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# CLT for $\beta$ ensembles at high-temperature, and for integrable systems: a transfer operator approach.

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## Abstract

In this paper, we prove a polynomial Central Limit Theorem for several integrable models, and for the  $\beta$  ensembles at high-temperatures with polynomial potential. Furthermore, we are able to relate the mean values, the variances and the correlations of the moments of these integrable systems with the one of the  $\beta$  ensembles. Moreover, we show that for several integrable models, the local functions' space-correlations decay exponentially fast.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Nagaev–Guivarc’h theory: a transfer operator approach</b>	<b>9</b>
<b>3</b>	<b>Application</b>	<b>14</b>
3.1	The Toda lattice, and the real $\beta$ ensemble at high-temperature . . . . .	15
3.1.1	The Toda chain’s currents . . . . .	18
3.2	The exponential Toda lattice, and the Laguerre $\beta$ ensemble at high-temperature .	22
3.3	The Volterra lattice, and the antisymmetric $\beta$ ensemble at high-temperature . . .	26
3.4	The defocusing Ablowitz-Ladik lattice, and the Circular $\beta$ ensemble at high-temperature . . . . .	28
3.5	The defocusing Schur flow, and the Jacobi $\beta$ ensemble at high-temperature . . .	31
3.6	The Itoh–Narita–Bogoyavleskii lattices . . . . .	33
<b>4</b>	<b>Technical Results</b>	<b>36</b>
<b>5</b>	<b>Conclusion and Outlooks</b>	<b>44</b>

## 1 Introduction

In this paper, we study eigenvalue fluctuations for several random matrix models related to some integrable dynamical systems and to the classical  $\beta$  ensembles in the high-temperature regime. Specifically, we consider random band matrices with fixed bandwidth and, under some mild assumptions, we prove a central limit theorem (CLT) for polynomial test functions for the empirical measure of the eigenvalues. In particular, we consider the following kind of matrices

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- Type 1-i) **Periodic Jacobi matrices**, which are periodic tridiagonal matrix of the form

$$\begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix},$$

for  $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$ ,  $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{R}_+^N$ .

- Type 1-ii) **Antisymmetric Bidiagonal Periodic matrices**:

$$\begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & -a_{N-1} & 0 \end{pmatrix},$$

for  $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}_+^N$ .

- Type 1-iii) **Periodic CMV (after Cantero, Moral and Velazquez) matrices**, which are  $2N \times 2N$  unitary matrices given by

$$\mathcal{E} = \mathcal{L}\mathcal{M},$$

where we define  $\mathcal{L}$  and  $\mathcal{M}$  in the following way. Let  $\mathbf{a} = (a_1, \dots, a_{2N})$  be complex numbers of the unit disk  $\mathbb{D}$ . Define the  $2 \times 2$  unitary matrix  $\Xi_j$

$$\Xi_j = \begin{pmatrix} \bar{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix}, \quad j = 1, \dots, 2N, \quad \rho_j = \sqrt{1 - |a_j|^2}. \quad (1.1)$$

Then,  $\mathcal{L}$  and  $\mathcal{M}$  are the  $2N \times 2N$  matrices

$$\mathcal{L} = \begin{pmatrix} \Xi_1 & & & & \\ & \Xi_3 & & & \\ & & \ddots & & \\ & & & \Xi_{2N-1} & \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} -a_{2N} & & & & \rho_{2N} \\ & \Xi_2 & & & \\ & & \ddots & & \\ & & & \Xi_{2N-2} & \\ \rho_{2N} & & & & \bar{a}_{2N} \end{pmatrix}.$$

The matrix  $\mathcal{E}$  is a pentadiagonal periodic matrix and is unitary.

- Type 1-iv) **Two diagonals periodic matrices** given by

$$\begin{pmatrix} 0 & a_1 & 0 & \dots & b_{N-r+1} & 0 & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & b_{N-r+2} & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & a_{r-1} & 0 & 0 & b_N \\ b_1 & 0 & \dots & \dots & 0 & a_r & 0 & 0 \\ 0 & b_2 & 0 & \dots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & a_{N-1} \\ a_N & 0 & \dots & b_{N-r} & \dots & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \downarrow N-r \text{ column} \\ \\ \\ \leftarrow r+1 \text{ row} \end{matrix}$$

Where  $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^N$ . In applications, we consider either  $a_1 = a_2 = \dots = a_N = 1$  or  $b_1 = b_2 = \dots = b_N = 1$ .

We also consider the non-periodic counterparts of the previous matrices. More specifically:

- Type 2-i) **Jacobi matrices**, which are symmetric tridiagonal matrices

$$\begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ & & & b_{N-1} & a_N \end{pmatrix},$$

where  $\mathbf{a} \in \mathbb{R}^N$  and  $\mathbf{b} \in \mathbb{R}_+^{N-1}$ .

- Type 2-ii) **Bidiagonal Antisymmetric matrices**:

$$\begin{pmatrix} 0 & a_1 & 0 & & \\ -a_1 & 0 & a_2 & \ddots & \\ 0 & -a_2 & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & a_{N-1} \\ & & 0 & -a_{N-1} & 0 \end{pmatrix},$$

for  $\mathbf{a} \in \mathbb{R}_+^{N-1}$ .

- Type 2-iii) **CMV matrices**,  $2N \times 2N$  unitary matrices of the form

$$\mathfrak{E} = \mathfrak{L}\mathfrak{M},$$

where

$$\mathfrak{L} = \text{diag}(\Xi_0, \Xi_2, \Xi_4, \dots, \Xi_{2N}) \quad \text{and} \quad \mathfrak{M} = \text{diag}(\Xi_1, \Xi_3, \Xi_5, \dots, \Xi_{2N-1}),$$

and the blocks  $\Xi_j$ ,  $j = 1, \dots, 2N - 1$  are defined in (1.1), while  $\Xi_0 = (1)$  and  $\Xi_{2N} = (\bar{a}_{2N})$  are  $1 \times 1$  matrices.

The periodic matrices that we consider are the *Lax matrices* of some integrable models. These are particular dynamical systems that are Liouville integrable, and their integrability is proved obtaining a Lax pair  $(L, A)$  [48] representation of the model, meaning that the equations of motions for each of these systems are equivalent to the following linear system for some matrices  $L, A$

$$\dot{L} \equiv \frac{dL}{dt} = [L; A] = LA - AL.$$

This formulation is useful since it implies that  $\{\text{Tr}(L^k)\}_{k=1}^N$  are a system of independent constants of motion  $\left(\frac{d}{dt}\text{Tr}(L^k) = 0\right)$  for the system at hand, so the system is integrable in classical sense. We call these quantities *conserved fields*.

Specifically, the Toda lattice [70] and the Exponential Toda lattice [32] have as Lax matrix a periodic Jacobi matrix, the Volterra lattice [32] has an antisymmetric periodic one, the Ablowitz-Ladik lattice [2] and the Schur flow [30] have a periodic CMV one, and the family of Itoh–Narita–Bogoyavlenskii [9] lattices have a bidiagonal periodic one.

We endow the periodic matrices  $L$  of type 1 with the so-called *Generalized Gibbs Ensemble* of the corresponding dynamical system. The main property of these measures is that they are invariant with respect to the dynamics of the corresponding integrable system. These Generalized Gibbs Ensembles have the form

$$\mu_N^{(1)} = \frac{1}{Z_N^{(1)}(\alpha, G)} \left( \prod_{j=1}^N F(x_j, \alpha) \right) e^{-\text{Tr}(G(L))} d\mathbf{x}, \quad (1.2)$$

where  $\mathbf{x} = (\mathbf{a}, \mathbf{b})$  for Jacobi matrices,  $\mathbf{x} = \mathbf{a}$  for antisymmetric matrix,  $\mathbf{x} = \mathbf{a}$  for periodic CMV matrices,  $\mathbf{x} = \mathbf{a}$  (resp.  $\mathbf{x} = \mathbf{b}$ ) for bidiagonal periodic matrices if  $b_1 = \dots = b_N = 1$  (resp. if  $a_1 = \dots = a_N = 1$ ) and  $G$  is a real-valued continuous function.

The non-periodic matrices of type 2 are related to the classical  $\beta$  ensembles, indeed both the real  $\beta$  ensemble and the Laguerre  $\beta$  ensemble [19] can be represented through a Jacobi matrix, the circular and the Jacobi ensemble have a representation in terms of CMV matrices [45], and the Antisymmetric  $\beta$  ensemble has a representation in terms of an antisymmetric matrix [20]. Specifically, we consider these ensembles in the high-temperature regime, meaning that the parameter  $\beta$  is not fixed, but scales with the matrix size  $N$  as  $\beta = \frac{2\alpha}{N}$ ,  $\alpha \in \mathbb{R}_+$ . The joint density for the entries of the matrix representation reads

$$\mu_N^{(2)} = \frac{1}{Z_N^{(2)}(\alpha, G)} \left( \prod_{j=1}^{N-1} F \left( x_j, \alpha \left( 1 - \frac{j}{N} \right) \right) \right) R(x_N) e^{-\text{Tr}(G(L))} d\mathbf{x}, \quad (1.3)$$

where  $\mathbf{x} = (\mathbf{a}, \mathbf{b})$  for tridiagonal matrices,  $\mathbf{x} = \mathbf{a}$  for the antisymmetric one and  $\mathbf{x} = \mathbf{a}$  for CMV one.

As we already mentioned, we focus on the fluctuations (or *linear statistics*) around the equilibrium measure of these general models, where we choose the functions  $F$  and  $G$  so that the partition functions

$$\begin{aligned} Z_N^{(1)}(\alpha, G) &= \int_{X^N} \left( \prod_{j=1}^N F(x_j, \alpha) \right) e^{-\text{Tr}(G(L))} d\mathbf{x} \\ Z_N^{(2)}(\alpha, G) &= \int_{X^N} \left( \prod_{j=1}^{N-1} F \left( x_j, \alpha \left( 1 - \frac{j}{N} \right) \right) \right) R(x_N) e^{-\text{Tr}(G(L))} d\mathbf{x} \end{aligned}$$

are finite for all  $N$ . Here  $X$  is a subset of  $\mathbb{R}^d$  ( $\mathbb{C}$  being identified with  $\mathbb{R}^2$ ).

Specifically, we study the fluctuations of polynomial test functions  $Q(z)$ , i.e.

$$\int_{\mathbb{C}} Q d\nu_N - \int_{\gamma} Q d\nu, \quad (1.4)$$

where  $\nu_N$  is the empirical measure of eigenvalues of  $L$  given by

$$\nu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j(L)}. \quad (1.5)$$

Here the  $\lambda_j(L)$  are the eigenvalues of  $L$  and  $\delta_x$  is the Dirac delta function centred at  $x$ ,  $\nu$  is the equilibrium measure (or *density of states*) of the system and  $\gamma$  is the support of the measure  $\nu$ . In this paper, we are able to analyse the random variable (1.4) for polynomial potentials  $G$ , using a transfer operator technique.

The study of spectral properties of Lax matrices of integrable models was initiated by Spohn in [66], see also [51]. In this paper, the author investigated time correlation functions for the Toda lattice. Applying the theory of *Generalized Hydrodynamic* [17], Spohn argued that they have a ballistic behaviour, meaning that they have symmetrically located peaks, which travel

in opposite directions at constant speed and decay as  $t^{-1}$  when  $t \rightarrow \infty$ . To obtain this result, Spohn had to compute the density of states of the Toda Lax matrix; he did it by connecting the Generalized Gibbs Ensemble of the Toda lattice to the real  $\beta$  ensemble in the high-temperature regime [4]. After that, the Generalized Gibbs ensemble for the Ablowitz-Ladik lattice and the Schur flow were connected to the Circular  $\beta$  ensemble and the Jacobi  $\beta$  ensemble in the high-temperature regime [27, 37] respectively by one of the present authors and T. Grava [34] and, independently, by H. Spohn [67]. For all these models, a large deviation principle for their mean density of states were developed in [35, 53]. Furthermore, in [32] the authors were able to connect the classical Gibbs ensemble for the Exponential Toda lattice and the Volterra one to the Laguerre [19] and to the Antisymmetric  $\beta$  ensemble [20] respectively.

As we mentioned, our study does not only involve integrable systems, but also the classical  $\beta$  ensembles. Specifically, we study the random variable (1.4) for these ensembles in the so-called high-temperature regime. The study of these quantities was initiated by Johansson in [41], where the author obtained a CLT for the Gaussian Unitary Ensemble with polynomial potential, then generalized for other models and other values of  $\beta$  in [11, 21, 63], and, more recently, in [12], where the authors obtained also a rigidity result for the eigenvalues of the  $\beta$  ensembles. We mention also the work [57], where the author obtained a CLT for the Gaussian  $\beta$  ensemble in the high-temperature regime for a quadratic potential, the work [37] where the authors obtained a CLT for the Circular  $\beta$  ensemble at high-temperature using a normal approximation method, and the recent paper [22] where the authors obtained a CLT for the real  $\beta$  ensemble in the high-temperature regime for general confining potentials. Another relevant paper that it is worth mentioning is [7], where the authors studied Coulomb gases in dimension  $d \geq 2$  and studied the local laws for any temperature regime. Finally, we recall also the recent work [14, 15] where the authors obtained a super-exponential bound for the convergence for the moments of the CUE, COE and CSE to a Gaussian vector.

**Statement of the results.** We come to precise statements of the main results of the present paper. We consider the previously mentioned family of random matrices and make the following assumptions.

**Hypotheses 1.1.** *The following hypotheses are valid throughout the paper:*

HP 1.  $X \subseteq \mathbb{R}^d$ ;

HP 2.  $F(x, \eta)$  is such that for any  $\eta > 0$   $F(\cdot, \eta) \in C^1(X)$ , and for any  $x \in X$ ,  $F(x, \cdot) \in C^\infty((0, +\infty))$ ;

HP 3.  $F(x, \eta) > 0$  almost surely for  $x \in X$ ,  $\eta \in (0, \infty)$ ;

HP 4.  $F(\cdot, \eta) \in L^1(X, \mathcal{B}) \cap L^2(X, \mathcal{B})$  for all  $\eta \in (0, \infty)$ , and  $\partial_\eta F(\cdot, \eta) \in L^1(X, \mathcal{B}) \cap L^2(X, \mathcal{B})$ ; moreover there exist a  $c \in \mathbb{N}$  and a compact set  $\mathcal{O} \subseteq X$  such that

- $\|F(x, \eta)\|_2 = O(\eta^{-c})$
- There exists  $d > 0$  such that for all  $\eta > 0$ ,  $\int_{\mathcal{O}} F(x, \eta) dx \geq d\eta^{-c}$

HP 5. The function  $\text{Tr}(G(L))$ , where  $L$  is one of the matrices of type 1 or 2, is **circular**, meaning that there exists some  $k \in \mathbb{N}_{\geq 1}$  and two functions  $W : X^k \times X^k \rightarrow \mathbb{C}$ ,  $W_1 : X^\ell \times X^k \rightarrow \mathbb{C}$ ,  $\widetilde{W}_1 : X^k \rightarrow \mathbb{C}$ ,  $\widetilde{W}_2 : X^\ell \rightarrow \mathbb{C}$ , such that writing  $N = kM + \ell$  with  $M \geq 0$  and  $0 \leq \ell \leq k - 1$ , we have

$$\mathrm{Tr}(G(L)) = \begin{cases} \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) + W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1) & \text{for type 1} \\ \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) + \widetilde{W}_1(\mathbf{x}_1) + W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) \\ + \widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell}) & \text{for type 2} \end{cases},$$

here  $\mathbf{x}_j = (x_{(j-1)k+1}, \dots, x_{jk})$ . In this case, we say that  $W$  is the **seed** of  $G$ , and  $W_1, \widetilde{W}_1, \widetilde{W}_2$  are the **weeds**.

HP 6. The real parts of  $W, W_1, \widetilde{W}_1, \widetilde{W}_2$  are lower bounded. Furthermore,  $\exp(-W) \in L^2(X^k \times X^k)$ ,  $\exp(-W_1) \in L^2(X^k \times X^\ell \times X^k)$ ,  $\exp(-\widetilde{W}_1) \in L^2(X^k)$ , and  $\exp(-\widetilde{W}_2) \in L^2(X^\ell)$

HP 7. Both integrals

$$\int_{X^k} \left( \partial_\eta \prod_{q=1}^k F(x_q, \eta) \right)^2 e^{-2W(\mathbf{x}, \mathbf{y})} d\mathbf{x} d\mathbf{y},$$

$$\int_{X^k} \left( \partial_\eta \prod_{q=1}^k F \left( x_q, \eta \left( 1 - \frac{(j-1)k+q}{N} \right) \right) \right)^2 e^{-2W(\mathbf{x}, \mathbf{y})} d\mathbf{x} d\mathbf{y}$$

are finite.

HP 8.  $R(x) \in L^2(X) \cap C^\infty(X)$ , and  $R(x) > 0$  for all  $x \in X$ , or  $R(x) = \delta_y(x)$  for some  $y \in X$ .

Here  $L^p(X, \mathcal{B})$  is the usual  $L^p$  space.

**Remark 1.1.** The definition of circular function and seeds was introduced in this context in [29].

Under these assumptions, we are able to prove our main theorem:

**Theorem 1.2.** Under hypotheses 1.1. Consider  $\mu_N^{(1)}, \mu_N^{(2)}$  (1.2)-(1.3), and let  $(S_N)_{N \geq 1}$  be a sequence of real random variables such that there exists a function  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\mathrm{Tr}(H(L))$  is circular, it satisfies HP 6., and

$$\mathbb{E}_1 [e^{-itS_N}] = \frac{Z_N^{(1)}(\alpha, G + itH)}{Z_N^{(1)}(\alpha, G)}, \quad \mathbb{E}_2 [e^{-itS_N}] = \frac{Z_N^{(2)}(\alpha, G + itH)}{Z_N^{(2)}(\alpha, G)},$$

are finite, here  $\mathbb{E}_1[\cdot], \mathbb{E}_2[\cdot]$  are the mean values taken with respect to  $\mu_N^{(1)}, \mu_N^{(2)}$  respectively. Furthermore, let  $W, h$  be the seeds of  $\mathrm{Tr}(G(L))$  and  $\mathrm{Tr}(H(L))$  respectively, and assume that  $\int_{X^{2k}} |h(\mathbf{x}, \mathbf{y})|^n e^{-2W(\mathbf{x}, \mathbf{y})} d\mathbf{x} d\mathbf{y}$  for  $n = 2, 4, 6$  are finite. Then, there exist two continuous functions

$$A(x) : \mathbb{R}_+ \longrightarrow \mathbb{R},$$

$$\sigma^2(x) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+,$$

such that under  $\mu_N^{(1)}$  (1.2)

$$(S_N - NA(\alpha)) / \sqrt{N}$$

converges to a Gaussian distribution  $\mathcal{N}(0, \sigma^2(\alpha))$  as  $N$  tends to infinity, and under  $\mu_N^{(2)}$  (1.3),

$$\left( S_N - N \int_0^1 A(\alpha x) dx \right) / \sqrt{N}$$

converges to a Gaussian distribution  $\mathcal{N}(0, \int_0^1 \sigma^2(\alpha x) dx)$  as  $N$  tends to infinity. Furthermore, defining the free energies  $\mathcal{F}_N^{(1)}(\alpha, G), \mathcal{F}_N^{(2)}(\alpha, G)$  as

$$\begin{aligned}\mathcal{F}_N^{(1)}(\alpha, G) &= - \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left( Z_N^{(1)}(\alpha, G) \right), \\ \mathcal{F}_N^{(2)}(\alpha, G) &= - \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left( Z_N^{(2)}(\alpha, G) \right),\end{aligned}\tag{1.6}$$

then

$$\begin{aligned}i. \quad \mathcal{F}^{(1)}(\alpha, G) &= \partial_\alpha (\alpha \mathcal{F}^{(2)}(\alpha, G)) & iii. \quad \int_0^1 A(\alpha x) &= i \partial_t \mathcal{F}^{(2)}(\alpha, G + itH)|_{t=0} \\ ii. \quad A(\alpha) &= i \partial_t \mathcal{F}^{(1)}(\alpha, G + itH)|_{t=0} & iv. \quad \sigma^2(\alpha) &= \partial_t^2 \mathcal{F}^{(1)}(\alpha, G + itH)|_{t=0} \\ & & v. \quad \int_0^1 \sigma^2(\alpha x) dx &= \partial_t^2 \mathcal{F}^{(2)}(\alpha, G + itH)|_{t=0}\end{aligned}$$

**Remark 1.3.** In the central part of the proof we introduce a family of operators acting on  $L^2(X^k)$  by

$$\mathcal{L}_{t,\alpha} f(\mathbf{y}) = \int_{X^k} f(\mathbf{x}) \prod_{j=1}^k F(x_j) e^{-(W+itH)(\mathbf{x},\mathbf{y})} d\mathbf{x} d\mathbf{y}.$$

Because of our assumptions, each  $\mathcal{L}_{t,\alpha}$  is Hilbert-Schmidt, meaning that the kernel

$$(\mathbf{x}, \mathbf{y}) \mapsto \prod_{j=1}^k F(x_j) e^{-(W+itH)(\mathbf{x},\mathbf{y})}$$

is in  $L^2(X^k \times X^k)$ , thus  $\mathcal{L}_{t,\alpha}$  is compact, and, as we show in the proof of Theorem 2.1,  $\mathcal{L}_{0,\alpha}$  has a simple dominating eigenvalue. Hypothesis HP 7. and the assumption that the integrals

$$\int_{X^{2k}} |h(\mathbf{x}, \mathbf{y})|^n e^{-2W(\mathbf{x},\mathbf{y})} d\mathbf{x} d\mathbf{y} \quad n = 2, 4, 6$$

are finite is merely to ensure that  $(\alpha, t) \mapsto \mathcal{L}_{t,\alpha}$  is regular. In particular, it is differentiable with respect to  $\alpha$ , and three times differentiable with respect to  $t$  as an operator valued function, which in turn ensures that for  $t$  small enough, the operator  $\mathcal{L}_{t,\alpha}$  has a simple, dominating eigenvalue  $\lambda(t, \alpha)$  and that  $t \mapsto \lambda(t, \alpha)$  is three times differentiable. Note that we only use the existence of a second derivative with respect to  $t$  in the proof of the main theorem, but use the existence of a third derivative in the proof of the Berry-Esseen bound, Theorem 4.5. We use the differentiability with respect to  $\alpha$  in the proof of Theorem 2.2.

