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Abstract We present here a Godunov-type scheme to simulate one-dimensional flows in a nozzle with variable cross-section. The method relies on the construction of a relaxation Riemann solver designed to handle all types of flow regimes, from subsonic to supersonic flows, as well as resonant transonic flows. Some computational results are also provided, in which this relaxation method is compared with the classical Rusanov scheme and a modified Rusanov scheme.

Key words: Relaxation scheme, Godunov-type scheme, resonant transonic flows.
MSC2010: 76M12, 76H05, 76S05, 65M12

1 Introduction

In this paper, we are interested in the numerical approximation of the solutions of a model describing one-dimensional barotropic flows in a nozzle. In this model, ρ and w are respectively the density and the velocity of the fluid while α stands for the cross-section of the nozzle, which is assumed to be constant in time. Under the classical assumption that α is small with respect to a characteristic length in the mainstream direction, the flow can be supposed to be one-dimensional and described by the following set of partial differential equations:

$$\begin{array}{l} \partial_t \alpha = 0, \\ \partial_t(\alpha\rho) + \partial_x(\alpha\rho w) = 0, \\ \partial_t(\alpha\rho w) + \partial_x(\alpha\rho w^2 + \alpha p(\tau)) - p(\tau)\partial_x\alpha = 0, \end{array} \qquad t > 0,\ x \in \mathbb{R}, \qquad (1)$$

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where $\tau = \rho^{-1}$ is the specific volume and $\tau \mapsto p(\tau)$ is a barotropic pressure law (satisfying $p'(\tau) < 0$ and $p''(\tau) > 0$). System (1) takes the condensed form:

$$\partial_t \mathbb{U} + \partial_x \mathbf{f}(\mathbb{U}) + \mathbf{c}(\mathbb{U})\partial_x \mathbb{U} = 0, \tag{2}$$

where the state vector is $\mathbb{U} = (\alpha, \alpha\rho, \alpha\rho w)^T$. The solutions are sought in the phase space of positive solutions defined as

$$\Omega = \{\mathbb{U} = (\alpha, \alpha\rho, \alpha\rho w)^T \in \mathbb{R}^3, \alpha > 0, \alpha\rho > 0\}. \tag{3}$$

We recall the properties of this model:

- **Property 1.1 (Hyperbolicity)** *System (1) admits, for $\mathbb{U}$ in Ω, the following eigenvalues*

$$\lambda_0(\mathbb{U}) = 0, \qquad \lambda_1(\mathbb{U}) = w - c(\tau), \qquad \lambda_2(\mathbb{U}) = w + c(\tau), \tag{4}$$

  where $c(\tau) = \tau\sqrt{-p'(\tau)}$. The system is hyperbolic (i.e. the corresponding eigenvectors span $\mathbb{R}^3$) if and only if $|w| \neq c(\tau)$. Besides, the fields associated with the λ_1 and λ_2 eigenvalues are genuinely non-linear while the field associated with λ_0 is linearly degenerate.
- **Property 1.2 (Entropy)** *The entropy solutions of system (1) satisfy the following inequality in the weak sense*

$$\partial_t(\alpha\rho\mathscr{E}) + \partial_x(\alpha\rho\mathscr{E}w + \alpha p(\tau)w) \leq 0 \tag{5}$$

  where $\mathscr{E} = \frac{w^2}{2} + e(\tau)$ is the total energy and where the function $\tau \mapsto e(\tau)$ is given by $e'(\tau) = -p(\tau)$.

