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Kaïs Ammari, Islam Boussaada, Silviu-Iulian Niculescu, Sami Tliba

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MULTIPLICITY MANIFOLDS AS AN OPENING TO PRESCRIBE EXPONENTIAL DECAY: AUTO-REGRESSIVE BOUNDARY FEEDBACK IN WAVE EQUATION STABILIZATION

KAÏS AMMARI, ISLAM BOUSSAADA, SILVIU-IULIAN NICULESCU, AND SAMI TLIBA

ABSTRACT. Exploring a more than 70 years old idea about the minimization of the spectral abscissa of linear functional differential equations, a series of recent works highlighted the insights that multiple spectral values may bring in the characterization of the decay rate for the solution of such dynamical systems. In fact, it has been shown that a spectral value of sufficiently high multiplicity tends to be dominant, in what is now known as the *multiplicity-induced-dominancy* (MID) property. When it is valid, this property can be remarkably helpful in the control of dynamical systems governed by functional differential equations or even some classes of partial differential equations. Beyond its simplicity, what sets it from other control methods is the valuable quantitative advantage it provides by prescribing the exact solution's decay rate. Since then, many works have been dedicated to studying the extent of the MID as well as its use in practical control applications. In this paper, apart from the extension of the MID property to difference functional equations with multiple delays, we study the case when the MID fails. In fact, despite the invalidity of the MID property, we emphasize the interest of forcing a spectral value multiplicity to derive a sharp estimate of the corresponding rightmost spectral value. To demonstrate the effectiveness of the proposed methodology, we consider the stabilization problem of the wave equation with an auto-regressive boundary feedback. By using an appropriate finite element modeling, a numerical simulation of the boundary control for the wave equation case is performed to illustrate these results through the example of vibrations' control of a long drill pipe submitted to a shock-like disturbance. The time responses show the effectiveness of the proposed approach and mainly that the decay rate can be arbitrarily selected.

CONTENTS

1. Introduction	2
2. On the control structure and strategy: MID property and on beyond	4
3. Problem Statement	10
4. Well-posedness of problem (3.10)	12
5. Asymptotic behavior	16
6. A spectral formulation via D'Alembert's formula	19
7. A parametric stability analysis via a spectral approach	20

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7.1. On the MID paradigm	21
7.2. Beyond the MID property in output feedback stabilization of the wave equation in (3.10)	24
8. Application: Longitudinal vibration control of a long drill pipe transmission line	27
8.1. Description of the problem	27
8.2. Finite element modeling	29
8.3. Numerical developments and physical intuitive interpretations	30
8.4. P3 δ Software	31
Acknowledgement	34
References	34

1. Introduction

Since the work of d'Alembert published more than 275 years ago and devoted to the vibrating string problem, the one-dimensional (two-way) wave equation is a subject of recurring interest, and there is a large literature on the qualitative and quantitative properties of its solutions. Surprisingly, such a simple model can be derived in a large variety of physical settings, and represents the simplest example of second-order hyperbolic partial differential equations (PDEs) model. It plays more than an important role in continuum/quantum mechanics (disturbances space-traveling over time), geophysics (*e.g.*, modeling seismic waves), oceanography physics (dynamics of the ocean waves), electrical engineering (see, for instance, models representing lossless propagation phenomena through transmission lines) and control engineering (*e.g.* vibration control problems), to cite only a few.

In this context, the existing links between hyperbolic PDEs and delay-differential equations of *neutral* type should be pointed out*. More precisely, as shown by Abolinia and Myshkis [1] and Cooke and Krumme [37] in the 60s, integrating along the characteristics of the boundary value problems for hyperbolic PDE leads to such neutral DDEs-based models. Interestingly, such results seem to insist (for the first time) on the functional character of the corresponding (neutral delay-differential) equations generated by the DAlembert method in the sense that they are considered as legitimate but autonomous mathematical objects. Furthermore, the properties of solutions of the mixed initial-boundary value problem can be established by using the results obtained for the functional equation and a representation formula of the solutions of the mixed problem in terms of the solutions of the associated equation.

In the sequel, we will exploit both model representations in the context of boundary control of the wave equation, and we will explicitly emphasize the existing links between the model parameters and the largest multiplicity of the roots of the corresponding characteristic function.

*see, *e.g.*, [46] and the references therein for appropriate definitions and classification of DDEs

In both retarded and neutral delay-differential equations (DDEs) cases, the generic maximal allowable multiplicity of a characteristic root induces dominance, that is, it simply represents the rightmost characteristic root. The characterization of such a property called *multiplicity induced dominance* (MID) can be found in [63, 19].

In controlling the wave equation, the proposed control law includes appropriate past information from both input and output signals and we called it an *auto-regressive* system due to the way the signals appear in the system's dynamic. From the control point of view, the corresponding controller called QPB (from quasipolynomial-based) is easy to implement. Furthermore, it belongs to the class of low-complexity infinite-dimensional controllers since on the one hand it includes a reduced number of parameters, and it is infinite-dimensional due to the presence of the delay terms in the model, on the other hand. Such a (linear) controller needs to be well-posed and *formally stable*[†]. In fact, it was emphasized in [58] that neutral systems that are not formally stable cannot be stabilized in the sense of bounded-input/bounded-output. As a matter of fact, in the 70s, [16] pointed out the existence of such a *sensitivity phenomenon*[‡] with respect to “small” delay perturbations, and such a problem was largely covered in the open literature during the period 80s-90s. For instance, amongst others, conditions for guaranteeing stability robustness of the closed-loop systems can be found in [56, 57], where the authors proved the robustness in terms of spectral radius of some appropriate well-posed and regular transfer function.

As expected, if the dynamics of the closed-loop system include more than one delay, the problem becomes hard to deal with. For example, some strange and unexpected phenomena can appear and, in particular, the *sensitivity* of the (exponential) stability with respect to the *delays ratio* is one of the interesting cases treated in the open literature is of utmost importance within the control framework and amply treated in the open literature. By analogy with the interference phenomenon in Physics[§], such a phenomenon is called *delay interference*, and it was first observed and discussed by MacDonald in the 80s [60]. In the retarded case, the characterization of such a property and its links with the delays-independent stability is proposed in [66]. Finally, although the extension to neutral systems is not explicitly discussed, [66] addresses also the delay interference problem in the case of the continuous-time delay-difference equation including multiple delays, and a few examples are presented. Surprisingly, such a phenomenon also appears in control theory as the controller structure may induce delay interference for the closed-loop system. For instance, in controlling the wave equation, [83] proposed a boundary control law including a single delay block, and despite the simplicity of the controller, the closed-loop system is subject to delay interference, and it lacks robustness with respect to the delays.

[†]that is all the poles of its transfer function have negative real part

[‡]an infinitesimal “small” delay can destabilize a dynamical system

[§]In Physics, it represents the combination of two or more wave motions to form a resultant wave in which the displacement is reinforced or canceled.

The contribution of the paper is threefold. First, the use of a QPB controller combined with the idea of forcing the multiplicity allows for avoiding the delay sensitivity phenomena mentioned above (with respect to "small" delays or/and delays ratio). In other words, the controller depicts *robustness properties* with respect to the QPB delay parameter. Second, by exploiting the delay as a *control parameter*, an explicit estimation of the decay rate is derived and such an estimation corresponds to the generic MID case for some values of the delay parameter. It should be mentioned that due to the structure of the continuous-time delay-difference equation, forcing multiplicity does not necessarily lead to the dominance but to some explicit bounds regarding the location of the rightmost characteristic root. Third, the QPB controller is explicitly applied to the longitudinal vibration control of a long drill pipe transmission line and a numerical study shows the effectiveness of the method. If the use of QPB in vibration control was already proposed in the literature by some of the authors of this paper [29, 82], it should be mentioned that its application to the wave equation represents a novelty.