In the central part of our paper, we show how to use the previous result to obtain a polynomial CLT for the integrable models that we mentioned, and for the classical  $\beta$  ensemble in the high-temperature regime. Specifically, we use the previous theorem with  $G$  and  $H$  polynomials and

$$S_N = \frac{1}{N} \text{Tr} (H(L)) \stackrel{(1.5)}{=} \int_{\mathbb{C}} H(x) d\nu_N(x).$$

The expectations we want to compute then reduce to

$$\mathbb{E}_j [e^{-itS_N}] = \frac{Z_N^{(j)}(\alpha, G + itH)}{Z_N^{(j)}(\alpha, G)}, \quad j = 1, 2.$$

Furthermore, as a by-product, we are also able to compute the so-called *susceptibility matrix*  $\mathcal{C}$  for integrable models. This is the matrix of the space-correlation functions of the conserved fields, i.e.



$$\mathcal{C}_{m,n} = \lim_{N \rightarrow \infty} \frac{1}{N} (\mathbb{E}_1 [\text{Tr}(L^m) \text{Tr}(L^n)] - \mathbb{E}_1 [\text{Tr}(L^m)] \mathbb{E}_1 [\text{Tr}(L^n)]) ,$$

where  $L$  is the Lax matrix of the integrable system at hand and the mean values are taken with respect to the corresponding Generalized Gibbs ensemble. The computation of such quantities is relevant to obtain the decay of the correlation functions for these integrable systems, as it is shown by Spohn in [65]. In particular, we can prove the following:

**Theorem 1.4.** *Under the same hypotheses as Theorem 1.2. Consider  $\mu_N^{(1)}$  (1.2) and define the free energy  $\mathcal{F}^{(1)}(\alpha, G)$  as in (1.6), then*

$$\mathcal{C}_{m,n} = \partial_{t_1} \partial_{t_2} \left( \mathcal{F}^{(1)}(\alpha, G + it_1 x^m + it_2 x^n) \right) \Big|_{t_1=t_2=0} . \quad (1.7)$$

**Remark 1.5.** *In view of Theorem 1.2, we can rewrite (1.7) as*

$$\mathcal{C}_{m,n} = \partial_\alpha \partial_{t_1} \partial_{t_2} \left( \alpha \left( \mathcal{F}^{(2)}(\alpha, G + it_1 x^m + it_2 x^n) \right) \right) \Big|_{t_1=t_2=0} .$$

*In our context, the previous equality implies that we can compute the susceptibility matrix of the integrable systems that we are considering in terms of just the free energy of the corresponding classical  $\beta$  ensemble in the high-temperature regime.*

Finally, considering the type 1 measures (1.2), we investigate the space-correlations for *local* functions, meaning that they depend only on a finite number (independent of  $N$ ) of consecutive variables, proving the following

**Theorem 1.6** (Decay of correlations). *Let  $W$  be the seed of  $\text{Tr}(G(L))$  and  $I, J : X^k \rightarrow \mathbb{R}$  two local functions such that  $\int_{X^k \times X^k} \left| I(\mathbf{x}) \prod_{i=1}^k F(x_i, \alpha) e^{-W(\mathbf{x}, \mathbf{y})} \right|^2 d\mathbf{x} d\mathbf{y} < \infty$ , and analogously for  $J(\mathbf{x})$ . Write  $N = kM + \ell$ , and let  $j \in \{1, \dots, M\}$ . Then there exists some  $0 < \mu < 1$  such that*

$$\mathbb{E}_1 [I(\mathbf{x}_1) J(\mathbf{x}_j)] - \mathbb{E}_1 [I(\mathbf{x}_1)] \mathbb{E}_1 [J(\mathbf{x}_j)] = O(\mu^{M-j} + \mu^j) .$$

In particular, this result implies that the space-correlations between two local functions acting on two different parts of the chain decay exponentially fast according to the distance between the set of particles they are acting on. In section 3.1, we use the previous result to rigorously justify the assumption of H. Spohn on the decay of space-correlations between the *local conserved fields* and their currents [68]; we mention also that one can follow exactly the same reasoning for all the other integrable systems that we consider.

The paper is organized as follows, in section 2 we state the theoretical results that lead to the proofs of Theorems 1.2 and 1.4. In section 3 we show how to apply our results to obtain a central limit theorem for several integrable systems and for the corresponding  $\beta$  ensembles in the high-temperature regime. A summary of these models can be found in Table 1. Specifically, we obtain a CLT for the Toda lattice and the real  $\beta$  ensemble, for the Exponential Toda lattice and the Laguerre  $\beta$  ensemble, the defocusing Ablowitz–Ladik lattice and the Circular  $\beta$  ensemble, the defocusing Schur flow and the Jacobi  $\beta$  ensemble, the Volterra lattice and the antisymmetric  $\beta$  ensemble, and for the families of Itoh–Narita–Bogoyavlenskii (INB) multiplicative, and additive lattices. Furthermore, we apply Theorem 1.6 to the Toda lattice to derive the limiting currents of the conserved fields. In section 4, we prove the technical results we used in section 2, we prove Theorem 1.6 and deduce a Berry-Esseen type bound for all the previously considered integrable models. Finally, in section 5 we give some conclusions and outlooks for future developments on this topic.

Integrable System (Type 1)	$\beta$ -ensemble at high-temperature (Type 2)
Toda lattice	Real
Defocusing Ablowitz-Ladik lattice	Circular
Exponential Toda lattice	Laguerre
Defocusing Schur flow	Jacobi
Volterra lattice	Antisymmetric

Table 1: Integrable systems and random matrix ensembles

## 2 Nagaev–Guivarc’h theory: a transfer operator approach

In this section, we prove Theorem 1.2-1.4, to do that we need to develop the fluctuations’ theory of Nagaev–Guivarc’h [31,36,38,55] through transfer operator methods, see for example [44,46,62].

The proof of these theorems is divided into 3 main parts. In the first one, we compute  $\mathbb{E}_1 [e^{-it\text{Tr}(H)}], \mathbb{E}_2 [e^{-it\text{Tr}(H)}]$  through transfer operator techniques. Since the proof of these results is technical, we postpone it to section 4. Our proof follows the same line as the corresponding one in [34]. In the second part, we prove a slight generalization of Nagaev–Guivarc’h theorems [31]. In the last part, we combine the previous two results to complete the proof of Theorem 1.2-1.4.

In view of the hypotheses 1.1, we consider the following decomposition of  $N = kM + \ell$ , in this notation, we can rewrite our measures as

$$\mu_{kM+\ell}^{(1)} = \frac{1}{Z_{kM+\ell}^{(1)}(\alpha, W)} \prod_{j=1}^{kM+\ell} F(x_j, \alpha) \prod_{j=1}^{M-1} \exp(-W(\mathbf{x}_j, \mathbf{x}_{j+1})) \quad (2.1)$$

$$\begin{aligned} & \times \exp(-W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1)) \prod_{j=1}^{kM+\ell} dx_j \\ \mu_{kM+\ell}^{(2)} &= \frac{1}{Z_{kM+\ell}^{(2)}(\alpha, W)} \prod_{j=1}^{kM+\ell-1} F\left(x_j, \alpha \left(1 - \frac{j}{kM+\ell}\right)\right) \prod_{j=1}^{M-1} \exp(-W(\mathbf{x}_j, \mathbf{x}_{j+1})) \quad (2.2) \\ & \times \exp\left(-\widetilde{W}_1(\mathbf{x}_1) - W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0)\right) \\ & \times \exp\left(-\widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell})\right) R(x_N) \prod_{j=1}^{kM+\ell} dx_j, \end{aligned}$$

Where  $\mathbf{x}_j = (x_{k(j-1)+1}, \dots, x_{kj})$  and the partition functions become:

$$Z_{kM+\ell}^{(1)}(\alpha, W) = \int_{X^{kM+\ell}} \prod_{j=1}^{kM+\ell} F(x_j, \alpha) \prod_{j=1}^{M-1} \exp(-W(\mathbf{x}_j, \mathbf{x}_{j+1})) \quad (2.3)$$

$$\begin{aligned} & \times \exp(-W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1)) \prod_{j=1}^{kM+\ell} dx_j \\ Z_{kM+\ell}^{(2)}(\alpha, W) &= \int_{X^{kM+\ell}} \prod_{j=1}^{kM+\ell-1} F\left(x_j, \alpha \left(1 - \frac{j}{kM+\ell}\right)\right) \prod_{j=1}^{M-1} \exp(-W(\mathbf{x}_j, \mathbf{x}_{j+1})) \quad (2.4) \\ & \times \exp\left(-W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) - \widetilde{W}_2(x_{kM+1}, \dots, x_{kM+\ell})\right) \\ & \times \exp\left(-\widetilde{W}_1(\mathbf{x}_1)\right) R(x_N) \prod_{j=1}^{kM+\ell} dx_j, \end{aligned}$$

we set  $Z_N^{(s)}(\alpha, W) \equiv Z_N^{(s)}(\alpha, G)$ ,  $s = 1, 2$ .

On the space  $L^2(X^k, \mathcal{B}^k)$  we introduce the standard scalar product for  $f, g \in L^2(X^k, \mathcal{B}^k)$  as

$$\langle f; g \rangle = \int_{X^k} f(x) \overline{g(x)} dx.$$

Furthermore, for  $I, J \subset \mathbb{R}$ , and  $E$  a normed space, denote by  $C^{s,d}(I \times J, E)$  the functions  $f : I \times J \rightarrow E$  that are  $C^s$  (respectively  $C^d$ ) with respect to the first (respectively the second) variable. If  $s = d$ , then we set  $C^d(I \times J, E) \equiv C^{d,d}(I \times J, E)$ , and if the normed space  $E = \mathbb{C}$  we just omit it.

**Transfer operator for partitions functions.** As we already stated, in the first part of the section, we apply the transfer operator method in order to compute  $\mathbb{E}_1 [e^{-itH}]$ ,  $\mathbb{E}_2 [e^{-itH}]$ . In particular, we prove the following theorems

**Theorem 2.1.** *Under Assumptions 1.1. Consider a real function  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Tr}(H(L))$  is circular, and let  $W$  be the seed of  $\text{Tr}(G(L) + itH(L))$ , thus  $W(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + itU(\mathbf{x}, \mathbf{y})$  for  $V, U : X^k \times X^k \rightarrow \mathbb{R}$  seeds of  $\text{Tr}(G(L))$ ,  $\text{Tr}(H(L))$ . Furthermore, assume that  $U \in L^d(X^{2k}, \exp(-2V))$ , with  $\mathbb{N} \ni d \geq 6$ . Then, there exists an  $\varepsilon > 0$ , and two complex valued functions  $\lambda(y, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$  and  $c_{k,\ell}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$  such that for all  $q \in \mathbb{N}$ :*

$$\mathbb{E}_1 [e^{-it\text{Tr}(H)}] = \frac{Z_{kM+\ell}^{(1)}(\alpha, t)}{Z_{kM+\ell}^{(1)}(\alpha, 0)} = c_{k,\ell}(\alpha, t) \lambda(\alpha, t)^{M-2} (1 + o(M^{-q})), \quad \text{as } M \rightarrow \infty,$$

for  $|t| < \varepsilon$ , here  $Z_{kM+\ell}^{(1)}(\alpha, t) \equiv Z_{kM+\ell}^{(1)}(\alpha, V + itU)$ . Furthermore,

$$\begin{aligned} \lambda(x, 0) &= 1, \\ c_{k,\ell}(x, 0) &= 1. \end{aligned}$$

Moreover, there exist two functions  $\tilde{c}_{k,\ell}(\alpha, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$  and  $\tilde{\lambda}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$  such that there exist two constants  $C_1, C_2 > 0$  such that for all  $q \in \mathbb{N}$ :

$$\begin{aligned} C_1 &< \tilde{c}_{k,\ell}(\alpha, t) < C_2, \\ \lambda(\alpha, t) &= \frac{\tilde{\lambda}(\alpha, t)}{\tilde{\lambda}(\alpha, 0)}, \\ Z_{kM+\ell}^{(1)}(\alpha, t) &= \tilde{c}_{k,\ell}(\alpha, t) \tilde{\lambda}(\alpha, t)^{M-2} (1 + o(M^{-q})). \end{aligned} \tag{2.5}$$

In the next theorem, we prove an analogue decomposition of the partition function for the second type of measure. This decomposition involves the same function  $\lambda(y, t)$  as in Theorem 2.1.

**Theorem 2.2.** *Under Assumptions 1.1. Consider a real function  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Tr}(H(L))$  is circular (HP 5.), and let  $W$  be the seed of  $\text{Tr}(G(L) + itH(L))$ , thus  $W(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + itU(\mathbf{x}, \mathbf{y})$  for  $V, U : X^k \times X^k \rightarrow \mathbb{R}$  seeds of  $\text{Tr}(G(L))$ ,  $\text{Tr}(H(L))$ . Furthermore, assume that  $U \in L^d(X^{2k}, \exp(-2V))$ , with  $\mathbb{N} \ni d \geq 6$ .*

*Then, there exists an  $\varepsilon > 0$  and  $c_{k,\ell,M}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$ , such that, with  $\lambda$  given by Theorem 2.1,*

$$\mathbb{E}_2 [e^{-it\text{Tr}(H)}] = \frac{Z_{kM+\ell}^{(2)}(\alpha, t)}{Z_{kM+\ell}^{(2)}(\alpha, 0)} = c_{k,\ell,M}(\alpha, t) \prod_{j=1}^{M-2} \lambda\left(\alpha \frac{j}{M}, t\right) (1 + o_M(1))$$

for  $|t| < \varepsilon$ . Furthermore,

$$\begin{aligned} \lambda(x, 0) &= 1 \\ \lim_{t \rightarrow 0} c_{k,\ell,M}(\alpha, t) &= 1 \quad \text{uniformly in } M \\ \text{the remainder } o_M(1) &\text{ is independent of } t \in [-\varepsilon, \varepsilon]. \end{aligned}$$

Moreover, there exist two functions  $\tilde{c}_{k,\ell,M}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ ,  $\tilde{\lambda}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ , and three constants  $C_1, C_2 > 0$  and  $p \in \mathbb{N}$  such that

$$\begin{aligned} C_1 N^p &< \tilde{c}_{k,\ell,M}(\alpha, t) < C_2 N^p, \\ \lambda(\alpha, t) &= \frac{\tilde{\lambda}(\alpha, t)}{\tilde{\lambda}(\alpha, 0)}, \\ Z_{kM+\ell}^{(2)}(\alpha, t) &= \tilde{c}_{k,\ell,M}(\alpha, t) \prod_{j=1}^{M-2} \tilde{\lambda}\left(\alpha \frac{j}{M}, t\right) (1 + o_M(1)). \end{aligned}$$

Since the proof of these results is technical, we postpone it to Section 4.

**Generalization of Nagaev–Guivarc’h method.** In this second part, we need to generalize some standard results from the fluctuation theory of Nagaev–Guivarc’h [31] to our situation. Specifically, we prove the following:

**Theorem 2.3.** *Let  $(X_n)_{n \geq 1}$  be a sequence of real random variables with partial sums  $(S_n)_{n \geq 1} \in \mathbb{R}$ . Assume that there exists  $\varepsilon > 0$ , two functions  $\lambda(t) \in C^1([0, \varepsilon])$ ,  $c(t) \in C^0([0, \varepsilon])$  and  $h_n(t) \in C^0([0, \varepsilon])$ , such that for all  $t \in [-\varepsilon, \varepsilon]$ , and all  $n \geq 1$  we have*

$$\mathbb{E} [e^{-itS_n}] = c(t)\lambda(t)^m (1 + h_n(t)), \quad (2.6)$$

Where  $\lim_{n \rightarrow \infty} n/m = k \in \mathbb{N}$ .

Furthermore, assume that:

- a. there exists  $A, \sigma^2 \in \mathbb{C}$  such that  $\lambda(t) = \exp(-iAt - \sigma^2 t^2/2 + o(t^3))$  as  $t \rightarrow 0$ ;
- b.  $h_n \xrightarrow{n \rightarrow \infty} 0$  uniformly in  $[-\varepsilon, \varepsilon]$ , and  $h_n(0) = 0$ ;
- c.  $c(0) = 1$ .

Then  $A \in \mathbb{R}$ ,  $\sigma^2 \geq 0$ , and  $(S_n - nA/k)/\sqrt{n}$  converges to a Gaussian distribution  $\mathcal{N}(0, \sigma^2/k)$  as  $n$  tends to infinity.

*Proof.* First, evaluating (2.6) at  $t = 0$ , we deduce that  $\lambda(0) = 1$ . Then, we use the asymptotic expansion of  $\lambda(t)$ , and properties b.-c. to prove that

$$\mathbb{E} \left[ \exp \left( -it \frac{S_n - mA}{n} \right) \right] \xrightarrow{n \rightarrow \infty} 1.$$

Thus, by Lévy theorem [73], we deduce that  $S_n/n - A/k$  converges in distribution to 0. So, since  $S_n$  is real, then  $A \in \mathbb{R}$ . Exploiting again the asymptotic expansion of  $\lambda(t)$  and properties b.-c., we show that  $\mathbb{E} \left[ \exp \left( -it \frac{S_n - mA}{\sqrt{n}} \right) \right]$  converges to the function  $\exp \left( -\frac{\sigma^2 t^2}{2k} \right)$ . By Lévy theorem [73], this must be the characteristic function of a real random variable, proving that  $\sigma^2 \geq 0$ , and that  $(S_n - nA/k)/\sqrt{n}$  converges to a Gaussian distribution  $\mathcal{N}(0, \sigma^2/k)$ .  $\square$

Further, we prove the following:

**Theorem 2.4.** *Let  $(X_n)_{n \geq 1}$  be a sequence of random variables with partial sums  $(S_n)_{n \geq 1} \in \mathbb{R}$ . Assume that there exists  $\varepsilon > 0$  and functions  $\lambda(x, t) \in C^{1,0}([0, 1] \times \mathbb{R})$ ,  $c_n(t) \in C^0(\mathbb{R})$ , and  $h_n(t)$  continuous in 0, such that for all  $t \in [-\varepsilon, \varepsilon]$ , and all  $n \geq 1$  we have*

$$\mathbb{E} [e^{-itS_n}] = c_n(t) \left( \prod_{j=1}^m \lambda(j/m, t) \right) (1 + h_n(t)),$$

where  $\lim_{n \rightarrow \infty} n/m = k$ .

Furthermore, assume that:

a. there exists two continuous functions  $A(x), \sigma^2(x) : [0, 1] \rightarrow \mathbb{C}$  such that

$$\lambda(x, t) = \exp(-iA(x)t - \sigma^2(x)t^2/2 + o(t^2)) \text{ as } t \rightarrow 0;$$

b.  $\|h_n\|_\infty \xrightarrow{n \rightarrow \infty} 0$  uniformly in  $[-\varepsilon, \varepsilon]$ , and  $h_n(0) = 0$ ;

c.  $c_n(0) = 1$  and  $\lim_{n \rightarrow \infty} c_n(t/\sqrt{n}) = \lim_{n \rightarrow \infty} c_n(t/n) = 1$ .

Then  $\int_0^1 A(x)dx \in \mathbb{R}$ ,  $\int_0^1 \sigma^2(x)dx \in \mathbb{R}_+$ , and  $\sqrt{n} \left( \frac{S_n}{n} - \frac{\int_0^1 A(x)dx}{k} \right)$  converges to a Gaussian distribution  $\mathcal{N} \left( 0, \frac{\int_0^1 \sigma^2(x)dx}{k} \right)$  as  $n$  tends to infinity.

*Proof.* First, let  $t = \tilde{t}/n$ , then by hypothesis c.

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ e^{-i\tilde{t}(S_n/n - \frac{1}{n} \sum_{\ell=1}^m A(\ell/m))} \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[ e^{-i\tilde{t} \left( S_n/n - \frac{\int_0^1 A(x)dx}{k} \right)} \right] = 1.$$

Thus, by Levy theorem,  $S_n/n \rightarrow \frac{\int_0^1 A(x)dx}{k}$  almost surely, thus, since  $S_n \in \mathbb{R}$ , this implies that  $\int_0^1 A(x)dx \in \mathbb{R}$ . Consider now  $t = \tilde{t}/\sqrt{n}$ , following the same reasoning one conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left[ e^{-i\tilde{t}\sqrt{n} \left( S_n/n - \frac{1}{n} \sum_{\ell=1}^m A(\ell/m) \right)} \right] &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ e^{-i\tilde{t}\sqrt{n} \left( S_n/n - \frac{\int_0^1 A(x)dx}{k} \right)} \right] \\ &= \lim_{n \rightarrow \infty} e^{-\frac{i^2}{2n} \sum_{\ell=1}^m \sigma^2(\ell/m)} = e^{-\frac{i^2}{2k} \int_0^1 \sigma^2(x)dx}, \end{aligned}$$

thus, by Lévy theorem [73],  $e^{-\frac{i^2}{2k} \int_0^1 \sigma^2(x)dx}$  must be the characteristic function of a real random variable, proving that  $\int_0^1 \sigma^2(x) \in \mathbb{R}_+$ .  $\square$

**Proof of Theorem 1.2-1.4.** We are now ready to prove Theorem 1.2-1.4, for convenience, we split the proof into two Lemmas, which combined give the full proof of our results.