The Godunov scheme for this model is difficult to implement because the Riemann problem for system (1) is hard to solve due to the non linearities of the pressure law (giving rise to the genuinely non-linear acoustic fields), to the absence of a satisfactory definition of the non-conservative product $p(\tau)\partial_x\alpha$ and to the resonance phenomenon that appears for transonic flows causing the model to lose hyperbolicity [5]. For these reasons, we rather follow the classical approach of [7] and design an approximate Riemann solver, relying on a relaxation method. With this end in view, the solutions of system (1) are approximated by the solutions of the following enlarged relaxation system in the limit of a vanishing positive parameter ε:

$$\begin{aligned}
&\partial_t \alpha^\varepsilon = 0,\\
&\partial_t(\alpha\rho)^\varepsilon + \partial_x(\alpha\rho w)^\varepsilon = 0, && t > 0,\ x \in \mathbb{R},\\
&\partial_t(\alpha\rho w)^\varepsilon + \partial_x(\alpha\rho w^2 + \alpha\pi(\tau, \mathscr{T}))^\varepsilon - \pi(\tau, \mathscr{T})^\varepsilon \partial_x\alpha^\varepsilon = 0,\\
&\partial_t(\alpha\rho\mathscr{T})^\varepsilon + \partial_x(\alpha\rho\mathscr{T}w)^\varepsilon = \frac{1}{\varepsilon}(\alpha\rho)^\varepsilon(\tau - \mathscr{T})^\varepsilon,
\end{aligned} \tag{6}$$

with a linearization of the pressure law given by $\pi(\tau, \mathscr{T}) = p(\mathscr{T}) + a^2(\mathscr{T} - \tau)$. The variable $\mathscr{T}$ is an additionnal unknown relaxing towards the specific volume τ in the limit $\varepsilon \searrow 0$, and the constant a is a numerical parameter that must be taken large

enough so as to guarantee the non-linear stability of the numerical approximation. The state vector for the relaxation system is $\mathbb{W} = (\alpha, \alpha\rho, \alpha\rho w, \alpha\rho\mathscr{T})^T$ and the solutions are sought in the phase space

$$\Omega^r = \{\mathbb{W} = (\alpha, \alpha\rho, \alpha\rho w, \alpha\rho\mathscr{T})^T \in \mathbb{R}^4, \alpha > 0, \alpha\rho > 0, \alpha\rho\mathscr{T} > 0\}. \quad (7)$$

The following property motivates the introduction of this relaxation system

Property 1.3 (Hyperbolicity) *The convective part of (6) admits, for $\mathbb{W}$ in Ω^r, the following eigenvalues*

$$\sigma_0(\mathbb{W}) = 0, \qquad \sigma_1(\mathbb{W}) = w - a\tau, \qquad \sigma_2(\mathbb{W}) = w, \qquad \sigma_3(\mathbb{W}) = w + a\tau. \quad (8)$$

The system is hyperbolic (i.e. the corresponding eigenvectors span $\mathbb{R}^4$) if and only if $|w| \neq a\tau$, and all the fields are linearly degenerate.

2 The Riemann problem for the relaxation system

In this section, we give the main ideas leading to the construction of solutions to the Riemann problem for the convective part of the relaxation system (6). Being given $\mathbb{W}_L$ and $\mathbb{W}_R$ two states in Ω^r, we look for solutions of

$$\begin{cases} \partial_t \mathbb{W} + \partial_x \mathbf{g}(\mathbb{W}) + \mathbf{d}(\mathbb{W})\partial_x \mathbb{W} = 0, \\ \mathbb{W}(x,0) = \mathbb{W}_L \quad \text{if} \quad x < 0 \quad \text{and} \quad \mathbb{W}_R \quad \text{if} \quad x > 0. \end{cases} \quad (9)$$

As all the characteristic fields are linearly degenerate, the solution turns out to be simpler to construct than a solution of the Riemann problem for the equilibrium system (1). Indeed, the solution is sought in the form of a self-similar function consisting in constant intermediate states separated by contact discontinuities. The linear degeneracy of the fields provides natural jump relations across each discontinuity and yields a set of equations eventually leading to the expessions of the wave speeds and intermediate states. However, some issues related to the resonance phenomenon still need to be handled with care (see [2] for details).