Finally, to the best of the authors' knowledge, such an approach was not considered in the literature and, as such, it also represents a novelty. We believe that the underlying ideas of the proposed methodology are useful in the study of the dynamical behavior of other classes of infinite-dimensional systems.

The remaining of the paper is organized as follows. Section 2 provides a brief overview on the MID property, as well as a comprehensive example illustrating the benefits of a multiplicity manifold in stability analysis and stabilization even when the MID fails. Section 3 is dedicated to recalling the recent results on the control of the wave equation with delay in the boundary and ends with the problem statement. For the sake of rigor, the well-posedness of the considered problem is provided in Section 4. Section 5 exhibits conditions under which the corresponding semigroup decays exponentially to the null steady state. A spectral formulation via DAlemberts formula is carried out in Section 6 where the system of partial differential equations is transformed into a functional difference equation. In Section 7, we present a summary of standard results on the stability of such difference equations and unfold the main results of the present paper. Namely, the MID property is investigated: in particular, values of the time delay which is multiple of the wave propagation time are addressed. Next, when the MID fails, we propose a general strategy to prescribe a lower bound for the solution's exponential decay rate. Finally, as an illustration of the proposed methodology, in Section 8 we provide a numerical study of the problem of longitudinal vibration control of a long drill pipe transmission line. The paper is concluded by a discussion on software perspectives.

2. On the control structure and strategy: MID property and on beyond

Since the 30s and the theory of servomechanisms proposed by Hazen [47], it is commonly accepted that, in most of practical situations, the presence of time-delays in dynamical processes is synonymous with instability and bad behaviors. However, there are cases where the delay in the input can induce a stabilizing effect on the dynamics of the system, and the idea of exploiting the *delay* as a *control parameter*

is not new (see, *e.g.*, [72, 67], and the references therein). For instance, at the end of the 70s, [81] introduced the so-called "proportional-minus-delay" (PMD) controller and showed that such a controller can replace the well-known PD-controller (quick responses to input changes) with less sensitivity to high-frequency noise. Such a controller was further implemented in various case studies including, amongst others, the well-known inverted pendulum (see, *e.g.*, [80, 26], to cite only a few).

In this work, we are exploiting a low-complexity controller structure called *QuasiPolynomial-Based controller* (QPB) which has been recently introduced in [29], the idea of which is close to the aforementioned PMD control in the sense that the controller structure is of PMD type however in both output and input signals, the delay being used as a design parameter to improve the overall behavior of the closed-loop system. In other words, given a generic dynamical system to control, modeled by the input-to-output mathematical model $y(t) = \mathcal{G}\{v(t)\}$ where $\mathcal{G}\{\cdot\}$ denotes the appropriate input-to-output mathematical operator of this system, $y(t) \in \mathbb{R}$ is its measured output and $v(t) \in \mathbb{R}$ its controlled input. For constructing the controlled input v at a given time value t , we need an appropriate past information from both the input v and the output y in addition to the present information given by the output. Hence, the QPB controller consists in the generic output feedback control law which is defined by the following continuous-time delay-difference equation relating y to v in the following fashion:

$$(2.1) \quad v(t) = -\alpha v(t - \tau) + \beta y(t) + \gamma y(t - \tau),$$

where α, β and γ are some real scalar parameters with the constraint $\gamma \neq \alpha\beta$ and τ is a positive scalar corresponding to the delay parameter of the QPB controller. Relation (2.1) is nothing but an auto-regressive equation, defining a dynamical system with its own actual and past behavior.

Lemma 2.1. *Given the control law in (2.1) for any $\tau \in \mathbb{R}$. If $\gamma = \alpha\beta$, then (2.1) is equivalent to*

$$(2.2) \quad v(t) = \beta y(t).$$

Proof. Suppose $\gamma = \alpha\beta$ for $\alpha, \beta \in \mathbb{R}$. Then (2.1) is equivalent to

$$v(t) - \beta y(t) = -\alpha(v(t - \tau) - \beta y(t - \tau)).$$

By defining $f(t) := v(t) - \beta y(t)$, the previous equation reads as

$$f(t) = -\alpha f(t - \tau).$$

Since this last relation holds for any $t, \tau \in \mathbb{R}$, the only solution for the above difference equation is $f(t) \equiv 0$, leading so to (2.2). Indeed, if $\alpha = 0$, then $f(t) = 0$ for all $t \in \mathbb{R}$. Now, for $\alpha \neq 0$, the previous equation is equivalent to having $f(t_1) = \alpha f(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. For any $t_3 \neq t_1, t_2$ one also has $f(t_1) = \alpha f(t_3)$, so that $f(t_3) = f(t_2) =: f_0 \in \mathbb{R}$. Since we also must have $f_0 = \alpha f_0$ with $\alpha \neq 0$, then $f_0 = 0$. Reciprocally, if (2.2) is true for all $t \in \mathbb{R}$, then for any $\tau \in \mathbb{R}$,

$$v(t - \tau) = \beta y(t - \tau).$$

By combining this last relation with (2.2), one gets (2.1) by defining $\gamma := \alpha\beta$. □

Despite its simplicity in terms of construction, such an infinite-dimensional controller induces good behaviors in closed loop and has proven its effectiveness through various applications in active vibration damping. Indeed, in some previous works, the authors have already considered the active vibration control of some specific mechanical structures described by the PDEs of continuous mechanical structures, *via* the same controller as the one proposed in this work. It was the case for a beam-like one in [29] and an axisymmetric membrane, both equipped with a piezoelectric sensor and a piezoelectric actuator that are collocated. However, the PDEs have been first discretized *via* the Finite Element Method in order to transform the systems' model into some finite-dimensional ones. Then, the same controller has been designed to achieve the vibrations' damping by using the MID paradigm that is described below. The results have proven the efficiency of such a control structure, motivating its use's extension directly to the PDE framework.

Having to adjust the four QPB parameters for achieving some control objectives may be seen as an undue difficulty. However, it offers degrees of freedom for performance purposes with the low cost of adding an auto-regressive term. Notice that the QPB parameters' tuning has always been done through an analytical qualitative/quantitative framework called *partial pole placement* (PPP) consisting in the stabilization of the closed-loop solutions with a prescribed admissible decay rate, which is a great advantage in practical applications. As a matter of fact, in all generality, assigning the exact location of a spectrum composed of infinitely many eigenvalues with only finitely many parameters is impossible. Nevertheless, the strategy of PPP consists in tuning the parameters in order to assign only finitely many eigenvalues, while guaranteeing that the rightmost eigenvalue is among them. Hence, the negativity of the assigned rightmost eigenvalue certifies the exponential stability. While tuning parameters to assign some eigenvalues is typically a trivial interpolation question, guaranteeing dominance of the chosen eigenvalues is a difficult problem.

Exploring some erstwhile ideas, present for instance in [73], the seminal works [27, 30] highlighted the fact that roots of quasipolynomials of high enough multiplicity tend to be dominant, in what came to be known as the *multiplicity-induced-dominancy* (MID) property. When available, this property can be helpful in the stabilization of time-delay systems since it suffices to select the system's free parameters in order to guarantee the existence of such a root of high multiplicity with a negative real part, and the MID property will ensure its dominance. While early works verified the MID property for some classes of time-delay systems, typically of low order, more recent papers, such as [29, 44, 20, 64, 21, 23, 15, 59, 28, 18], have extended those results to more general situations.