**Lemma 2.5.** *Under hypotheses 1.1. Consider  $\mu_{kM+\ell}^{(1)}, \mu_{kM+\ell}^{(2)}$  (2.1)-(2.2), and let  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Tr}(H(L))$  is circular (HP.5) with seed  $U$ , so that  $W + itU$  is the seed of  $\text{Tr}(G(L) + itH(L))$ . Let  $S_{kM+\ell} = \text{Tr}(H(L))$ . Then*

$$\mathbb{E}_1 [e^{-itS_{kM+\ell}}] = \frac{Z_{kM+\ell}^{(1)}(\alpha, t)}{Z_{kM+\ell}^{(1)}(\alpha, 0)}, \quad \mathbb{E}_2 [e^{-itS_{kM+\ell}}] = \frac{Z_{kM+\ell}^{(2)}(\alpha, t)}{Z_{kM+\ell}^{(2)}(\alpha, 0)},$$

where  $\mathbb{E}_1[\cdot], \mathbb{E}_2[\cdot]$  are the mean values taken with respect to  $\mu_{kM+\ell}^{(1)}, \mu_{kM+\ell}^{(2)}$  respectively. Furthermore, assume that  $U \in L^d(X^{2k}, \exp(-2W))$ , with  $\mathbb{N} \ni d \geq 3$ . Then, there exist four continuous functions

$$\begin{aligned} A(x) &: \mathbb{R}_+ \longrightarrow \mathbb{R}, & \tilde{A}(x) &: \mathbb{R}_+ \longrightarrow \mathbb{R}, \\ \sigma(x) &: \mathbb{R}_+ \longrightarrow \mathbb{R}_+, & \tilde{\sigma}(x) &: \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \end{aligned}$$

such that under  $\mu_{kM+\ell}^{(1)}$ ,

$$(S_{kM+\ell} - (kM + \ell)A(\alpha)/k) / \sqrt{kM + \ell}$$

converges to a Gaussian distribution  $\mathcal{N}(0, \sigma^2(\alpha)/k)$  as  $M$  tends to infinity, and under  $\mu_{kM+\ell}^{(2)}$ ,

$$(S_{kM+\ell} - (kM + \ell)\tilde{A}(\alpha)/k) / \sqrt{kM + \ell}$$

converges to a Gaussian distribution  $\mathcal{N}(0, \tilde{\sigma}^2(\alpha)/k)$  as  $M$  tends to infinity.

The proof of the previous result is a trivial application of Theorem 2.1-2.2-2.3-2.4. Furthermore, we can interpret the previous relations through the *free energies* of  $\mu_{kM+\ell}^{(1)}$ ,  $\mu_{kM+\ell}^{(2)}$  (2.1)-(2.2):

**Lemma 2.6.** *Under the same hypotheses and notation of Lemma 2.5. Consider the two measures  $\mu_{kM+\ell}^{(1)}$ ,  $\mu_{kM+\ell}^{(2)}$  (2.1)-(2.2), and define the free energies as*

$$\mathcal{F}^{(1)}(\alpha, W) = - \lim_{M \rightarrow \infty} \frac{\ln(Z_{kM+\ell}^{(1)}(\alpha, W))}{kM + \ell}, \quad \mathcal{F}^{(2)}(\alpha, W) = - \lim_{M \rightarrow \infty} \frac{\ln(Z_{kM+\ell}^{(2)}(\alpha, W))}{kM + \ell},$$

then, using the same notation as in Lemma 2.5, the following holds:

$$\begin{aligned} i. \quad \mathcal{F}^{(1)}(\alpha, W) &= \partial_\alpha (\alpha \mathcal{F}^{(2)}(\alpha, W)) & iii. \quad \tilde{A}(\alpha) &= ik \partial_t \mathcal{F}^{(2)}(\alpha, W + itU)|_{t=0} \\ ii. \quad A(\alpha) &= ik \partial_t \mathcal{F}^{(1)}(\alpha, W + itU)|_{t=0} & iv. \quad \sigma(\alpha) &= k \partial_t^2 \mathcal{F}^{(1)}(\alpha, W + itU)|_{t=0} \\ & & v. \quad \tilde{\sigma}(\alpha) &= k \partial_t^2 \mathcal{F}^{(2)}(\alpha, W + itU)|_{t=0} \end{aligned}$$

**Remark 2.7.** *The previous theorem implies that*

$$A(\alpha) = \partial_\alpha (\alpha \tilde{A}(\alpha)), \quad \sigma^2(\alpha) = \partial_\alpha (\alpha \tilde{\sigma}^2(\alpha)).$$

*Proof.* To prove *i.*, we can just compute the free energy of  $\mu_{kM+\ell}^{(1)}$ ,  $\mu_{kM+\ell}^{(2)}$  using Theorem 2.1-2.2. For  $\mathcal{F}^{(1)}(\alpha, W)$  we deduce immediately that

$$\begin{aligned} \mathcal{F}^{(1)}(\alpha, W) &= - \lim_{M \rightarrow \infty} \frac{1}{kM + \ell} \ln(Z_{kM+\ell}^{(1)}) \stackrel{(2.5)}{=} - \lim_{M \rightarrow \infty} \frac{M}{kM + \ell} \ln(\tilde{\lambda}(\alpha, 0)) \\ &= - \frac{1}{k} \ln(\tilde{\lambda}(\alpha, 0)). \end{aligned} \tag{2.7}$$

The proof for  $\mathcal{F}^{(2)}(\alpha, W)$  follows in the same way. We now prove *ii.* – *iv.* First, we notice that following the notation of Theorem 2.3 - 2.1 :

$$c(t) = c_{k,\ell}(\alpha, t), \quad h_n = 1 + o(M^{-a}), \quad \lambda(t) = \lambda(\alpha, t),$$

thus to compute explicitly  $A(\alpha, W)$ ,  $\sigma(\alpha, W)$  we have just to expand  $\lambda(\alpha, t)$  around  $t = 0$

$$\lambda(\alpha, t) = 1 + t \partial_t \lambda(\alpha, 0) + \frac{t^2}{2} \partial_t^2 \lambda(\alpha, 0) + o(t^3),$$

which implies that

$$\lambda(\alpha, t) = \exp \left( t \partial_t \ln(\lambda(\alpha, t))|_{t=0} + \frac{t^2}{2} \partial_t^2 \ln(\lambda(\alpha, t))|_{t=0} + o(t^3) \right). \quad (2.8)$$

This implies that  $A(\alpha) = -i \partial_t \ln(\lambda(\alpha, t))|_{t=0}$ ,  $\sigma^2(\alpha) = -\partial_t^2 \ln(\lambda(\alpha, t))|_{t=0}$ . From (2.5), we deduce that  $\partial_t \ln(\lambda(\alpha, t))|_{t=0} = \partial_t \ln(\tilde{\lambda}(\alpha, t))|_{t=0}$ , thus from the previous expressions and the explicit form of the free energy (2.7) we conclude.

To prove *iii. – v.* we proceed in the same way, thus following the notation of Theorem 2.4 - 2.2:

$$c_n(t) = c_{k,\ell,M}(\alpha, t), \quad h_n = 1 + o(1), \quad \lambda(j/M, t) = \lambda \left( \alpha \frac{j}{M}, t \right).$$

Thus, as in (2.8) except that  $\alpha \rightarrow \alpha \frac{j}{M}$ , we expand  $\lambda \left( \alpha \frac{j}{M}, t \right)$  around  $t = 0$ , leading to

$$\begin{aligned} \tilde{A}(\alpha) &= \partial_t \left( \int_0^1 \ln(\lambda(\alpha x, t)) dx \right) \Big|_{t=0}, \\ \tilde{\sigma}^2(\alpha) &= \partial_t^2 \left( \int_0^1 \ln(\lambda(\alpha x, t)) dx \right) \Big|_{t=0} \end{aligned}$$

which concludes the proof.  $\square$

**Remark 2.8.** We notice that the Lemma 2.6, and Lemma 2.5 imply that we can compute the expected values, and the variances of  $S_{kM+\ell}$  according to  $\mu_{kM+\ell}^{(1)}$ ,  $\mu_{kM+\ell}^{(2)}$  just computing derivatives of the corresponding free energy. This property is broadly used in the physics literature, but we lacked of a precise statement, and of a proof for the general result. Furthermore, we can compute the expected value, and the variance of  $S_{kM+\ell}$  according to  $\mu_{kM+\ell}^{(1)}$  starting from the corresponding values for  $\mu_{kM+\ell}^{(2)}$ . Thus, we have reduced all this problem to the computation of the free energy of  $\mu_{kM+\ell}^{(2)}$ .

The proof of both Theorem 1.2 and Theorem 1.4 follows from the four previous lemmas. Thus, we have completed the proof of our main theorems, and now we show how to apply them to some integrable models, and the  $\beta$  ensembles in the high-temperature regime.

### 3 Application

In this section, we show how to apply Theorem 1.2 to obtain a CLT for some integrable systems and for the classical  $\beta$  ensembles in the high-temperature regime. Namely, we prove a CLT for the systems of table 1.

Specifically, we are able to prove that all the integrable systems in table 1 in the periodic case have a Generalized Gibbs ensemble of the form  $\mu_{kM+\ell}^{(1)}$  (2.1), that is the reason of the label “type 1”. Meanwhile, the  $\beta$  ensembles at high-temperatures are characterized by a probability distribution of the form  $\mu_{kM+\ell}^{(2)}$  (2.2), that is the reason for the label “type 2”. In this way, we proved a further connection between the theory of integrable systems and Random Matrix Theory. Indeed, in view of Theorem 1.2 and Theorem 1.4, for any integrable system in the previous table, we can relate its free energy, moments, variances and covariances with the corresponding quantities of the random matrix model on the same line. Moreover, in the final part of this section, we consider the family of INB lattices that do not have a known  $\beta$  ensemble counterpart. Despite that, we are still able to derive the existence of a polynomial central limit theorem. Finally, applying Theorem 1.6, we are able to show that for the Toda lattice the space-correlations between the local conserved fields and the currents decay exponentially.

### 3.1 The Toda lattice, and the real $\beta$ ensemble at high-temperature

In this subsection, we focus on the Toda lattice, which is an integrable model, and its relation with the real  $\beta$  ensemble in the high-temperature regime. The connection between these two systems was first realized by Spohn in [66], see also [51, 65]. In this seminal paper, the author was able to compute the density of states for the Toda lattice when the initial data is sampled according to a Generalized Gibbs ensemble in terms of one of the Gaussian  $\beta$  ensemble in the high-temperature regime. This was further developed in [35] where the authors obtained a Large Deviations Principle for the Toda lattice, and they connect it to the one for the real  $\beta$  ensemble in the high-temperature regime. In this paper, we further develop this analysis, obtaining a CLT theorem for these two systems, and connecting them. This result is particularly relevant in the context of the so-called Generalized Hydrodynamics, a recent physical theory that allows computing the correlation functions for classical integrable models, for an introduction to the subject see [17, 68]. According to this theory, one of the main ingredients to compute the correlation functions for the integrable model at hand is to be able to calculate the correlation functions for the conserved fields at time 0. Thanks to our result, we are able to access these quantities. We show how to do it at the end of this subsection. We mention also the recent work [52], where the authors made molecular dynamics simulations of the correlation functions of the Toda lattice, and they compared them with the predictions of linear Generalized Hydrodynamics, showing an astonishing agreement.

**The Toda lattice.** The classical Toda chain [70] is the dynamical system described by the following Hamiltonian:

$$H_T(\mathbf{p}, \mathbf{q}) := \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{j=1}^N V_T(q_{j+1} - q_j), \quad V_T(x) = e^{-x} + x - 1,$$

with periodic boundary conditions  $q_{j+N} = q_j + \Omega \quad \forall j \in \mathbb{Z}, \Omega > 0$ . Its equations of motion take the form

$$\dot{q}_j = \frac{\partial H_T}{\partial p_j} = p_j, \quad \dot{p}_j = -\frac{\partial H_T}{\partial q_j} = V_T'(q_{j+1} - q_j) - V_T'(q_j - q_{j-1}), \quad j = 1, \dots, N. \quad (3.1)$$

It is well known that the Toda chain is an integrable system [39, 70], one way to prove it is to put the Toda equations in Lax pair form. This was obtained by Flaschka [25], and Manakov [50] through the following *non-canonical* change of coordinates:

$$a_j := -p_j, \quad b_j := e^{\frac{1}{2}(q_j - q_{j+1})} \equiv e^{-\frac{1}{2}r_j}, \quad 1 \leq j \leq N,$$

where  $r_j = q_{j+1} - q_j$  is the relative distance.

Defining the Lax operator  $L$  as the periodic Jacobi matrix [72]

$$L := \begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix}, \quad (3.2)$$



and the antisymmetric matrix  $B$

$$B := \begin{pmatrix} 0 & b_1 & 0 & \dots & -b_N \\ -b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & -b_{N-1} & 0 \end{pmatrix},$$

a straightforward calculation shows that the equations of motions (3.1) are equivalent to

$$\dot{L} = [B; L],$$

where  $[B; L] = BL - LB$  is the commutator of two matrices. This form implies that  $\text{Tr}(L^k)$ ,  $k = 1, \dots, N$  are constants of motions for the Toda lattice, so the system is integrable. We call these quantities *conserved fields*.

On the phase space  $\mathbb{R}^N \times \mathbb{R}_+^N$ , we introduce the *Generalized Gibbs Ensemble*

$$d\mu_T := \frac{1}{Z_N^T(\alpha, P)} \prod_{j=1}^N b_j^{2\alpha-1} \mathbb{1}_{b_j > 0} e^{-\text{Tr}(P(L))} d\mathbf{a} d\mathbf{b}, \quad (3.3)$$

where  $P(x)$  is a polynomial of even degree with positive leading coefficients, and  $\alpha > 0$  is a pressure parameter.

Our aim is to obtain a central limit theorem for the conserved fields when the initial data is sampled according to (3.3). So, we want to apply Theorem 1.2 to this model. To do that, we need some preparation. First, we recall the following result about the structure of the trace of periodic Jacobi matrices which was proved in [33]:

**Theorem 3.1** (cf. Theorem 3.1 [33]). *For any  $1 \leq m \leq N - 1$ , consider the matrix  $L$  given by (3.2). One has*

$$\text{Tr}(L^m) = \sum_{j=1}^N h_j^{(m)},$$

where  $h_j^{(m)} := [L^m]_{jj}$  is given explicitly by

$$h_j^{(m)}(\mathbf{p}, \mathbf{r}) = \sum_{(\mathbf{n}, \mathbf{q}) \in \mathcal{A}^{(m)}} \rho^{(m)}(\mathbf{n}, \mathbf{q}) \prod_{i=-\tilde{m}}^{\tilde{m}-1} a_{j+1}^{q_i} \prod_{i=-\tilde{m}+1}^{\tilde{m}-1} b_{j+i}^{2n_i},$$

where it is understood  $a_j \equiv a_{j \bmod N+1}$ ,  $b_j \equiv b_{j \bmod N+1}$  and  $\mathcal{A}^{(m)}$  is the set

$$\mathcal{A}^{(m)} := \left\{ (\mathbf{n}, \mathbf{q}) \in \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0^{\mathbb{Z}} : \begin{aligned} & \sum_{i=-\tilde{m}}^{\tilde{m}-1} (2n_i + q_i) = m, \\ & \forall i \geq 0, \quad n_i = 0 \Rightarrow n_{i+1} = q_{i+1} = 0, \\ & \forall i < 0, \quad n_{i+1} = 0 \Rightarrow n_i = q_i = 0 \end{aligned} \right\}.$$

The quantity  $\tilde{m} := \lfloor m/2 \rfloor$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and  $\rho^{(m)}(\mathbf{n}, \mathbf{q}) \in \mathbb{N}$  is given by

$$\rho^{(m)}(\mathbf{n}, \mathbf{q}) := \binom{n_{-1} + n_0 + q_0}{q_0} \binom{n_{-1} + n_0}{n_0} \prod_{\substack{i=-\tilde{m} \\ i \neq -1}}^{\tilde{m}-1} \binom{n_i + n_{i+1} + q_{i+1} - 1}{q_{i+1}} \binom{n_i + n_{i+1} - 1}{n_{i+1}}.$$

This Theorem immediately leads to the following Corollary

**Corollary 3.2.** *Fix  $m \in \mathbb{N}$ , and consider the matrix  $L$  (3.2). Then for  $N$  big enough, there exists some  $k = k(m) \in \mathbb{N}$ , and two polynomial functions  $V : \mathbb{R}^{2k} \times \mathbb{R}_+^{2k} \rightarrow \mathbb{R}$ ,  $V_1 : \mathbb{R}^{2k+\ell} \times \mathbb{R}_+^{2k+\ell} \rightarrow \mathbb{R}$  such that*

$$\mathrm{Tr}(L^m) = \sum_{j=1}^{M-1} V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1}) + V_1(\mathbf{a}_M, a_{kM+1}, \dots, a_{kM+\ell}, \mathbf{a}_1, \mathbf{b}_M, b_{kM+1}, \dots, b_{kM+\ell}, \mathbf{b}_1),$$

where  $N = kM + \ell$ ,  $\mathbf{a}_j = (a_{(j-1)k+1}, a_{(j-1)k+2}, \dots, a_{jk})$ , and similarly for  $\mathbf{b}_j$ .

**Remark 3.3.** *In other words, the function  $\mathrm{Tr}(L^m)$  is circular in the sense of Hypotheses 1.1, HP. 5. Furthermore, we notice that the local potential  $V(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2)$  is bounded from below, this can be proved using the explicit formula in Theorem 3.1 or applying the properties of super-Motzkin paths used for the proof of the theorem in [33].*

We apply the previous Corollary to the Gibbs measure of the Toda lattice (3.3), so it can be written as

$$\begin{aligned} d\mu_T &= \frac{1}{Z_N^T(\beta, P)} \prod_{j=1}^N b_j^{2\alpha-1} \mathbb{1}_{b_j > 0} \exp\left(-\sum_{j=1}^M V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1})\right. \\ &\quad \left.- V_1(\mathbf{a}_M, a_{kM+1}, \dots, a_{kM+\ell}, \mathbf{a}_1, \mathbf{b}_M, b_{kM+1}, \dots, b_{kM+\ell}, \mathbf{b}_1)\right) d\mathbf{a} d\mathbf{b}. \end{aligned}$$

We would like to apply Theorem 1.2 to the previous density with  $F(b) = b^{2\alpha-1}$ , and  $W = V$ , but in this case  $F \notin L^2(\mathbb{R}_+)$ , so we have to take care of this issue. To do it, we fix  $\varepsilon > 0$ , and consider the following measure

$$\begin{aligned} d\mu_T &= \frac{1}{Z_N^T(\beta, P)} \prod_{j=1}^N b_j^{2\alpha-1} e^{-\varepsilon(a_j^2 + b_j^2)} \mathbb{1}_{a_j > 0} \exp\left(-\sum_{j=1}^M V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1})\right. \\ &\quad \left.- V_1(\mathbf{a}_M, a_{kM+1}, \dots, a_{kM+\ell}, \mathbf{a}_1, \mathbf{b}_M, b_{kM+1}, \dots, b_{kM+\ell}, \mathbf{b}_1)\right. \\ &\quad \left.+ \varepsilon \sum_{j=1}^N a_j^2 + b_j^2\right) d\mathbf{a} d\mathbf{b}, \end{aligned}$$

this is exactly the same measure as before, but, following the notation of Theorem 1.2, we can now set

$$\begin{aligned} F(a, b, \alpha) &= b^{2\alpha-1} e^{-\varepsilon(a^2 + b^2)}, \\ W(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1}) &= V(\mathbf{a}_j, \mathbf{b}_j, \mathbf{a}_{j+1}, \mathbf{b}_{j+1}) - \frac{\varepsilon}{2} \sum_{n=1}^{2k} a_{(j-1)k+n}^2 + b_{(j-1)k+n}^2. \end{aligned}$$

These functions satisfy the hypotheses of Theorem 1.2, so we can apply it and deduce the following

**Corollary 3.4** (CLT for the Toda lattice). *Consider the Lax matrix  $L$  (3.2) of the Toda lattice distributed according to the Generalized Gibbs Ensemble (3.3), and assume that  $P(x)$  is a polynomial of even degree with positive leading order coefficient. Then, defining the Free energy  $\mathcal{F}_T(\alpha, P)$  as*

$$\mathcal{F}_T(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^T(\alpha, P)),$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(L^j) - \mathbb{E}[\text{Tr}(L^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2).$$

Where

$$\mathbb{E}[\text{Tr}(L^j)] = iN \partial_t \mathcal{F}_T(\alpha, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_T(\alpha, P + itx^j)|_{t=0}|.$$

Moreover, we can also apply Theorem 1.4 to compute the correlation between the conserved fields at time zero, indeed the theorem immediately implies that

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[\text{Tr}(L^j) \text{Tr}(L^n)] - \mathbb{E}[\text{Tr}(L^j)] \mathbb{E}[\text{Tr}(L^n)]}{N} = \partial_{t_1} \partial_{t_2} \mathcal{F}_T(\alpha, P + it_1 x^j + it_2 x^n)|_{t_1, t_2=0}, \quad (3.4)$$

where the mean value is taken with respect to the Gibbs measure of the Toda lattice (3.3). We notice that this implies that we can compute the susceptibility matrix of the Toda lattice (1.7) in terms of the derivative of the Free energy.

