C(2i+1)$. Let $e \in E(\Gamma \bullet \mathbb{H})$ such that $e \cap \{\varphi_C(2i), \varphi_C(2i+2)\} \neq \varnothing$ and $e \setminus V(C) \neq \varnothing$. By the definition of $\Gamma \bullet \mathbb{H}$, $|e \cap \{\varphi_C(2i), \varphi_C(2i+2)\}| = 1$, $|e \setminus V(C)| = 2$, and $(e \setminus V(C)) \cup \{\varphi_C(2i)\}, (e \setminus V(C)) \cup \{\varphi_C(2i+2)\} \in E(\Gamma \bullet \mathbb{H})$. Thus, $\{\varphi_C(2i), \varphi_C(2i+2)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(2i+1)$. Lastly, suppose that $w(C) \geq 2$, and consider $i \in \{1, \ldots, w(C)-1\}$. By (26), $\{\varphi_C(2i-1), \varphi_C(2i+1)\}$ is a module of $((\Gamma \bullet \mathbb{H})[V(C)]) - \varphi_C(2i)$. For every $e \in E(\Gamma \bullet \mathbb{H})$, if $e \cap \{\varphi_C(2i-1), \varphi_C(2i+1)\} \neq \varnothing$, then $e \subseteq V(C)$. Therefore, $\{\varphi_C(2i-1), \varphi_C(2i+1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(2i)$. □

Lemma 47. *Given $C \in \mathfrak{C}_{\text{even}}(\Gamma)$, the following assertions hold*

1. *$V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(0)$;*
2. *$V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(2w(C)-2), \varphi_C(2w(C)-1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(2w(C)-1)$;*
3. *if $w(C) \geq 2$, then for $m \in \{1, \ldots, 2w(C)-2\}$, $\{\varphi_C(m-1), \varphi_C(m+1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(m)$.*

Proof. For the first assertion, consider $e \in E(\Gamma \bullet \mathbb{H})$ such that $\varphi_C(1) \in e$. By definition of $\Gamma \bullet \mathbb{H}$, there exists $D \in \mathfrak{C}_{\text{odd}}(\Gamma)$ such that $e = \varphi_C(0)\varphi_C(1)\varphi_D(2k)$, where $0 \leq k \leq w(D)$. Consequently, there does not exist $e \in E((\Gamma \bullet \mathbb{H}) - \varphi_C(0))$ such that $\varphi_C(1) \in e$. It follows that $V(\Gamma \bullet \mathbb{H}) \setminus \{\varphi_C(0), \varphi_C(1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(0)$.

As in the proof of the second assertion of Lemma 46, the second assertion is deduced from the first one by using Remark 45.

For the third assertion, suppose that $w(C) \geq 2$. Consider $i \in \{1, \ldots, w(C)-1\}$. We prove that $\{\varphi_C(2i-1), \varphi_C(2i+1)\}$ is a module of $(\Gamma \bullet \mathbb{H}) - \varphi_C(2i)$. Let $e \in E((\Gamma \bullet \mathbb{H}) - \varphi_C(2i))$ such that $e \cap \{\varphi_C(2i-1), \varphi_C(2i+1)\} \neq \varnothing$ and $e \setminus \{\varphi_C(2i-1), \varphi_C(2i+1)\} \neq \varnothing$. By definition of $\Gamma \bullet \mathbb{H}$, there exist $D \in \mathfrak{C}_{\text{odd}}(\Gamma)$ and $k \in \{0, \ldots, w(D)\}$ such that

$$\begin{cases} e = \varphi_C(2j)\varphi_C(2i-1)\varphi_D(2k), \text{ where } j \in \{0, \ldots, i-1\} \\ \text{or} \\ e = \varphi_C(2j)\varphi_C(2i+1)\varphi_D(2k), \text{ where } j \in \{0, \ldots, i\}. \end{cases}$$

In the second instance, we have $j \in \{0, \ldots, i-1\}$ because $\varphi_C(2i) \notin e$. By the definition of $\Gamma \bullet \mathbb{H}$, we have $\varphi_C(2j)\varphi_C(2i-1)\varphi_D(2k), \varphi_C(2j)\varphi_C(2i+1)\varphi_D(2k) \in$