Since the multiplicity of spectral values is not important as such, instead of assigning a single root of high enough multiplicity, some recent works such as [17] have considered the assignment of several simple real roots and shown that, in several situations, the rightmost root among those assigned is dominant, consisting in what has been named the *coexisting-real-roots-induced-dominancy* (CRRID) property. Assigning several simple real roots instead of a single root of large multiplicity allows weaker constraints

in the control design and less sensitivity to parameters' variations or uncertainties, as explored in the applications in [24] or [82].

The main ingredient behind most of the proofs of the MID and the CRRID properties described in the aforementioned references is an integral representation of the corresponding quasipolynomial when the roots are assigned. In the case of the MID, when assigning roots of maximal possible multiplicity, such an integral representation turns out to be nothing but the well-known Kummer confluent hypergeometric function, as proved in [64]. In the case of the MID property with multiplicity smaller than the maximal one, the integral representation involves linear combinations of Kummer functions [23], while, for the CRRID properties, generalizations of hypergeometric functions are involved, see for instance [17]. Proofs of dominance then rely on results pertaining to the locations of roots of such special functions, such as those from [22, 84, 76].

Let us define the quasipolynomial $Q : \mathbb{C} \times \mathbb{R}_+^* \mapsto \mathbb{C}$ as

$$(2.3) \quad Q(s, \tau) := P_0(s) + P_\tau(s) e^{-\tau s},$$

where P_0 and P_τ are polynomials with real coefficients given by

$$(2.4) \quad P_0(s) = s^n + \sum_{k=0}^{n-1} a_k s^k, \quad P_\tau(s) = \sum_{k=0}^m \alpha_k s^k.$$

The degree of Q is defined as the sum of the degrees of the involved polynomials plus the number of delays, *i.e.*, $\deg(Q) := n + m + 1$, see for instance [20].

Definition 2.2. *We say that a characteristic root s_0 of Q satisfies the MID property if the following holds:*

- (i): *its algebraic multiplicity (denoted by $M(s_0)$) is larger than one;*
- (ii): *it is dominant in the sense that all the characteristic roots $s \in \mathbb{C}$ of (2.3) are located to the left of s_0 in \mathbb{C}_- , *i.e.*, s satisfies the condition $\Re(s) \leq \Re(s_0)$.*

In other words, s_0 is the rightmost root of the spectrum and defines the *spectral abscissa* of the quasipolynomial Q .

In the case where $M(s_0) = \deg(Q)$, it was shown in [64] (in the case where $m = n - 1$) and in [20] (in the general case $m \leq n$) that s_0 satisfies the MID property. This “limit” case is also called *generic MID* or GMID for short. In the case where $M(s_0) < \deg(Q)$ and the MID hold, those cases refer to the *Intermediate MID* or IMID for short.

Notice that even if these remarkable properties are not valid, the existence of multiple roots or the coexistence of negative roots may consist in breaches for the location of the spectral abscissa as can be seen in the following example borrowed from [28]. Let us revisit the following control problem where the aim is to establish a delayed-state-feedback controller capable of fast stabilizing and damping oscillations of

solutions of a given second-order system:

$$(2.5) \quad \begin{cases} \ddot{u}(t) + a_1 \dot{u}(t) + a_0 u(t) = v(t), & \text{where} \\ v(t) = -\alpha_0 u(t - \tau) - \alpha_1 \dot{u}(t - \tau). \end{cases}$$

More precisely, one is concerned with an open-loop system admitting one oscillating (possibly unstable) mode when the discriminant $\Delta = a_1^2 - 4a_0$ of the open-loop characteristic polynomial satisfies $\Delta < 0$. Notice that, by looking for the closed-loop system solution in the form $u(t) = e^{st}$ or simply by employing the Laplace transform, one gets the corresponding characteristic equation in the complex domain which reads:

$$(2.6) \quad \begin{cases} Q(s, \tau) = P_0(s) + P_1(s)e^{-\tau s}, & \text{where} \\ P_0(s) = s^2 + a_1 s + a_0 & \text{and} \quad P_1(s) = \alpha_1 s + \alpha_0. \end{cases}$$

It has been shown in [28, Theorem 4.1] that the only admissible quadruple spectral value is achieved at $s = s^*$ where

$$(2.7) \quad s^* = -\frac{a_1 + \sqrt{-2\Delta}}{2},$$

with $\tau = \tau^*$ and the system parameters satisfy $\tau^* = 2\sqrt{-\frac{2}{\Delta}}$, $\alpha_0 = \frac{(5\Delta - a_1\sqrt{-2\Delta})}{4} e^{s^*\tau^*}$, $\alpha_1 = -\frac{\sqrt{-2\Delta}}{2} e^{s^*\tau^*}$.

In this case, s^* corresponds to the spectral abscissa of the closed-loop (2.5) and consequently, it is said to satisfy the GMID property, see also [64, 20]. Further, in the same work, it has been shown in [28, Theorem 4.2] that the only admissible triple spectral values are

$$(2.8) \quad s_{\pm} = \frac{-\tau a_1 - 4 \pm \sqrt{8 + \tau^2 \Delta}}{2\tau},$$

which exist only for $\Delta \geq -8/\tau^2$ and are achieved when the control parameters satisfy:

$$(\star_{\pm}) \quad \begin{cases} \alpha_0 = \left(a_1 s_{\pm} + \frac{a_1^2}{2} - \frac{\Delta}{2} + \frac{6a_1 + 10s_{\pm}}{\tau} + \frac{6}{\tau^2} \right) e^{s_{\pm}\tau}, \\ \alpha_1 = \left(2s_{\pm} + a_1 + \frac{2}{\tau} \right) e^{s_{\pm}\tau}. \end{cases}$$

Interestingly, it has been shown that if, (\star_+) is satisfied then $s = s_+$ is the spectral abscissa corresponding to (2.6) that is the IMID is satisfied. Nonetheless, if (\star_-) is satisfied then s_- cannot be the spectral abscissa corresponding to (2.6) as it is always dominated by a single real root s_0 . Further, in the particular case of $\Delta^* = -\frac{8}{\tau^2}$, the triple root at s_- merges with the aforementioned s_0 giving rise to the quadruple spectral value s^* which is proven to be dominant. So that, as illustrated in Figure 1 in the study of the delayed PD controller in the problem of stabilization and quenching oscillations of the harmonic oscillator (no damping) with natural frequency $\omega_0 = \sqrt{2}$

$$(2.9) \quad \begin{cases} \ddot{u}(t) + 2u(t) = v(t) & \text{where} \\ v(t) = -\alpha_0 u(t - \tau) - \alpha_1 \dot{u}(t - \tau), \end{cases}$$

and controllers gains satisfying (\star_-) , thanks to the continuity of the spectrum location with respect to parameters variation, for instance by decreasing delay $\tau = \tau^* - \epsilon$ with $\epsilon \in (0, 0.255)$, one is able to monitor