### 3.1.1 The Toda chain's currents

Since the conserved fields are local quantities, they must satisfy a local conservation law. Following the notation of [69], we define

$$Q_j^{[n],N} = L_{j,j}^n,$$

where  $L \in \text{Mat}(N, \mathbb{R})$  is (3.2). We can easily compute the evolution equation for such quantities as

$$\frac{d}{dt} Q_j^{[n],N} = (BL^n - L^n B) = b_{j-1} L_{j,j-1}^n - b_j L_{j+1,j}^n.$$

Defining  $J_j^{[n],N} = b_{j-1} L_{j,j-1}^n$ , we have

$$\frac{d}{dt} Q_j^{[n],N} = J_j^{[n],N} - J_{j+1}^{[n],N}$$

and we say that  $J_j^{[n],N}$  is the current of the local conserved field  $Q_j^{[n],N}$ . In particular, defining the matrix  $L^\downarrow$  as

$$L_{i,j}^\downarrow = \begin{cases} L_{i,j} & \text{if } j < i \text{ or } i = 1, j = N \\ 0 & \text{otherwise} \end{cases}$$

we can recast the previous definition as

$$J_j^{[n],N} = (L^n L^\downarrow)_{j,j}.$$

We notice that both  $Q_j^{[n],N}$  and  $J_j^{[n],N}$  depend on time, and we adopt the convention that if not explicitly written the evaluation is at time 0. Furthermore, we define

$$Q^{[n],N} = \sum_{j=1}^N Q_j^{[n],N}, \quad J^{[n],N} = \sum_{j=1}^N J_j^{[n],N}, \quad (3.5)$$

and we refer to  $Q^{[n],N}$  as the  $n^{\text{th}}$ -conserved field, and to  $J^{[n],N}$  as the  $n^{\text{th}}$ -total current.

The evaluation of the expected values of both the currents  $J_j^{[n],N}$  and the total current  $J^{[n],N}$  according to the Generalized Gibbs ensemble (3.3) is one of the crucial steps to apply the theory of Generalized Hydrodynamics to the Toda lattice, as it is explained in [69]. In this manuscript, the author used some heuristic arguments to explicitly derive the expression for these quantities, here we rigorously justify his argument applying Theorem 1.6.

First, we extend the definition of  $Q_j^{[n],N}$  and  $J_j^{[n],N}$  for  $n = 0$ , setting  $Q_j^{[0],N} = r_j$ , and  $J_j^{[0],N} = -p_j = -Q_j^{[1],N}$ . We notice that  $\sum_{j=1}^N J_j^{[0],N} = -\sum_{j=1}^N Q_j^{[1],N}$  is still a conserved field. We are now in position to show how to compute the limiting Toda average current

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[ J^{[n],N} \right],$$

in terms of the susceptibility matrix (1.7) of the Toda chain, so in particular of the derivative of the Free energy (3.4). Indeed, we prove the following:

**Lemma 3.5.** *Consider the Lax matrix  $L$  (3.2) of the Toda lattice distributed according to the Generalized Gibbs Ensemble (3.3), and assume that  $P(x)$  is a polynomial of even degree with positive leading order coefficient. Then, for any fixed  $n \in \mathbb{N}$ , and  $\alpha \in \mathbb{R}_+$  defining the total currents  $J^{[n],N}$  as in (3.5) we have the following equality*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[ J^{[n],N} \right] = \int_0^\alpha \partial_{t_1} \partial_{t_2} \mathcal{F}_T(s, P + it_1 x + it_2 x^n) |_{t_1, t_2=0} ds.$$

*Proof.* In view of the cyclic structure of the measure  $\mu_T$  and of the total current, we deduce that

$$\frac{1}{N} \mathbb{E} \left[ J^{[n],N} \right] = \mathbb{E} \left[ J_1^{[n],N} \right].$$

Furthermore, for any fixed  $N$ , we deduce, by differentiating with respect to the parameter  $\alpha$ , the following equality

$$\partial_\alpha \mathbb{E} \left[ J_1^{[n],N} \right] = -\text{Cov} \left( J_1^{[n],N} ; \sum_{j=1}^N r_j \right) = -\sum_{j=1}^N \text{Cov} \left( J_1^{[n],N} ; Q_j^{[0],N} \right), \quad (3.6)$$

where we defined for any functions  $f, g \in L^2(X^N, \mu_T)$

$$\text{Cov}(f; g) = \mathbb{E}[fg] - \mathbb{E}[f] \mathbb{E}[g].$$

We show now that the following limits coincide

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N \text{Cov} \left( J_1^{[n],N} ; Q_j^{[0],N} \right) = \lim_{N \rightarrow \infty} \sum_{j=1}^N \text{Cov} \left( J_1^{[0],N} ; Q_{N-j+2}^{[n],N} \right). \quad (3.7)$$

Indeed, for any  $n, m \geq 0$  and  $t \in \mathbb{R}$

$$\begin{aligned} \text{Cov} \left( J_{j+1}^{[n],N}(t) - J_j^{[n],N}(t) ; Q_1^{[m],N}(0) \right) &= -\frac{d}{dt} \text{Cov} \left( Q_j^{[n],N}(t) ; Q_1^{[m],N}(0) \right) \\ &= -\frac{d}{dt} \text{Cov} \left( Q_1^{[n],N}(0) ; Q_{N-j+2}^{[m],N}(-t) \right) \\ &= \text{Cov} \left( Q_1^{[n],N}(0) ; J_{N-j+3}^{[m],N}(-t) - J_{N-j+2}^{[m],N}(-t) \right), \end{aligned} \quad (3.8)$$

where we used that  $s \mapsto Q_j^{[n],N}(t+s)Q_1^{[m],N}(s)$  is constant in law under the Toda dynamic, and the periodicity of the matrix  $L$  (3.2). Denoting the difference operator  $\partial_j f(j) = f(j+1) - f(j)$ , equation (3.8) shows that

$$\partial_j \left( \text{Cov} \left( J_j^{[n],N}(t) ; Q_1^{[m],N}(0) \right) - \text{Cov} \left( Q_1^{[n],N}(0) ; J_{N-j+2}^{[m],N}(-t) \right) \right) = 0$$

Evaluating the previous expression at  $t = 0$ , we deduce that there is some constant  $c_N$ , independent of  $j$ , such that

$$\text{Cov} \left( J_j^{[n],N} ; Q_1^{[m],N} \right) - \text{Cov} \left( Q_1^{[n],N} ; J_{N-j+2}^{[m],N} \right) = c_N.$$

Furthermore, since both  $Q_j^{[n],N}$ , and  $J_j^{[m],N}$  are local quantities, in view of Theorem 1.6, we deduce that  $\lim_{N \rightarrow \infty} N c_N = 0$ . So, evaluating the previous expression for  $m = 0$ , we deduce (3.7). Thus, in the large  $N$  limit, we can recast (3.6) as

$$\lim_{N \rightarrow \infty} \partial_\alpha \mathbb{E} \left[ J_1^{[n],N} \right] = - \lim_{N \rightarrow \infty} \sum_{j=1}^N \text{Cov} \left( J_1^{[0],N} ; Q_j^{[n],N} \right) = \lim_{N \rightarrow \infty} \sum_{j=1}^N \text{Cov} \left( Q_1^{[1],N} ; Q_j^{[n],N} \right).$$

Moreover, in view of the periodicity properties of the conserved fields and (3.4)

$$\lim_{N \rightarrow \infty} \partial_\alpha \mathbb{E} \left[ J_1^{[n],N} \right] = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Cov} \left( Q^{[1],N} ; Q^{[n],N} \right) = \partial_{t_1} \partial_{t_2} \mathcal{F}_T(\alpha, P + it_1 x + it_2 x^n) |_{t_1, t_2 = 0}.$$

Noticing that  $\lim_{\alpha \rightarrow 0} \mathbb{E} \left[ J_1^{[n],N} \right] = 0$ , and that we can always uniformly bound  $\mathbb{E} \left[ J_1^{[n],N} \right]$  by a constant independent of  $N$ , the previous equation implies that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ J_1^{[n],N} \right] = \int_0^\alpha \partial_{t_1} \partial_{t_2} \mathcal{F}_T(s, P + it_1 x + it_2 x^n) |_{t_1, t_2 = 0} ds.$$

So, we conclude.  $\square$

**The real  $\beta$ -ensemble in the high-temperature regime.** The real  $\beta$ -ensemble is the probability measure on  $\mathbb{R}^N$  given by

$$d\mathbb{P}_H(\lambda_1, \dots, \lambda_N) = \frac{1}{\mathfrak{Z}_N^H(\beta, P)} \prod_{i < j} |\lambda_j - \lambda_i|^\beta e^{-\sum_{j=1}^N P(\lambda_j)} d\boldsymbol{\lambda}, \quad (3.9)$$

where  $\beta > 0$  and  $P$  is a continuous function such that the partition function

$$\mathfrak{Z}_N^H(\beta, P) = \int_{\mathbb{R}^N} \prod_{i < j} |\lambda_j - \lambda_i|^\beta e^{-\sum_{j=1}^N P(\lambda_j)} d\boldsymbol{\lambda}$$

is finite. This is the case if  $P$  grows to infinity fast enough, namely if for some  $\beta' > \max(1, \beta)$ ,

$$\liminf_{|x| \rightarrow \infty} \frac{P(x)}{N \beta' \ln |x|} > 1,$$

see [6, equation (2.6.2)].

Dumitriu and Edelman showed in [18] that the  $\beta$ -ensemble admits a tridiagonal representation

$$H = \begin{pmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & b_{N-1} & a_N \end{pmatrix}, \quad (3.10)$$

where the entries of the matrix are distributed according to the following probability measure

$$d\mu_H = \frac{1}{Z_N^H(\beta, P)} \prod_{j=1}^{N-1} b_j^{\beta(N-j)-1} \mathbb{1}_{b_j \geq 0} \exp(-\text{Tr}(P(H))) d\mathbf{a}d\mathbf{b}. \quad (3.11)$$

Then, the eigenvalues of  $H$  are distributed according to  $d\mathbb{P}_H$  (3.9). An important example is the case  $P(x) = x^2/2$  for which we recover the classical *Gaussian  $\beta$  Ensemble*, see [6, Section 2.5], and the distribution  $\mu_H$  factorizes in the following way: the entries of  $H$  can be seen to be independent (modulo the symmetry of the matrix), Gaussian  $\mathcal{N}(0, 1)$  on the diagonal, and the law of the off-diagonal elements is given by renormalized chi variables

$$b_j \sim \frac{1}{\sqrt{2}} \chi_{(N-j)\beta},$$

where the variable  $X$  is  $\chi_\kappa$ -distributed if its law is given by the density function

$$f(x) = \frac{x^{\kappa-1} e^{-x^2/2}}{2^{\kappa/2-1} \Gamma(\kappa/2)}.$$

We are interested in the so-called *high-temperature regime* for this model, specifically, we are interested in the infinite size  $N$  limit, in such a way that  $\beta = \frac{2\alpha}{N}$  for some  $\alpha > 0$ . In this regime the probability distribution (3.11) becomes

$$d\mu_H = \frac{1}{Z_N^H(\beta)} \prod_{j=1}^{N-1} b_j^{2\alpha(1-\frac{j}{N})-1} \mathbb{1}_{b_j \geq 0} \exp(-\text{Tr}(P(H))) d\mathbf{a}d\mathbf{b}.$$

This regime has drawn a lot of attention from the random matrix and statistical physics communities lately. Introducing the empirical measure by

$$d\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i},$$

this model was first considered in [4], where the authors were able to compute the limiting empirical measure for this model when  $P(x) = x^2/2$ . Recently, Garcia-Zelada showed in [28] that under a general choice of  $P$ , the sequence of empirical measures satisfies a large deviation principle with strictly convex rate function, ensuring the convergence of  $\hat{\mu}_N$ . Although the limiting measure is not explicit, its density  $\rho_\alpha^P$  satisfies for almost every  $x$  the nonlinear equation

$$P(x) - 2\alpha \int_{\mathbb{R}} \log|x-y| \rho_\alpha^P(y) dy + \log \rho_\alpha^P(x) = \lambda_\alpha^P$$

for some constant  $\lambda_\alpha^P$ , see [35, Lemma 3.2] for example.

The fluctuations of the eigenvalues in the bulk and at the edge of a configuration were studied for example in [8, 47, 57, 58, 61]. These fluctuations were shown to be described by Poisson statistics in this regime. With the choice  $P(x) = x^2/2$ , Nakano and Trinh proved in [57] a Central Limit theorem for this ensemble, namely they proved that for smooth enough  $f: \mathbb{R} \rightarrow \mathbb{R}$ , the random variables

$$\sqrt{N} \left( \int_{\mathbb{R}} f d\hat{\mu}_N - \int_{\mathbb{R}} f \rho_\alpha^P dx \right)$$

converge towards a centred Gaussian variable with variance depending both on  $\alpha$  and  $P$ . In [22], the authors showed this central limit theorem for general confining potentials and smooth enough, decaying at infinity test functions. In this paper, we consider the case where  $P$  is a

polynomial of even degree  $\geq 2$ . We deduce here from Section 2 a central limit theorem for polynomial test functions.

Indeed, in view of Corollary 3.2, following the same reasoning as in the case of Toda lattice, we can apply Theorem 1.2 to the real  $\beta$  ensemble in the high-temperature regime, thus we deduce that

**Corollary 3.6** (CLT for Gaussian  $\beta$  ensemble). *Consider the matrix representation (3.10) of the real  $\beta$  ensemble in the high-temperature regime, and let  $P(x)$  be a polynomial of even degree with positive leading order coefficient. Then, defining the Free energy  $\mathcal{F}_H(\alpha, P)$  as*

$$\mathcal{F}_H(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^T(\alpha, P)),$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(H^j) - \mathbb{E}[\text{Tr}(H^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(H^j)] = iN \partial_t \mathcal{F}_H(\alpha, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_H(\alpha, P + itx^j)|_{t=0}|.$$

Thus, we obtained a central limit theorem for the real  $\beta$  ensemble in the high-temperature regime with polynomial potential.

Furthermore, we are in place to apply the second part of our result; indeed, we deduce the following identities

$$\begin{aligned} \partial_\alpha(\alpha \partial_t \mathcal{F}_H(\alpha, P + itx^j)|_{t=0}) &= \partial_t \mathcal{F}_T(\alpha, P + itx^j)|_{t=0}, \\ \partial_\alpha(\alpha \partial_t^2 \mathcal{F}_H(\alpha, P + itx^j)|_{t=0}) &= \partial_t^2 \mathcal{F}_T(\alpha, P + itx^j)|_{t=0} \end{aligned}$$

so we are able to compute both the moments and their variances of the Toda lattice starting from the one of the real  $\beta$  ensemble at high-temperature.

**Remark 3.7.** *Applying the second part of Theorem 1.2, we deduce the following equality valid for the currents of the Toda lattice:*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ J_1^{[n], N} \right] = \int_0^\alpha \partial_{t_1} \partial_{t_2} \mathcal{F}_T(s, P + it_1 x + it_2 x^n)|_{t_1, t_2=0} ds = \alpha \partial_{t_1} \partial_{t_2} \mathcal{F}_H(\alpha, P + it_1 x + it_2 x^n)|_{t_1, t_2=0}.$$

### 3.2 The exponential Toda lattice, and the Laguerre $\beta$ ensemble at high-temperature

In this subsection, we focus on the Exponential Toda lattice and its relation with the Laguerre  $\beta$  ensemble in the high-temperature regime [27]. These two systems were considered in [32]. In this paper, the authors considered the classical Gibbs ensemble for the Exponential Toda lattice and were able to compute the density of states for this model connecting it to the Laguerre  $\alpha$  ensemble [51], which is related to the classical  $\beta$  one in the high-temperature regime. Here we consider both the Generalized Gibbs ensemble for the integrable lattice and the Laguerre  $\beta$  ensemble at high-temperature with polynomial potential, and we obtain a CLT for both systems, furthermore, we connect the two in the same way as we did for the Toda lattice and the real  $\beta$  ensemble.

**The exponential Toda lattice.** The exponential Toda lattice is the Hamiltonian system on  $\mathbb{R}^{2N}$  described by the Hamiltonian

$$H_E(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^N e^{-p_j} + \sum_{j=1}^N e^{q_j - q_{j+1}}, \quad p_j, q_j \in \mathbb{R}, \quad (3.12)$$

with canonical Poisson bracket. Here, we consider periodic boundary conditions

$$q_{j+N} = q_j + \Omega, \quad p_{j+N} = p_j, \quad \forall j \in \mathbb{Z},$$

and  $\Omega \geq 0$  is an arbitrary constant. The equations of motion are given in Hamiltonian form as

$$\begin{aligned} \dot{q}_j &= \frac{\partial H_E}{\partial p_j} = -e^{-p_j}, \\ \dot{p}_j &= -\frac{\partial H_E}{\partial q_j} = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}. \end{aligned} \quad (3.13)$$

Following [32], we perform the non-canonical change of coordinates

$$x_j = e^{-\frac{p_j}{2}}, \quad y_j = e^{\frac{q_j - q_{j+1}}{2}} = e^{-\frac{r_j}{2}}, \quad r_j = q_{j+1} - q_j, \quad j = 1, \dots, N,$$

to obtain a Lax Pair for this system. Indeed, in these variables, the Hamiltonian (3.12) transform into

$$H_E(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^N (x_j^2 + y_j^2),$$

and the Hamilton's equations (3.13) become

$$\dot{x}_j = \frac{x_j}{2} (y_j^2 - y_{j-1}^2), \quad \dot{y}_j = \frac{y_j}{2} (x_{j+1}^2 - x_j^2), \quad j = 1, \dots, N, \quad (3.14)$$

where  $x_{N+1} = x_1$ ,  $y_0 = y_N$ .

Let us introduce the matrices  $L, A \in \text{Mat}(N)$  as

$$\begin{aligned} L &= \begin{pmatrix} x_1^2 + y_N^2 & x_1 y_1 & & & x_N y_N \\ x_1 y_1 & x_2^2 + y_1^2 & x_2 y_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x_{N-1} y_{N-1} \\ x_N y_N & & & x_{N-1} y_{N-1} & x_N^2 + y_{N-1}^2 \end{pmatrix}, \\ A &= \frac{1}{2} \begin{pmatrix} 0 & x_1 y_1 & & & -x_N y_N \\ -x_1 y_1 & 0 & x_2 y_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x_{N-1} y_{N-1} \\ x_N y_N & & & -x_{N-1} y_{N-1} & 0 \end{pmatrix}, \end{aligned} \quad (3.15)$$

The system of equations (3.14) admits the Lax representation

$$\dot{L} = [A, L].$$

Hence, the quantities  $H_m = \text{Tr}(L^m)$ ,  $m = 1, \dots, N$  are constants of motion as well as the eigenvalues of  $L$ . For this integrable model, we define the generalized Gibbs ensemble as

$$d\mu_{ET} = \frac{1}{Z_N^{H_E}(\alpha, \gamma, P)} \prod_{j=1}^N x_j^{2\frac{\alpha}{\gamma}-1} y_j^{2\alpha-1} \mathbb{1}_{x_j \geq 0} \mathbb{1}_{y_j \geq 0} e^{-\text{Tr}(P(L))} dx dy, \quad (3.16)$$



where  $\alpha, \gamma > 0$ , and  $P$  is a real valued polynomial with positive leading coefficient.  $Z_N^{HE}(\alpha, \gamma, P)$  is the normalization constant.

**Remark 3.8.** *The definition of our Gibbs ensemble is slightly different from the one given in [32], indeed there the authors were considering just the classical Gibbs ensemble for this model, so the case  $P(x) = x/2$ .*

We notice that the structure of (3.16) resembles the one of  $\mu_{kM+\ell}^{(1)}$  (2.1), thus we want to apply Theorem 1.2. To do this, we have to identify the functions  $F, W$ . First, as an application of Theorem 3.1, we obtain the following corollary

**Corollary 3.9.** *Fix  $m \in \mathbb{N}$ , and consider the matrix  $L$  (3.15). Then for  $N$  big enough, there exists some  $k = k(m) \in \mathbb{N}$ , and two polynomial functions  $V : \mathbb{R}_+^{2k} \times \mathbb{R}_+^{2k} \rightarrow \mathbb{R}$  and  $V_1 : \mathbb{R}_+^{2k+\ell} \times \mathbb{R}_+^{2k+\ell} \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} \text{Tr}(L^m) &= \sum_{j=1}^M V(\mathbf{x}_j, \mathbf{y}_j, \mathbf{x}_{j+1}, \mathbf{y}_{j+1}) \\ &\quad + V_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1, \mathbf{y}_M, y_{kM+1}, \dots, y_{kM+\ell}, \mathbf{y}_1), \end{aligned}$$

where  $N = kM + \ell$ .