We show that the solutions can be expressed in terms of the physical data $\mathbb{V}_L = (\rho_L, w_L, \mathscr{T}_L)$ and $\mathbb{V}_R = (\rho_R, w_R, \mathscr{T}_R)$ (*i.e.* all the initial data excluding the cross-section α) and of the ratio of left and right initial sections $\nu := \frac{\alpha_L}{\alpha_R}$. More precisely, we introduce the following quantities depending only on $(\mathbb{V}_L, \mathbb{V}_R)$

$$w^\sharp := \frac{1}{2}(w_L + w_R) - \frac{1}{2a}(\pi_R - \pi_L), \quad (10)$$

$$\tau_L^\sharp := \tau_L + \frac{1}{a}(w^\sharp - w_L) = \tau_L + \frac{1}{2a}(w_R - w_L) - \frac{1}{2a^2}(\pi_R - \pi_L), \quad (11)$$

$$\tau_R^\sharp := \tau_R - \frac{1}{a}(w^\sharp - w_R) = \tau_R + \frac{1}{2a}(w_R - w_L) + \frac{1}{2a^2}(\pi_R - \pi_L), \quad (12)$$

where $w^\sharp$ has the dimension of a speed and $\tau_L^\sharp, \tau_R^\sharp$ the dimension of specific volumes. These quantities appear in the explicit expressions of the solutions and it can be proved that these specific volumes need to be positive in order to guarantee the positivity of the solutions. In the numerical applications however, a will be chosen large for stability matters (see section 4) and it will always be possible to impose the positivity of $\tau_L^\sharp$ and $\tau_R^\sharp$ by taking a large enough.

The main result of this section is the existence theorem for the Riemann problem.

Theorem 2.1 *Let $\mathbb{W}_L$ and $\mathbb{W}_R$ be two positive states in Ω^r. Assume that a is such that $\tau_L^\sharp > 0$ and $\tau_R^\sharp > 0$. Then the Riemann problem (9) admits a positive self-similar solution whatever the ratio $\nu = \frac{\alpha_L}{\alpha_R}$ is.*

Sketch of the proof *(see [2] for details).* The proof consists in the effective construction of a solution. For the relaxation system, the eigenvalues are not naturally ordered because of the existence of a standing wave, and a resonance phenomenon does appear for transonic flows. Therefore, in order to construct solutions, we investigate all admissible wave configurations (including sonic and supersonic ones) and for each admissible ordering of the eigenvalues, we determine sufficient conditions on the initial states $\mathbb{W}_L$ and $\mathbb{W}_R$ for the solution to have this particular ordering. Eventually, we check *a posteriori* that the determined conditions totally cover the whole space of initial conditions $\Omega^r \times \Omega^r$. □

Figure 1 represents the map of the admissible solutions given by Theorem 2.1 with respect to the initial states $\mathbb{W}_L$ and $\mathbb{W}_R$. The right part of the chart corresponds to the solutions with positive material speed, while the left part depicts the symmetric configurations with negative material speed.

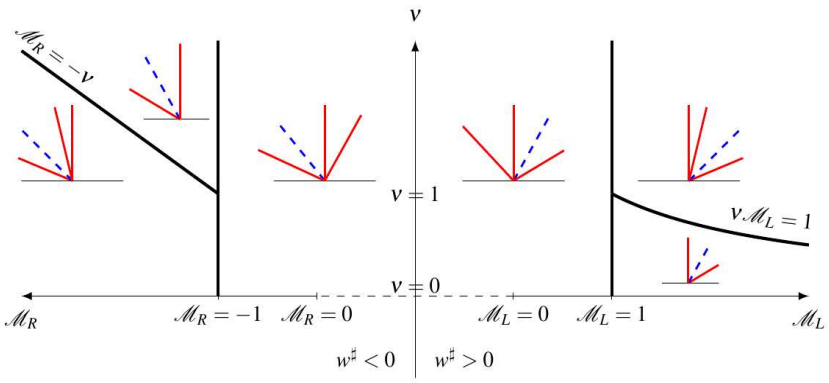

Fig. 1 Wave configuration of the solution of the Riemann problem (9) with respect to $\mathbb{W}_L$ and $\mathbb{W}_R$. $\mathscr{M}_L = \frac{w_L}{a\tau_L}$ and $\mathscr{M}_R = \frac{w_R}{a\tau_R}$ are the Mach numbers of the initial left and right states $\mathbb{W}_L$ and $\mathbb{W}_R$. The material wave is represented by a dashed line.