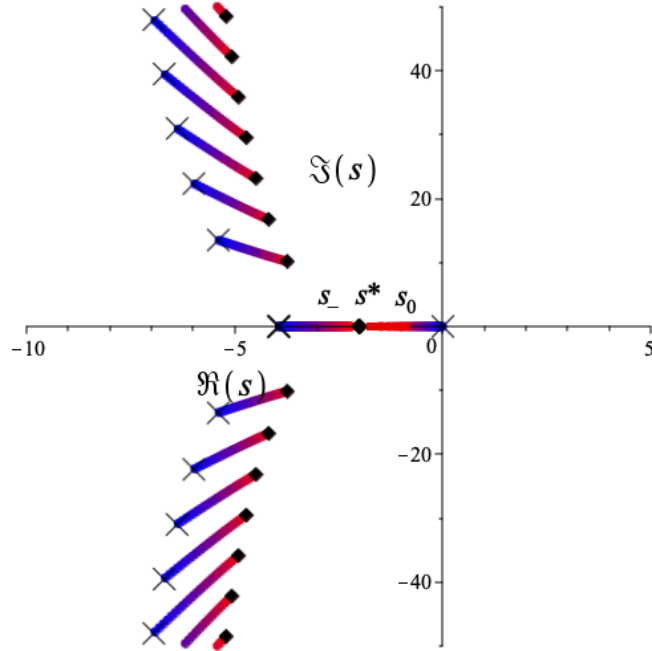


FIGURE 1. The spectrum distribution of (2.5) with $a_0 = 2$, $a_1 = 0$ and control parameters given by (\star_-) with respect to delay variation (from red to blue) by continuously decreasing the delay value from $\tau^* = 1$, where the nominal spectrum distribution is represented by filled diamonds (\blacklozenge) and achieved for the delay value $\tau = \tau^* = 1$ for which the quadruple spectral value s^* corresponds to the spectral abscissa, to τ_{crit} , where the end spectrum distribution is represented by the cross symbol (\times) which is achieved for the delay value $\tau = \tau_{crit}$ for which the rightmost spectral value is $s_0 = 0$ (loss of the exponential stability). This gives a sensitivity chart of the spectrum distribution with respect to delay variation illustrating a remarkable splitting of the abscissa s^* on the real axis into a breakaway branch s_0 , which becomes itself the rightmost spectral value, moving towards the origin of the complex plane and a dominated locked triple spectral value s_- branch moving in the opposite direction.

the closed-loop system solution's decay rate s_0 (spectral abscissa location) for a delay range $\tau \in (\tau_{crit}, \tau^*)$ where τ_{crit} corresponds to the delay value achieving $s_0 = 0$.

This provides a comprehensive example for which one is able to assign a lower-bound of the closed-loop solution's decay rate, even though the multiple spectral root is not the spectral abscissa.

3. Problem Statement

We study the boundary stabilization of a wave equation in $(0, 1) \subset \mathbb{R}$ including auto-regressive boundary conditions given by:

$$(3.10) \quad \begin{cases} u_{tt}(x, t) - u_{xx}(x, t) = 0, & x \in (0, 1), t > 0, \\ u(0, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (0, 1), \\ u_x(1, t) = -\alpha u_x(1, t - \tau) + \beta u_t(1, t) + \gamma u_t(1, t - \tau), & t > 0, \\ u_t(1, t - \tau) = 0, \quad u_x(1, t - \tau) = 0, & t \in (0, \tau). \end{cases}$$

The constant $\tau > 0$ is a time delay, α , β and γ are real numbers corresponding to the QPB control law's gains and the initial data u_0 , u_1 are given functions belonging to suitable spaces that will be precised later.

It is worth noticing that any choice of the gains such that $\gamma = \alpha\beta$ must be rejected because it amounts to having a boundary condition (3.10) that would be reduced to

$$(3.11) \quad u_x(1, t) = \beta u_t(1, t).$$

Such a case has already been considered previously by several works, as for example in [32, 53, 54], and is known as a *damping injection term*.

The problem of boundary stabilization of the wave equation with time delay in the output feedback loop has been considered in several works; see for instance [13] for the problem of damping torsional vibrations in a drilling system and [85] for a special emphasis on time delay equal to a multiple of the wave propagation time. More precisely, in [85], the system (3.10) has been considered with $\alpha = \beta = 0$. Some remarkable and unexpected properties have been pointed out. In particular, a stabilizing effect of even multiples of the wave propagation time has been exhibited. However, odd multiples induce unstable closed-loop system dynamics. Furthermore, a lack of robustness to small perturbations in time delay has been reported.

The purpose of this paper is to study problem (3.10), in which two delayed terms act in the boundary conditions. To our best knowledge, the auto-regressive term weighted by coefficient α is the main novelty in such a problem. In recent years, one very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems. Many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [2, 3, 40, 68, 69, 70] for more details).

As it has been proved by Datko [39, Example 3.5], systems of the form

$$(3.12) \quad \begin{cases} w_{tt}(x, t) - w_{xx}(x, t) - aw_{xt}(x, t) = 0, & x \in (0, 1), t > 0, \\ w(0, t) = 0, \quad w_x(1, t) = -kw_t(1, t - \tau), & t > 0, \end{cases}$$

where a , k and τ are positive constants become unstable for arbitrarily small values of τ and any values of a and k . In (3.12) and even in the presence of the strong damping $-aw_{xxt}$, without any other damping, the overall structure can be unstable. This was one of the main motivations for considering problem (3.10). Of course, the structures of problems (3.10) and (3.12) are different due to the nature of the boundary conditions in each problem.

Subsequently, Datko et al [40] treated the following one-dimensional problem:

$$(3.13) \quad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2 u(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) = -ku_t(1,t-\tau), & t > 0, \end{cases}$$

which models the vibrations of a string clamped at one end and free at the other, where $u(x,t)$ is the displacement of the string. Also, the string is controlled by a boundary control force (with a delay) at the free end. They showed that, if the positive constants a and k satisfy

$$k \frac{e^{2a} + 1}{e^{2a} - 1} < 1,$$

then the delayed feedback system (3.13) is stable for all sufficiently small delays. On the other hand, if

$$k \frac{e^{2a} + 1}{e^{2a} - 1} > 1,$$

then there exists a dense open set $D \subset (0, \infty)$ such that for each $\tau \in D$, system (3.13) admits exponentially unstable solutions.

It is well known that in the absence of delay in (3.13), that is, for $\tau = 0$, the system is uniformly asymptotically stable under the condition $a^2 + k^2 > 0$ and the total energy of the solution satisfies for all $t > 0$,

$$(3.14) \quad E(t, u) := \int_0^1 (u_t^2 + u_x^2 + a^2 u^2) dx \leq C E(0, u) e^{-\alpha t}$$

for some positive constant α ; see [33] for more details.

In [68], the authors examined a wave equation system with a linear boundary damping term exhibiting a delay. Namely, they considered the following system

$$(3.15) \quad \begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ u(x, t) = 0, & x \in \gamma_0, \quad t > 0, \\ \frac{\partial u}{\partial \nu}(x, t) = \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) & x \in \gamma_1, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u_t(x, 0) = u_1(x), & x \in \Omega, \\ u_t(x, t - \tau) = g_0(x, t - \tau), & x \in \Omega, \quad \tau \in (0, 1), \end{cases}$$

and proved, under the assumption

$$(3.16) \quad \mu_2 < \mu_1$$

(which means that the weight of the feedback with delay is smaller than the one without), that the null stationary state is exponentially stable. On the contrary, if (3.16) does not hold, they produced a sequence of delays for which the corresponding solution of (3.15) is unstable. The main approach used in [68], is an observability inequality obtained by means of a Carleman estimate.

The case of time-varying delay (*i.e.* $\tau = \tau(t)$ is a function depending on t) has been studied by Nicaise, Valein and Fridman [71] in one space dimension. In their work, an exponential stability result was given under the condition:

$$(3.17) \quad \mu_2 < \sqrt{1-d} \mu_1,$$

where d is a constant satisfying

$$(3.18) \quad \tau'(t) \leq d < 1, \forall t > 0.$$

Delay effects arise in many applications and practical problems and it is well known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay (*e.g.* [53, 54, 38, 40]).