As in the Toda lattice case, if we naively set  $F(x, y) = x^{2\frac{\alpha}{\gamma}-1} y^{2\alpha-1}$ , this would not fit in the hypotheses of our theorem, since this is not an  $L^2(\mathbb{R}_+^2)$  function. As in the previous case, we have just to consider a slight modification of the measure:

$$d\mu_{ET} = \frac{1}{Z_N^{HE}(\alpha, \gamma, P)} \prod_{j=1}^N x_j^{2\frac{\alpha}{\gamma}-1} y_j^{2\alpha-1} \exp\left(-\varepsilon \frac{x_j^2 + y_j^2}{2}\right) \mathbb{1}_{x_j \geq 0} \mathbb{1}_{y_j \geq 0} e^{-\text{Tr}(P(L)) + \varepsilon \frac{x_j^2 + y_j^2}{2}} dx dy,$$

for fixed  $\varepsilon > 0$ , but small. In this way, defining  $F(x, y, \alpha) = x^{2\frac{\alpha}{\gamma}-1} y^{2\alpha-1} \exp(-\varepsilon \frac{x^2 + y^2}{2})$ , and  $W(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) = V(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) - \frac{\varepsilon}{2} \sum_{j=1}^{2k} x_j^2 + y_j^2$  we are in the same hypotheses as Theorem 1.2, thus we deduce the following corollary

**Corollary 3.10** (CLT for the Exponential Toda lattice). *Consider the Lax matrix  $L$  (3.15) of the Exponential Toda lattice distributed according to the Generalized Gibbs Ensemble (3.16). Then, defining the Free energy  $\mathcal{F}_{HE}(\alpha, \gamma, P)$  as*

$$\mathcal{F}_{ET}(\alpha, \gamma, P) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^{HE}(\alpha, \gamma, P)),$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(L^j) - \mathbb{E}[\text{Tr}(L^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(L^j)] = iN \partial_t \mathcal{F}_{ET}(\alpha, \gamma, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{ET}(\alpha, \gamma, P + itx^j)|_{t=0}|$$

**The Laguerre  $\beta$  ensemble in the high-temperature regime.** The Laguerre  $\beta$  ensemble is a random matrix ensemble introduced by Dumitriu and Edelman in [18]. It has the following matrix representation

$$Q = \begin{pmatrix} x_1^2 & x_1 y_1 & & & & \\ x_1 y_1 & x_2^2 + y_1^2 & x_2 y_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & x_{N-1} y_{N-1} \\ & & & & x_{N-1} y_{N-1} & x_N^2 + y_{N-1}^2 \end{pmatrix}, \quad (3.17)$$

where the entries of  $Q$  are distributed according to

$$d\mu_L = \frac{1}{Z_N^L(\beta)} \prod_{j=1}^N x_j^{\beta(M-j+1)-1} \mathbb{1}_{x_j \geq 0} \prod_{j=1}^{N-1} y_j^{\beta(N-j)-1} \mathbb{1}_{y_j \geq 0} \exp(-\text{Tr}(P(Q))) \, dx dy, \quad (3.18)$$

where  $M$  is such that  $\lim_{N \rightarrow \infty} N/M = \gamma \in (0, 1]$ , and  $P$  can be any continuous function such that the partition function is well-defined, for our purpose we consider  $P(x)$  to be a polynomial.

The remarkable property of this ensemble is that it is possible to explicitly compute the joint eigenvalue density as

$$d\mathbb{P}_L = \frac{1}{\mathfrak{Z}_N^L(\beta, P)} \prod_{j=1}^N \lambda_j^{\frac{\beta}{2}(M-N+1)-1} \mathbb{1}_{\lambda_j \geq 0} \prod_{j < i} |\lambda_j - \lambda_i|^\beta e^{-\sum_{j=1}^N P(\lambda_j)} d\boldsymbol{\lambda}.$$

We are interested in the so-called high-temperature limit, i.e. when  $\beta = \frac{2\alpha}{N}$ ,  $\alpha \in \mathbb{R}_+$ , which was considered in [5], where the authors were able to compute the density of states for the particular case  $P(x) = x/2$ .

In this regime, the density (3.18) takes the form

$$d\mu_L = \frac{1}{Z_N^L(\alpha, \gamma, P)} \prod_{j=1}^N x_j^{\frac{2\alpha}{\gamma}(1-\frac{j+1}{N})-1} \mathbb{1}_{x_j \geq 0} \prod_{j=1}^{N-1} y_j^{2\alpha(1-\frac{j}{N})-1} \mathbb{1}_{y_j \geq 0} \exp(-\text{Tr}(P(Q))) \, dx dy.$$

The structure of this density resembles the one of  $d\mu_{kM+\ell}^{(2)}$ (2.2), indeed proceeding as in the case of the Exponential Toda lattice, we deduce the following corollary

**Corollary 3.11** (CLT for Laguerre  $\beta$  ensemble). *Consider the matrix representation (3.17) of the Laguerre  $\beta$  ensemble in the high-temperature regime, and let  $P(x)$  be a real polynomial of degree at least 1. Then, defining the Free energy  $\mathcal{F}_L(\alpha, \gamma, P)$  as*

$$\mathcal{F}_L(\alpha, \gamma, P) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^L(\alpha, \gamma, P)),$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(Q^j) - \mathbb{E}[\text{Tr}(Q^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(Q^j)] = iN \partial_t \mathcal{F}_L(\alpha, \gamma, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_L(\alpha, \gamma, P + itx^j)|_{t=0}|$$

Which is the perfect analogue of the result for the Exponential Toda lattice. Furthermore, we are in position to apply the second part of our result, indeed we can deduce the following identities

$$\begin{aligned} \partial_\alpha(\alpha \partial_t \mathcal{F}_L(\alpha, \gamma, P + itx^j)|_{t=0}) &= \partial_t \mathcal{F}_{ET}(\alpha, \gamma, P + itx^j)|_{t=0}, \\ \partial_\alpha(\alpha \partial_t^2 \mathcal{F}_L(\alpha, \gamma, P + itx^j)|_{t=0}) &= \partial_t^2 \mathcal{F}_{ET}(\alpha, \gamma, P + itx^j)|_{t=0}, \end{aligned}$$

thus, we can compute all the quantities involved in the previous theorems from the Free Energy of the Laguerre ensemble.

### 3.3 The Volterra lattice, and the antisymmetric $\beta$ ensemble at high-temperature

In this subsection, we focus on the Volterra lattice and its relation with the Antisymmetric  $\beta$  ensemble [20] in the high-temperature regime [27]. These two systems were considered in [32]. In this paper, the authors considered the classical Gibbs ensemble for the Volterra lattice and were able to compute the density of states for this model connecting it to the Antisymmetric  $\alpha$  ensemble [27], which is related to the classical  $\beta$  one introduced by Dumitriu and Forrester [20].

**The Volterra Lattice.** The Volterra lattice (or *discrete KdV equation*) is the following systems of  $N$  coupled ODEs

$$\dot{a}_j = a_j (a_{j+1} - a_{j-1}), \quad j = 1, \dots, N, \quad (3.19)$$

here  $a_j \in \mathbb{R}_+$  for  $j = 1, \dots, N$ , and we consider periodic boundary conditions  $a_j = a_{j+N}$  for all  $j \in \mathbb{Z}$ . Volterra introduced it to study evolution of populations in a hierarchical system of competing species. This system was considered by Kac and van Moerbeke in [42], who solved it explicitly using a discrete version of the inverse scattering transform introduced by Flaschka [24].

Introducing on the phase space  $\mathbb{R}_+^N$  the following Poisson bracket

$$\{a_j, a_i\}_{\text{Volt}} = a_j a_i (\delta_{i,j+1} - \delta_{i,j-1}),$$

and defining the Hamiltonian  $H_1 = \sum_{j=1}^N a_j$ , we can rewrite the equations of motion (3.19) in Hamiltonian form as

$$\dot{a}_j = \{a_j, H_1\}_{\text{Volt}}. \quad (3.20)$$

An elementary constant of motion for the system is  $H_0 = \prod_{j=1}^N a_j$  which is independent of  $H_1$ .

The Volterra lattice is a completely integrable system, and it admits several equivalents *Lax representations*, see e.g. [32, 42, 54]. We use the one presented in [32]. Specifically, we introduce the matrices  $L, A \in \text{Mat}(\mathbb{R}, N)$  as

$$L = \begin{pmatrix} 0 & \sqrt{a_1} & & & -\sqrt{a_N} \\ -\sqrt{a_1} & 0 & \sqrt{a_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{a_{N-1}} \\ \sqrt{a_N} & & & -\sqrt{a_{N-1}} & 0 \end{pmatrix}, \quad (3.21)$$

$$A = \frac{1}{2} \sum_{j=1}^N \sqrt{a_j a_{j+1}} (E_{j,j+2} - E_{j+2,j}),$$

where  $E_{r,s}$  is defined as  $(E_{r,s})_{ij} = \delta_r^i \delta_s^j$  and  $E_{j+N,i} = E_{j,i+N} = E_{j,i}$ . Then, it follows that the equations of motion (3.20) are equivalent to

$$\dot{L} = [L; A].$$

In view of this Lax pair, we know that  $\text{Tr}(L^k)$  are constant of motion for the model.

Following [32], we introduce the Generalized Gibbs Ensemble of the Volterra lattice (3.19) as

$$d\mu_{\text{Volt}}(\mathbf{a}) = \frac{e^{\text{Tr}(P(L))} \prod_{j=1}^N a_j^{\alpha-1} \mathbb{1}_{a_j > 0} d\mathbf{a}}{Z_N^{\text{Volt}}(\alpha, P)}, \quad (3.22)$$

where  $\alpha > 0$ ,  $P(x)$  is a polynomial of the form  $P(x) = (-1)^j x^{2j} + \text{l.o.t}$ , otherwise the previous measure is not normalizable, moreover, we notice that, in view of the antisymmetric nature of

$L$ ,  $\text{Tr}(L^{2j+1}) = 0$ . For this reason, we perform the change of coordinates  $\sqrt{a_j} = x_j$ , where we take just the positive root, so the previous measure read

$$d\mu_{\text{Volt}}(\mathbf{x}) = \frac{e^{\text{Tr}(P(L))} \prod_{j=1}^N x_j^{2\alpha-1} \mathbb{1}_{x_j>0} d\mathbf{x}}{Z_N^{\text{Volt}}(\alpha, P)}.$$

This Generalized Gibbs ensemble resembles the structure of  $\mu_{kM+\ell}^{(1)}$  (2.1), we have just to identify  $F, W$ . We notice that it is possible to generalize Theorem 3.1 also for the antisymmetric situation, so we deduce the following Corollary:

**Corollary 3.12.** *Fix  $m \in \mathbb{N}$ , and consider the matrix  $L$  (3.21). Then for  $N$  big enough, there exists a  $k = k(m) \in \mathbb{N}$ , and two polynomial functions  $V : \mathbb{R}_+^k \times \mathbb{R}_+^k \rightarrow \mathbb{R}$ ,  $V_1 : \mathbb{R}_+^k \times \mathbb{R}_+^\ell \times \mathbb{R}_+^k \rightarrow \mathbb{R}$  such that*

$$\text{Tr}(L^m) = \sum_{j=1}^M V(\mathbf{x}_j, \mathbf{x}_{j+1}) + V_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1),$$

where  $N = kM + \ell$ .

Thus, following the same kind of reasoning as in the Toda lattice, section 3.1, and the Exponential Toda lattice, section 3.2, we deduce the following:

**Corollary 3.13** (CLT for Volterra lattice). *Consider the Lax matrix  $L$  (3.21) of the Volterra lattice distributed according to the Generalized Gibbs Ensemble (3.22). Then, defining the Free energy  $\mathcal{F}_{\text{Volt}}(\alpha, P)$  as*

$$\mathcal{F}_{\text{Volt}}(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^{\text{Volt}}(\alpha, P)),$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(L^j) - \mathbb{E}[\text{Tr}(L^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(L^j)] = iN \partial_t \mathcal{F}_{\text{Volt}}(\alpha, P - itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{\text{Volt}}(\alpha, P - itx^j)|_{t=0}|.$$

**The Antisymmetric  $\beta$  ensemble in the high-temperature regime** The Antisymmetric  $\beta$  ensemble is a random matrix ensemble introduced by Dumitriu and Forrester in [20]; it has the following matrix representation

$$Q = \begin{pmatrix} 0 & x_1 & & & & \\ -x_1 & 0 & x_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & x_{N-1} & \\ & & & -x_{N-1} & 0 & \end{pmatrix}, \quad (3.23)$$

and the entries of the matrix  $Q$  are distributed according to

$$d\mu_{AG} = \frac{1}{Z_N^{AG}(\beta, P)} \prod_{j=1}^{N-1} x_j^{\beta(N-j)-1} \mathbb{1}_{x_j \geq 0} \exp(\text{Tr}(P(Q))) d\mathbf{x}, \quad (3.24)$$

here  $P(x)$  can be any function that makes (3.24) normalizable, but for our purpose we will consider  $P(x)$  polynomial of the form  $P(x) = (-1)^j x^{2j} + \text{l.o.t.}$

As in the previous cases, we are interested in the high-temperature regime for this model, so we set  $\beta = \frac{2\alpha}{N}$ , and we rewrite the previous density as

$$d\mu_{AG} = \frac{1}{Z_N^{AG}(\alpha, P)} \prod_{j=1}^{N-1} x_j^{\alpha(1-\frac{j}{N})-1} \mathbb{1}_{x_j \geq 0} \exp(\text{Tr}(P(Q))) dx. \quad (3.25)$$

This regime was introduced in [32], where the author computed the density of states for this model in the case  $P(x) = x^2/2$ . The structure of this last density (3.25) resembles the one of  $\mu_{kM+\ell}^{(2)}(2.2)$ , indeed proceeding as in the case of the Volterra lattice, we deduce the following corollary

**Corollary 3.14** (CLT for Antisymmetric  $\beta$  ensemble). *Consider the matrix representation (3.23) of the Antisymmetric  $\beta$  ensemble in the high-temperature regime, endowed with the probability distribution  $d\mu_{AG}$  (3.25), and let  $P(x)$  be a polynomial of the form  $P(x) = (-1)^j x^{2j} + l.o.t.$ . Then, defining the Free energy  $\mathcal{F}_{AG}(\alpha, P)$  as*

$$\mathcal{F}_{AG}(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^{AG}(\alpha, P)),$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(Q^j) - \mathbb{E}[\text{Tr}(Q^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2).$$

where

$$\mathbb{E}[\text{Tr}(Q^j)] = iN \partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{AG}(\alpha, P - itx^j)|_{t=0}.$$

Which is the perfect analogue of the result for the Volterra lattice.

**Remark 3.15.** *In view of Theorem 1.2, we deduce the following identities*

$$\begin{aligned} \partial_\alpha(\alpha \partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)|_{t=0}) &= \partial_t \mathcal{F}_{AG}(\alpha, P - itx^j)|_{t=0}, \\ \partial_\alpha(\alpha \partial_t^2 \mathcal{F}_{AG}(\alpha, P - itx^j)|_{t=0}) &= \partial_t^2 \mathcal{F}_{AG}(\alpha, P - itx^j)|_{t=0} \end{aligned}$$

### 3.4 The defocusing Ablowitz-Ladik lattice, and the Circular $\beta$ ensemble at high-temperature

In this subsection, we focus on the defocusing Ablowitz-Ladik lattice, and its relation to the Circular  $\beta$  ensemble at high-temperature [34, 37, 69]. This relation was highlighted by one of the present authors and T. Grava [34], and independently by H. Spohn [69]. In these papers, the authors were able to characterize the density of states of the Ablowitz-Ladik lattice in terms of the one of the Circular  $\beta$  ensemble in the high-temperature regime. Moreover, in [34] the authors were able to compute explicitly the density of states in the case of linear potential in terms of the solution of the Double Confluent Heun Equation [16] highlighting a connection with the Painlevé equations [26, 49]. In [53], the two present authors obtained a large deviations principles for the empirical spectral measure for any continuous and bounded potential.

**The defocusing Ablowitz-Ladik lattice.** The defocusing Ablowitz-Ladik (dAL) lattice is defined by the following system of nonlinear equations

$$i\dot{a}_j = -(a_{j+1} + a_{j-1} - 2a_j) + |a_j|^2(a_{j-1} + a_{j+1}), \quad (3.26)$$

where  $a_j(t) \in \mathbb{C}$ . We assume  $N$ -periodic boundary conditions  $a_{j+N} = a_j$ , for all  $j \in \mathbb{Z}$ . The dAL lattice was introduced by Ablowitz and Ladik [2, 3] as the spatial integrable discretization

of the defocusing cubic nonlinear Schrödinger Equation for the complex function  $\psi(x, t)$ ,  $x \in S^1$  and  $t \in \mathbb{R}$ :

$$i\partial_t\psi(x, t) = -\partial_x^2\psi(x, t) + 2|\psi(x, t)|^2\psi(x, t).$$

As for the others dynamical systems that we considered, the dAL is an integrable system. Its integrability was proved by Ablowitz and Ladik by discretizing the  $2 \times 2$  Zakharov-Shabat Lax pair [1, 2] of the cubic nonlinear Schrödinger equation. Furthermore, Nenciu and Simon [60, 64] constructed a new Lax pair for this lattice. Following their construction we double the size of the chain according to the periodic boundary condition, thus we consider a chain of  $2N$  particles  $a_1, \dots, a_{2N}$  such that  $a_j = a_{j+N}$  for  $j = 1, \dots, N$ . Define the  $2 \times 2$  unitary matrix  $\Xi_j$

$$\Xi_j = \begin{pmatrix} \bar{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix}, \quad j = 1, \dots, 2N, \quad \rho_j = \sqrt{1 - |a_j|^2} \quad (3.27)$$

and the  $2N \times 2N$  matrices

$$\mathcal{M} = \begin{pmatrix} -a_{2N} & & & & & & & \rho_{2N} \\ & \Xi_2 & & & & & & \\ & & \Xi_4 & & & & & \\ & & & \ddots & & & & \\ & & & & \Xi_{2N-2} & & & \\ \rho_{2N} & & & & & & & \bar{a}_{2N} \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} \Xi_1 & & & & & \\ & \Xi_3 & & & & \\ & & \ddots & & & \\ & & & \Xi_{2N-1} & & \end{pmatrix}.$$

Now let us define the unitary Lax matrix

$$\mathcal{E} = \mathcal{L}\mathcal{M}, \quad (3.28)$$

that has the structure of a 5-band periodic diagonal matrix. The matrix  $\mathcal{E}$  is a periodic CMV matrix [13]. The equations of motion (3.26) are equivalent to the following Lax equation for the matrix  $\mathcal{E}$ :

$$\dot{\mathcal{E}} = i[\mathcal{E}, \mathcal{E}^+ + (\mathcal{E}^+)^\dagger],$$

where  $\dagger$  stands for hermitian conjugate and

$$\mathcal{E}_{j,k}^+ = \begin{cases} \frac{1}{2}\mathcal{E}_{j,j} & j = k \\ \mathcal{E}_{j,k} & k = j + 1 \bmod 2N \text{ or } k = j + 2 \bmod 2N \\ 0 & \text{otherwise.} \end{cases}$$

Since the matrix  $\mathcal{E}$  is a periodic band matrix with fixed bandwidth, we can follow the same reasoning as in the previous cases and conclude the following

**Lemma 3.16.** *Fix  $m \in \mathbb{N}$ , and consider the matrix  $\mathcal{E}$  (3.28). Then for  $N$  big enough, there exists a  $k = k(m) \in \mathbb{N}$ , and two polynomials  $V : \mathbb{C}_+^k \times \mathbb{C}_+^k \rightarrow \mathbb{R}$ , and  $V : \mathbb{C}_+^k \times \mathbb{C}_+^\ell \times \mathbb{C}_+^k \rightarrow \mathbb{R}$  such that*

$$\text{Tr}(\mathcal{E}^m) = \sum_{j=1}^M V(\mathbf{a}_j, \mathbf{a}_{j+1}) + V_1(\mathbf{a}_M, \mathbf{a}_{kM+1}, \dots, \mathbf{a}_{kM+\ell}, \mathbf{a}_1),$$

where  $N = kM + \ell$ .

Following [34, 53, 69], we notice that the quantity  $K_0 = \prod_{j=1}^N (1 - |a_j|^2)$  is conserved, so this means that if  $|a_j(0)| < 1$  for all  $j = 1, \dots, N$  then  $|a_j(t)| < 1$  for all  $j = 1, \dots, N$  for all  $t \in \mathbb{R}$ , so we can consider  $\mathbb{D}^N$  as our phase space, here  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ . On this phase space, we introduce the Generalized Gibbs ensemble for the defocusing AL lattice as

$$d\mu_{dAL} = \frac{\prod_{j=1}^N (1 - |a_j|^2)^{\alpha-1} \mathbb{1}_{a_j \in \mathbb{D}} \exp(-\text{Tr}(P(\mathcal{E}))) d^2 \mathbf{a}}{Z_N^{dAL}(\alpha, P)}, \quad (3.29)$$

where  $P$  is a real-valued Polynomial, meaning that there exists a polynomial  $\tilde{P}$  such that  $P = \Re(\tilde{P})$ . In view of Lemma 3.16, we are in the hypotheses of Theorem 1.2, thus we deduce the following:

**Corollary 3.17** (CLT for defocusing Ablowitz–Ladik lattice). *Consider the Lax matrix  $\mathcal{E}$  (3.28) of the defocusing Ablowitz–Ladik lattice distributed according to the Generalized Gibbs Ensemble (3.29). Then, defining the Free energy  $\mathcal{F}_{dAL}(\alpha, P)$  as*

$$\mathcal{F}_{dAL}(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{\ln(Z_N^{dAL}(\alpha, P))}{2N},$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(\mathcal{E}^j) - \mathbb{E}[\text{Tr}(\mathcal{E}^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(\mathcal{E}^j)] = iN \partial_t \mathcal{F}_{dAL}(\alpha, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_{dAL}(\alpha, P + itx^j)|_{t=0}|.$$

**The circular  $\beta$  ensemble at high-temperature.** The circular  $\beta$  ensemble was introduced by Killip and Nenciu in [45]; as the other  $\beta$  ensembles that we considered, it possesses a matrix representation. Consider the two block diagonal matrices

$$\mathfrak{M} = \text{diag}(\Xi_1, \Xi_3, \Xi_5, \dots) \quad \text{and} \quad \mathfrak{L} = \text{diag}(\Xi_0, \Xi_2, \Xi_4, \dots),$$

where the block  $\Xi_j$ ,  $j = 1, \dots, N-1$  are defined in (3.27), while  $\Xi_0 = (1)$  and  $\Xi_N = (\bar{\alpha}_N)$  are  $1 \times 1$  matrices. Then, we define  $\mathfrak{E}$  as follows

$$\mathfrak{E} = \mathfrak{L}\mathfrak{M}. \quad (3.30)$$

The entries of this matrix are distributed according to

$$d\mu_C = \frac{\prod_{j=1}^{N-1} (1 - |a_j|^2)^{\beta(N-j)-1} \mathbb{1}_{a_j \in \mathbb{D}} \exp(-\text{Tr}(P(\mathfrak{E}))) \prod_{j=1}^{N-1} d^2 a_j \frac{da_N}{ia_N}}{Z_N^{dAL}(\beta, P)}.$$

As for the other  $\beta$  ensembles, one can explicitly compute the joint eigenvalue density for this ensemble as

$$d\mathbb{P}_C = \frac{1}{\mathfrak{Z}_N^C(\beta, P)} \prod_{j < \ell} |e^{i\theta_j} - e^{i\theta_\ell}|^\beta \mathbb{1}_{\theta_j \in \mathbb{T}} e^{-\sum_{j=1}^N P(e^{i\theta_j})} d\boldsymbol{\theta},$$

here  $\mathbb{T} = [-\pi, \pi)$ ,  $e^{i\theta_j}$  are the eigenvalues of  $\mathfrak{E}$ , and  $P$  can be any continuous function that makes the measure normalizable. We restrict our attention to the class of real polynomial  $P(z)$ .

