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# A FUNCTIONAL LIMIT THEOREM FOR LATTICE OSCILLATING RANDOM WALK

May 11, 2023

Marc Peigné, Tran Duy Vo <sup>(1)</sup>

## Abstract

The paper is devoted to an invariance principle for Kemperman's model of oscillating random walk on  $\mathbb{Z}$ . This result appears as an extension of the invariance principal theorem for classical random walks on  $\mathbb{Z}$  or reflected random walks on  $\mathbb{N}_0$ . Relying on some natural Markov sub-process which takes into account the oscillation of the random walks between  $\mathbb{Z}^-$  and  $\mathbb{Z}^+$ , we first construct an aperiodic sequence of renewal operators acting on a suitable Banach space and then apply a powerful theorem proved by S. Gouëzel.

Keywords and phrases: oscillating random walk, invariance principle, skew Brownian motion, renewal sequences of operators, Markov chain

## 1 Model and setting

### 1.1 Introduction

Consider two independent sequences of i.i.d. discrete random variables  $(\xi_n)_{n \geq 1}$  and  $(\xi'_n)_{n \geq 1}$ , defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and with respective distributions  $\mu$  and  $\mu'$ . For any fixed  $\alpha \in [0, 1]$ , the oscillating random walk  $\mathcal{X}^{(\alpha)} = (X_n^{(\alpha)})_{n \geq 0}$

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is defined recursively by

$$X_{n+1}^{(\alpha)} = \begin{cases} X_n^{(\alpha)} + \xi_n & \text{if } X_n^{(\alpha)} \leq -1, \\ \eta_n & \text{if } X_n^{(\alpha)} = 0, \\ X_n^{(\alpha)} + \xi'_n & \text{if } X_n^{(\alpha)} \geq 1, \end{cases} \quad (1.1)$$

where  $\eta_n := B_n \xi_n + (1 - B_n) \xi'_n$  and  $(B_n)_{n \geq 0}$  is a sequence of i.i.d. Bernoulli random variables (independent of  $(\xi_n)$  and  $(\xi'_n)$ ) with  $\mathbb{P}[B_i = 1] = \alpha = 1 - \mathbb{P}[B_i = 0]$ .

When we want to emphasize the dependence in  $\mu$  and  $\mu'$  of this oscillating process, we denote it by  $\mathcal{X}^{(\alpha)}(\mu, \mu')$ .

This spatially non-homogeneous random walk was first introduced by KEMPERMAN [12] to model discrete-time diffusion in one dimensional space with three different media  $\mathbb{Z}^+$  and  $\mathbb{Z}^-$  and a barrier  $\{0\}$ . Whenever the process  $\mathcal{X}^{(\alpha)}(\mu, \mu')$  stays on the negative half line, its excursion is directed by the jumps  $\xi_n$  until it reaches the positive half line; then, it continues being directed by the jumps  $\xi'_n$  until returning in the negative half line and so on. After each visit of the origin, the increment is governed by the distribution of  $\eta_n$ , which is a convex combination of  $\mu$  and  $\mu'$ . Our considering system is referred to as a special case when the barrier is degenerated as a single point; in general context, it may be any determined interval  $[a, b] \cap \mathbb{Z}$  which passes through the origin, see [13] for instance. Another interesting variant has been studied by MADRAS and TANNY in [15] dealing with an oscillating random walk with a moving barrier at some constant speed. Although this basically leads to differences in its long-term behaviour compared to (1.1), we may be able to trace this model back to (1.1) by using some appropriate translations for its random increments.

In the present paper, we prove an invariant principle for  $\mathcal{X}^{(\alpha)}(\mu, \mu')$  towards the skew Brownian motion  $(B_t^\gamma)_{t > 0}$  on  $\mathbb{R}$  with parameter  $\gamma \in [0, 1]$ . The diffusion  $(B_t^\gamma)_{t > 0}$  is obtained from the standard Brownian process by independently altering the signs of the excursions away from 0, each excursion being positive with probability  $\gamma$  and negative with probability  $1 - \gamma$ . By [18], its heat kernel is given by: for any  $x, y \in \mathbb{R}$  and  $t > 0$ ,

$$p_t^\gamma(x, y) := p_t(x, y) + (2\gamma - 1) \text{sign}(y) p_t(0, |x| + |y|),$$

where  $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$  is the transition density of the Brownian motion.

Throughout this paper, we suppose that the following general assumptions always hold:

**H1**  $(\xi_n)_{n \geq 1}$  and  $(\xi'_n)_{n \geq 1}$  are independent sequences of i.i.d.  $\mathbb{Z}$ -valued random variables, with finite variances  $\sigma^2$  and  $\sigma'^2$ , respectively.

**H2** Both distributions  $\mu$  and  $\mu'$  are centered (i.e.  $\mathbb{E}[\xi_n] = \mathbb{E}[\xi'_n] = 0$ ).

**H3** Both distributions  $\mu$  and  $\mu'$  are strongly aperiodic on  $\mathbb{Z}$ , i.e. their supports are not included in  $b + a\mathbb{Z}$  for any  $a > 1$  and  $b \in \{0, \dots, a - 1\}$ .

**H4**  $\mathbb{E}[(\xi_n^+)^3] + \mathbb{E}[(\xi_n^-)^3] < +\infty$ , where  $\xi_n^+ := \max\{0, \xi_n\}$  and  $\xi_n^- := \max\{0, -\xi_n\}$ .

We denote by  $S = (S_n)_{n \geq 1}$  (resp.  $S' = (S'_n)_{n \geq 1}$ ) the random walk defined by  $S_0 = 0$  and  $S_n = \xi_1 + \dots + \xi_n$  for  $n \geq 1$  (resp.  $S'_0 = 0$  and  $S'_n = \xi'_1 + \dots + \xi'_n$  for  $n \geq 1$ ). Let  $(\ell_i)_{i \geq 0}$  be the sequence of strictly ascending ladder epochs associated with  $S$  and defined recursively by  $\ell_0 = 0$  and, for  $i \geq 1$ ,

$$\ell_{i+1} := \inf\{k > \ell_i \mid S_k > S_{\ell_i}\}$$

(with the convention  $\inf \emptyset = +\infty$ ). We also consider the sequence of descending ladder epochs  $(\ell'_i)_{i \geq 0}$  of  $S'$ , defined as follows,

$$\ell'_0 = 0, \quad \text{and} \quad \ell'_{i+1} := \inf\{k > \ell'_i \mid S'_k < S'_{\ell'_i}\}, \quad \text{for any } i \geq 1.$$

Under hypothesis **H2**, it holds  $\mathbb{P}[\limsup_{n \rightarrow +\infty} S_n = +\infty] = \mathbb{P}[\liminf_{n \rightarrow +\infty} S'_n = -\infty] = 1$ ; hence, all the random variables  $\ell_i$  and  $\ell'_i$  are  $\mathbb{P}$ -a.s. finite. In addition, both sequences  $(\ell_{i+1} - \ell_i)_{i \geq 0}$  and  $(S_{\ell_{i+1}} - S_{\ell_i})_{i \geq 0}$  contain i.i.d. random elements with distributions of  $\ell_1$  and  $S_{\ell_1}$ , respectively; the same property holds for  $(\ell'_{i+1} - \ell'_i)_{i \geq 0}$  and  $(S'_{\ell'_{i+1}} - S'_{\ell'_i})_{i \geq 0}$ . Consequently, processes  $(\ell_i)_{i \geq 0}$ ,  $(S_{\ell_i})_{i \geq 0}$ ,  $(\ell'_i)_{i \geq 0}$  and  $(S'_{\ell'_i})_{i \geq 0}$  are all random walks with i.i.d. increments.

We denote  $\mu_+$  the distribution of  $S_{\ell_1}$  and  $\mathcal{U}_+$  its potential defined by  $\mathcal{U}_+ := \sum_{n \geq 0} (\mu_+)^{*n}$ . Similarly  $\mu'_-$  denotes the distribution of  $S'_{\ell'_1}$  and  $\mathcal{U}'_- := \sum_{n \geq 0} (\mu'_-)^{*n}$ .

In particular, the oscillating random walk  $\mathcal{X}^{(\alpha)}$  visits  $\mathbb{Z}^-$  and  $\mathbb{Z}^+$  infinitely often; in order to control the excursions inside each of these two half lines, it is natural to consider the following stopping times  $\tau^S(-x)$ ,  $\tau^{S'}(x)$  with  $x \geq 1$ , associated with  $S$  and  $S'$  respectively and defined by

$$\tau^S(-x) := \inf\{n \geq 1 \mid -x + S_n \geq 0\}, \quad \text{and} \quad \tau^{S'}(x) := \inf\{n \geq 1 \mid x + S'_n \leq 0\}.$$

In the sequel, we focus on the ‘‘ascending renewal function’’  $h_a$  of  $S$  and the ‘‘de-

scending renewal function"  $h'_d$  of  $S'$  defined by

$$h_a(x) := \begin{cases} \mathcal{U}_+[0, x] = \sum_{i \geq 0} \mathbb{P}[S_{\ell_i} \leq x] & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h'_d(x) := \begin{cases} \mathcal{U}'_-[-x, 0] = \sum_{i \geq 0} \mathbb{P}[S'_{\ell'_i} \geq -x] & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by  $\check{h}_a$  the function  $x \mapsto h_a(-x)$ , it appears in the definition of the parameter  $\gamma$  below.

Both functions  $h_a$  and  $h'_d$  are increasing and satisfy  $h_a(x) = O(x)$  and  $h'_d(x) = O(x)$ . They appear crucially in the quantitative estimates of the fluctuations of  $S$  and  $S'$ ; see subsection 2.1 for precise statements.

Let us end this paragraph devoted to the presentation of quantities that play an important role in the rest of the paper.

- By classical results on 1-dimensional random walks [9], under hypotheses **H1** and **H2**, both constants  $c = \frac{\mathbb{E}[S_{\ell_1}]}{\sigma\sqrt{2\pi}}$  and  $c' = \frac{\mathbb{E}[-S_{\ell'_1}]}{\sigma'\sqrt{2\pi}}$  are finite.

- Under hypotheses **H1**, **H2** and **H3**, the ‘‘crossing sub-process’’  $\mathcal{X}_C^{(\alpha)}$  which corresponds to the sign changes of the process  $\mathcal{X}^{(\alpha)}$  is well defined (see section 3.1) and it is positive recurrent on its unique irreducible class. We denote by  $\nu$  its unique invariant probability measure on  $\mathbb{Z}$ .

## 1.2 Main result

**From now on, we fix**  $\alpha \in [0, 1]$  and consider the continuous and linearly interpolated version  $(X_{nt})$  of  $\mathcal{X}^{(\alpha)}$ , defined by: for any  $n \geq 1$  and  $t \in (0, 1]$ ,

$$X_{nt}^{(\alpha)} = \sum_{i=1}^n \left( X_{[nt]}^{(\alpha)} + (nt - [nt]) \times J_{[nt]+1} \right) \mathbb{1}_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}(t),$$

where

$$J_{[nt]+1} := \begin{cases} \xi_{[nt]+1} & \text{if } X_{[nt]}^{(\alpha)} \leq -1 \\ \eta_{[nt]+1} & \text{if } X_{[nt]}^{(\alpha)} = 0 \\ \xi'_{[nt]+1} & \text{if } X_{[nt]}^{(\alpha)} \geq 1. \end{cases}$$

We also set

$$X^{(\alpha, n)}(t) := \begin{cases} \frac{X_{nt}^{(\alpha)}}{\sigma\sqrt{n}} & \text{if } X_{nt} \leq 0, \\ \frac{X_{nt}^{(\alpha)}}{\sigma'\sqrt{n}} & \text{if } X_{nt} \geq 0. \end{cases}$$

The main result of this paper is the following one.