Let us also cite the recent work of Ammari and Nicaise, [9], in which the authors performed a complete study of the stabilization of elastic systems by collocated feedback with or without delay, see also [3, 10, 12, 11, 8, 7, 6, 4, 5].

4. Well-posedness of problem (3.10)

In this section, we shall first transform the delay boundary conditions by introducing a new unknown. Then, a semigroup approach and the Lumer-Phillips' theorem shall be used to prove the existence and the uniqueness of the solution of the problem (3.10).

Note that the well-posedness of the evolution equations with delay is not always obtained. Recently, Dreher, Quintilla and Racke have shown some ill-posedness results for a wide range of evolution equations with a delay term [42].

In order to prove the global existence and the uniqueness of the solution of problem (3.10), we shall first transform the problem (3.10) to the problem (4.22) by making the change of variables (4.19), and then we shall use the semigroup approach to prove the existence of the unique solution of problem (4.22).

To overcome the problem of the boundary delay, we introduce the new variables:

$$(4.19) \quad z(\rho, t) := u_t(1, t - \tau\rho), \quad w(\rho, t) := u_x(1, t - \tau\rho), \quad \rho \in (0, 1), \quad t > 0.$$

Then, we have

$$(4.20) \quad \tau z_t(\rho, t) + z_\rho(\rho, t) = 0, \text{ in } (0, 1) \times (0, +\infty),$$

$$(4.21) \quad \tau w_t(\rho, t) + w_\rho(\rho, t) = 0, \text{ in } (0, 1) \times (0, +\infty).$$

Hence, problem (3.10) is equivalent to:

$$(4.22) \quad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in (0,1), t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(1,t) + \alpha w(1,t) = \beta u_t(1,t) + \gamma z(1,t), & t > 0 \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, & \rho \in (0,1), t > 0 \\ \tau w_t(\rho,t) + w_\rho(\rho,t) = 0, & \rho \in (0,1), t > 0 \\ z(0,t) = u_t(1,t), w(0,t) = u_x(1,t), & t > 0 \\ u(x,0) = u_0(x), & x \in (0,1), \\ u_t(x,0) = u_1(x), & x \in (0,1), \\ z(\rho,0) = 0, w(\rho,0) = 0, & \rho \in (0,1). \end{cases}$$

The first natural question is the existence of solutions for the problem (4.22). In the following, we shall give a sufficient condition guaranteeing that the above problem is well-posed.

For this purpose, we use a semigroup formulation of the initial-boundary value problem (4.22). If we denote $V := (u, u_t, z, w)^T$, we define the energy space:

$$\mathcal{H} := H_\ell^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1),$$

where

$$H_\ell^1(0,1) := \{u \in H^1(0,1) / u(0) = 0\},$$

and (\cdot, \cdot) denotes the scalar product in $L^2(0,1)$, i.e. $(u, v) = \int_0^1 u(x)v(x)dx$.

Clearly, \mathcal{H} is a Hilbert space with respect to the inner product

$$(4.23) \quad \langle V_1, V_2 \rangle_{\mathcal{H}} = \int_0^1 u_x^1 u_x^2 dx + \int_0^1 v^1 v^2 dx + \xi_1 \int_0^1 z^1 z^2 d\rho + \xi_2 \int_0^1 w^1 w^2 d\rho$$

for $V_1 = (u^1, v^1, z^1, w^1)^T$, $V_2 = (u^2, v^2, z^2, w^2)^T$ and $\xi_1, \xi_2 > 0$ nonnegative real numbers to be defined later.

Therefore, if $V_0 \in \mathcal{H}$ and $V \in \mathcal{H}$, the problem (4.22) is formally equivalent to the following abstract evolution equation in the Hilbert space \mathcal{H} :

$$(4.24) \quad \begin{cases} V'(t) = \mathcal{A}V(t), & t > 0, \\ V(0) = V_0, \end{cases}$$

where ' denotes the derivative with respect to time t , $V_0 := (u_0, u_1, 0, 0)^T$ and the operator \mathcal{A} is defined by:

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \\ -\tau^{-1} z_\rho \\ -\tau^{-1} w_\rho \end{pmatrix}.$$

The domain of \mathcal{A} is the set of $V = (u, v, z, w)^T$ such that:

$$(4.25) \quad (u, v, z, w)^T \in \left(H_\ell^1(0, 1) \cap H^2(0, 1)\right) \times H_\ell^1(0, 1) \times H^1(0, 1) \times H^1(0, 1),$$

$$(4.26) \quad v(1) = z(0), \quad u_x(1) = w(0), \quad u_x(1) + \alpha w(1) = \beta v(1) + \gamma z(1).$$

If $(\alpha, \beta, \gamma, \xi_1, \xi_2)$ satisfies the following conditions:

$$(4.27) \quad 0 < \alpha < 1, \quad \beta < \gamma \sqrt{1 + \frac{2\alpha^2}{1 - \alpha^2}} < 0, \quad 2\tau \left(\frac{\alpha^2 \gamma^2}{\beta(\alpha^2 - 1)} - \frac{\gamma^2}{2\beta} \right) < \xi_1 < -\beta\tau, \quad \xi_2 = -\frac{\tau}{\beta},$$

then, the well-posedness of problem (4.22) is ensured by:

Theorem 4.1. *Let $V_0 \in \mathcal{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of problem (4.24). Moreover, if $V_0 \in \mathcal{D}(\mathcal{A})$, then*

$$V \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

Proof. The proof of the existence and the uniqueness of the solution of problem (4.24) relies on the semigroup approach and the Lumer-Phillips' theorem.

Indeed, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$. By definition of the operator \mathcal{A} and the scalar product of \mathcal{H} , we have:

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = \int_0^1 u_x v_x dx + \int_0^1 v u_{xx} dx - \frac{\xi_1}{\tau} \int_0^1 z z_\rho d\rho - \frac{\xi_2}{\tau} \int_0^1 w w_\rho d\rho.$$

By Green's formula we obtain:

$$(4.28) \quad \langle \mathcal{A}V, V \rangle_{\mathcal{H}} = u_x(1)v(1) - \frac{\xi_1}{2\tau} (z^2(1) - z^2(0)) - \frac{\xi_2}{2\tau} (w^2(1) - w^2(0)).$$

Hence, equation (4.28) becomes:

$$(4.29) \quad \begin{aligned} \langle \mathcal{A}V, V \rangle_{\mathcal{H}} &= (-\alpha w(1) + \beta z(0) + \gamma z(1))z(0) - \frac{\xi_1}{2\tau} (z^2(1) - z^2(0)) - \frac{\xi_2}{2\tau} (w^2(1) - w^2(0)) \\ &= -\frac{\xi_1}{2\tau} z^2(1) - \frac{\xi_2}{2\tau} w^2(1) + \left(\beta + \frac{\xi_1}{2\tau}\right) z^2(0) + \frac{\xi_2}{2\tau} w^2(0) - \alpha w(1)z(0) + \gamma z(1)z(0) \\ &= -\frac{\xi_1}{2\tau} z^2(1) - \frac{\xi_2}{2\tau} w^2(1) + \left(\beta + \frac{\xi_1}{2\tau}\right) z^2(0) + \frac{\xi_2}{2\tau} (-\alpha w(1) + \beta z(0) + \gamma z(1))^2 \\ &\quad - \alpha w(1)z(0) + \gamma z(1)z(0) \\ &= -\frac{\xi_1}{2\tau} z^2(1) - \frac{\xi_2}{2\tau} w^2(1) + \left(\beta + \frac{\xi_1}{2\tau}\right) z^2(0) + \frac{\xi_2}{2\tau} \alpha^2 w^2(1) + \frac{\xi_2}{2\tau} \beta^2 z^2(0) + \frac{\xi_2}{2\tau} \gamma^2 z^2(1) \\ &\quad - \frac{\xi_2 \alpha \gamma}{\tau} w(1)z(1). \end{aligned}$$