We are interested in the high-temperature limit for this ensemble [34, 69], so we set  $\beta = \frac{2\alpha}{N}$ , obtaining

$$d\mu_C = \frac{\prod_{j=1}^{N-1} (1 - |a_j|^2)^{2\alpha(1-\frac{j}{N})-1} \mathbb{1}_{a_j \in \mathbb{D}} \exp(-\text{Tr}(P(\mathfrak{E}))) \prod_{j=1}^{N-1} d^2 a_j \frac{da_N}{ia_N}}{Z_N^{dAL}(\alpha, P)}. \quad (3.31)$$

So, in view of Lemma 3.16, we are in the hypotheses of Theorem 1.2, so we deduce the following

**Corollary 3.18** (CLT for Circular  $\beta$  ensemble). *Consider the matrix representation (3.30) of the Circular  $\beta$  ensemble in the high-temperature regime, endowed with the probability distribution  $d\mu_C$  (3.31), and let  $P(x)$  be a real-valued polynomial. Then, defining the Free energy  $\mathcal{F}_C(\alpha, P)$  as*

$$\mathcal{F}_C(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{\ln(Z_N^C(\alpha, P))}{2N},$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(\mathfrak{E}^j) - \mathbb{E}[\text{Tr}(\mathfrak{E}^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(\mathfrak{E}^j)] = iN \partial_t \mathcal{F}_C(\alpha, \gamma, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_C(\alpha, P + itx^j)|_{t=0}.$$

**Remark 3.19.** *We notice that*

- *Hardy and Lambert in [37] already proved a CLT theorem for the Circular  $\beta$  ensemble in the high-temperature regime for a wider class of functions and potentials than we can consider with our result. Nevertheless, we highlight the fact that in our case we can explicitly compute the means, and the variances in terms of the Free energy.*
- *The following identities hold in view of the last part of Theorem 1.2*

$$\begin{aligned} \partial_\alpha(\alpha \partial_t \mathcal{F}_C(\alpha, P + itx^j)|_{t=0}) &= \partial_t \mathcal{F}_{dAL}(\alpha, P + itx^j)|_{t=0}, \\ \partial_\alpha(\alpha \partial_t^2 \mathcal{F}_C(\alpha, P + itx^j)|_{t=0}) &= \partial_t \mathcal{F}_{dAL}(\alpha, P + itx^j)|_{t=0}. \end{aligned}$$

*This relation was already proved in [34] with the same kind of argument that we followed.*

### 3.5 The defocusing Schur flow, and the Jacobi $\beta$ ensemble at high-temperature

In this subsection, we focus on the defocusing Schur flow [30], and its relation to the Jacobi  $\beta$  ensemble at high-temperature [27]. This relation was first noticed in [69], and then the two present authors obtained a large deviations principles for the empirical spectral measure for the defocusing Schur flow, and they were able to link it to the one of the Jacobi  $\beta$  ensemble in the high-temperature regime [53].

**The defocusing Schur flow.** The defocusing Schur flow is the system of ODEs [30]

$$\dot{a}_j = \rho_j^2(a_{j+1} - a_{j-1}), \quad \rho_j = \sqrt{1 - |a_j|^2},$$

and, as before, we consider periodic boundary conditions, namely  $a_j = a_{j+N}$  for all  $j \in \mathbb{Z}$ .

We notice that, if one chooses an initial data such that  $a_j(0) \in \mathbb{R}$  for all  $j = 1, \dots, N$ , then  $a_j(t) \in \mathbb{R}$  for all times. Moreover, it is straightforward to verify that  $K_0 = \prod_{j=1}^N (1 - |a_j|^2)$  is conserved along the Schur flow. This implies that we can choose as phase space for the Schur flow the  $N$ -cube  $\mathbb{I}^N$ , where  $\mathbb{I} := (-1, 1)$ . Furthermore, it was shown in [30], that the Schur flow has the same Lax matrix as the focusing Ablowitz–Laddik lattice.

Following [32, 69], on  $\mathbb{I}^N$  we define the finite volume limit GGE as

$$d\mu_{dS}(\mathbf{a}) = \frac{\prod_{j=1}^N (1 - a_j^2)^{\alpha-1} \mathbb{1}_{\mathbf{a}_j \in \mathbb{I}} \exp(-\text{Tr}(P(\mathcal{E}))) d\mathbf{a}}{Z_N^{dS}(\alpha, P)}, \quad (3.32)$$

where  $P(x) : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial. Thanks to Lemma 3.16, we can apply Theorem 1.2 obtaining a CLT theorem for the defocusing Schur flow





considering orthogonal polynomials, obtaining an explicit recurrence relation for the limiting variance.

Again, thanks to Lemma 3.16, we can apply Theorem 1.2 deducing the following

**Corollary 3.21** (CLT for Jacobi  $\beta$  ensemble in the high-temperature). *Consider the matrix representation  $\mathfrak{E}$  (3.30) of the Jacobi  $\beta$  ensemble in the high-temperature regime (3.34). Then, defining the Free energy  $\mathcal{F}_J(\alpha, P)$  as*

$$\mathcal{F}_J(\alpha, P) = - \lim_{N \rightarrow \infty} \frac{\ln(Z_N^J(\alpha, P))}{N},$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\lim_{N \rightarrow \infty} \frac{\text{Tr}(\mathfrak{E}^j) - \mathbb{E}[\text{Tr}(\mathfrak{E}^j)]}{\sqrt{N}} \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\mathbb{E}[\text{Tr}(\mathfrak{E}^j)] = iN \partial_t \mathcal{F}_J(\alpha, P + itx^j)|_{t=0}, \quad \sigma^2 = |\partial_t^2 \mathcal{F}_J(\alpha, P + itx^j)|_{t=0}.$$

**Remark 3.22.** *We notice that for  $N$  even, for  $\mathbf{a} + \mathbf{b} = -1 + \frac{\beta}{4}$  we can apply the final part of Theorem 1.2, thus we deduce that*

$$\begin{aligned} \partial_\alpha(\alpha \partial_t \mathcal{F}_J(\alpha, P + itx^j)|_{t=0}) &= \partial_t \mathcal{F}_{dS}(\alpha, P + itx^j)|_{t=0}, \\ \partial_\alpha(\alpha \partial_t^2 \mathcal{F}_J(\alpha, P + itx^j)|_{t=0}) &= \partial_t^2 \mathcal{F}_{dS}(\alpha, P + itx^j)|_{t=0} \end{aligned}$$

### 3.6 The Itoh–Narita–Bogoyavleskii lattices

In this section, we apply our results to two families of integrable lattices with short-range interaction that generalize the Volterra one (3.19). These families are described in [10] (see also [9, 40, 59]).

One is called *additive Itoh–Narita–Bogoyavleskii (INB)  $r$ -lattice* and is defined by the following equations

$$\dot{a}_i = a_i \left( \sum_{j=1}^r a_{i+j} - \sum_{j=1}^r a_{i-j} \right), \quad i = 1, \dots, N, \quad N \geq r \in \mathbb{N}. \quad (3.35)$$

The second family is called the *multiplicative Itoh–Narita–Bogoyavleskii (INB)  $r$ -lattice* and is defined by the equations

$$\dot{a}_i = a_i \left( \prod_{j=1}^r a_{i+j} - \prod_{j=1}^r a_{i-j} \right), \quad i = 1, \dots, N, \quad N \geq r \in \mathbb{N}. \quad (3.36)$$

In both cases we consider the periodicity condition  $a_{j+N} = a_j$ . We notice that setting  $r = 1$ , we recover in both cases the Volterra lattice. Moreover, both families admit the KdV equation as continuum limits, see [10].

In both cases the interaction is short-range, but in the additive case (3.35) the nonlinearity is quadratic as in the Volterra lattice, instead in the multiplicative one (3.36) it is of polynomial order.

As we already mentioned, both families are integrable for all  $r \in \mathbb{N}$ , indeed both families admits a Lax pair formulation. For the additive INB lattice (3.35), it reads

$$\begin{aligned}
L^{(+,r)} &= \sum_{i=1}^N (a_{i+r} E_{i+r,i} + E_{i,i+1}) \tag{3.37} \\
&= \begin{pmatrix} 0 & 1 & 0 & \cdots & a_{N-r} & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & a_{N-r+1} & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & a_{N-r+2} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ a_{r+1} & 0 & \cdots & \cdots & 0 & 1 & 0 & 0 \\ 0 & a_{r+2} & 0 \cdots & \cdots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 1 \\ 1 & 0 & \cdots & a_{N-r-1} & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \downarrow N-r \text{ column} \\ \\ \\ \\ \leftarrow r+1 \text{ row} \end{array} \\
A^{(+,r)} &= \sum_{i=1}^N \left( \sum_{j=0}^r a_{i+j} \right) E_{i,i} + E_{i,i+r+1},
\end{aligned}$$

we recall that we are always considering periodic boundary conditions, so for all  $j \in \mathbb{Z}$ ,  $a_{j+N} = a_j$  and  $E_{i,j+N} = E_{i+N,j} = E_{i,j}$ . In this notation, the equations of motion (3.35) are equivalent to

$$\dot{L}^{(+,r)} = [L^{(+,r)}; A^{(+,r)}].$$

Analogously, the multiplicative INB  $r$ -lattices have a Lax Pair formulation, which reads

$$\begin{aligned}
L^{(\times,r)} &= \sum_{i=1}^N (a_i E_{i,i+1} + E_{i+r,i}), \tag{3.38} \\
&= \begin{pmatrix} 0 & a_1 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 1 & 0 & \cdots & \cdots & 0 & a_r & 0 & 0 \\ 0 & 1 & 0 \cdots & \cdots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & a_{N-1} \\ a_N & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \downarrow N-r \text{ column} \\ \\ \\ \\ \leftarrow r+1 \text{ row} \end{array} \\
A^{(\times,r)} &= \sum_{i=1}^N \left( \prod_{j=0}^r a_{i+j} \right) E_{i,i+r+1}.
\end{aligned}$$

Following the construction made in [32], where the authors numerically computed the density of states for these two families of lattices, we introduce the generalized Gibbs ensemble for these models as

$$d\mu_{+,r} = \frac{\exp(-\text{Tr}(P(L^{(+,r)}))) \prod_{j=1}^N a_j^{\alpha-1} \mathbb{1}_{a_j \geq 0} da}{Z_N^{(+,r)}(\alpha, P)}, \tag{3.39}$$

$$d\mu_{\times,r} = \frac{\exp(-\text{Tr}(P(L^{(\times,r)}))) \prod_{j=1}^N a_j^{\alpha-1} \mathbb{1}_{a_j \geq 0} da}{Z_N^{(\times,r)}(\alpha, P)}, \tag{3.40}$$

where  $P(x)$  is a polynomial. Moreover, enforcing the result of [32]

**Lemma 3.23.** *Fix  $\ell \in \mathbb{N}$ . Then for  $N$  large enough*

$$\mathrm{Tr} \left( (L^{(+,r)})^\ell \right) = \mathrm{Tr} \left( (L^{(\times,r)})^\ell \right) = 0,$$

if  $\ell$  is not an integer multiple of  $r + 1$ .

we can consider just the polynomials  $P(x)$  such that  $P(x) = x^{j(r+1)} + \text{l.o.t.}$  for some  $j \in \mathbb{N}$ . Due to the local structure of  $L^{(+,r)}, L^{(\times,r)}$ , one can deduce the following:

**Lemma 3.24.** *Fix  $m \in \mathbb{N}$ , and consider the matrices  $L^{(+,r)}, L^{(\times,r)}$  (3.37)-(3.38). Then for  $N$  big enough, there exist  $k^{(+,r)} = k^{(+,r)}(m), k^{(\times,r)} = k^{(\times,r)}(m) \in \mathbb{N}$ , and four polynomial functions  $V^{(+,r)} : \mathbb{R}_+^{k^{(+,r)}} \times \mathbb{R}_+^{k^{(+,r)}} \rightarrow \mathbb{R}$ ,  $V^{(\times,r)} : \mathbb{R}_+^{k^{(\times,r)}} \times \mathbb{R}_+^{k^{(\times,r)}} \rightarrow \mathbb{R}$ ,  $V_1^{(+,r)} : \mathbb{R}_+^{k^{(+,r)}} \times \mathbb{R}_+^{\ell^{(+,r)}} \times \mathbb{R}_+^{k^{(+,r)}} \rightarrow \mathbb{R}$ ,  $V_1^{(\times,r)} : \mathbb{R}_+^{k^{(\times,r)}} \times \mathbb{R}_+^{\ell^{(\times,r)}} \times \mathbb{R}_+^{k^{(\times,r)}} \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} \mathrm{Tr} \left( (L^{(+,r)})^m \right) &= \sum_{j=1}^{M^{(+,r)}} V^{(+,r)}(\mathbf{x}_j, \mathbf{x}_{j+1}) \\ &\quad + V_1^{(+,r)}(\mathbf{x}_{M^{(+,r)}}, x_{k^{(+,r)}M^{(+,r)}+1}, \dots, x_{k^{(+,r)}M^{(+,r)}+\ell^{(+,r)}}, \mathbf{x}_1), \\ \mathrm{Tr} \left( (L^{(\times,r)})^m \right) &= \sum_{j=1}^{M^{(\times,r)}} V^{(\times,r)}(\mathbf{x}_j, \mathbf{x}_{j+1}) \\ &\quad + V_1^{(\times,r)}(\mathbf{x}_{M^{(\times,r)}}, x_{k^{(\times,r)}M^{(\times,r)}+1}, \dots, x_{k^{(\times,r)}M^{(\times,r)}+\ell^{(\times,r)}}, \mathbf{x}_1), \end{aligned}$$

where  $N = k^{(+,r)}M^{(+,r)} + \ell^{(+,r)} = k^{(\times,r)}M^{(\times,r)} + \ell^{(\times,r)}$ .

Thus, proceeding as we have done for the others systems previously considered, we obtain the following:

**Corollary 3.25** (CLT for INB lattices). *Consider the Lax matrices  $L^{(+,r)}, L^{(\times,r)}$  (3.37)-(3.38) of the additive and multiplicative INB lattices respectively distributed according to their Generalized Gibbs Ensemble (3.39)-(3.40). Then, defining the Free energies  $\mathcal{F}_{+,r}(\alpha, P), \mathcal{F}_{\times,r}(\alpha, P)$  as*

$$\begin{aligned} \mathcal{F}_{+,r}(\alpha, P) &= - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^{(+,r)}(\alpha, P)), \\ \mathcal{F}_{\times,r}(\alpha, P) &= - \lim_{N \rightarrow \infty} \frac{1}{N} \ln(Z_N^{(\times,r)}(\alpha, P)), \end{aligned}$$

for all  $j \in \mathbb{N}$  fixed, we have the following weak limit

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\mathrm{Tr} \left( (L^{(+,r)})^{(r+1)j} \right) - \mathbb{E} \left[ \mathrm{Tr} \left( (L^{(+,r)})^{(r+1)j} \right) \right]}{\sqrt{N}} &\rightarrow \mathcal{N}(0, \sigma_{+,r}^2), \\ \lim_{N \rightarrow \infty} \frac{\mathrm{Tr} \left( (L^{(\times,r)})^{(r+1)j} \right) - \mathbb{E} \left[ \mathrm{Tr} \left( (L^{(\times,r)})^{(r+1)j} \right) \right]}{\sqrt{N}} &\rightarrow \mathcal{N}(0, \sigma_{\times,r}^2), \end{aligned}$$

where

$$\mathbb{E} \left[ \mathrm{Tr} \left( (L^{(+,r)})^{(r+1)j} \right) \right] = iN \partial_t \mathcal{F}_{+,r}(\alpha, P + itx^{(r+1)j})|_{t=0}, \quad \sigma_{+,r}^2 = |\partial_t^2 \mathcal{F}_{+,r}(\alpha, P + itx^{(r+1)j})|_{t=0}|,$$

$$\mathbb{E} \left[ \mathrm{Tr} \left( (L^{(\times,r)})^{(r+1)j} \right) \right] = iN \partial_t \mathcal{F}_{\times,r}(\alpha, P + itx^{(r+1)j})|_{t=0}, \quad \sigma_{\times,r}^2 = |\partial_t^2 \mathcal{F}_{\times,r}(\alpha, P + itx^{(r+1)j})|_{t=0}|.$$

**Remark 3.26.** *We recall that in [32], it was shown that the density of states for this model has support on the complex plane, but despite that all the moments of the Generalized Gibbs ensemble are reals. Furthermore, in this case, we lack a  $\beta$  ensemble to compare with.*

## 4 Technical Results

In this section, we prove the technical results that we used to prove our main Theorems 1.2-1.4, the proof follows the same line as the proof of [34, Proposition 4.2], and we prove Theorem 1.6, whose proof uses the same machinery as the latter proofs. In the last part, we prove a Berry-Esseen type bound for the type 1 measure  $\mu_N^{(1)}$ . We start by proving Theorem 2.1 and Theorem 2.2.

To prove these results, we follow the same ideas as in [31, Theorem 2.4]. In particular, we enforce the following proposition, which can be easily deduced from [31, Proposition 2.3]:

**Proposition 4.1.** *Let  $\lambda(0)$  be an isolated eigenvalue of the operator  $\mathcal{L}_0$  with multiplicity one, and assume that the family of operators  $t \rightarrow \mathcal{L}_t$  depends on  $t$  in a  $C^d$  way, with  $d \geq 3$ . Then,  $\lambda(t)$ , the corresponding eigenprojection  $\pi_t$  and its eigenfunction  $\varphi_t$  are  $C^d$  with respect to  $t$ .*

*Moreover, assume that the rest of the spectrum of  $\mathcal{L}_0$  is contained in a disk of radius  $|\lambda(0)| - \delta$ . Writing  $\mathcal{Q}_t = (I - \pi_t)\mathcal{L}_t$ , so that  $\mathcal{L}_t = \lambda(t)\pi_t + \mathcal{Q}_t$ . For any  $r > |\lambda(0)| - \delta$ , there exists a constant  $C > 0$  independent of  $t, n$  such that  $\|\mathcal{Q}_t^n\| \leq Cr^n$  for all  $n \in \mathbb{N}$ .*

Applying the previous proposition, we can prove both Theorem 2.1 and Theorem 2.2. For the reader's convenience, we report the two statements here.