**Theorem 1.1.** *Assume that hypotheses **H1**–**H4** are satisfied. Then, as  $n \rightarrow +\infty$ , the normalized stochastic process  $\{X^{(\alpha, n)}(t), t \in [0, 1]\}_{n \geq 1}$  converges weakly in the space of continuous function  $C([0, 1])$  to the skew Brownian motion  $W_\gamma := \{W_\gamma(t), t \in [0, 1]\}$  with parameter  $\gamma = \frac{c'\nu(h'_d)}{c\nu(\check{h}_a) + c'\nu(h'_d)}$ .*

Let us clarify the value of the parameter  $\gamma$  in two peculiar cases of (1.1).

- When  $\mu = \mu'$ , the chain  $\mathcal{X}^{(\alpha)}$  is an ordinary random walk on  $\mathbb{Z}$  directed by the unique type of jumps  $(\xi_n)_{n \geq 1}$  and the limit diffusion  $W_\gamma$  is the Brownian motion. In this case, the parameter  $\gamma$  equals  $\frac{1}{2}$  since the sequences of ladder heights  $(S_{\ell_i})_{i \geq 1}$  and  $(-S'_{\ell'_i})_{i \geq 1}$  coincide.

- When  $\mu(x) = \mu'(-x)$  for any  $x \in \mathbb{Z}$ , the random walk  $\mathcal{X}^{(\alpha)}$  is the so-called “anti-symmetric random walk” (or “reflected random walk” as usual), which appears in several works, see for instance [8] and [17]. By setting  $\xi_n = -\xi'_n$ , the behaviour of the chain  $\mathcal{X}^{(\alpha)}$  on positive and negative half lines, respectively, are mirror images of each other. Hence, we may “glue” them together to get an unifying Markov chain on  $\mathbb{Z}^+ \cup \{0\}$  receiving  $\{0\}$  as its reflecting boundary. Accordingly,  $\gamma = 1$  in this case and it matches perfectly with the result in [16], which states that the normalized reflected random walk (constructed as above) converges weakly in  $C([0, 1])$  towards the absolute value of the standard Brownian motion.

- Notice that the limit process  $W_\gamma$  does not depend on  $\alpha \in [0, 1]$ . Henceforth, we fix  $\alpha$  and set  $\mathcal{X}^{(\alpha)} = \mathcal{X}$  in order to simplify the notations.

### 1.3 Notations

We set  $\mathbb{Z} := \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$ ,  $\mathbb{N} := \{1, 2, 3, \dots\}$  and  $\bar{\mathbb{D}}$  the closed unit ball in  $\mathbb{C}$ . Given two positive real sequences  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  and  $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$ , we write as usual

- $a_n \sim b_n$  if  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ ,

- $a_n \approx b_n$  if  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ ,
- $a_n = O(b_n)$  if  $\limsup_{n \rightarrow \infty} a_n/b_n < +\infty$ ,
- $a_n = o(b_n)$  if  $\lim_{n \rightarrow \infty} a_n/b_n = 0$ ,
- $\mathbf{a} \preceq \mathbf{b}$  if  $a_n \leq c b_n$  for some constant  $c > 0$ .

The paper is organized as follows. In Section 2, we recall some important estimates in the theory of fluctuations of random walks; we introduce in particular the renewal functions associated with 1-dimensional random walks and relative conditional limit theorems. These helpful tools appear in Section 4 to compute the multi-dimensional distribution of the limit process. The center of gravity of the paper is Section 3 where we adapt the approach used in [16] in the case of the reflected random walk (with proper adjustments to derive Corollary 3.5 and to determine the parameter  $\gamma$  later on). The last two sections are devoted to the proof of Theorem 1.1 and the appendix.

## 2 Auxiliary results for random walks

In this section, we present some classical results on fluctuations of random walk on  $\mathbb{Z}$ .

### 2.1 Asymptotic estimates for fluctuations of a random walk

The following statement summarizes classical results on fluctuations of random walks which are used below at various places (for instance, see Proposition 11 in [6], Theorem A in [14] et al).

**Lemma 2.1.** (Asymptotic property) Under assumptions **H1**–**H3**, for any  $x, y \geq 1$ , it holds, as  $n \rightarrow \infty$ ,

$$a) \mathbb{P}[\tau^S(-x) > n] \sim 2c \frac{h_a(x)}{\sqrt{n}}, \quad \text{and} \quad \mathbb{P}[\tau^{S'}(x) > n] \sim 2c' \frac{h'_a(x)}{\sqrt{n}};$$

$$b) \mathbb{P}[\tau^S(-x) > n, -x + S_n = -y] \sim \frac{1}{\sigma\sqrt{2\pi}} \frac{h_a(x)h_d(y)}{n^{3/2}},$$

$$\text{and} \quad \mathbb{P}[\tau^{S'}(x) > n, x + S'_n = y] \sim \frac{1}{\sigma'\sqrt{2\pi}} \frac{h'_d(x)h'_a(y)}{n^{3/2}};$$

where  $h_d$  (resp.  $h'_a$ ) is the descending (resp. ascending) renewal function associated with the random walk  $S$  (resp.  $S'$ ).

$$c) \mathbb{P}[\tau^S(-x) = n] \sim c \frac{h_a(x)}{n^{3/2}}, \quad \text{and} \quad \mathbb{P}[\tau^{S'}(x) = n] \sim c' \frac{h'_d(x)}{n^{3/2}};$$

**Lemma 2.2.** (*Upper bound*) For any  $n \geq 1$ , it holds

$$a) \mathbb{P}[\tau^S(-x) > n] \preceq \frac{1+x}{\sqrt{n}}, \quad \text{and} \quad \mathbb{P}[\tau^{S'}(x) > n] \preceq \frac{1+x}{\sqrt{n}};$$

$$b) \mathbb{P}[\tau^S(-x) > n, -x + S_n = -y] \preceq \frac{(1+x)(1+y)}{n^{3/2}},$$

$$\text{and} \quad \mathbb{P}[\tau^{S'}(x) > n, x + S'_n = y] \preceq \frac{(1+x)(1+y)}{n^{3/2}};$$

$$c) \mathbb{P}[\tau^S(-x) = n] \preceq \frac{1+x}{n^{3/2}}, \quad \text{and} \quad \mathbb{P}[\tau^{S'}(x) = n] \preceq \frac{1+x}{n^{3/2}}.$$

As a direct consequence of b) in Lemmas 2.1 and 2.2, for any  $x \geq 1$  and  $w \geq 0$ ,

$$\mathbb{P}[\tau^S(-x) = n, -x + S_n = w] \preceq \frac{1+x}{n^{3/2}} \sum_{z \geq w+1} z\mu(z). \quad (2.1)$$

Indeed, for any  $n \geq 1$ ,

$$\begin{aligned} & \mathbb{P}[\tau^S(-x) = n, -x + S_n = w] \\ &= \sum_{y \geq 1} \mathbb{P}[\tau^S(-x) = n, -x + S_{n-1} = -y, -y + \xi_n = w] \\ &= \sum_{y \geq 1} \mathbb{P}[\tau^S(-x) > n-1, -x + S_{n-1} = -y, -y + \xi_n = w] \\ &= \sum_{y \geq 1} \mathbb{P}[\tau^S(-x) > n-1, -x + S_{n-1} = -y] \mu(y+w) \\ &\preceq \frac{1+x}{n^{3/2}} \underbrace{\sum_{y \geq 1} (1+y)\mu(y+w)}_{\preceq \sum_{z \geq w+1} z\mu(z)}. \end{aligned}$$

## 2.2 Conditional limit theorems

It is worth remarking some necessary limit theorems which are very helpful for us to control the fluctuation of excursions between two consecutive successive crossing times and contribute significantly to reduce the complexity when dealing with multidimensional distribution of these excursions. Now, assume that  $\mathbb{E}[\xi'_1] = 0$  and  $\mathbb{E}[(\xi'_1)^2] < +\infty$  and let  $(S'(t))_{t \geq 0}$  be the continuous time process constructed from the sequence  $(S'_n)_{n \geq 0}$  by using the linear interpolation between the values at integer points.



By Lemma 2.3 in [1], for  $x \geq 1$ , the rescaled process  $\left(\frac{x + S'_{[nt]}}{\sigma' \sqrt{n}}, t \in [0, 1]\right)$  conditioning on the event  $[\tau^{S'}(x) > n]$  converges weakly on  $C([0, 1], \mathbb{R})$  towards the Brownian meander. In other words, for any bounded Lipschitz continuous function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  and any  $t \in (0, 1]$  and  $x \geq 1$ ,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \psi \left( \frac{x + S'_{[nt]}}{\sigma' \sqrt{n}} \right) \mid \tau^{S'}(x) > [nt] \right] = \frac{1}{t} \int_0^{+\infty} \psi(u) u \exp \left( -\frac{u^2}{2t} \right) du. \quad (2.2)$$

Let us also state the Caravena-Chaumont's result about random bridges conditioned to stay positive in the discrete case. Roughly speaking, as  $n \rightarrow +\infty$ , for any starting point  $x \geq 1$  and any ending point  $y \geq 1$ , the random bridge of the random walk  $S$ , starting at  $x$ , ending at  $y$  at time  $n$  and conditioned to stay positive until time  $n$ , after a linear interpolation and a diffusive rescaling, converges in distribution on  $C([0, 1], \mathbb{R})$  towards the normalized Brownian excursion  $\mathcal{E}^+$ :

$$\left( \left( \frac{S'_{[nt]}}{\sigma' \sqrt{n}} \right)_{t \in [0, 1]} \mid \tau^{S'}(x) > [nt], S'_n = y \right) \xrightarrow{\mathcal{L}} \mathcal{E}^+, \quad \text{as } n \rightarrow +\infty.$$

More precisely, for any  $x, y \geq 1, 0 < s < t \leq 1$  and any bounded Lipschitz continuous function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \psi \left( \frac{x + S'_{[ns]}}{\sigma' \sqrt{n}} \right) \mid \tau^{S'}(x) > [nt], x + S'_{[nt]} = y \right] \\ = \int_0^{+\infty} 2\psi(u\sqrt{s}) \exp \left( -\frac{u^2}{2\frac{s}{t}\frac{t-s}{t}} \right) \frac{u^2}{\sqrt{2\pi \frac{s^3}{t^3} \frac{(t-s)^3}{t^3}}} du. \end{aligned} \quad (2.3)$$

### 3 Crossing times and renewal theory

In order to analyse the asymptotic behavior of the process  $\mathcal{X}$ , we decompose  $X_n$  as a sum of successive excursions in  $\mathbb{Z}^-$  or  $\mathbb{Z}^+$ . It is therefore interesting to introduce the sequence  $\mathbf{C} = (C_k)_{k \geq 0}$  of ‘‘crossing times’’, i.e. times at which the process  $\mathcal{X}$

changes its sign: more precisely,  $C_0 = 0$  and, for any  $k \geq 0$ ,

$$C_{k+1} := \begin{cases} \inf\{n > C_k \mid X_{C_k} + (\xi_{C_{k+1}} + \dots + \xi_n) \geq 0\} & \text{if } X_{C_k} \leq -1, \\ C_k + 1 & \text{if } X_{C_k} = 0, \\ \inf\{n > C_k \mid X_{C_k} + (\xi'_{C_{k+1}} + \dots + \xi'_n) \leq -1\} & \text{if } X_{C_k} \geq 1. \end{cases} \quad (3.1)$$

Under hypothesis **H2**, the random times  $C_k$  are  $\mathbb{P}$ -a.s. finite and form a sequence of finite stopping times with respect to the canonical filtration  $(\sigma(\xi_k, \xi'_k) \mid k \leq n)_{n \geq 1}$ .