$E((\Gamma\bullet\mathbb{H})-\varphi_C(2i))$. Thus, $\{\varphi_C(2i-1),\varphi_C(2i+1)\}$ is a module of $(\Gamma\bullet\mathbb{H})-\varphi_C(2i)$. Lastly, consider $i \in \{0,\dots,w(C)-2\}$. We prove that $\{\varphi_C(2i),\varphi_C(2i+2)\}$ is a module of $(\Gamma\bullet\mathbb{H})-\varphi_C(2i+1)$. Let $e \in E((\Gamma\bullet\mathbb{H})-\varphi_C(2i+1))$ such that $e\cap\{\varphi_C(2i),\varphi_C(2i+2)\}\neq\varnothing$ and $e\setminus\{\varphi_C(2i),\varphi_C(2i+2)\}\neq\varnothing$. By the definition of $\Gamma\bullet\mathbb{H}$, there exist $D\in\mathfrak{C}_{\mathrm{odd}}(\Gamma)$ and $k\in\{0,\dots,w(D)\}$ such that

$$\begin{cases} e=\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k), \text{ where } j\in\{i,\dots,w(C)-1\} \\ \text{or} \\ e=\varphi_C(2i+2)\varphi_C(2j+1)\varphi_D(2k), \text{ where } j\in\{i+1,\dots,w(C)-1\}. \end{cases}$$

In the first instance, we have $j\in\{i+1,\dots,w(C)-1\}$ because $\varphi_C(2i+1)\notin e$. By the definition of $\Gamma\bullet\mathbb{H}$, we have $\varphi_C(2i)\varphi_C(2j+1)\varphi_D(2k),\varphi_C(2i+2)\varphi_C(2j+1)\varphi_D(2k)\in E((\Gamma\bullet\mathbb{H})-\varphi_C(2i+1))$. Thus, $\{\varphi_C(2i),\varphi_C(2i+2)\}$ is a module of $(\Gamma\bullet\mathbb{H})-\varphi_C(2i+1)$. □

Proof of Theorem 17. To begin, suppose that $\Gamma\bullet\mathbb{H}$ is critical. In particular, $\Gamma\bullet\mathbb{H}$ is prime. By Proposition 18, Assertions (C1) and (C2) hold. For Assertion (C3), consider $v\in V_1(\Gamma)$. We have $\{v\}\in\mathfrak{C}_1(\Gamma)$. Suppose that $\mathbb{H}-\{v\}$ is connected. Since $\Gamma\bullet\mathbb{H}$ is critical, $(\Gamma\bullet\mathbb{H})-v$ is decomposable. We have

$$(\Gamma\bullet\mathbb{H})-v=(\Gamma-v)\bullet(\mathbb{H}-\{v\}).$$

Since $\mathbb{H}-\{v\}$ is connected, it follows from Proposition 18 applied to $(\Gamma\bullet\mathbb{H})-v$ that there exists a nontrivial module $\mathbb{M}_{\{v\}}$ of $\mathbb{H}-\{v\}$ such that $\mathbb{M}_{\{v\}}\subseteq\mathfrak{C}_1(\Gamma-v)$. Since $\mathfrak{C}_1(\Gamma-v)=\mathfrak{C}_1(\Gamma)\setminus\{\{v\}\}$, we have $\mathbb{M}_{\{v\}}\subseteq\mathfrak{C}_1(\Gamma)\setminus\{\{v\}\}$.

Conversely, suppose that Assertions (C1), (C2), and (C3) hold. Since Assertions (C1) and (C2) hold, it follows from Proposition 18 that $\Gamma\bullet\mathbb{H}$ is prime. Furthermore, it follows from Assertion (C3) and Proposition 18 that $(\Gamma\bullet\mathbb{H})-v$ is decomposable for each $v\in V_1(\Gamma)$. Lastly, consider $v\in V(\Gamma)\setminus V_1(\Gamma)$. There exists $C\in\mathfrak{C}(\Gamma)\setminus\mathfrak{C}_1(\Gamma)$ such that $v\in V(C)$. It follows from Lemmas 46 and 47 that $(\Gamma\bullet\mathbb{H})-v$ is decomposable. Consequently, $\Gamma\bullet\mathbb{H}$ is critical. □

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