**Theorem 4.2.** *Under Assumptions 1.1. Consider a real function  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Tr}(H(L))$  is circular (HP 5.), and let  $W$  be the seed of  $\text{Tr}(G(L) + itH(L))$ , thus  $W(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + itU(\mathbf{x}, \mathbf{y})$  for  $V, U : X^k \times X^k \rightarrow \mathbb{R}$  the seeds of  $\text{Tr}(G(L)), \text{Tr}(H(L))$ . Furthermore, assume that  $U \in L^{2d}(X^{2k}, \exp(-2V))$ , with  $\mathbb{N} \ni d \geq 3$ . Then, there exists an  $\varepsilon > 0$ , and two complex valued functions  $\lambda(y, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ , and  $c_{k,\ell}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$  such that for all  $q \in \mathbb{N}$  :*

$$\mathbb{E}_1 \left[ e^{-it\text{Tr}(H(L))} \right] = \frac{Z_{kM+\ell}^{(1)}(\alpha, t)}{Z_{kM+\ell}^{(1)}(\alpha, 0)} = c_{k,\ell}(\alpha, t)\lambda(\alpha, t)^{M-2} (1 + o(M^{-q})) , \quad \text{as } M \rightarrow \infty ,$$

for  $|t| < \varepsilon$ , here  $Z_{kM+\ell}^{(1)}(\alpha, t) \equiv Z_{kM+\ell}^{(1)}(\alpha, V + itU)$ . Furthermore,

$$\begin{aligned} \lambda(x, 0) &= 1 \\ c_{k,\ell}(x, 0) &= 1 . \end{aligned}$$

Moreover, there exist two functions  $\tilde{c}_{k,\ell}(\alpha, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$  and  $\tilde{\lambda}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$  such that there exist two constants  $C_1, C_2 > 0$  such that for all  $q \in \mathbb{N}$ :

$$\begin{aligned} C_1 &< \tilde{c}_{k,\ell}(\alpha, t) < C_2 , \\ \lambda(\alpha, t) &= \frac{\tilde{\lambda}(\alpha, t)}{\tilde{\lambda}(\alpha, 0)} , \\ Z_{kM+\ell}^{(1)}(\alpha, t) &= \tilde{c}_{k,\ell}(\alpha, t)\tilde{\lambda}(\alpha, t)^{M-2} (1 + o(M^{-q})) . \end{aligned}$$

*Proof.* Define the kernel operator (depending on  $k \in \mathbb{N}$ ,  $\alpha > 0$  and  $t \in \mathbb{R}$ )  $\mathcal{L}_{t,\alpha} : L^2(X^k) \rightarrow L^2(X^k)$  as

$$\mathcal{L}_{t,\alpha} f(\mathbf{y}) = \int_{X^k} f(\mathbf{x}) \prod_{q=1}^k F(x_q, \alpha) e^{-W(\mathbf{y}, \mathbf{x})} d\mathbf{x} . \quad (4.1)$$

Then, for all  $k \in \mathbb{N}$ ,  $\alpha > 0$  and  $t \in \mathbb{R}$ ,  $\mathcal{L}_{t,\alpha}$  is a Hilbert-Schmidt operator [43], meaning that the function  $(\mathbf{x}, \mathbf{y}) \mapsto \prod_{q=1}^k F(x_q, \alpha) e^{-V(\mathbf{y}, \mathbf{x})}$  is  $L^2(X^k \times X^k)$ , and so it is compact. Moreover, since the kernel is positive, we can apply a generalization of Jentzsch's theorem [74, Theorem

137.4] in combinations with Proposition 4.1 deducing that there exist two functions  $(y, t) \mapsto \tilde{\lambda}(y, t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon, \varepsilon])$ ,  $(y, t) \mapsto \varphi(\cdot, y, t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon, \varepsilon], L^2(X^k))$ , and an operator  $\mathcal{Q}_t : L^2(X^k) \rightarrow L^2(X^k)$  such that

$$\mathcal{L}_{t,\alpha}\phi(\mathbf{y}) = \tilde{\lambda}(\alpha, t) \langle \phi, \varphi(\cdot, \alpha, t) \rangle \varphi(\mathbf{y}, \alpha, t) + \mathcal{Q}_t\phi(\mathbf{y}), \quad \forall \phi \in L^2(X^k)$$

where  $\tilde{\lambda}(y, 0) > 0$ ,  $\varphi(\mathbf{x}, y, 0) > 0$  is the associated eigenfunction of  $\mathcal{L}_{t,\alpha}$  with  $\|\varphi\|_2 = 1$  and there exists a  $\delta > 0$  such that  $\|\mathcal{Q}_t\| \leq |\tilde{\lambda}(\alpha, t)| - \delta$ , denoting by  $\langle \cdot, \cdot \rangle$  the standard scalar product in  $L^2(X^k)$ .

For  $\mathbf{x} \in X^k$  define  $G_{\mathbf{x}}(\mathbf{y})$  as

$$G_{\mathbf{x}}(\mathbf{y}) = \begin{cases} \int_{X^\ell} \prod_{j=kM+1}^{kM+\ell} F(x_j, \alpha) \exp(-W_1(\mathbf{y}, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x})) \prod_{j=kM+1}^{kM+\ell} dx_j, & \ell > 0, \\ \exp(-W(\mathbf{y}, \mathbf{x})), & \ell = 0, \end{cases} \quad (4.2)$$

and the linear operator  $\mathcal{S}_t : L^2(X^k \times X^k) \rightarrow \mathbb{C}$  as

$$\mathcal{S}_t\varphi = \int_{X^k \times X^k} \prod_{j=1}^{2k} F(x_j, \alpha) \exp(-W(\mathbf{x}_1, \mathbf{x}_2)) \varphi(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2, \quad (4.3)$$

we notice that  $\|\mathcal{S}_t\| \leq c\|F\|_2^{2k}$ , and so it is continuous.

In this notation, we can recast (2.3), applying  $\mathcal{S}_t$  to  $(\mathbf{x}, \mathbf{y}) \mapsto (\mathcal{L}_{t,\alpha}^{M-2} G_{\mathbf{x}})(\mathbf{y})$ , as

$$Z_{kM+\ell}^{(1)}(\alpha, t) = \mathcal{S}_t \left( (\mathcal{L}_{t,\alpha}^{M-2} G_{\mathbf{x}_1})(\mathbf{x}_2) \right) = \tilde{\lambda}^{M-2}(\alpha, t) \mathcal{S}_t(\langle \varphi(\cdot, \alpha, t); G_{\mathbf{x}_1} \rangle \varphi(\mathbf{x}_2, \alpha, t)) + \mathcal{S}_t(\mathcal{Q}_t^{M-2} G_{\mathbf{x}_1}(\mathbf{x}_2)),$$

where here and in the sequel, if  $h \in L^2(X^k \times X^k)$ , we write abusively  $\mathcal{S}_t(h(\mathbf{x}, \mathbf{y}))$  for  $\mathcal{S}_t(h)$ . Defining

$$c_{k,\ell}(\alpha, t) = \frac{\mathcal{S}_t(\langle \varphi(\cdot, \alpha, t); G_{\mathbf{x}_1} \rangle \varphi(\mathbf{x}_M, \alpha, t))}{\mathcal{S}_0(\langle \varphi(\cdot, \alpha, 0); G_{\mathbf{x}_1} \rangle \varphi(\mathbf{x}_M, \alpha, 0))},$$

$$\lambda(\alpha, t) = \frac{\tilde{\lambda}^{M-2}(\alpha, t)}{\tilde{\lambda}^{M-2}(\alpha, 0)},$$

and since in view of Proposition 4.1  $\|\mathcal{Q}_t^n\| \leq (|\tilde{\lambda}(t)| - \delta)^n$  we conclude.  $\square$

**Theorem 4.3.** *Under Assumptions 1.1. Consider a real function  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Tr}(H(L))$  is circular, and let  $W$  be the seed of  $\text{Tr}(G(L) + itH(L))$ , thus  $W(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + itU(\mathbf{x}, \mathbf{y})$  for  $V, U : X^k \times X^k \rightarrow \mathbb{R}$  seeds of  $\text{Tr}(G(L))$ ,  $\text{Tr}(H(L))$ . Furthermore, assume that  $U \in L^{2d}(X^{2k}, \exp(-2V))$ , with  $\mathbb{N} \ni d \geq 3$ . Then there exists an  $\varepsilon > 0$  and two scalar functions  $\lambda(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon], \mathbb{C})$ ,  $c_{k,\ell,M}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$ , such that*

$$\mathbb{E}_2 \left[ e^{-it\text{Tr}(H(L))} \right] = \frac{Z_{kM+\ell}^{(2)}(\alpha, t)}{Z_{kM+\ell}^{(2)}(\alpha, 0)} = c_{k,\ell,M}(\alpha, t) \prod_{j=1}^{M-2} \lambda\left(\alpha \frac{j}{M}, t\right) (1 + o_M(1)) \quad (4.4)$$

for  $|t| < \varepsilon$ . Furthermore,

$$\lambda(x, 0) = 1$$

$$\lim_{t \rightarrow 0} c_{k,\ell,M}(\alpha, t) = 1 \quad \text{uniformly in } M$$

the remainder  $o_M(1)$  is independent of  $t \in [-\varepsilon, \varepsilon]$ .

Moreover, there exist two functions  $\tilde{c}_{k,\ell,M}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ ,  $\tilde{\lambda}(\alpha, t) \in C^{1,d}(\mathbb{R}_+ \times [-\varepsilon, \varepsilon])$ , and three constants  $C_1, C_2 > 0$  and  $p \in \mathbb{N}$  such that

$$\begin{aligned} C_1 N^p &< \tilde{c}_{k,\ell,M}(\alpha, t) < C_2 N^p, \\ \lambda(\alpha, t) &= \frac{\tilde{\lambda}(\alpha, t)}{\tilde{\lambda}(\alpha, 0)}, \\ Z_{kM+\ell}^{(2)}(\alpha, t) &= \tilde{c}_{k,\ell,M}(\alpha, t) \prod_{j=1}^{M-2} \tilde{\lambda}\left(\alpha \frac{j}{M}, t\right) (1 + o_M(1)). \end{aligned}$$

*Proof.* Define the family of kernel operators (depending on  $k \in \mathbb{N}$ ,  $\alpha > 0$  and  $t \in \mathbb{R}$ )  $\mathcal{L}_{t,\alpha}^{(j)} : L^2(X^k) \rightarrow L^2(X^k)$  as

$$\mathcal{L}_{t,\alpha}^{(j)} f(\mathbf{y}) = \int_{X^k} f(\mathbf{x}) \prod_{q=1}^k F\left(x_q, \alpha \left(1 - \frac{(j-1)k+q}{N}\right)\right) e^{-W(\mathbf{y}, \mathbf{x})} d\mathbf{x}.$$

Then, for all  $k \in \mathbb{N}$ ,  $\alpha > 0$  and  $t \in \mathbb{R}$ , the kernel of  $\mathcal{L}_{t,\alpha}^{(j)}$  is in  $L^2(X^k \times X^k)$ , thus it is a Hilbert-Schmidt operator, and so it is compact. Moreover, since for  $t = 0$  the kernel is positive, we can apply a generalization of Jentzsch's theorem [74, Theorem 137.4] in combinations with Proposition 4.1 deducing that there exist two functions  $\hat{\lambda} = \hat{\lambda}(y, t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon, \varepsilon])$ ,  $(y, t) \mapsto \varphi(\cdot, y, t) \in C^{1,d}(\mathbb{R}^+ \times [-\varepsilon, \varepsilon], L^2(X^k))$  and an operator  $\mathcal{Q}_t^{(j)} : L^2(X^k) \rightarrow L^2(X^k)$  such that  $\forall \phi \in L^2(X^k)$ ,  $\forall |t| < \varepsilon$ ,

$$\mathcal{L}_{t,\alpha}^{(j)} \phi(\mathbf{y}) = \hat{\lambda}\left(\alpha \left(1 - \frac{j}{M}\right), t\right) \pi_t^{(j)} \phi(\mathbf{y}) + \mathcal{Q}_t^{(j)} \phi(\mathbf{y}) \quad (4.5)$$

with

$$\pi_t^{(j)} \phi(\mathbf{y}) = \left\langle \phi; \varphi\left(\cdot, \alpha \left(1 - \frac{j}{M}\right), t\right) \right\rangle \varphi\left(\mathbf{y}, \alpha \left(1 - \frac{j}{M}\right), t\right),$$

where  $\hat{\lambda}\left(\alpha \left(1 - \frac{j}{M}\right), t\right)$  is the biggest eigenvalue (in modulus) of  $\mathcal{L}_{t,\alpha}^{(j)}$ ,  $\hat{\lambda}(y, 0) > 0$ ,  $\varphi(\cdot, y, 0) > 0$ ,  $\|\varphi(\cdot, y, t)\|_2 = 1$  and there exists a  $\delta_j > 0$  such that  $\|\mathcal{Q}_t^{(j)}\| \leq |\hat{\lambda}\left(\alpha \left(1 - \frac{j}{M}\right), t\right)| - \delta_j$ , and we recall that we denote by  $\langle \cdot, \cdot \rangle$  the standard scalar product in  $L^2(X^k)$ . Furthermore, with  $\tilde{\lambda}$  the function of Theorem 4.2, we have

$$\hat{\lambda}\left(\alpha \left(1 - \frac{j}{M}\right), t\right) = \tilde{\lambda}\left(\alpha \left(1 - \frac{j}{M}\right), t\right) + O\left(\frac{1}{M}\right), \quad (4.6)$$

Where the  $O\left(\frac{1}{M}\right)$  term is uniform in  $t \in (-\varepsilon, \varepsilon)$ . Indeed, recalling that  $\mathcal{L}_{t,\alpha}$  is defined in (4.1), by the integrability assumptions on  $U$  and on  $\partial_\alpha F$  (HP 7. of Assumptions 1.1), we have

$$\|\mathcal{L}_{t,\alpha}^{(j)} - \mathcal{L}_{t,\alpha(1-j/M)}\| \leq C_t \frac{\alpha}{M} \quad (4.7)$$

where  $C_t \geq 0$  is bounded on  $[-\varepsilon, \varepsilon]$ . We then deduce (4.6) by applying Proposition 4.1.

Define the function  $h_t$  on  $X^k$  by

$$\begin{aligned} h_t(\mathbf{x}) &= \int_{X^{k+\ell}} \prod_{j=1}^{k+\ell-1} F\left(x_{k(M-1)+j}, \alpha \left(1 - \frac{j+k(M-1)}{kM+\ell}\right)\right) \exp(-W(\mathbf{x}, \mathbf{x}_M)) \times \\ &\times \exp(-W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) - W(x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0)) R(x_N) \prod_{j=1}^{k+\ell} dx_{k(M-1)+j}. \end{aligned} \quad (4.8)$$

Note that in view of assumptions 1.1,  $\|h_t\|_\infty = O(N^{\mathbf{c}(\ell+k)})$ .

We recall that  $W = V + itU$  so  $h_t$  depends on  $t$ , and the linear operator  $\mathcal{S}_t : L^2(X^k) \rightarrow \mathbb{C}$  as

$$\mathcal{S}_t(\psi) = \int_{X^k} \prod_{j=1}^k F\left(x_j, \alpha\left(1 - \frac{j}{kM + \ell}\right)\right) e^{-W(0, \dots, 0, \mathbf{x})} \psi(\mathbf{x}) \prod_{j=1}^k dx_j.$$

We notice that, again in view of the assumptions 1.1 the operator  $\mathcal{S}_t$  is uniformly bounded in  $k, \ell$  for all  $t \in \mathbb{R}$ , and it is continuous in  $t$  in the operator norm sense.

In this notation, we can rewrite  $Z_{kM+\ell}^{(2)}(\alpha, t)$  as

$$Z_{kM+\ell}^{(2)}(\alpha, t) = \mathcal{S}_t\left(\mathcal{L}_{t,\alpha}^{(2)} \dots \mathcal{L}_{t,\alpha}^{(M-1)} h_t\right) = \mathcal{S}_t\left(\prod_{j=2}^{M-1} \mathcal{L}_{t,\alpha}^{(j)} h_t\right)$$

Applying the decomposition (4.5), it follows that we can decompose the previous expression as

$$\begin{aligned} Z_{kM+\ell}^{(2)}(\alpha, t) &= \prod_{j=2}^{M-1} \hat{\lambda}_t\left(\alpha\left(1 - \frac{j}{M}\right)\right) \mathcal{S}_t\left(\pi_t^{(2)} \dots \pi_t^{(M-1)} h_t\right) \\ &\quad + \mathcal{S}_t\left(\mathcal{L}_{t,\alpha}^{(2)} \dots \mathcal{L}_{t,\alpha}^{(M-2)} \mathcal{Q}_t^{(M-1)} h_t\right) \\ &\quad + \sum_{n=2}^{M-2} \prod_{j=n+1}^{M-1} \hat{\lambda}_t\left(\alpha\left(1 - \frac{j}{M}\right)\right) \mathcal{S}_t\left(\mathcal{L}_{t,\alpha}^{(2)} \dots \mathcal{L}_{t,\alpha}^{(n-1)} \mathcal{Q}_t^{(n)} \pi_t^{(n+1)} \dots \pi_t^{(M-1)} h_t\right), \end{aligned} \tag{4.9}$$

where we arranged the terms of the product of the  $\mathcal{L}_{t,\alpha}^{(j)}$ 's by order of the first appearance from the right of a factor  $\mathcal{Q}_t^{(j)}$  (the first term being the product where no  $\mathcal{Q}_t^{(j)}$  appears). We notice that

$$\mathcal{S}_t\left(\pi_t^{(2)} \dots \pi_t^{(M-1)} h_t\right) = \langle \varphi_t^{1/M}; h_t \rangle \prod_{i=2}^{M-2} \langle \varphi_t^{\alpha(1-i/M)}; \varphi_t^{\alpha(1-(i+1)/M)} \rangle \mathcal{S}_t\left(\varphi_t^{\alpha(1-\frac{2}{M})}\right),$$

where we set  $\varphi_t^{\alpha(1-i/M)} \equiv \varphi(\cdot, \alpha(1-i/M), t)$  to shorten the notation. Furthermore, the ratio

$$\prod_{i=2}^{M-1} \frac{\langle \varphi_t^{\alpha(1-\frac{i}{M})}; \varphi_t^{\alpha(1-\frac{i+1}{M})} \rangle}{\langle \varphi_0^{\alpha(1-\frac{i}{M})}; \varphi_0^{\alpha(1-\frac{i+1}{M})} \rangle}$$

converges uniformly to 1 in  $M \geq 1$ ,  $t \in (-\varepsilon, \varepsilon)$ . This is due to the fact that

$$\langle \varphi_t^{\alpha(1-\frac{i}{M})}; \varphi_t^{\alpha(1-\frac{i+1}{M})} \rangle = 1 + O\left(\frac{\alpha}{M}\right),$$

because of (4.7) and Proposition 4.1, thus the product

$$\prod_{i=2}^{M-1} \langle \varphi_t^{\alpha(1-\frac{i}{M})}; \varphi_t^{\alpha(1-\frac{i+1}{M})} \rangle$$

stays bounded below and above uniformly on  $M \geq 1$ ,  $t \in (-\varepsilon, \varepsilon)$ .

Denoting the first term of (4.9) by  $f(\alpha, t)$ , and the second and third terms by  $g_1(\alpha, t)$  and  $g_2(\alpha, t)$ , we can rewrite (4.4) as



$$\frac{Z_{kM+\ell}^{(2)}(\alpha, t)}{Z_{kM+\ell}^{(2)}(\alpha, 0)} = \frac{f(\alpha, t)}{f(\alpha, 0)} \left( \frac{1 + \frac{g_1(\alpha, t)}{f(\alpha, t)} + \frac{g_2(\alpha, t)}{f(\alpha, t)}}{1 + \frac{g_1(\alpha, 0)}{f(\alpha, 0)} + \frac{g_2(\alpha, 0)}{f(\alpha, 0)}} \right).$$

Thus, to prove our result we need to show that there exist 3 constants  $c_1, c_2, c_3$  independent of  $M$  such that for all  $t \in (-\varepsilon, \varepsilon)$ ,

$$\left| \frac{g_1(\alpha, t)}{f(\alpha, t)} \right| \leq c_1, \quad (4.10)$$

$$\left| \frac{g_2(\alpha, t)}{f(\alpha, t)} \right| \leq c_2, \quad (4.11)$$

$$\left| \frac{\langle \varphi_t^{(1/M)}; h_t \rangle \mathcal{S}_t \varphi_t^{\alpha(1-\frac{2}{M})}}{\langle \varphi_0^{(1/M)}; h_0 \rangle \mathcal{S}_0 \varphi_0^{\alpha(1-\frac{2}{M})}} \right| \leq c_3. \quad (4.12)$$

If we are able to show this, then defining

$$c_{k,\ell,M}(\alpha, t) = \frac{\langle \varphi_t^{(1/M)}; h_t \rangle \mathcal{S}_t \varphi_t^{\alpha(1-\frac{2}{M})}}{\langle \varphi_0^{(1/M)}; h_0 \rangle \mathcal{S}_0 \varphi_0^{\alpha(1-\frac{2}{M})}} \prod_{i=2}^{M-1} \frac{\langle \varphi_t^{\alpha(1-\frac{i}{M})}; \varphi_t^{\alpha(1-\frac{i+1}{M})} \rangle}{\langle \varphi_0^{\alpha(1-\frac{i}{M})}; \varphi_0^{\alpha(1-\frac{i+1}{M})} \rangle},$$

$$\lambda(y, t) = \frac{\tilde{\lambda}(y, t)}{\tilde{\lambda}(y, 0)},$$

we obtain (4.4) with the wanted properties. Notice that in the definition of  $\lambda$  we took  $\frac{\tilde{\lambda}(y, t)}{\tilde{\lambda}(y, 0)}$  instead of  $\frac{\hat{\lambda}(y, t)}{\hat{\lambda}(y, 0)}$ . This is indeed possible because of equation (4.6).

First, we focus on (4.11). The term  $g_2(\alpha, t)$  is given by

$$\langle \varphi^{(1/M)}; h_t \rangle \sum_{n=2}^{M-2} \prod_{j=n+1}^{M-1} \hat{\lambda} \left( \alpha \left( 1 - \frac{j}{M} \right), t \right) \prod_{i=1}^{M-n-2} \langle \varphi_t^{\frac{\alpha i}{M}}; \varphi_t^{\frac{\alpha(i+1)}{M}} \rangle \mathcal{S}_t \left( \mathcal{L}_{t,\alpha}^{(2)} \dots \mathcal{L}_{t,\alpha}^{(n-1)} \mathcal{Q}_t^{(n)} \varphi_t^{\alpha(1-\frac{n+1}{M})} \right).$$

Because  $\varphi(\mathbf{x}, y, t)$  is regular with respect to  $y$ , we deduce that there exists a function  $(y, t) \mapsto \psi(\cdot, y, t) \in C^\infty(\mathbb{R}^+ \times [-\varepsilon, \varepsilon], L^2(X^k))$  with  $\|\psi_t^{\alpha(1-n/M)}\|_2$  uniformly bounded in  $n, M$  and  $t$  such that

$$\mathcal{Q}_t^n \left( \varphi_t^{\alpha(1-\frac{n+1}{M})} \right) = \mathcal{Q}_t^{(n)} \left( \varphi_t^{\alpha(1-\frac{n}{M})} \right) + \frac{1}{M} \mathcal{Q}_t^{(n)} \psi_t^{\alpha(1-n/M)} = \frac{1}{M} \mathcal{Q}_t^{(n)} \psi_t^{\alpha(1-n/M)}.$$

given this equality, it is trivial to prove (4.11), recalling that for any  $t, j$ ,  $\frac{1}{\hat{\lambda}_t^{(j)}} \mathcal{L}_{t,\alpha}^{(j)}$  has operator norm smaller than one.