### 3.1 On the crossing sub-process $\mathcal{X}_{\mathcal{C}}$

We denote  $\mathcal{X}_{\mathcal{C}} := (X_{C_k})_{k \geq 0}$  the **crossing sub-process of  $\mathcal{X}$** , which plays an important role in this paper.

**Lemma 3.1.** *The sub-process  $\mathcal{X}_{\mathcal{C}}$  is a time-homogeneous Markov chain on  $\mathbb{Z}$  with transition kernel  $\mathcal{C} = (\mathcal{C}(x, y))_{x, y \in \mathbb{Z}}$  given by*

$$\mathcal{C}(x, y) = \begin{cases} \sum_{t=0}^{-x-1} \mu_+(y - x - t) \mathcal{U}_+(t) & \text{if } x \leq -1 \text{ and } y \geq 0, \\ \alpha \mu(y) + (1 - \alpha) \mu'(y) & \text{if } x = 0 \text{ and } y \in \mathbb{Z}, \\ \sum_{t=-x+1}^0 \mu'_-(y - x - t) \mathcal{U}'_-(t) & \text{if } x \geq 1 \text{ and } y \leq 0, \end{cases} \quad (3.2)$$

*Proof.* The Markov property is obvious from the above definition.

Now, we compute  $\mathcal{C}(x, y)$  for any  $x \leq -1$  and  $y \geq 0$  (other cases are similar) as follows. Noticing that the first crossing time  $C_1$  belongs  $\mathbb{P}$ -a.s. to the set  $\{\ell_k \mid k \geq 1\}$  and that the sequence  $(S_{\ell_k})_{k \geq 1}$  is increasing, we may write

$$\begin{aligned} \mathcal{C}(x, y) &= \sum_{k \geq 1} \mathbb{P}[x + S_{\ell_{k-1}} \leq -1, x + S_{\ell_k} = y] \\ &= \sum_{k \geq 1} \sum_{t=0}^{-x-1} \mathbb{P}[S_{\ell_{k-1}} = t] \mathbb{P}[S_{\ell_k} - S_{\ell_{k-1}} = y - x - t] \\ &= \sum_{t=0}^{-x-1} \mathbb{P}[S_{\ell_1} = y - x - t] \sum_{i \geq 0} \mathbb{P}[S_{\ell_i} = t] \\ &= \sum_{t=0}^{-x-1} \mu_+(y - x - t) \mathcal{U}_+(t). \end{aligned}$$

□

By Theorems 2.1 and 2.2 in [19], under hypothesis **H3**, the oscillating random walk  $\mathcal{X}$  is irreducible on  $\mathbb{Z}$ ; furthermore, when **H2** holds, the crossing sub-process  $\mathcal{X}_{\mathbf{C}}$  is well defined and it is irreducible, aperiodic and positive recurrent on its unique essential class  $\mathcal{I}_{\mathbf{C}}(X_0)$ . Notice that this essential class can be a proper subset of  $\mathbb{Z}$ ; it occurs for instance when the support of  $\mu$  is bounded from above or the one of  $\mu'$  is bounded from below. Nevertheless, it admits a unique invariant probability measure  $\nu$  supported by  $\mathcal{I}_{\mathbf{C}}(X_0)$ . Also in [19], the explicit expression of  $\nu$  is only known when  $\alpha \in \{0, 1\}$  and the support of  $\mu$  (resp.  $\mu'$ ) is included in  $\mathbb{Z}^+$  (resp. in  $\mathbb{Z}^-$ ). However, the existence of  $\nu$  is enough for our purpose regardless of its exact formula.

Another point to insist on here is that any excursion between two consecutive crossing times is uniquely governed by  $S$  or  $S'$ ; thus, all the results obtained in the previous section can be applied. The decomposition technique that exploits this fact is classical and extremely efficient in controlling the varying excursions over time of Markov processes; for example, we use it in the last section to estimate the convergence of finite dimensional distribution. As a direct application, we can prove that the strong law of large numbers still holds for the chain  $\mathcal{X}$ .

**Lemma 3.2.** *Assume that  $\mathbb{E}[|\xi_n|] + \mathbb{E}[|\xi'_n|] < +\infty$  and  $\mathbb{E}[\xi_n] = \mathbb{E}[\xi'_n] = 0$ . Then, it holds*

$$\lim_{n \rightarrow +\infty} \frac{X_n}{n} = 0 \quad \mathbb{P}\text{-a.s.}$$

*Proof.* We decompose  $X_n$  as  $X_n = X_n \mathbb{1}_{\{X_n \geq 1\}} + X_n \mathbb{1}_{\{X_n \leq -1\}}$ .

Let us estimate the first term. For any  $n \geq 1$ , there exists a random integer  $k(n) \geq 0$  such that  $C_{k(n)} \leq n < C_{k(n)+1}$ ; notice that the condition  $X_n \geq 1$  yields  $X_{C_{k(n)}} \geq 1$ . Hence, we get

$$\begin{aligned} 0 \leq \frac{X_n \mathbb{1}_{\{X_n \geq 1\}}}{n} &= \frac{X_{C_{k(n)}} + S'_n - S'_{C_{k(n)}}}{n} \\ &\leq \frac{\max\{X_0, \xi_{C_{k(n)}}\}}{n} + \frac{S'_n}{n} - \frac{S'_{C_{k(n)}}}{C_{k(n)}} \frac{C_{k(n)}}{n} \\ &\leq \frac{\max\{X_0, \xi_{C_{k(n)}}\}}{n} + \frac{S'_n}{n} + \left| \frac{S'_{C_{k(n)}}}{C_{k(n)}} \right|. \end{aligned}$$

By the strong law of large numbers, the different terms on the right-hand side above

converges  $\mathbb{P}$ -a.s. to 0; so does  $\frac{X_n \mathbb{1}_{\{X_n \geq 1\}}}{n}$ . The second term is treated in the same way.  $\square$

### 3.2 On aperiodic renewal sequences of operators

Let  $(\mathbb{Z}^{\otimes \mathbb{N}}, (\mathcal{P}(\mathbb{Z}))^{\otimes \mathbb{N}}, \mathcal{X}, (\mathbb{P}_x)_{x \in \mathbb{Z}}, \theta)$  be the canonical space, i.e. the space of trajectories associated with the Markov chain  $\mathcal{X}$ . For any  $x \in \mathbb{Z}$ , we probability measure  $\mathbb{P}_x$  is the conditional probability with respect to the event  $[X_0 = x]$ , we denote by  $\mathbb{E}_x$  the corresponding conditional expectation. The operator  $\theta$  is the classical shift transformation defined by  $\theta((x_k)_{k \geq 0}) = (x_{k+1})_{k \geq 0}$  for any  $(x_k)_{k \geq 0} \in \mathbb{Z}^{\otimes \mathbb{N}}$ .

In this section, we study the behavior as  $n \rightarrow +\infty$  of the sequence

$$\Sigma_n(x, y) = \sum_{k=1}^{+\infty} \mathbb{P}_x[C_k = n, X_n = y],$$

for any  $x, y \in \mathbb{Z}$ . Since the position at time  $C_k$  may vary, so that the excursions of  $\mathcal{X}$  between two successive crossing times are not independent, it thus motivates us to take into account the long-term behaviours of these quantities and express them in terms of operators related to the crossing sub-process  $\mathcal{X}_{\mathcal{C}}$ . For this purpose, we apply a general renewal theorem due to S. Gouëzel [10]. This theorem relies on the decomposition of the operator  $\mathcal{C}$  using a sequence of operators  $(\mathcal{C}_n)_{n \geq 1}$  acting on some Banach space and that are not so difficult to deal with.

It is natural in our context to deal with the operators  $\mathcal{C}_n = (\mathcal{C}_n(x, y))_{x, y \in \mathbb{Z}}, n \geq 1$ , defined by: for any  $x, y \in \mathbb{Z}$  and any  $n \geq 1$ ,

$$\mathcal{C}_n(x, y) := \mathbb{P}_x[C_1 = n, X_n = y].$$

The relation  $\mathcal{C}(x, y) = \sum_{n \geq 1} \mathcal{C}_n(x, y)$  is straightforward. We also pay attention to the case  $x = 0$ , that is  $\mathcal{C}_1(0, y) = \mathbb{P}_0[X_1 = y] = \alpha \mu(y) + (1 - \alpha) \mu'(y)$  and  $\mathcal{C}_n(0, y) = 0$  if  $n \geq 2$ .

For a function  $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$ , we formally set

$$\mathcal{C}_n \varphi(x) := \sum_{y \in \mathbb{Z}: xy \leq 0} \mathcal{C}_n(x, y) \varphi(y) = \mathbb{E}_x[\varphi(X_n), C_1 = n] \quad \text{if } x \in \mathbb{Z} \setminus \{0\},$$

and  $\mathcal{C}_1 \varphi(0) = \sum_{y \in \mathbb{Z}} \mathcal{C}_1(0, y) \varphi(y) = \mathbb{E}_0[\varphi(X_1)]$  and  $\mathcal{C}_n \varphi(0) = 0$  if  $n \geq 2$ . The quantity  $\mathcal{C}_n \varphi(x)$  is well defined for instance when  $\varphi \in L^\infty(\mathbb{Z})$ . Other Banach spaces can be

considered; under moment assumptions, we describe below the action of the  $\mathcal{C}_n$  on a bigger Banach space  $\mathcal{B}$ , more suitable to the situation as explained a little further on.

Notice that  $\mathcal{C}_n(x, y) = \mathcal{C}_n \mathbb{1}_{\{y\}}(x)$  for any  $x, y \in \mathbb{Z}$ , which yields, by induction,

$$\begin{aligned} \Sigma_n(x, y) &= \sum_{k=1}^{+\infty} \mathbb{P}_x[C_k = n, X_n = y] \\ &= \sum_{k=1}^{+\infty} \sum_{j_1 + \dots + j_k = n} \mathbb{P}_x[C_1 = j_1, C_2 - C_1 = j_2, \dots, C_k - C_{k-1} = j_k, X_n = y] \\ &= \sum_{k=1}^{+\infty} \sum_{j_1 + \dots + j_k = n} \mathcal{C}_{j_1} \dots \mathcal{C}_{j_k} \mathbb{1}_{\{y\}}(x). \end{aligned} \quad (3.3)$$

As announced above, we apply a result of S. Gouezel, stated in a general framework [10], that of *aperiodic renewal sequence of operator*, i.e. sequences  $(\mathcal{C}_n)_{n \geq 1}$  of operators acting on a Banach space  $\mathcal{B}$  and satisfying the following conditions:

- the operators  $\mathcal{C}_n, n \geq 1$ , act on  $\mathcal{B}$  and  $\sum_{n \geq 1} |\mathcal{C}_n|_{\mathcal{B}} < +\infty$  ;
- the operator  $\mathcal{C}(z) := \sum_{n \geq 1} z^n \mathcal{C}_n$ , defined for any  $z \in \overline{\mathbb{D}}$ , satisfies

R1-  $\mathcal{C}(1)$  has a simple eigenvalue at 1 (with corresponding eigenprojector  $\Pi$ ) and the rest of its spectrum is contained in a disk of radius  $< 1$ ;

R2- for any complex number  $z \in \overline{\mathbb{D}} \setminus \{1\}$ , the spectral radius of  $\mathcal{C}(z)$  is  $< 1$ ;

R3- for any  $n \geq 1$ , the real number  $r_n$  defined by  $\Pi \mathcal{C}_n \Pi = r_n \Pi$  is  $\geq 0$ .