3 Numerical approximation

In this section, we derive a numerical scheme from the relaxation approximation introduced in section 1, the aim being to approximate the weak solutions of a Cauchy problem associated with system (1):

$$\begin{cases} \partial_t \mathbb{U} + \partial_x \mathbf{f}(\mathbb{U}) + \mathbf{c}(\mathbb{U})\partial_x \mathbb{U} = 0, \\ \mathbb{U}(x,0) = \mathbb{U}_0(x). \end{cases} \tag{13}$$

Let Δx be a space step and Δt a time step. The space is partitioned into cells $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} C_j$ with $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$, where $x_{j+\frac{1}{2}} = (j+\frac{1}{2})\Delta x$ are the cell interfaces. At the discrete times $t^n = n\Delta t$, the solution of (13) is approximated on each cell C_j by a constant value denoted by $\mathbb{U}_j^n = \left(\alpha_j^n, (\alpha\rho)_j^n, (\alpha\rho w)_j^n\right)^T$. We now describe the two-step splitting method associated with the relaxation system (6) in order to calculate the values of the approximate solution at time t^{n+1} $(\mathbb{U}_j^{n+1})_{j\in\mathbb{Z}}$ from those at time t^n.

Step 1: Time evolution $(t^n \to t^{n+1,-})$

We first introduce the piecewise constant approximate solution of the relaxation system at time t^n: $x \mapsto \mathbb{W}(x,t^n) = \mathbb{W}_j^n$ in C_j with $\mathbb{W}_j^n = \left(\alpha_j^n, (\alpha\rho)_j^n, (\alpha\rho w)_j^n, (\alpha\rho\mathscr{T})_j^n\right)$, where $\mathscr{T}_j^n := \tau_j^n$, *i.e.* $\mathbb{W}_j^n$ is at equilibrium. Then, the following Cauchy problem is **exactly solved** for $t \in [0,\Delta t]$ with Δt small enough (see condition (15) below)

$$\begin{cases} \partial_t \widetilde{\mathbb{W}} + \partial_x \mathbf{g}(\widetilde{\mathbb{W}}) + \mathbf{d}(\widetilde{\mathbb{W}})\partial_x \widetilde{\mathbb{W}} = 0, \\ \widetilde{\mathbb{W}}(x,0) = \mathbb{W}(x,t^n). \end{cases} \tag{14}$$

Since the initial condition $x \mapsto \mathbb{W}(x,t^n)$ is piecewise constant, the exact solution of (14) is obtained by gluing together the solutions of the Riemann problems set at each cell interface $x_{j+\frac{1}{2}}$, provided that these solutions do not interact during the period Δt, *i.e.* provided the following classical CFL condition

$$\frac{\Delta t}{\Delta x} \max_{\mathbb{W}} |\sigma_i(\mathbb{W})| < \frac{1}{2}, \ i \in \{0,...,3\}, \tag{15}$$

for all $\mathbb{W}$ under consideration. More precisely, if (x,t) is in $[x_j, x_{j+1}] \times [0,\Delta t]$, then

$$\widetilde{\mathbb{W}}(x,t) = \mathbb{W}_r\left(\frac{x - x_{j+1/2}}{t}; a_{j+1/2}, \mathbb{W}_j^n, \mathbb{W}_{j+1}^n\right), \tag{16}$$

where $(x,t) \mapsto \mathbb{W}_r\left(\frac{x}{t}; a, \mathbb{W}_L, \mathbb{W}_R\right)$ is the self-similar solution of the Riemann problem constructed in section 1, which clearly depends on the local choice of the parameter a. Then, in order to define a piecewise constant approximate solution at time $t^{n+1,-}$, the solution $\widetilde{\mathbb{W}}(x,t)$ is averaged on each cell C_j at time Δt:

$$\mathbb{W}(x,t^{n+1,-}) = \mathbb{W}_j^{n+1,-} := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \widetilde{\mathbb{W}}(x,\Delta t)dx, \quad \forall x \in C_j, \quad \forall j \in \mathbb{Z}. \tag{17}$$