For (4.10), it suffices to show that there exists a constant  $c_2$  independent of  $M$  such that

$$\left\| \frac{\mathcal{Q}_t^{(M-1)} h_t}{\langle \varphi_t^{(1/M)}; h_t \rangle} \right\| \leq c_2. \quad (4.13)$$

From the assumptions, (4.5) and the definition of  $h_t$  (4.8), we deduce that there exists a constant  $d_1$  such that

$$\left\| \mathcal{Q}_j^{(M-1)} h_t \right\| \leq d_1 \left( \lambda \left( \frac{\alpha}{M} \right) - \delta_1 \right) M^{c(k+\ell)},$$

on the other hand, in view of the previous proof and the assumptions, we conclude that, for  $t$  small enough, there exists a constant  $d_2$  such that

$$\left| \left\langle \varphi_t^{(1/M)}, h_t \right\rangle \right| \geq d_2 M^{c(k+\ell)}. \quad (4.14)$$

Indeed, for  $t = 0$ ,  $\left\langle \varphi_0^{(1/M)}, h_0 \right\rangle$  is given by

$$\int_{X^{2k+\ell}} \varphi_0^{(1/M)}(\mathbf{x}) \prod_{j=1}^{k+\ell-1} F \left( x_{k(M-1)+j}, \alpha \left( 1 - \frac{k(M-1)+j}{kM+\ell} \right) \right) r(\mathbf{x}, \mathbf{x}_M, \tilde{\mathbf{x}}) d\mathbf{x} d\mathbf{x}_M d\tilde{\mathbf{x}},$$

where we denoted  $\tilde{\mathbf{x}} = (x_{kM+1}, \dots, x_{kM+\ell})$  and

$$r(\mathbf{x}, \mathbf{x}_M, \tilde{\mathbf{x}}) = e^{-W(\mathbf{x}, \mathbf{x}_M) - W(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0) - W(x_{kM+1}, \dots, x_{kM+\ell}, 0, \dots, 0)}.$$

By Assumptions 1.1 HP.4, and positivity of  $\varphi_0^{(1/M)}$ ,

$$\left\langle \varphi_0^{(1/M)}, h_0 \right\rangle \geq (d(N/\alpha)^c)^{k+\ell-1} \int_{\mathcal{O}} \varphi_0^{(1/M)}(\mathbf{x}) d\mathbf{x} \inf_{\mathcal{O}^{2k+\ell}} r \geq c_{k,\ell} M^{k+\ell} \int_{\mathcal{O}} \varphi_0^{(1/M)}(\mathbf{x}) d\mathbf{x}.$$

By continuity of  $\eta \mapsto \varphi_0^\eta$ , this last integral converges to  $\int_{\mathcal{O}} \varphi_0^{(1/M)}(\mathbf{x}) d\mathbf{x} > 0$ , thus we conclude for the case  $t = 0$ . Finally, we conclude on (4.14) for  $t$  small enough by continuity.

Combining the two previous estimates, and setting  $p = c(k + \ell)$  we deduce (4.13), which leads to (4.11). The proof of (4.12) is analogous, thus we conclude.  $\square$

We now turn on the proof of Theorem 1.6, which we rewrite here for convenience.

**Theorem 4.4** (Decay of correlations). *Let  $W$  be the seed of  $\text{Tr}(G(L))$  and  $I, J : X^k \rightarrow \mathbb{R}$  two local functions such that  $\int_{X^k \times X^k} \left| I(\mathbf{x}) \prod_{i=1}^k F(x_i, \alpha) e^{-W(\mathbf{x}, \mathbf{y})} \right|^2 d\mathbf{x} d\mathbf{y} < \infty$ , and analogously for  $J(\mathbf{x})$ . Write  $N = kM + \ell$ , and let  $j \in \{1, \dots, M\}$ . Then there exists some  $0 < \mu < 1$  such that*

$$\mathbb{E}_1 [I(\mathbf{x}_1) J(\mathbf{x}_j)] - \mathbb{E}_1 [I(\mathbf{x}_1)] \mathbb{E}_1 [J(\mathbf{x}_j)] = O(\mu^{M-j} + \mu^j).$$

*Proof.* Let  $\mathcal{L} = \mathcal{L}_{0,\alpha}$  with  $\mathcal{L}_{t,\alpha}$  given by (4.1). Furthermore, define  $\mathcal{L}^{(J)}$

$$\mathcal{L}^{(J)} \phi(\mathbf{y}) = \int_{X^k} \phi(\mathbf{x}) \prod_{i=1}^k F(x_i, \alpha) J(\mathbf{x}) e^{-V(\mathbf{y}, \mathbf{x})} = \mathcal{L}(J\phi)(\mathbf{y}),$$

and  $\mathcal{L}^{(I)}$  analogously. With  $G_{\mathbf{x}}^{(I)}(\mathbf{y}) = I(\mathbf{x}) G_{\mathbf{x}}(\mathbf{y})$ ,  $G$  given in (4.2), we have for  $j \geq 3$

$$\begin{aligned} \mathbb{E}_1 [I(\mathbf{x}_1) J(\mathbf{x}_j)] &= \frac{\mathcal{S}_0 \left( \left( \mathcal{L}^{M-j} \mathcal{L}^{(J)} \mathcal{L}^{j-3} G_{\mathbf{x}_1}^{(I)} \right) (\mathbf{x}_2) \right)}{\mathcal{S}_0(\mathcal{L}^{M-2} G_{\mathbf{x}_1}(\mathbf{x}_2))} \\ &= \frac{\tilde{\lambda}^{M-j}(\alpha, 0) \mathcal{S}_0(\pi_0 \mathcal{L}^{(J)} \mathcal{L}^{j-3} G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) + O(\tilde{\lambda}^{j-3} r^{M-j})}{\tilde{\lambda}^{M-2}(\alpha, 0) \mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)) + O(r^{M-2})}, \end{aligned}$$

where  $\mathcal{S}_t$  is defined in (4.3), and we used the decomposition

$$\mathcal{L}_0^k = \tilde{\lambda}^k(\alpha, 0) \pi_0 + \mathcal{Q}_0^k,$$

where  $\pi_0$  is the orthogonal projection on the (one dimensional) eigenspace associated with  $\tilde{\lambda}(\alpha, 0)$ , and  $\mathcal{Q}_0$  is an operator such that  $\|\mathcal{Q}_0^k\| \leq Cr^k$  for some  $0 < r < \tilde{\lambda}$ . Similarly,

$$\mathcal{S}_0(\pi_0 \mathcal{L}^{(J)} \mathcal{L}^{j-3} G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) = \tilde{\lambda}^{j-3}(\alpha, 0) \mathcal{S}_0(\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) + O(r^{j-3}).$$

We deduce

$$\mathbb{E}_1 [I(\mathbf{x}_1)J(\mathbf{x}_j)] = \frac{\mathcal{S}_0((\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}^{(I)})(\mathbf{x}_2)) + O((r/\tilde{\lambda})^{M-j} + (r/\tilde{\lambda})^{j-3})}{\tilde{\lambda} \left( \mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)) + O((r/\tilde{\lambda})^{M-2}) \right)}.$$

Similarly, we deduce

$$\mathbb{E}_1 [I(\mathbf{x}_1)] \mathbb{E}_1 [J(\mathbf{x}_j)] = \frac{\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) \mathcal{S}_0(\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)) + O((r/\tilde{\lambda})^{M-j} + (r/\tilde{\lambda})^{j-3})}{\tilde{\lambda} \left( \mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2))^2 + O((r/\tilde{\lambda})^{M-2}) \right)}.$$

By a direct computation, recalling that  $\pi_0 \phi = \langle \varphi_1, \phi \rangle \varphi_1$  where  $\varphi_1$  is the eigenfunction associated with  $\tilde{\lambda}$ , we deduce the following

$$\mathcal{S}_0 \left( \pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2) \right) = \langle \mathcal{L}^{(J)} \varphi_1, \varphi_1 \rangle \int \langle G_{\mathbf{x}}, \varphi_1 \rangle I(\mathbf{x}) F(\mathbf{x}) \varphi_1(\mathbf{x}) d\mathbf{x},$$

$$\mathcal{S}(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2)) = \int \langle G_{\mathbf{x}}, \varphi_1 \rangle \varphi_1(\mathbf{x}) F(\mathbf{x}) d\mathbf{x}, \quad \mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) = \int \langle G_{\mathbf{x}}, \varphi_1 \rangle I(\mathbf{x}) F(\mathbf{x}) \varphi_1(\mathbf{x}) d\mathbf{x},$$

and

$$\mathcal{S}_0 \left( \pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2) \right) = \langle \mathcal{L}^{(J)} \varphi_1, \varphi_1 \rangle \int \langle G_{\mathbf{x}}, \varphi_1 \rangle F(\mathbf{x}) \varphi_1(\mathbf{x}) d\mathbf{x}.$$

These formulas imply that

$$\mathcal{S}_0((\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}^{(I)})(\mathbf{x}_2)) = \frac{\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}^{(I)}(\mathbf{x}_2)) \mathcal{S}_0(\pi_0 \mathcal{L}^{(J)} \pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2))}{\mathcal{S}_0(\pi_0 G_{\mathbf{x}_1}(\mathbf{x}_2))},$$

and so

$$\mathbb{E}_1 [I(\mathbf{x}_1)J(\mathbf{x}_j)] - \mathbb{E}_1 [I(\mathbf{x}_1)] \mathbb{E}_1 [J(\mathbf{x}_j)] = O((r/\tilde{\lambda})^{M-j} + (r/\tilde{\lambda})^{j-3}).$$

□

Finally, we prove a Berry-Esseen bound type theorem for the measure  $\mu_{kM+\ell}^{(1)}$ :

**Theorem 4.5.** *Under Hypotheses 1.1. Consider the measure  $\mu_{kM+\ell}^{(1)}$ ,  $G$  satisfying assumptions 1.1 and  $H : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Tr}(H(L))$  is cyclic (HP. 5) with seed  $h$  and weed  $h_1$  such that  $h, h_1 \in L^d(X^{2k}, \exp(-2W))$ , with  $\mathbb{N} \ni d \geq 6$ , so that*

$$\mathbb{E}_1 \left[ e^{-it \text{Tr}(H(L))} \right] = \frac{Z_{kM+\ell}^{(1)}(\alpha, G + itH)}{Z_{kM+\ell}^{(1)}(\alpha, G)}.$$

Then, there exists  $A \in \mathbb{R}$ ,  $\sigma, C > 0$  such that if  $Y \sim \mathcal{N}(0, \sigma^2)$  we have for any interval  $J$  of the real line

$$\left| \mathbb{P} \left( \left[ \text{Tr}(H(L)) - (kM + \ell)A \right] / \sqrt{(kM + \ell)} \in J \right) - \mathbb{P}(Y \in J) \right| \leq \frac{C}{\sqrt{(kM + \ell)}}.$$

*Proof.* We adapt the arguments of [31, Theorem 3.7]. By [23, Lemma XVI.3.2], there exists a constant  $C$  such that for any  $X$  real random variable, and  $Y$  Gaussian random variable, for any interval  $J \subset \mathbb{R}$  and for any  $T > 0$ , we have

$$|\mathbb{P}(X \in J) - \mathbb{P}(Y \in J)| \leq C \int_0^T \frac{|\mathbb{E}[e^{-itX}] - e^{-\sigma^2 t^2/2}|}{t} dt + \frac{C}{T}.$$

We take  $X = (\text{Tr}(H(L)) - (kM + \ell)A) / \sqrt{kM + \ell}$ . We are going to show that, taking  $T = \varepsilon \sqrt{kM + \ell}$  for some small enough  $\varepsilon$ , the last integral remains bounded by  $\frac{C_{k,\ell}}{\sqrt{kM + \ell}}$ , where  $C_{k,\ell}$  is a constant depending on  $k, \ell$ . Recall  $N = kM + \ell$ . By Theorem 1.2, there exists an  $A \in \mathbb{R}, \sigma > 0$  such that as  $N$  goes to infinity  $X$  converges to  $\mathcal{N}(0, \sigma^2)$ . Since  $t^{-1}$  is not integrable at 0, we consider the special interval  $[0, N^{-1}]$ . In this interval, we have the following estimate, denoting by  $W$  the seed of  $\text{Tr}(G(L))$ :

$$\begin{aligned} \left| \mathbb{E}[e^{-itX}] - e^{-it\sqrt{N}A} \right| &\stackrel{(2.4)}{=} \frac{|Z_{kM+\ell}^{(1)}(\alpha, G + i\frac{t}{\sqrt{N}}H) - Z_{kM+\ell}^{(1)}(\alpha, G)|}{Z_{kM+\ell}^{(1)}(\alpha, G)} \\ &= \frac{1}{Z_{kM+\ell}^{(1)}(\alpha, G)} \left| \sum_{p=1}^{M-1} \int_{X^{kM+\ell}} \mathfrak{F}(\mathbf{x}) \prod_{j=1}^{p-1} e^{i\frac{t}{\sqrt{N}}h(\mathbf{x}_j, \mathbf{x}_{j+1})} \left( e^{i\frac{t}{\sqrt{N}}h(\mathbf{x}_p, \mathbf{x}_{p+1})} - 1 \right) \prod_{j=1}^N dx_j \right. \\ &\quad \left. + \int_{X^{kM+\ell}} \mathfrak{F}(\mathbf{x}) \left( e^{i\frac{t}{\sqrt{N}}h_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1)} - 1 \right) \right|, \end{aligned}$$

with the convention that the empty product is equal to one. Here we defined

$$\begin{aligned} \mathfrak{F}(\mathbf{x}) &= \prod_{j=1}^{kM+\ell} F(x_j, \alpha) \exp \left( - \sum_{j=1}^{M-1} W(\mathbf{x}_j, \mathbf{x}_{j+1}) - W_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1) \right) \\ &\quad \times \exp \left( \frac{it}{\sqrt{N}} h_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1) \right), \end{aligned}$$

where  $h_1$  is the weed of  $H$  in the sense of HP 5 of Hypotheses 1.1. Thus, since  $|e^{i\frac{t}{\sqrt{N}}h(\mathbf{x}_p, \mathbf{x}_{p+1})} - 1| \leq |h(\mathbf{x}_p, \mathbf{x}_{p+1})|N^{-1/2}t$ , we deduce the following inequality

$$\left| \mathbb{E}_1[e^{-itX}] - e^{-it\sqrt{N}A} \right| \leq \mathbb{E}_1[|h(\mathbf{x}_1, \mathbf{x}_2)|] t\sqrt{N} + \frac{t}{\sqrt{N}} \mathbb{E}_1[|h_1(\mathbf{x}_M, x_{kM+1}, \dots, x_{kM+\ell}, \mathbf{x}_1)|], \quad (4.15)$$

and this last term is by assumption bounded by  $Ct\sqrt{N}$  for some  $C$  independent of  $N$  and  $t$ . Thus integrating for  $t \in [0, N^{-1}]$  we deduce the following

$$\begin{aligned} &\int_0^{\frac{1}{N}} \frac{|\mathbb{E}_1[e^{-itX}] - e^{-\sigma^2 t^2/2}|}{t} dt \\ &\leq \int_0^{\frac{1}{N}} \frac{|\mathbb{E}_1[e^{-itX}] - e^{-it\sqrt{N}A}| + |e^{-it\sqrt{N}A} - 1| + |1 - e^{-\sigma^2 t^2/2}|}{t} dt \\ &\stackrel{(4.15)}{\leq} \int_0^{\frac{1}{N}} \frac{C\sqrt{N}t + t\sqrt{N}A + \sigma^2 t^2/2}{t} dt \leq \frac{C_1}{\sqrt{N}}, \end{aligned}$$

for some constant  $C_1$ .

We now consider the integral on  $[1/N, \varepsilon\sqrt{N}]$ . Here we use the spectral decomposition of  $\mathbb{E}_1[e^{itX}]$ . Since  $h \in L^d(X^{2k}, \exp(-2W))$  for some  $d \geq 6$ , we deduce (following Remark 1.3) applying Proposition 4.1, and from Theorem 2.1, that there exist two continuous functions  $p(t) \in C^0([-\varepsilon, \varepsilon])$  and  $c_{k,\ell}(y, t) \in C^{1,d}(\mathbb{R} \times [-\varepsilon, \varepsilon])$  for some  $\varepsilon > 0$ , such that  $c_{k,\ell}(y, 0) = 1$  and  $\|p\|_\infty < +\infty$ , such that for  $q \geq 1$

$$\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|\mathbb{E}_1[e^{-itX}] - e^{-\sigma^2 t^2/2}|}{t} dt = \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|c_{k,\ell}(\alpha, t/\sqrt{N})e^{-\sigma^2 t^2/2 + t^3 p(t/\sqrt{N})/\sqrt{N}}(1 + o(N^{-q})) - e^{-\sigma^2 t^2/2}|}{t} dt,$$

thus we have the following estimate

$$\begin{aligned} \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|\mathbb{E}_1[e^{-itX}] - e^{-\sigma^2 t^2/2}|}{t} dt &\leq \|c_{k,\ell}(\alpha, \cdot)(1 + o(M^{-q}))\|_{\infty, [0, \varepsilon]} \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|(1 - e^{t^3 p(t/\sqrt{N})/\sqrt{N}})e^{-\sigma^2 t^2/2}|}{t} dt \\ &\quad + \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|1 - c_{k,\ell}(\alpha, t/\sqrt{N})(1 + o(M^{-q}))| e^{-\sigma^2 t^2/2}}{t} dt, \end{aligned}$$

where  $\|\cdot\|_{\infty, [0, \varepsilon]}$  in the  $L^\infty$  norm on  $[0, \varepsilon]$ .

We notice that  $\|c_{k,\ell}(\alpha, \cdot)(1 + o(M^{-q}))\|_{\infty, [0, \varepsilon]}$  is uniformly bounded in  $N$ . Moreover,

$$\begin{aligned} \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{e^{-\sigma^2 t^2/2}}{t} |e^{t^3 p(t/\sqrt{N})/\sqrt{N}} - 1| dt &\leq \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{e^{-\sigma^2 t^2/2}}{t\sqrt{N}} e^{t^3 \|p\|_{\infty, [0, \varepsilon]}/\sqrt{N}} t^3 \|p\|_{\infty, [0, \varepsilon]} dt \\ &\leq \int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{e^{-\sigma^2 t^2/2}}{\sqrt{N}} e^{t^2 \varepsilon \|p\|_{\infty, [0, \varepsilon]}} t^2 \|p\|_{\infty, [0, \varepsilon]} dt, \end{aligned}$$

where in the first inequality we used the bound  $|e^x - 1| \leq |x|e^{|x|}$ . Since for  $\varepsilon$  small enough  $\|p\|_\infty \varepsilon < \sigma^2/4$ , thus integrating, we deduce that

$$\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|e^{-\sigma^2 t^2/2 + t^3 p(t/\sqrt{N})/\sqrt{N}} - e^{-\sigma^2 t^2/2}|}{t} dt = O\left(\frac{1}{\sqrt{N}}\right).$$

To conclude, we have to show that the last integral is of order  $N^{-1/2}$ . Since  $c_{k,\ell}(\alpha, t)$  is  $C^1$  in  $t$ , and  $c_{k,\ell}(\alpha, 0) = 1$ , it is easy to deduce that there exists a constant  $C$  such that

$$\int_{\frac{1}{N}}^{\varepsilon\sqrt{N}} \frac{|1 - c_{k,\ell}(\alpha, t/\sqrt{N})(1 + o(M^{-q}))| e^{-\sigma^2 t^2/2}}{t} dt \leq \frac{C}{\sqrt{N}}$$

so we conclude.  $\square$

## 5 Conclusion and Outlooks

In this paper, we proved a general Central Limit Theorem type result and we apply it to several models in random matrix theory and integrable systems. By doing this, we strengthen the connection between these two subjects. Specifically, we could connect the expected values and the variances of the moments of each classical  $\beta$  ensemble in the high-temperature regime with one specific integrable model, see Table 1.

The results that we have obtained are relevant for two main reasons. Under the random matrix theory perspective, we were able to develop a general framework to prove polynomial

central limit theorems for the classical  $\beta$  ensemble in the high-temperature regime, based on their band matrix representation and on the transfer operator technique. Under the integrable systems' theory point of view, our result enables the explicit computation of the so-called *susceptibility* matrix, which is a fundamental object in the theory of Generalized Hydrodynamics in order to compute the correlation functions for integrable models. Furthermore, we are able to prove rigorously the exponential decay of correlation for short-range interacting systems with polynomial potential.

It would be fascinating to generalize our result to a wider class of potential and functions and to obtain a Berry-Esseen bound for the classical  $\beta$  ensemble in the high-temperature regime. Furthermore, defining a new  $\beta$  ensemble related to the INB lattice would be interesting. Finally, we point out that it would be interesting to obtain large deviation principles for the Exponential Toda lattice and the Volterra one in the spirit of [35, 53].

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