Condition R2 implies that, for any  $z \in \overline{\mathbb{D}} \setminus \{1\}$ , the operator  $I - \mathcal{C}(z)$  is invertible on  $\mathcal{B}$  and

$$(I - \mathcal{C}(z))^{-1} = \sum_{k \geq 0} \mathcal{C}(z)^k = \sum_{k \geq 0} \left( \sum_{j \geq 1} \mathcal{C}_j z^j \right)^k = \sum_{n \geq 0} H_n z^n$$

with  $H_0 = I$  and  $H_n = \sum_{k=1}^{+\infty} \sum_{j_1 + \dots + j_k = n} \mathcal{C}_{j_1} \dots \mathcal{C}_{j_k}$ . The above identity, called the *renewal equation*, is of fundamental importance to understand the asymptotics of  $H_n$  in the non-commutative setting; in particular, the equality (3.3) yields  $\Sigma_n(x, y) = H_n \mathbb{1}_{\{y\}}(x)$  so that the asymptotic behaviour of  $(\Sigma_n(x, y))_{n \geq 1}$  is related to that of  $(H_n)_{n \geq 1}$ .

By [10], if the sequence  $(\mathcal{C}_n)_{n \geq 1}$  satisfies the following additive assumptions

$$\text{R4}(\ell, \beta). \quad |\mathcal{C}_n|_{\mathcal{B}} \leq C \frac{\ell(n)}{n^{1+\beta}},$$

$$\text{R5}(\ell, \beta). \quad \sum_{j>n} r_j \sim \frac{\ell(n)}{n^\beta},$$

where  $C > 0$ ,  $\beta \in (0, 1)$  and  $\ell$  is a slowly varying function, then the sequence  $(n^{1-\beta}\ell(n)H_n)_{n \geq 1}$  converges in  $\mathcal{B}$  to the operator  $d_\beta \Pi$ , with  $d_\beta = \frac{1}{\pi} \sin \beta\pi$ .

In the next subsection, we introduce some Banach space  $\mathcal{B}$  in order to be able to apply this general result.

### 3.3 Spectral property of the transition matrix $\mathcal{C} = (\mathcal{C}(x, y))_{x, y \in \mathbb{Z}}$

The operator  $\mathcal{C}$  acts on the space  $L^\infty(\mathbb{Z})$  of bounded functions on  $\mathbb{Z}$ . By the following lemma, it satisfies some strong spectral property on this space.

**Lemma 3.3.** *Assume **H1**– **H3** hold. Then, the infinite matrix  $\mathcal{C}$  satisfies the Doeblin condition and therefore, it is a quasi-compact operator on  $L^\infty(\mathbb{Z})$ , the space of bounded functions on  $\mathbb{Z}$ . Furthermore, the eigenvalue 1 is simple, with associated eigenvector  $\mathbf{1}$ , and the rest of the spectrum is included in a disk of radius  $< 1$ .*

*Proof.* Under the above assumptions, the positive random variable  $S_{\ell_1}$  has finite first moment; hence, by the renewal theorem,

$$\lim_{t \rightarrow +\infty} \mathcal{U}_+(t) = \frac{1}{\mathbb{E}[S_{\ell_1}]} > 0.$$

The above convergence readily implies  $\delta := \inf_{z \in \mathbb{Z}^+} \mathcal{U}_+(z) > 0$ .

Consequently, by (3.2), for any  $x \leq -1$  and  $y \geq 0$ ,

$$\mathcal{C}(x, y) \geq \mu_+(y+1) \mathcal{U}_+(-x-1) \geq \delta \mu_+(y+1).$$

In the same vein, one gets  $\mathcal{C}(x, y) \geq \delta' \mu'_-(y-1)$  for any  $x \geq 1$  and  $y \leq 0$  with  $\delta' := \inf_{z \in \mathbb{Z}^-} \mathcal{U}'_-(z) > 0$ . Hence, it is easy to show that there exists a probability measure  $\mathbf{m}$  s.t. for any  $x \in \mathbb{Z}$ ,

$$\mathcal{C}(x, \cdot) \geq \delta \mathbf{m}(\cdot),$$

which immediately implies the quasi-compactness of  $\mathcal{C}$ . The control of the peripheral spectrum readily follows.  $\square$

Thanks to this lemma, we could believe that hypothesis R1 is satisfied by the sequence  $(\mathcal{C}_n)_{n \geq 1}$  acting on  $L^\infty(\mathbb{Z})$  since  $\mathcal{C}(1) = \mathcal{C}$ . Unfortunately, it holds  $\sum_{n \geq 1} |\mathcal{C}_n|_\infty = +\infty$ . Indeed, it holds  $|\mathcal{C}_n|_\infty = \sup_{x \in \mathbb{Z}} \mathbb{P}[C_1 = n]$ ; now, if we assume for instance  $x \leq 1$ , it holds  $\mathbb{P}_x[C_1 = n] = \mathbb{P}[\tau^S(x) = n]$  with

$$(i) \quad \mathbb{P}[\tau^S(x) = n] = O(1/n),$$

and

$$(ii) \quad \liminf_{n \rightarrow +\infty} n \mathbb{P}[\tau^S(x_n) = n] > 0 \text{ when } x_n \asymp \sqrt{n} \text{ }^{(2)}$$

(see Lemma 5 and Theorem (B) [7]). Consequently  $|\mathcal{C}_n| \asymp 1/n$ .

Thus, we have to choose another Banach space  $\mathcal{B}$ . By (3.1), it is clear that  $C_{k+1} = \tau_S(X_{C_k})$  when  $X_{C_k} \leq -1$  and  $C_{k+1} = \tau_{S'}(X_{C_k})$  when  $X_{C_k} \geq 1$ . Consequently, the behaviour as  $n \rightarrow +\infty$  of the  $k^{\text{th}}$ -term  $\mathbb{P}_x[C_k = n, X_n = y]$  of the sum  $\Sigma_n(x, y)$  is closely related to the distributions of  $\tau_S$  and  $\tau_{S'}$ ; in particular, by Lemma 2.1, its dependence on  $y$  is expressed in terms of  $h_a(y)$  and  $h'_d(y)$ . This explains why we have to choose a Banach space on which the action of  $\mathcal{C}$  has “nice” spectral properties - as compactness or quasi-compactness - and also does contain these functions  $h_a$  and  $h'_d$ . The fact that they are both sublinear leads us to examine the action of  $\mathcal{C}$  on the space  $\mathcal{B}$  of complex valued functions on  $\mathbb{Z}$  defined by

$$\mathcal{B} := \left\{ f : \mathbb{Z} \rightarrow \mathbb{C} : |f|_{\mathcal{B}} := \sup_{x \in \mathbb{Z}} \frac{|f(x)|}{1 + |x|} < +\infty \right\}.$$

Apparently, both functions  $h_a$  and  $h'_d$  belong to  $\mathcal{B}$  since  $h_a(x) = O(x)$  and  $h'_d(x) = O(x)$  for arbitrary  $x \geq 1$ . Furthermore, the map  $\mathcal{C}$  acts on  $\mathcal{B}$  as a compact operator whose spectrum can be easily controlled.

**Proposition 3.4.** *Assume that hypotheses **H1**–**H4** hold. Then,*

- (i) *The map  $\mathcal{C}$  acts on  $\mathcal{B}$  and  $\mathcal{C}(\mathcal{B}) \subset L^\infty(\mathbb{Z})$ .*
- (ii)  *$\mathcal{C}$  is a compact operator on  $\mathcal{B}$  with spectral radius  $\rho_{\mathcal{B}} = 1$  and with the unique and simple dominant eigenvalue 1.*
- (iii) *The rest of the spectrum of  $\mathcal{C}$  on  $\mathcal{B}$  is contained in a disk of radius  $< 1$ .*

*Consequently, the operator  $\mathcal{C}$  on  $\mathcal{B}$  may be decomposed as*

$$\mathcal{C} = \Pi + Q$$

---

<sup>2</sup>the notation  $f(n) \asymp g(n)$  means that there exists  $K > 0$  s.t.  $\frac{1}{K}g(n) \leq f(n) \leq Kg(n)$  for  $n$  large enough.

where

- $\Pi$  is the eigenprojector from  $\mathcal{B}$  to  $\mathbb{C}$  corresponding to the eigenvalue 1 and  $\Pi(\phi) = \nu(\phi)\mathbf{1}$ , where  $\nu$  is the unique  $\mathcal{C}$ -invariant probability measure on  $\mathbb{Z}$ ;
- the spectral radius of  $Q$  on  $\mathcal{B}$  is  $< 1$ ;
- $\Pi Q = Q\Pi = 0$ .

*Proof.* i) Note that  $\mathcal{U}'_-(t) = \sum_{n \geq 0} \mathbb{P}[S'_{\ell'_n} = t] = \mathbb{P}[\exists n \geq 0 : S'_{\ell'_n} = t] \leq 1$ . For any  $\varphi \in \mathcal{B}$  and  $x \geq 1$ , we have

$$\begin{aligned} |\mathcal{C}\varphi(x)| &\leq \sum_{y \leq 0} \sum_{t=-x+1}^0 \mu'_-(y-x-t) |\varphi(y)| \\ &\leq |\varphi|_{\mathcal{B}} \sum_{y \leq 0} (1+|y|) \mu'_-(y-1) \\ &\leq |\varphi|_{\mathcal{B}} \left( \mathbb{E}[|S'_{\ell'_1}|] + \frac{1}{2} \mathbb{E}[|S'_{\ell'_1}|^2] \right), \end{aligned}$$

which is finite since  $\mathbb{E}[(\xi'_n)^3] < +\infty$  (see [5]). Other cases can be estimated in the same way and yield

$$|\mathcal{C}\varphi|_{\mathcal{B}} \leq |\mathcal{C}\varphi|_{\infty} \leq |\varphi|_{\mathcal{B}} \left( \mathbb{E}[|S'_{\ell'_1}|] + \frac{1}{2} \mathbb{E}[|S'_{\ell'_1}|^2] \right) < +\infty. \quad (3.4)$$

ii) By (3.4), the operator  $\mathcal{C}$  acts continuously from  $\mathcal{B}$  into  $L^\infty(\mathbb{Z})$ ; since the inclusion map  $i : L^\infty(\mathbb{Z}) \hookrightarrow \mathcal{B}$  is compact, the operator  $\mathcal{C}$  is also compact on  $\mathcal{B}$ .