To treat the last terms in the above equation, Young's inequality gives, for all $\varepsilon > 0$:

$$(4.30) \quad \langle \mathcal{A}V, V \rangle_{\mathcal{H}} \leq \left(-\frac{\xi_1}{2\tau} + \frac{\xi_2}{2\tau} \gamma^2 + \varepsilon \frac{\xi_2}{\tau} |\alpha \gamma|\right) z^2(1) + \left(\beta + \frac{\xi_1}{2\tau} + \frac{\xi_2}{2\tau} \beta^2\right) z^2(0) + \left(-\frac{\xi_2}{2\tau} + \frac{\xi_2}{2\tau} \alpha^2 + \frac{\xi_2}{2\varepsilon} \frac{|\alpha \gamma|}{\tau}\right) w^2(1).$$

According to condition (4.27) and by choosing $\frac{\alpha \gamma}{\alpha^2 - 1} < \varepsilon < \left(\frac{\xi_1}{2\tau} + \frac{\gamma^2}{2\beta}\right) \frac{\beta}{\alpha \gamma}$, we obtain

$$(4.31) \quad \langle \mathcal{A}V, V \rangle_{\mathcal{H}} \leq 0.$$

Thus the operator \mathcal{A} is dissipative.

Now, we want to show that $\forall \lambda > 0$, $\lambda I - \mathcal{A}$ is surjective. To this end, it is clear that it suffices to show that $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$.

For $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, let $V = (u, v, z, w)^T \in \mathcal{D}(\mathcal{A})$ be a solution of

$$(\lambda I - \mathcal{A})V = F,$$

where

$$(4.32) \quad \lambda u - v = f_1,$$

$$(4.33) \quad \lambda v - u_{xx} = f_2,$$

$$(4.34) \quad \lambda z + \frac{1}{\tau} z_\rho = f_3,$$

$$(4.35) \quad \lambda w + \frac{1}{\tau} w_\rho = f_4.$$

To find $V = (u, v, z, w)^T \in \mathcal{D}(\mathcal{A})$ solution of the system (4.32)-(4.35), we suppose that u is determined with the appropriate regularity. Then, from (4.32), we get:

$$(4.36) \quad v = \lambda u - f_1.$$

Therefore, from the compatibility condition on γ_1 AND equations (4.26), we determine $z(0)$ and $w(0)$ by:

$$(4.37) \quad z(0) = v(1) = \lambda u(1) - f_1(1), \quad w(0) = u_x(1).$$

Thus, from equations (4.34)-(4.35), z and w are given, respectively, by:

$$(4.38) \quad \begin{cases} z(\rho) = (\lambda u(1) - f_1(1)) e^{-\lambda \rho \tau} + e^{-\lambda \rho \tau} \int_0^\rho e^{\lambda \rho s} f_3(s) ds, & \rho \in (0, 1), \\ w(\rho) = u_x(1) e^{-\lambda \rho \tau} + e^{-\lambda \rho \tau} \int_0^\rho e^{\lambda \rho s} f_4(s) ds, & \rho \in (0, 1). \end{cases}$$

Consequently, knowing u , we may deduce v by equation (4.36), and z and w by system (4.38).

We recall that since $V = (u, v, z, w)^T \in \mathcal{D}(\mathcal{A})$, from equations (4.33) and (4.36), u must satisfy:

$$(4.39) \quad \lambda^2 u - u_{xx} = f_2 + \lambda f_1, \quad \text{in } (0, 1)$$

with the boundary conditions

$$(4.40) \quad \begin{cases} u(0) = 0, \\ (1 + \alpha e^{-\lambda \tau}) u_x(1) - \lambda (\beta + \gamma e^{-\lambda \tau}) u(1) = \\ -(\beta + \gamma e^{-\lambda \tau}) f(1) + \int_0^1 e^{-\lambda(\tau-s)} (-\alpha f_4(s) + \gamma f_3(s)) ds. \end{cases}$$

Using the previous expression, we get:

$$(4.41) \quad u(x) = \begin{cases} \left\{ \frac{-(\beta + \gamma e^{-\lambda \tau}) f(1) + \int_0^1 e^{-\lambda(\tau-s)} (-\alpha f_4(s) + \gamma f_3(s)) ds}{\lambda(1 + \alpha e^{-\lambda \tau}) \operatorname{ch}(\lambda) - \lambda(\beta + \gamma e^{-\lambda \tau}) \operatorname{sh}(\lambda)} + \right. \\ \left. - \frac{\int_0^1 ((1 + \alpha e^{-\lambda \tau}) + (\beta + \gamma e^{-\lambda \tau}) \operatorname{sh}(\lambda(1-y))) (f_2(y) + \lambda f_1(y)) dy}{\lambda(1 + \alpha e^{-\lambda \tau}) \operatorname{ch}(\lambda) - \lambda(\beta + \gamma e^{-\lambda \tau}) \operatorname{sh}(\lambda)} \right\} \operatorname{sh}(x) + \\ \left. \frac{1}{\lambda} \int_0^x \operatorname{sh}(\lambda(x-y)) (f_2(y) + \lambda f_1(y)) dy, \quad x \in (0, 1). \right.$$

As a result, $u \in H^2(0, 1) \cap H_\ell^1(0, 1)$ verifies (4.39)-(4.40), so we recover u and v . Next, by (4.38) we obtain z, w . Thus, we have found $V = (u, v, z, w)^T \in \mathcal{D}(\mathcal{A})$ solution of $(\lambda I - \mathcal{A})V = F$.

The operator \mathcal{A} generates a C_0 semigroup of contractions $e^{t\mathcal{A}}$ on \mathcal{H} . Owing to the Lumer-Phillips' theorem, there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of the problem (4.24). This completes the proof of Theorem 4.1. \square

5. Asymptotic behavior

In this section, we show that under condition (4.27), the semigroup $e^{t\mathcal{A}}$ decays exponentially to the null steady state. To this end, our technique is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

Theorem 5.1. *Suppose that condition (4.27) holds. Then, there exist constants $C, \omega > 0$ such that, for all $V_0 \in \mathcal{H}$, the semigroup $e^{t\mathcal{A}}$ satisfies the following estimate*

$$(5.42) \quad \|e^{t\mathcal{A}} V_0\|_{\mathcal{H}} \leq C e^{-\omega t} \|V_0\|_{\mathcal{H}}, \quad \forall t > 0.$$

Proof of theorem 5.1. We shall use the following frequency domain theorem for uniform stability from Huang-Prüss [49, 75] of a C_0 semigroup of contractions on a Hilbert space:

Lemma 5.2. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space \mathcal{H} satisfies*

$$\|e^{t\mathcal{L}} U_0\|_{\mathcal{H}} \leq C e^{-\theta t} \|U_0\|_{\mathcal{H}}$$

for some constant $C > 0$ and for $\theta > 0$ if, and only if,

$$(5.43) \quad \rho(\mathcal{L}) \supset \{i\delta \mid \delta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$(5.44) \quad \limsup_{\delta \in \mathbb{R}_+, |\delta| \rightarrow \infty} \|(i\delta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty,$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator \mathcal{L} .