Step 2: Instantaneous relaxation $(t^{n+1,-} \to t^{n+1})$

The second step consists in sending ε to zero instantaneously in the piecewise constant function $\mathbb{W}(x,t^{n+1,-})$ obtained at the end of the first step. This amounts to imposing $\mathscr{T}_j^{n+1} := \tau_j^{n+1}$, thus we have

$$\mathbb{W}_j^{n+1} = \left(\alpha_j^{n+1,-}, (\alpha\rho)_j^{n+1,-}, (\alpha\rho w)_j^{n+1,-}, \alpha_j^{n+1,-}\right)^T. \tag{18}$$

Finally, the new cell value at time t^{n+1} of the approximate solution reads

$$\mathbb{U}_j^{n+1} = \left(\alpha_j^{n+1,-}, (\alpha\rho)_j^{n+1,-}, (\alpha\rho w)_j^{n+1,-}\right)^T. \tag{19}$$

We can prove that this two-step relaxation method can be equivalently rewritten in the form of a Godunov-type finite volume scheme [7].

4 Non-linear stability of the scheme

Non-linear stability issues are usually dealt with through a so-called *discrete entropy inequality*, which is the discrete counterpart of the entropy inequality (5) satisfied by the weak solutions of the model. We have the following definition:

Definition 4.1 *We say that a numerical scheme satisfies a discrete entropy inequality if there exists a numerical entropy flux $G(\mathbb{U}_L,\mathbb{U}_R)$ which is consistent with the exact entropy flux $\mathscr{G} = \alpha\rho\mathscr{E}w + \alpha p(\tau)w$ (in the sense that $G(\mathbb{U},\mathbb{U}) = \mathscr{G}(\mathbb{U})$ for all $\mathbb{U}$) such that, under some CFL condition, the discrete values $(\mathbb{U}_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ computed by the scheme automatically satisfy*

$$(\alpha\rho\mathscr{E})(\mathbb{U}_j^{n+1}) - (\alpha\rho\mathscr{E})(\mathbb{U}_j^n) + \frac{\Delta t}{\Delta x}(G(\mathbb{U}_j^n,\mathbb{U}_{j+1}^n) - G(\mathbb{U}_{j-1}^n,\mathbb{U}_j^n)) \leq 0. \tag{20}$$

As seen in section 3, under the CFL condition (15), the different Riemann problems at each interface do not interact and the parameter $a = a_{j+\frac{1}{2}}$ can be chosen locally interface by interface. Usually, if $a_{j+\frac{1}{2}}$ is large enough, so as to satisfy a so-called Whitham condition (see [1]), then a discrete entropy inequality (20) is guaranteed. In order to define $a_{j+\frac{1}{2}}$, we propose a weak Whitham-like condition that handles the resonance phenomenon and still guarantees a discrete entropy inequality under the CFL condition (15) (see [2] for details).

5 Numerical tests

In this section, we run the relaxation scheme described in section 3 on a Riemann problem that contains the standing wave associated with the constant cross-section α, a left-going λ_1-rarefaction wave, a sonic right-going λ_1-rarefaction wave and a right-going λ_2-shock. The chosen pressure law is an ideal gas barotropic pressure law $p(\tau) = \tau^{-\gamma}$, with $\gamma = 3$. The left and right initial conditions are given by $\alpha_L = 3.0$, $\rho_L = 1.0$, $w_L = 0$, $\alpha_R = 1.0$, $\rho_R = 0.1$, and $w_R = 0$. The outcome of the relaxation method is compared with two other numerical schemes. The first one is the classical Rusanov scheme where the cross-section α is preserved throughout time:

$$\alpha_j^{n+1} := \alpha_j^n. \tag{21}$$

The second one is a modification of the Rusanov scheme that consists in applying the scheme to the whole state vector $\mathbb{U}$ (including the cross-section α) causing α to be dissipated:

$$\alpha_j^{n+1} := \alpha_j^n - \frac{\Delta t}{\Delta x}\left(q^n_{j+\frac{1}{2}} - q^n_{j-\frac{1}{2}}\right), \tag{22}$$