Let us now compute the spectral radius  $\rho_{\mathcal{B}}$  of  $\mathcal{C}$ . The fact that  $\mathcal{C}$  is a stochastic matrix yields  $\rho_{\mathcal{B}} \geq 1$ . To prove  $\rho_{\mathcal{B}} \leq 1$ , it suffices to show that  $\mathcal{C}$  has bounded powers on  $\mathcal{B}$ . For any  $n \geq 1$  and  $x \in \mathbb{Z}$ ,

$$|\mathcal{C}^n \varphi(x)| \leq \sum_{y \in \mathbb{Z}} \mathcal{C}^{n-1}(x, y) |\mathcal{C}\varphi(y)| \leq |\mathcal{C}\varphi|_{\infty} \sum_{y \in \mathbb{Z}} \mathcal{C}^{n-1}(x, y) = |\mathcal{C}\varphi|_{\infty}.$$

Together with (3.4), it implies

$$|\mathcal{C}^n \varphi|_{\mathcal{B}} \leq |\mathcal{C}^n \varphi|_{\infty} \leq |\mathcal{C}\varphi|_{\infty} \leq |\varphi|_{\mathcal{B}} \left( \mathbb{E}[|S'_{\ell'_1}|] + \frac{1}{2} \mathbb{E}[|S'_{\ell'_1}|^2] \right).$$

Hence  $|\mathcal{C}^n|_{\mathcal{B}} \leq \mathbb{E}[|S'_{\ell'_1}|] + \frac{1}{2} \mathbb{E}[|S'_{\ell'_1}|^2]$  for any  $n \geq 1$  and  $\rho_{\mathcal{B}} = \lim_{n \rightarrow +\infty} |\mathcal{C}^n|_{\mathcal{B}}^{1/n} \leq 1$ .

Let us now control the peripheral spectrum of  $\mathcal{C}$ . Let  $\theta \in \mathbb{R}$  and  $\psi \in \mathcal{B}$  such that  $\mathcal{C}\psi = e^{i\theta}\psi$ . Obviously, the function  $\psi$  is bounded and  $|\psi| \leq \mathcal{C}|\psi|$ . Consequently,



$|\psi|_\infty - |\psi|$  is non-negative and super-harmonic (i.e.  $\mathcal{C}(|\psi|_\infty - |\psi|) \leq |\psi|_\infty - |\psi|$ ) on the unique irreducible class  $\mathcal{I}_{\mathbf{C}}(X_0)$  of  $\mathcal{X}$ . By the classical denumerable Markov chains theory, it is thus constant on  $\mathcal{I}_{\mathbf{C}}(X_0)$  which follows that  $|\psi|$  is constant on  $\mathcal{I}_{\mathbf{C}}(X_0)$ .

Without loss of generality, we may assume  $|\psi(x)| = 1$  for any  $x \in \mathcal{I}_{\mathbf{C}}(X_0)$ , i.e.  $\psi(x) = e^{i\phi(x)}$  for some  $\phi(x) \in \mathbb{R}$ . We may rewrite the equality  $\mathcal{C}\psi = e^{i\theta}\psi$  as

$$\forall x \in \mathcal{I}_{\mathbf{C}}(X_0) \quad \sum_{y \in \mathcal{I}_{\mathbf{C}}(X_0)} \mathcal{C}(x, y) e^{i(\phi(y) - \phi(x))} = e^{i\theta}.$$

Note that  $\mathcal{C}(x, y) > 0$  for all  $x, y \in \mathcal{I}_{\mathbf{C}}(X_0)$ ; by convexity, one readily gets  $e^{i\theta} = e^{i(\phi(y) - \phi(x))}$  for such points  $x, y$ . Taking  $x = y \in \mathcal{I}_{\mathbf{C}}(X_0)$ , we thus obtain  $e^{i\theta} = 1$ .

In particular, the function  $\psi$  is harmonic on  $\mathcal{I}_{\mathbf{C}}(X_0)$ , hence constant on this set, by *Liouville's theorem*. Furthermore, for any  $x \in \mathbb{Z}$ , it holds  $\mathcal{C}(x, y) > 0 \iff y \in \mathcal{I}_{\mathbf{C}}(X_0)$ ; consequently, for any fixed  $y_0 \in \mathcal{I}_{\mathbf{C}}(X_0)$  and any  $x \in \mathbb{Z}$ ,

$$\psi(x) = \mathcal{C}\psi(x) = \sum_{y \in \mathcal{I}_{\mathbf{C}}(X_0)} \mathcal{C}(x, y) \psi(y) = \psi(y_0).$$

Therefore, the function  $\psi$  is constant on  $\mathbb{Z}$ .

iii) This is a direct consequence of (ii). □

### 3.4 A renewal limit theorem for the sequence of crossing times

The main goal of this part is to prove the following statement.

**Proposition 3.5.** *For any  $x, y \in \mathbb{Z}$ ,*

$$\lim_{n \rightarrow +\infty} \sqrt{n} \Sigma_n(x, y) = \frac{\nu(y)}{2\pi \left( c\nu(\check{h}_a) + c'\nu(h'_d) \right)}.$$

This is a consequence of the fact that  $(\mathcal{C}_n)_{n \geq 1}$  is an aperiodic renewal sequence of operators on  $\mathcal{B}$  satisfying R4 and R5 ( with  $\beta = 1/2$  and  $\ell$  constant).

The fact that all the  $\mathcal{C}_n, n \geq 1$ , act on  $\mathcal{B}$  and  $\sum_{n \geq 1} |\mathcal{C}_n|_{\mathcal{B}} < +\infty$  is a consequence of the following lemma.

**Lemma 3.6.** *Under hypotheses **H1**–**H4**, for any  $n \geq 1$ , the operator  $\mathcal{C}_n$  acts on  $\mathcal{B}$  and*

$$|\mathcal{C}_n|_{\mathcal{B}} = O\left(\frac{1}{n^{3/2}}\right).$$

*Proof.* By (2.1), for any  $x \geq 1$  and  $\phi \in \mathcal{B}$ ,

$$\begin{aligned} |\mathcal{C}_n\phi(-x)| &\leq \sum_{w \geq 0} \mathbb{P}_{-x}[C_1 = n; X_n = w] |\phi(w)| \\ &= \sum_{w \geq 0} \mathbb{P}[\tau^S(-x) = n, -x + S_n = w] |\phi(w)| \\ &\preceq \frac{1+x}{n^{3/2}} \sum_{w \geq 0} \left( \sum_{z \geq w+1} z\mu(z) \right) |\phi(w)| \\ &\leq \frac{1+x}{n^{3/2}} |\phi|_{\mathcal{B}} \underbrace{\sum_{w \geq 0} (1+w) \left( \sum_{z \geq w+1} z\mu(z) \right)}_{\preceq \sum_{z \geq 1} z^3 \mu(z) = \mathbb{E}[(\xi_1^+)^3]}. \end{aligned}$$

$$\text{Similarly, } |\mathcal{C}_n\phi(x)| \leq \frac{1+x}{n^{3/2}} |\phi|_{\mathcal{B}} \underbrace{\sum_{w \leq 0} (1+|w|) \left( \sum_{z \leq w-1} |z|\mu'(z) \right)}_{\preceq \sum_{z \leq -1} |z|^3 \mu'(z) = \mathbb{E}[(\xi_1^-)^3]}.$$

Moreover,  $|\mathcal{C}_1\phi(0)| \preceq |\phi|_{\mathcal{B}}$  and  $\mathcal{C}_n\phi(0) = 0$  for all  $n \geq 2$ . This completes the proof.  $\square$

Condition R1 coincides with the statement of Proposition 3.4. Similarly, R2 and R3 correspond to assertions *i*) and *ii*) of the next proposition. Consequently,  $(\mathcal{C}_n)_{n \geq 1}$  is an aperiodic renewal sequence of operators.

**Proposition 3.7.** *Suppose that **H1**–**H4** are satisfied. Then the sequence  $(\mathcal{C}_n)_{n \geq 1}$  holds the following properties*

- i) The spectral radius  $\rho_{\mathcal{B}}(z)$  of  $\mathcal{C}(z)$  is strictly less than 1 for  $z \in \overline{\mathbb{D}} \setminus \{1\}$ .*

ii) For any  $n \geq 1$ , it holds  $\Pi \mathcal{C}_n \Pi = r_n \Pi$  with

$$r_n := \nu(\mathcal{C}_n \mathbf{1}) = \sum_{x \in \mathbb{Z}} \nu(x) \mathbb{P}_x[C_1 = n] \geq 0.$$

$$iii) \sum_{j > n} r_j \sim \frac{2(c\nu(\check{h}_a) + c'\nu(h'_d))}{\sqrt{n}} \text{ as } n \rightarrow +\infty.$$

*Proof.* i) The argument is close to the one used to prove Proposition 3.4. For any  $z \in \overline{\mathbb{D}} \setminus \{1\}$ , the operator  $\mathcal{C}(z)$  is compact on  $\mathcal{B}$  with spectral radius  $\rho_{\mathcal{B}}(z) \leq 1$ . We now prove  $\rho_{\mathcal{B}}(z) \neq 1$  by contraposition. Suppose  $\rho_{\mathcal{B}}(z) = 1$ ; in other words, there exist  $\theta \in \mathbb{R}$  and  $\varphi \in \mathcal{B}$  such that  $\mathcal{C}(z)\varphi = e^{i\theta}\varphi$ . Since  $\mathcal{C}$  is bounded from  $\mathcal{B}$  into  $L^\infty(\mathbb{Z})$  and  $0 \leq |\varphi| \leq \mathcal{C}|\varphi|$ , the function  $|\varphi|$  is  $\mathcal{C}$ -superharmonic, bounded and thus constant on its essential class  $\mathcal{I}_{\mathcal{C}}(X_0)$ .

Without loss of generality, we can suppose that  $|\varphi| = 1$  on  $\mathcal{I}_{\mathcal{C}}(X_0)$ ; equivalently,  $\varphi(x) = e^{i\phi(x)}$  for some function  $\phi : \mathcal{I}_{\mathcal{C}}(X_0) \rightarrow \mathbb{R}$ . For any  $x \in \mathcal{I}_{\mathcal{C}}(X_0)$ , we get

$$\mathcal{C}(z)\varphi(x) = e^{i\theta}\varphi(x) \iff \sum_{n \geq 1} \sum_{y \in \mathcal{I}_{\mathcal{C}}(X_0)} z^n e^{i(\phi(y) - \phi(x))} \mathbb{P}_x[C_1 = n; X_n = y] = e^{i\theta},$$

with  $\sum_{n \geq 1} \sum_{y \in \mathcal{I}_{\mathcal{C}}(X_0)} \mathbb{P}_x[C_1 = n; X_n = y] = 1$ . By convexity, it readily holds  $z^n e^{i(\phi(y) - \phi(x))} = e^{i\theta}$  for all  $x, y \in \mathcal{I}_{\mathcal{C}}(X_0)$  and  $n \geq 1$ . By taking  $x = y$ , we obtain  $z^n = e^{i\theta}$  for all  $n \geq 1$ ; consequently  $z = 1$ , contradiction.

ii) For any  $\phi \in \mathcal{B}$  and  $n \geq 1$ ,

$$\Pi \mathcal{C}_n \Pi \phi = \nu(\phi) \Pi(\mathcal{C}_n \mathbf{1}) = \nu(\phi) \mathbf{1} \sum_{z \in \mathbb{Z}} \nu(z) \mathbb{P}_z[C_1 = n] = r_n \Pi(\phi)$$

$$\text{with } r_n = \sum_{z \in \mathbb{Z}} \nu(z) \mathbb{P}_z[C_1 = n] \geq 0.$$

iii) On the one hand, by Proposition 3.4, the eigenprojector  $\Pi$  acts on  $\mathcal{B}$ ; thus, since  $\check{h}_a \in \mathcal{B}$ , it holds  $\nu(\check{h}_a) < +\infty$ . On the other hand, the support  $\mathcal{I}_{\mathcal{C}}(X_0)$  of  $\nu$  intersects  $\mathbb{Z}^-$  and the support of  $\check{h}_a$  equals  $\mathbb{Z}^-$ ; hence  $\nu(\check{h}_a) > 0$ . Similarly  $0 < \nu(h'_d) < +\infty$ .