First, we consider the point spectrum of \mathcal{A} .

Lemma 5.3. *The spectrum of \mathcal{A} contains no point on the imaginary axis.*

Proof. Since \mathcal{A} admits a compact resolvent, its spectrum $\sigma(\mathcal{A})$ consists only of eigenvalues of \mathcal{A} . We shall show that the equation

$$(5.45) \quad \mathcal{A}Z = i\delta Z$$

with $Z = (u, v, z, w)^T \in \mathcal{D}(\mathcal{A})$ and $\delta \in \mathbb{R}$ admits only the trivial solution.

Equation (5.45) reads as follows :

$$(5.46) \quad i\delta u - v = 0,$$

$$(5.47) \quad i\delta v - u_{xx} = 0,$$

$$(5.48) \quad i\delta z + \tau^{-1}z_\rho = 0,$$

$$(5.49) \quad i\delta w + \tau^{-1}w_\rho = 0.$$

Taking the inner product of (5.45) with Z , inequality (4.30) entails :

$$z(0) = 0, \quad w(1) = 0 \text{ and } z(1) = 0.$$

Thus, we have $z = 0$ and $w = 0$, and since $u_x(1) = w(0)$, $v(1) = z(0)$, we also obtain $u = 0$ and $v = 0$. As a consequence, the only solution of (5.45) is the trivial one. \square

The following lemma shows that (5.44) holds with $\mathcal{L} = \mathcal{A}$.

Lemma 5.4. *The resolvent operator of \mathcal{A} satisfies condition (5.44).*

Proof. Suppose that condition (5.44) is false. By the Banach-Steinhaus Theorem (see [31]), there exist a sequence of real numbers $\delta_n \rightarrow +\infty$ and a sequence of vectors $Z_n = (u_n, v_n, z_n, w_n)^t \in \mathcal{D}(\mathcal{A})$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$(5.50) \quad \|(i\delta_n I - \mathcal{A})Z_n\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

i.e.,

$$(5.51) \quad i\delta_n u_n - v_n \equiv f_n \rightarrow 0 \quad \text{in } H_\ell^1(0, 1),$$

$$(5.52) \quad i\delta_n v_n - u_n'' \equiv g_n \rightarrow 0 \quad \text{in } L^2(0, 1),$$

$$(5.53) \quad i\delta_n z_n + \frac{1}{\tau} \partial_\rho z_n \equiv h_n \rightarrow 0 \quad \text{in } L^2(0, 1),$$

$$(5.54) \quad i\delta_n w_n + \frac{1}{\tau} \partial_\rho w_n \equiv k_n \rightarrow 0 \quad \text{in } L^2(0, 1).$$

Our goal is to derive from (5.50) that $\|Z_n\|_{\mathcal{H}}$ converges to zero, in order to obtain a contradiction.

First step. Notice that we have

$$(5.55) \quad \|(i\delta_n I - \mathcal{A})Z_n\|_{\mathcal{H}} \geq |\Re(\langle (i\delta_n I - \mathcal{A})Z_n, Z_n \rangle_{\mathcal{H}})|.$$

Then, by inequality (4.30) and limit (5.50), we derive

$$(5.56) \quad z_n(1) \rightarrow 0, \quad v_n(1) = z_n(0) \rightarrow 0, \quad w_n(1) \rightarrow 0.$$

This further leads, by limit (5.51) and the trace theorem, to

$$(5.57) \quad \delta_n u_n(1) \rightarrow 0.$$

Moreover, since $Z_n \in \mathcal{D}(\mathcal{A})$, by limits (5.56) we get

$$(5.58) \quad u'_n(1) \rightarrow 0.$$

Recalling that

$$(5.59) \quad z_n(\rho) = z_n(0) e^{-i\delta_n \tau \rho} + \tau \int_0^\rho e^{-i\delta_n \tau(\rho-s)} h_n(s) ds,$$

$$(5.60) \quad w_n(\rho) = w_n(1) e^{-i\delta_n \tau(\rho-1)} - \tau \int_\rho^1 e^{-i\delta_n \tau(\rho-s)} k_n(s) ds,$$

limits (5.56) imply that

$$(5.61) \quad z_n \rightarrow 0 \text{ in } L^2(0, 1), w_n \rightarrow 0 \text{ in } L^2(0, 1).$$

Second step. Now, let us express v_n as a function of u_n from equation (5.51) and substitute it into (5.52) to get

$$(5.62) \quad -\delta_n^2 u_n - u''_n = i\delta_n f_n + g_n.$$

Next, we take the inner product of the above equation (5.62) with xu' in $L^2(0, 1)$, we obtain that

$$(5.63) \quad \int_0^1 (-\delta_n^2 u_n(x) - u''_n(x)) x u'(x) dx = \int_0^1 (i\delta_n f_n(x) + g_n(x)) x u'(x) dx = \\ \int_0^1 g_n(x) x u'_n(x) dx - \int_0^1 i\delta_n f'_n(x) x u_n(x) dx - \int_0^1 i\delta_n f_n(x) u_n(x) dx + i\delta_n u_n(1) f_n(1).$$

It is clear that the right-hand side of equality (5.63) converges to zero since f_n, g_n converge to zero in H^1 and L^2 , respectively. By a straightforward calculation, we deduce that

$$\int_0^1 -\delta_n^2 u_n(x) x u'_n(x) dx = \frac{1}{2} \int_0^1 \delta_n^2 u_n^2(x) dx - \frac{\delta_n^2}{2} u_n^2(1)$$

and

$$\int_0^1 -u''_n(x) x u'_n(x) dx = \frac{1}{2} \int_0^1 (u'_n)^2(x) dx - \frac{(u'_n(1))^2}{2}.$$

This leads to

$$\int_0^1 \delta_n^2 u_n^2(x) dx + \int_0^1 (u'_n)^2(x) dx - \delta_n^2 u_n^2(1) - (u'_n(1))^2 \longrightarrow 0.$$

So, limits (5.57), (5.58) and (5.61), obviously contradict $\|Z_n\|_{\mathcal{A}} = 1$. \square

The two hypotheses of Lemma 5.2 are proved by Lemma 5.4. Then inequality (5.42) holds which ends the proof of Theorem 5.1. \square

6. A spectral formulation via D'Alembert's formula

In order to express the problem into a form more suited for control purposes with a spectral approach, the wave equation in system (3.10) is now considered through the D'Alembert formula which allows the solution of the wave equation to be written as:

$$(6.64) \quad u(x, t) = \frac{1}{2} (u_0(x+t) + u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} u_1(\xi) d\xi.$$