with $q^n_{j+\frac{1}{2}} = -r(\mathbb{U}_j^n)(\alpha^n_{j+1} - \alpha^n_j)$ where the scalar $r(\mathbb{U}_j^n)$ is the maximal value of the spectral radius of the Jacobian matrices $(\nabla \mathbf{f} + \mathbf{c})(\mathbb{U}_k^n)$ for $k = j, j+1$. In Figure 2, we can see that, due to a smoothing effect, the dissipation of the cross-section α provides a notable improvement for the Rusanov scheme (see [4] and [8] for different approaches to improve the Rusanov scheme). The L^1-norm of the error on α, at the final time T, vanishes as the space step Δx goes to zero (with $\Delta t/\Delta x$ constant) with the order $\mathscr{O}(\Delta x^{1/2})$.

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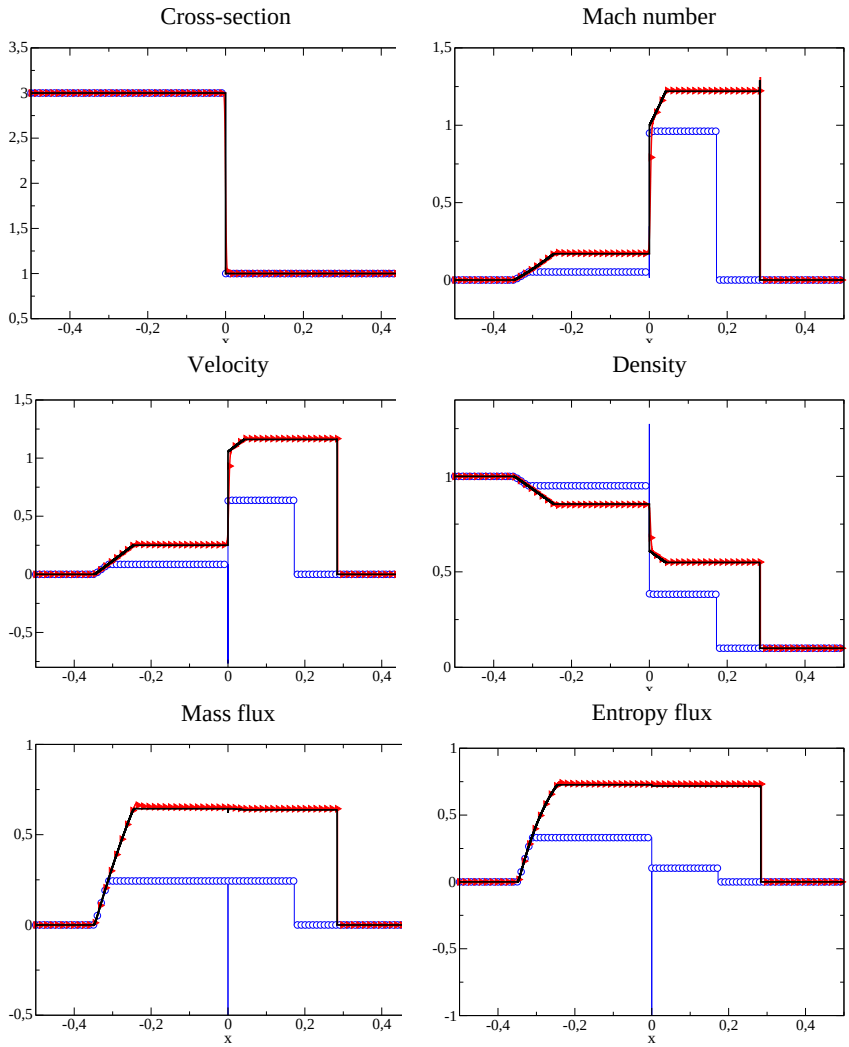

Fig. 2 Solution of the Riemann problem at time $T = 0.2$. Space step $\Delta x = 10^{-5}$. Straight line: relaxation scheme, circles: classical Rusanov scheme, triangles: Rusanov scheme with dissipation of the cross-section.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.