Now, let us write

$$\begin{aligned}
\sum_{j>n} r_j &= \sum_{j>n} \sum_{x \in \mathbb{Z}} \nu(x) \mathbb{P}_x[C_1 = j] \\
&= \sum_{x \in \mathbb{Z}} \nu(x) \mathbb{P}_x[C_1 > n] \\
&= \sum_{x \leq -1} \nu(x) \mathbb{P}[\tau^S(x) > n] + \sum_{x \geq 1} \nu(x) \mathbb{P}[\tau^{S'}(x) > n] \quad (\text{since } \mathbb{P}_0[C_1 = 1] = 1) \\
&\sim \frac{2c}{\sqrt{n}} \sum_{x \leq -1} \nu(x) h_a(-x) + \frac{2c'}{\sqrt{n}} \sum_{x \geq 1} \nu(x) h'_d(x) \\
&= 2 \frac{c \nu(\check{h}_a) + c' \nu(h'_d)}{\sqrt{n}}.
\end{aligned}$$

□

Finally, combining lemma 3.6 and proposition 3.7 *iii*), we see that conditions R4 and R5 are satisfied with  $\ell = \text{const} = 2(c \nu(\check{h}_a) + c' \nu(h'_d))$  and  $\beta = 1/2$ .

Consequently, by [10], the sequence  $(\sqrt{n}H_n)_{n \geq 1}$  converges in  $\mathcal{B}$  to the operator  $\mathbf{c}^{-1}\Pi$  with  $\mathbf{c} = 2\pi(c\nu(\check{h}_a) + c'\nu(h'_d))$ . Formally, one may write

$$\sup_{n \geq 1} \left| \sqrt{[ns]} H_{[ns]} - \mathbf{c}^{-1}\Pi \right|_{\mathcal{B}} \longrightarrow 0, \quad \text{as } n \rightarrow +\infty. \quad (3.5)$$

## 4 Proof of Theorem 1.1

### Part I. Convergence of finite dimensional distribution

For  $m \geq 1$ , let  $\{\varphi_i : \mathbb{R} \rightarrow \mathbb{R} \mid i = \overline{1, \dots, m}\}$  be a sequence of bounded and Lipschitz continuous functions with corresponding Lipschitz coefficients  $Lip(\varphi_i)$ . Assume that the time sequence  $\{t_i\}_{1 \leq i \leq m}$  is strictly increasing with values in  $(0, 1]$  and  $t_0 = 0$ . In this part, we prove that

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x \left[ \prod_{i=1}^m \varphi_i \left( X^{(n)}(t_i) \right) \right] = \int_{\mathbb{R}^m} \prod_{i=1}^m \varphi_i(u_i) p_{t_i - t_{i-1}}^\gamma(u_{i-1}, u_i) du_1 \dots du_m,$$

where  $u_0 = 0$ .

Without loss of generality, we assume  $\sigma = \sigma'$  to reduce unnecessary complexity associated with subcases and suppose  $x \geq 1$ .

Case m=1

We first notice that  $\mathbb{E}_x[\varphi_1(X^{(n)}(t_1))] \approx \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \right]$  since

$$\begin{aligned} \left| \mathbb{E}_x[\varphi_1(X^{(n)}(t_1))] - \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \right] \right| &\leq Lip(\varphi_1) \mathbb{E}_x \left[ \left| X^{(n)}(t_1) - \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right| \right] \\ &\leq Lip(\varphi_1) \frac{\mathbb{E}[|\xi_{[nt_1]+1}|] + \mathbb{E}[|\eta_{[nt_1]+1}|] + \mathbb{E}[|\xi'_{[nt_1]+1}|]}{\sigma\sqrt{n}}, \end{aligned}$$

which tends to 0 as  $n \rightarrow +\infty$ . Now, we can decompose  $\mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \right]$  as

$$\underbrace{\mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right), X_{[nt_1]} > 0 \right]}_{A^+(n)} + \underbrace{\mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right), X_{[nt_1]} < 0 \right]}_{A^-(n)}.$$

- Estimate  $A^+(n)$

$$\begin{aligned} A^+(n) &\approx \sum_{k=1}^{[nt_1]-1} \sum_{\ell \geq 1} \sum_{y \geq 1} \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right), C_\ell = k, X_k = y, y + \xi'_{k+1} > 0, \right. \\ &\quad \left. \dots, y + \xi'_{k+1} + \dots + \xi'_{[nt_1]} > 0 \right] \\ &= \sum_{k=1}^{[nt_1]-1} \sum_{y \geq 1} \mathbb{E} \left[ \varphi_1 \left( \frac{y + \xi'_{k+1} + \dots + \xi'_{[nt_1]}}{\sigma\sqrt{n}} \right), \tau^{S'}(y) > [nt_1] - k \right] \\ &\quad \left( \sum_{\ell \geq 1} \mathbb{P}_x[C_\ell = k; X_k = y] \right) \\ &= \sum_{k=1}^{[nt_1]-1} \sum_{y \geq 1} \Sigma_k(x, y) \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-k}}{\sigma\sqrt{n}} \right) \mid \tau^{S'}(y) > [nt_1] - k \right] \\ &\quad \mathbb{P}[\tau^{S'}(y) > [nt_1] - k]. \end{aligned}$$

For any  $0 \leq s \leq 1$  and  $n \geq 1$ , let  $f_n$  be the function defined by

$$f_n(s) := n \sum_{y \geq 1} \Sigma_{[ns]}(x, y) \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[ns]}}{\sigma\sqrt{n}} \right) \mid \tau^{S'}(y) > [nt_1] - [ns] \right] \mathbb{P}[\tau^{S'}(y) > [nt_1] - [ns]]$$

if  $s \in \left[ \frac{k}{n}, \frac{k+1}{n} \right)$  for  $2 \leq k \leq [nt_1] - 4$ , and  $f_n(s) = 0$  if  $s \in \left[ 0, \frac{2}{n} \right) \cup \left[ \frac{[nt_1]-3}{n}, t_1 \right)$ .

Hence,

$$A^+(n) = \int_0^{t_1} f_n(s) ds + O\left(\frac{1}{\sqrt{n}}\right).$$

For  $n \geq 1$  and any  $y \geq 1$ , one may write  $f_n(s) = \sum_{y \geq 1} a_n(y)b_n(y)$ , where

$$\begin{aligned} a_n(y) &:= n \Sigma_{[ns]}(x, y) \mathbb{P}[\tau^{S'}(y) > [nt_1] - [ns]], \\ b_n(y) &:= \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1] - [ns]}}{\sigma \sqrt{n}} \right) \mid \tau^{S'}(y) > [nt_1] - [ns] \right]. \end{aligned}$$

To compute the point convergence of  $f_n(s)$ , we need the two following lemmas

**Lemma 4.1.** *Let  $(a_n(y))_{y \geq 1}, (b_n(y))_{y \geq 1}$  be real sequences satisfying*

1.  $a_n(y) \geq 0$  and  $\lim_{n \rightarrow +\infty} \sum_{y \geq 1} a_n(y) = A$ ,
2.  $\lim_{n \rightarrow +\infty} b_n(y) = B$  and  $\sup_{n \in \mathbb{Z}^+, y \geq 1} |b_n(y)| < +\infty$ .

Then

$$\lim_{n \rightarrow +\infty} \sum_{y \geq 1} a_n(y)b_n(y) = AB.$$

**Lemma 4.2.** *Let us set, for any  $0 < s < t < 1$  and  $x \in \mathbb{Z}$ ,*

$$\Sigma_n^\pm(x, s, t) := n \sum_{\ell \geq 1} \mathbb{P}_x[C_\ell = [ns], \pm X_{[ns]} > 0, C_{\ell+1} > [nt]].$$

Then it holds

$$\lim_{n \rightarrow +\infty} \Sigma_n^+(x, s, t) = \frac{\gamma}{\pi \sqrt{s(t-s)}} \text{ and } \lim_{n \rightarrow +\infty} \Sigma_n^-(x, s, t) = \frac{1-\gamma}{\pi \sqrt{s(t-s)}}.$$

Furthermore,

$$\Sigma_n^\pm(x, s, t) \preceq \frac{1 + |x|}{\sqrt{s(t-s)}}.$$

We omit the first lemma while the technical proof of the second one is deferred to the appendix. Let us now continue our computation by using lemma 4.1. Note that  $\sum_{y \geq 1} a_n(y) = \Sigma_n^+(x, s, t_1)$  and  $|b_n(y)| \leq |\varphi|_\infty$ . On the one hand, from the lemmas

4.1 and 4.2 together with (2.2), it yields

$$\lim_{n \rightarrow +\infty} f_n(s) = \frac{\gamma}{\pi \sqrt{s(t_1 - s)}} \int_0^{+\infty} \varphi_1(z\sqrt{t_1 - s})z \exp\left(\frac{-z^2}{2}\right) dz,$$

and  $f_n$  is also integrable. Indeed,

$$\sup_n |f_n(s)| \preceq \frac{1 + |x|}{\sqrt{s(t_1 - s)}} |\varphi|_\infty \in L^1([0, t_1]).$$

Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} A^+(n) &= \frac{\gamma}{\pi} \int_0^{t_1} \frac{1}{\sqrt{s(t_1 - s)}} \left( \int_0^{+\infty} \varphi_1(z\sqrt{t_1 - s})z \exp\left(\frac{-z^2}{2}\right) dz \right) ds \\ &\stackrel{u:=z\sqrt{t_1-s}}{=} \frac{\gamma}{\pi} \int_0^{+\infty} \varphi_1(u)u \left( \int_0^{t_1} \frac{1}{\sqrt{s(t_1 - s)^3}} \exp\left(\frac{-u^2}{2(t_1 - s)}\right) ds \right) du \\ &\stackrel{x:=\frac{s}{t_1-s}}{=} \frac{\gamma}{\pi} \int_0^{+\infty} \varphi_1(u)u \left[ \frac{1}{t_1} \exp\left(\frac{-u^2}{2t_1}\right) \underbrace{\int_0^{+\infty} \frac{1}{\sqrt{x}} \exp\left(\frac{-u^2}{2t_1}x\right) dx}_{\frac{\sqrt{2\pi t_1}}{u}} \right] du \\ &= \gamma \int_0^{+\infty} \varphi_1(u) \frac{2 \exp(-u^2/2t_1)}{\sqrt{2\pi t_1}} du. \end{aligned} \tag{4.1}$$

Similarly,

$$\lim_{n \rightarrow +\infty} A^-(n) = (1 - \gamma) \int_{-\infty}^0 \varphi_1(u) \frac{2 \exp(-u^2/2t_1)}{\sqrt{2\pi t_1}} du. \tag{4.2}$$

Summing both sides of (4.1) and (4.2), we thus obtain

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x[\varphi_1(X_{t_1}^{(n)})] = \int_{\mathbb{R}} \varphi_1(u) p_{t_1}^\gamma(0, u) du = \int_{\mathbb{R}} \tilde{\varphi}_1(u) \frac{2 \exp(-u^2/2t_1)}{\sqrt{2\pi t_1}} du,$$

where  $\tilde{\varphi}_1(u) = \gamma \varphi_1(u) \mathbb{1}_{(0, +\infty)}(u) + (1 - \gamma) \varphi_1(u) \mathbb{1}_{(-\infty, 0)}(u)$ .

Case m=2

For given  $s \in (0, 1]$  and  $n \geq 1$ , let  $\kappa$  be the first crossing time after time  $[ns]$  defined by

$$\kappa := \kappa(n, s) = \min\{k > [ns] : X_{[ns]} X_k \leq 0\}.$$

As in the case  $m = 1$ , it holds

$$\mathbb{E}_x[\varphi_1(X^{(n)}(t_1))\varphi_2(X^{(n)}(t_2))] \approx \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{X_{[nt_2]}}{\sigma\sqrt{n}} \right) \right],$$

and the right hand side term may be decomposed as  $A_1^\pm(n) + A_2^\pm(n)$ , where

$$A_1^\pm(n) := \sum_{k_2=[nt_1]+1}^{[nt_2]} \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{X_{[nt_2]}}{\sigma\sqrt{n}} \right) \mathbb{1}_{[\kappa=k_2]} \mathbb{1}_{[\pm X_{[nt_1]} > 0]} \right],$$

and

$$A_2^\pm(n) := \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{X_{[nt_2]}}{\sigma\sqrt{n}} \right) \mathbb{1}_{[\kappa > [nt_2]]} \mathbb{1}_{[\pm X_{[nt_1]} > 0]} \right].$$

• Estimate  $A_1^+(n)$

$$\begin{aligned} A_1^+(n) &\approx \sum_{k_1=1}^{[nt_1]-1} \sum_{k_2=[nt_1]+1}^{[nt_2]} \sum_{\ell \geq 1} \sum_{y \geq 1} \sum_{z \geq 1} \sum_{w \leq 0} \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{X_{[nt_2]}}{\sigma\sqrt{n}} \right), C_\ell = k_1, \right. \\ &\quad \left. X_{k_1} = y, y + \xi'_{k_1+1} > 0, \dots, y + \xi'_{k_1+1} + \dots + \xi'_{k_2-2} > 0, \right. \\ &\quad \left. y + \xi'_{k_1+1} + \dots + \xi'_{k_2-1} = z, z + \xi'_{k_2} = w \right] \\ &= \sum_{k_1=1}^{[nt_1]-1} \sum_{k_2=[nt_1]+1}^{[nt_2]} \sum_{\ell \geq 1} \sum_{y \geq 1} \sum_{z \geq 1} \sum_{w \leq 0} \mathbb{E}_x \left[ \varphi_1 \left( \frac{y + \xi'_{k_1+1} + \dots + \xi'_{[nt_1]}}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{X_{[nt_2]}}{\sigma\sqrt{n}} \right), \right. \\ &\quad \left. C_\ell = k_1, X_{k_1} = y, y + \xi'_{k_1+1} > 0, \dots, y + \xi'_{k_1+1} + \dots + \xi'_{k_2-2} > 0, \right. \\ &\quad \left. y + \xi'_{k_1+1} + \dots + \xi'_{k_2-1} = z, z + \xi'_{k_2} = w \right] \\ &= \sum_{k_1=1}^{[nt_1]-1} \sum_{y \geq 1} \Sigma_{k_1}(x, y) \sum_{k_2=[nt_1]+1}^{[nt_2]} \sum_{z \geq 1} \sum_{w \leq 0} \mathbb{E}_w \left[ \varphi_2 \left( \frac{X_{[nt_2]-k_2}}{\sigma\sqrt{n}} \right) \right] \\ &\quad \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-k_1}}{\sigma\sqrt{n}} \right) \mid \tau^{S'}(y) > k_2 - k_1 - 1, y + S'_{k_2-k_1-1} = z \right] \\ &\quad \mathbb{P}[\tau^{S'}(y) > k_2 - k_1 - 1, y + S'_{k_2-k_1-1} = z] \mathbb{P}[\xi'_1 = w - z]. \end{aligned}$$



For any  $2 \leq k_1 < [nt_1] - 6$  and  $[nt_1] < k_2 \leq [nt_2]$ , we set

$$g_n(s_1, s_2) = n^2 \sum_{y \geq 1} \Sigma_{[ns_1]}(x, y) \sum_{z \geq 1} \sum_{w \leq 0} \mathbb{E}_w \left[ \varphi_2 \left( \frac{X_{[nt_2]-[ns_2]}}{\sigma \sqrt{n}} \right) \right] \\ \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[ns_1]}}{\sigma \sqrt{n}} \right) \mid \tau^{S'}(y) > [ns_2] - [ns_1] - 1, y + S'_{[ns_2]-[ns_1]-1} = z \right] \\ \mathbb{P}[\tau^{S'}(y) > [ns_2] - [ns_1] - 1, y + S'_{[ns_2]-[ns_1]-1} = z] \mathbb{P}[\xi'_1 = w - z]$$

if  $(s_1, s_2) \in [\frac{k_1}{n}, \frac{k_1+1}{n}] \times [\frac{k_2}{n}, \frac{k_2+1}{n}]$  and be equal to 0 for other values of  $k_1$  s.t.  $0 \leq k_1 \leq [nt_1]$ .

Hence,

$$A_1^+(n) = \int_0^{t_1} \int_{t_1}^{t_2} g_n(s_1, s_2) ds_1 ds_2 + O\left(\frac{1}{\sqrt{n}}\right).$$

Moreover, by (2.3), we have

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[ns_1]}}{\sigma \sqrt{n}} \right) \mid \tau^{S'}(y) > [ns_2] - [ns_1] - 1, y + S'_{[ns_2]-[ns_1]-1} = z \right] \\ = \int_0^{+\infty} 2\varphi_1(u' \sqrt{t_1 - s_1}) \exp\left(\frac{-u'^2}{2 \frac{t_1 - s_1}{s_2 - s_1} \frac{s_2 - t_1}{s_2 - s_1}}\right) \frac{u'^2}{\sqrt{2\pi \frac{(t_1 - s_1)^3 (s_2 - t_1)^3}{(s_2 - s_1)^3 (s_2 - s_1)^3}}} du' \\ = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} \varphi_1(u) \exp\left(\frac{-u^2}{2 \frac{(t_1 - s_1)(s_2 - t_1)}{s_2 - s_1}}\right) \frac{u^2}{\sqrt{\frac{(t_1 - s_1)^3 (s_2 - t_1)^3}{(s_2 - s_1)^3}}} du.$$

Using the limit result in one-dimensional case, one gets

$$\lim_{n \rightarrow +\infty} \mathbb{E}_w \left[ \varphi_2 \left( \frac{X_{[nt_2]-[ns_2]}}{\sigma \sqrt{n}} \right) \right] = \int_{\mathbb{R}} \tilde{\varphi}_2(v) \frac{2 \exp\left(\frac{-v^2}{2(t_2 - s_2)}\right)}{\sqrt{2\pi(t_2 - s_2)}} dv$$

with  $\tilde{\varphi}_2(v) = \gamma \varphi_2(v) \mathbb{1}_{(0, +\infty)}(v) + (1 - \gamma) \varphi_2(v) \mathbb{1}_{(-\infty, 0)}(v)$ .

We set, for any  $n \geq 1$ ,

$$a_n(y, z, w) := n^2 \Sigma_{[ns_1]}(x, y) \mathbb{P}[\tau^{S'}(y) > [ns_2] - [ns_1] - 1, y + S'_{[ns_2]-[ns_1]-1} = z] \mathbb{P}[\xi'_1 = w - z], \\ b_n(y, z, w) := \mathbb{E}_w \left[ \varphi_2 \left( \frac{X_{[nt_2]-[ns_2]}}{\sigma \sqrt{n}} \right) \right] \\ \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[ns_1]}}{\sigma \sqrt{n}} \right) \mid \tau^{S'}(y) > [ns_2] - [ns_1] - 1, y + S'_{[ns_2]-[ns_1]-1} = z \right].$$

Let us state the asymptotic behaviour of the related quantities, which allows us to determine the point convergence of the function  $g_n(s_1, s_2)$ . The proof will be shown in the next section.

**Lemma 4.3.** *For any  $0 < s < t < 1$  and any  $x \in \mathbb{Z}$ , we set*

$$\hat{\Sigma}_n^\pm(x, s, t) := n^2 \sum_{\ell \geq 1} \mathbb{P}_x[C_\ell = [ns], \pm X_{[ns]} > 0, C_{\ell+1} = [nt]].$$

*Then each quantity converges to a constant independent of  $x$  as  $n \rightarrow +\infty$ ; in particular,*

$$\lim_{n \rightarrow +\infty} \hat{\Sigma}_n^+(x, s, t) = \frac{\gamma}{2\pi \sqrt{s(t-s)^3}}, \text{ and } \lim_{n \rightarrow +\infty} \hat{\Sigma}_n^-(x, s, t) = \frac{1-\gamma}{2\pi \sqrt{s(t-s)^3}}.$$

*Moreover,*

$$\hat{\Sigma}_n^\pm(x, s, t) \leq \frac{1 + |x|}{\sqrt{s(t-s)^3}}.$$

It is clear that  $\sum_{y \geq 1} \sum_{z \geq 1} \sum_{w \leq 0} a_n(y, z, w) = \hat{\Sigma}_n^+(x, s_1, s_2) \xrightarrow{n \rightarrow +\infty} \frac{\gamma}{2\pi \sqrt{s_1(s_2 - s_1)^3}}$  by lemma 4.3 and  $|b_n(y, z, w)| \leq |\varphi_1|_\infty |\varphi_2|_\infty$ . Then by lemma 4.1, one gets

$$\lim_{n \rightarrow +\infty} g_n(s_1, s_2) = \frac{\gamma}{\pi^2} \int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi_1(u) \tilde{\varphi}_2(v) \frac{e^{\frac{-v^2}{2(t_2-s_2)}} u^2 \exp\left(\frac{-u^2}{2\frac{(t_1-s_1)(s_2-t_1)}{s_2-s_1}}\right)}{\sqrt{s_1(t_1-s_1)^3 (s_2-t_1)^3 (t_2-s_2)}} dudv,$$

and also  $\sup_n |g_n(s_1, s_2)| \leq \frac{1 + |x|}{\sqrt{s_1(s_2 - s_1)^3}} |\varphi_1|_\infty |\varphi_2|_\infty \in L^1([0, s_1] \times [s_1, s_2])$ .

Finally, we apply again the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} A_1^+(n) \\
&= \frac{\gamma}{\pi^2} \int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi_1(u) \tilde{\varphi}_2(v) \left( \int_0^{t_1} \int_{t_1}^{t_2} \frac{e^{\frac{-v^2}{2(t_2-s_2)}} u^2 \exp\left(\frac{-u^2}{2\frac{(t_1-s_1)(s_2-t_1)}{s_2-s_1}}\right)}{\sqrt{s_1(t_1-s_1)^3(s_2-t_1)^3(t_2-s_2)}} ds_1 ds_2 \right) dudv \\
&= \frac{\gamma}{\pi^2} \frac{\sqrt{2\pi t_1}}{t_1} \int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi_1(u) \tilde{\varphi}_2(v) |u| \left( \int_{t_1}^{t_2} \frac{e^{\frac{-u^2 s_2}{2t_1(s_2-t_1)}} e^{\frac{-v^2}{2(t_2-s_2)}}}{\sqrt{(t_2-s_2)(s_2-t_1)^3}} ds_2 \right) dudv \\
&= \frac{2\gamma}{\pi \sqrt{t_1(t_2-t_1)}} \int_0^{+\infty} \int_{-\infty}^{+\infty} \varphi_1(u) \tilde{\varphi}_2(v) e^{-\frac{u^2 t_2 + v^2 t_1 + 2|uv|t_1}{t_1(t_2-t_1)}} dudv.
\end{aligned}$$

More precisely, it can be rewritten as

$$\begin{aligned}
\lim_{n \rightarrow +\infty} A_1^+(n) &= \frac{2\gamma^2}{\pi \sqrt{t_1(t_2-t_1)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(u) \varphi_2(v) e^{-\frac{u^2}{2t_1}} e^{-\frac{(u+v)^2}{2(t_2-t_1)}} dudv \\
&\quad + \frac{2\gamma(1-\gamma)}{\pi \sqrt{t_1(t_2-t_1)}} \int_0^{+\infty} \int_{-\infty}^0 \varphi_1(u) \varphi_2(v) e^{-\frac{u^2}{2t_1}} e^{-\frac{(u-v)^2}{2(t_2-t_1)}} dudv.
\end{aligned}$$

In this calculation, we used the classical integral

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \exp\left(-\lambda_1 x - \frac{\lambda_2}{x}\right) dx = \sqrt{\frac{\pi}{\lambda_1}} e^{-2\sqrt{\lambda_1 \lambda_2}}$$

with  $\lambda_1 > 0$  and  $\lambda_2 \geq 0$ . Similarly,

$$\begin{aligned}
\lim_{n \rightarrow +\infty} A_1^-(n) &= \frac{2(1-\gamma)^2}{\pi \sqrt{t_1(t_2-t_1)}} \int_{-\infty}^0 \int_{-\infty}^0 \varphi_1(u) \varphi_2(v) e^{-\frac{u^2}{2t_1}} e^{-\frac{(u+v)^2}{2(t_2-t_1)}} dudv \\
&\quad + \frac{2\gamma(1-\gamma)}{\pi \sqrt{t_1(t_2-t_1)}} \int_{-\infty}^0 \int_0^{+\infty} \varphi_1(u) \varphi_2(v) e^{-\frac{u^2}{2t_1}} e^{-\frac{(u-v)^2}{2(t_2-t_1)}} dudv.
\end{aligned}$$

- Estimate  $A_2^+(n)$

$$\begin{aligned}
A_2^+(n) &= \sum_{k=1}^{[nt_1]-1} \sum_{\ell \geq 1} \sum_{y \geq 1} \mathbb{E}_x \left[ \varphi_1 \left( \frac{X_{[nt_1]}}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{X_{[nt_2]}}{\sigma \sqrt{n}} \right), C_\ell = k, X_k = y, \right. \\
&\quad \left. y + \xi'_k > 0, \dots, y + \xi'_k + \dots + \xi'_{[nt_1]} > 0, \dots, y + \xi'_k + \dots + \xi'_{[nt_2]} > 0 \right] \\
&= \sum_{k=1}^{[nt_1]-1} \sum_{y \geq 1} \Sigma_k(x, y) \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-k}}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{y + S'_{[nt_2]-k}}{\sigma \sqrt{n}} \right), \tau^{S'}(y) > [nt_2] - k \right].
\end{aligned}$$

For any  $r \in (0, t_1)$ , we denote

$$\begin{aligned}
g_n(r) &:= n \sum_{y \geq 1} \Sigma_{[nr]}(x, y) \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[nr]}}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{y + S'_{[nt_2]-[nr]}}{\sigma \sqrt{n}} \right) \mid \tau^{S'}(y) > [nt_2] - [nr] \right] \\
&\quad \mathbb{P}[\tau^{S'}(y) > [nt_2] - [nr]].
\end{aligned}$$

In the same way, we set

$$a_n(x, y) := n \Sigma_{[nr]}(x, y) \mathbb{P}[\tau^{S'}(y) > [nt_2] - [nr]],$$

and

$$b_n(y) := \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[nr]}}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{y + S'_{[nt_2]-[nr]}}{\sigma \sqrt{n}} \right) \mid \tau^{S'}(y) > [nt_2] - [nr] \right].$$

Note that  $\sum_{y \geq 1} a_n(x, y) = \Sigma_n^+(x, r, t_2)$ . Furthermore, it follows from Theorem 3.2 in [3] and Theorems 2.23 and 3.4 in [11] that

$$\begin{aligned}
\lim_{n \rightarrow +\infty} b_n(y) &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi_1 \left( \frac{y + S'_{[nt_1]-[nr]} \sqrt{[nt_2] - [nr]}}{\sigma \sqrt{[nt_2] - [nr]} \sqrt{n}} \right) \right. \\
&\quad \left. \varphi_2 \left( \frac{y + S'_{[nt_2]-[nr]} \sqrt{[nt_2] - [nr]}}{\sigma \sqrt{n} \sqrt{n}} \right) \mid \tau^{S'}(y) > [nt_2] - [nr] \right] \\
&= \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(u) \varphi_2(v) \frac{\sqrt{t_2 - r}}{\sqrt{(t_1 - r)^3}} u e^{\frac{-u^2}{2(t_1 - r)}} \\
&\quad \left( e^{-\frac{(u-v)^2}{2(t_2 - t_1)}} - e^{-\frac{(u+v)^2}{2(t_2 - t_1)}} \right) dudv.
\end{aligned}$$

It immediately yields

$$\begin{aligned} \lim_{n \rightarrow +\infty} A_2^+(n) &= \frac{\gamma}{\pi \sqrt{2\pi(t_2 - t_1)}} \int_0^{t_1} \left( \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(u) \varphi_2(v) \right. \\ &\quad \left. \frac{\sqrt{t_2 - r}}{\sqrt{r(t_2 - r)(t_1 - r)^3}} u e^{\frac{-u^2}{2(t_1 - r)}} \left( e^{-\frac{(u-v)^2}{2(t_2 - t_1)}} - e^{-\frac{(u+v)^2}{2(t_2 - t_1)}} \right) dudv \right) dr \\ &= \frac{\gamma}{\pi \sqrt{t_1(t_2 - t_1)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(u) \varphi_2(v) e^{\frac{-u^2}{2t_1}} \left( e^{-\frac{(u-v)^2}{2(t_2 - t_1)}} - e^{-\frac{(u+v)^2}{2(t_2 - t_1)}} \right) dudv. \end{aligned}$$

Analogously,

$$\lim_{n \rightarrow +\infty} A_2^-(n) = \frac{1 - \gamma}{\pi \sqrt{t_1(t_2 - t_1)}} \int_{-\infty}^0 \int_{-\infty}^0 \varphi_1(u) \varphi_2(v) e^{\frac{-u^2}{2t_1}} \left( e^{-\frac{(u-v)^2}{2(t_2 - t_1)}} - e^{-\frac{(u+v)^2}{2(t_2 - t_1)}} \right) dudv.$$

Hence, we conclude

$$\lim_{n \rightarrow +\infty} \mathbb{E}_x \left[ \varphi_1(X^{(n)}(t_1)) \varphi_2(X^{(n)}(t_2)) \right] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi_1(u) \varphi_2(v) p_{t_1}^\gamma(0, u) p_{t_2 - t_1}^\gamma(u, v) dudv.$$

Case m=3

It can be done similarly by induction.

## **Part II. Tightness of the sequence $\{X^{(n)}\}$**

Let us recall the modulus of continuity of a function  $f : [0, 1] \rightarrow \mathbb{R}$  is defined by

$$\omega_f(\delta) := \sup\{|f(t) - f(s)| : t, s \in [0, 1] \text{ s.t. } |t - s| \leq \delta\}.$$

Using Theorems 7.1 and 7.3 in [2] together with the fact that  $\omega_{X^{(n)}}(\delta) \leq \omega_S(\delta) + \omega_{S'}(\delta)$ , we finally achieve our result.

# 5 Appendix

## 5.1 Proof of Lemma 4.2.

*Proof.* For each  $z \in \mathbb{Z}^+$ , it is clear that the function  $g_n(z) = \sqrt{n} \mathbb{P}_z[\mathcal{C}_1 > n]$  is bounded from above by  $h'_d(z)$  by Lemma 2.2 and the sequence  $(g_n)_{n \geq 1}$  is thus

bounded in  $\mathcal{B}$ . Furthermore, the strong Markov property yields

$$\begin{aligned}
\Sigma_n^+(x, s, t) &= n \sum_{\ell \geq 1} \mathbb{E}_x \left[ \mathbb{P}_{X_{C_\ell}} [\mathcal{C}_1 \circ \theta^{C_\ell} > [nt] - [ns]], C_\ell = [ns], X_{[ns]} > 0 \right] \\
&= \sqrt{\frac{n^2}{[ns]([nt] - [ns])}} \sqrt{[ns]} \sum_{\ell \geq 1} \mathbb{E}_x \left[ g_{[nt]-[ns]}(X_{C_\ell}), C_\ell = [ns], X_{[ns]} > 0 \right] \\
&= \frac{1 + o(n)}{\sqrt{s(t-s)}} \sqrt{[ns]} H_{[ns]} \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) (x),
\end{aligned}$$

where  $\theta$  is the classical shift operator.

To prove  $\sqrt{[ns]} H_{[ns]} \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) (x) \rightarrow \frac{\gamma}{\pi}$  as  $n \rightarrow +\infty$ , we first notice that

$$\left| \sqrt{[ns]} H_{[ns]} \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) (x) - \frac{\gamma}{\pi} \right| \leq \Delta_1(n) + \Delta_2(n),$$

with

$$\begin{aligned}
\Delta_1(n) &:= \left| \sqrt{[ns]} H_{[ns]} \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) (x) - \mathbf{c}^{-1} \Pi \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) \right|, \\
\Delta_2(n) &:= \mathbf{c}^{-1} \left| \Pi \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) - 2c' \nu(h'_d) \right|.
\end{aligned}$$

By (3.5), one gets

$$\Delta_1(n) \leq (1 + |x|) \left| \sqrt{[ns]} H_{[ns]} - \mathbf{c}^{-1} \Pi \right|_{\mathcal{B}} |g_{[nt]-[ns]}|_{\mathcal{B}} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Moreover, by Lemma 2.1 and the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} \left| \Pi \left( g_{[nt]-[ns]} \mathbb{1}_{(0,+\infty)} \right) - 2c' \nu(h'_d) \right| = 0,$$

and thus  $\Delta_2(n) \rightarrow 0$  as  $n \rightarrow +\infty$ .  $\square$

## 5.2 Proof of Lemma 4.3.

*Proof.* Repeat the above argument with the function  $\tilde{g}_n(z) = n^{3/2} \mathbb{P}_z[C_1 = n]$ .  $\square$

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