Introducing the variable θ as follows:

$$(6.65) \quad \begin{cases} \theta(x) = -\frac{1}{2} u_0(-x) + \frac{1}{2} \int_0^{-x} u_1(s) ds & \text{for } x \in (-1, 0), \\ \theta(x) = \frac{1}{2} u_0(x) + \frac{1}{2} \int_0^x u_1(s) ds & \text{for } x \in (0, 1), \end{cases}$$

enables express u in terms of θ :

$$(6.66) \quad u(x, t) = \theta(x+t) - \theta(t-x),$$

so that

$$(6.67) \quad u_t(x, t) = \theta'(x+t) - \theta'(-x+t),$$

$$(6.68) \quad u_x(x, t) = \theta'(x+t) + \theta'(-x+t).$$

Thus, the control function on the boundary $x = 1$, say $v(t) := u_x(1, t)$, is then given by

$$(6.69) \quad v(t) = \theta'(1+t) + \theta'(-1+t).$$

By defining $\eta := 1+t$, one gets:

$$(6.70) \quad v(\eta-1) = \theta'(\eta) + \theta'(\eta-2),$$

so that

$$(6.71) \quad v(\eta-1-\tau) = \theta'(\eta-\tau) + \theta'(\eta-2-\tau),$$

Furthermore, the feedback control law's expression in (3.10) becomes:

$$(6.72) \quad v(\eta-1) + \alpha v(\eta-1-\tau) = \beta (\theta'(\eta) - \theta'(\eta-2)) + \gamma (\theta'(\eta-\tau) - \theta'(\eta-2-\tau)).$$

Hence, one can consider the expression of v in equations (6.70) and (6.71) and substitute them into equation (6.72). As a consequence, in the case where $\beta \neq 1$, we get

$$(6.73) \quad \theta'(\eta) + \frac{1+\beta}{1-\beta} \theta'(\eta-2) + \frac{\alpha-\gamma}{1-\beta} \theta'(\eta-\tau) + \frac{\alpha+\gamma}{1-\beta} \theta'(\eta-2-\tau) = 0.$$

which is a continuous time-difference equation that is exclusively expressed in terms of $\theta'(x)$.

Remark 6.1. Note that the case $\beta = 1$ is not consistent with the dissipativity conditions established in (4.27) in Section 4. Therefore, $\beta \neq 1$ is the only case considered in the sequel.

7. A parametric stability analysis via a spectral approach

Let $s \in \mathbb{C}$ be the Laplace variable of the Laplace Transform operator and the delay such that $\tau \in \mathbb{R}_+^*$. Applying the Laplace transform to equation (6.73), one gets the following characteristic function:

$$(7.74) \quad Q(s, \tau) := 1 + \frac{1+\beta}{1-\beta} e^{-2s} + \frac{\alpha-\gamma}{1-\beta} e^{-\tau s} + \frac{\alpha+\gamma}{1-\beta} e^{-(\tau+2)s}.$$

The exponential stability of the closed-loop system in (6.73) can be achieved if, and only if, the roots of (7.74) have a strictly negative real part. Moreover, a decay-rate of σ , with $\sigma > 0$, for $\theta'(\eta)$ converging to zero can be obtained if, and only if, $-\sigma$ is an upper-bound on the real part of any root of (7.74). In fact, the well-known Hale-Silkowski criterion completely characterizes the exponential stability of (6.73), see for instance [46, Chapter 9, Theorem 6.1] and for further refinement and generalization of the above result see [35]. As a matter of fact, since the three involved delays in equation (6.73) (2, τ , and $\tau+2$) are rationally dependent, then one can transform (6.73) into an equivalent matrix equation involving only two delays 2 and τ . By denoting

$$(7.75) \quad f(t) := \begin{pmatrix} \theta'(t) \\ \theta'(t-2) \end{pmatrix}, \quad A := \begin{pmatrix} \frac{1+\beta}{\beta-1} & 0 \\ \frac{\alpha+\gamma}{\beta-1} & 0 \end{pmatrix}, \quad B := \begin{pmatrix} \frac{\alpha-\gamma}{\beta-1} & \frac{\alpha+\gamma}{\beta-1} \\ 0 & 0 \end{pmatrix},$$

equation (6.73) is equivalent to the system:

$$(7.76) \quad f(t) = A f(t-2) + B f(t-\tau).$$

Furthermore, if the delays τ and 2 are rationally independent, then using [45, Chapter 9, Theorem 6.1] one concludes that (7.76) is stable locally in the delays if, and only if,

$$\sup_{\varphi_1, \varphi_2 \in [0, 2\pi]} \mu(A e^{i\varphi_1} + B e^{i\varphi_2}) < 1,$$

where $\mu(\cdot)$ designates the spectral radius of a given square matrix.

This entails the following necessary and sufficient stability conditions:

$$\begin{cases} 1 + \frac{1+\beta}{\beta-1} > \left| \frac{2\alpha}{\beta-1} \right|, \\ 1 - \frac{1+\beta}{\beta-1} > \left| \frac{2\gamma}{\beta-1} \right|, \end{cases}$$

which reduce for $\beta < 0$ to the following set of necessary and sufficient conditions:

$$\begin{cases} |\gamma| < 1, \\ |\alpha| < -\beta. \end{cases}$$

However, if the delays τ and 2 are rationally dependent, then one can further transform (7.76) into an equivalent augmented matrix equation and again using [45, Chapter 9, Theorem 6.1] one can obtain explicit necessary and sufficient conditions.

Despite the importance of the result recalled above from a qualitative point of view, it does not unfortunately provide any information on the solution decay rate.

In the following section, we shall exploit the manifold of spectral values' multiplicities to get some insights on the solutions' decay rates.

7.1. On the MID paradigm . Notice that, in the case of a quasipolynomial with more than one exponential term, such as the one under consideration in (7.74), the characterization of the MID property remains an open problem, to which we partially contribute in the present work.

As emphasized in Section 2, the MID property consists in conditions under which a multiple spectral value is dominant. More precisely, in this section, we shall provide some configurations in which the GMID applies; this corresponds to the dominance of spectral values with a multiplicity which is equal to the degree of the considered quasipolynomial. Notice that such a degree may vary when some coefficients are set to be zero or when some delays are set to be equal. In particular, the case $\tau = 2$ will be considered separately since it allows to decrease the degree of the quasipolynomial Q .

In this case, the quasipolynomial Q reads

$$(7.77) \quad Q(s, 2) = 1 + \frac{(1 + \beta + \alpha - \gamma)e^{-2s}}{1 - \beta} + \frac{(\alpha + \gamma)e^{-4s}}{1 - \beta}$$

which admits a degree equal to two for $\gamma \neq \alpha$ and $\gamma \neq \alpha + \beta + 1$. We are able to enunciate the following:

Theorem 7.1. *Consider the quasipolynomial $Q(\cdot, \tau)$ given by (7.74) and let $\tau = 2$.*

A given real number s_0 is a double root of (7.77) if, and only if,

$$(7.78) \quad \begin{cases} \alpha = \frac{\gamma(2e^{-2s_0} - e^{-4s_0} - 1) + 2}{e^{-4s_0} - 2e^{-2s_0} - 1}, \\ \beta = \frac{(2\gamma - 1)e^{-4s_0} - 2e^{-2s_0} - 1}{e^{-4s_0} - 2e^{-2s_0} - 1}. \end{cases}$$

If (7.78) is satisfied then the GMID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot, 2)$ given by (7.77). Furthermore, all zeros of (7.77) are double and lie on the vertical axis $\Re(s) = s_0$.

Proof. Since the degree of the quasipolynomial (7.77) is equal to 2, let us investigate its admissible double roots. Let $s_0 \in \mathbb{R}$ be a double root of (7.77); that is $Q(s_0, 2) = Q_s(s_0, 2) = 0$. Thanks to the linearity of the two equations obtained in the parameters α and β , one easily deduces conditions (7.78). By substituting the resulting conditions into the expression of $Q(\cdot, 2)$ one gets for arbitrary γ :

$$Q(s, 2) = \left(e^{-2(s-s_0)} - 1 \right)^2,$$

which means that each root of $Q(\cdot, 2)$ is necessarily double. Furthermore, let $s = x + i\omega \neq s_0$ be a root of $Q(\cdot, 2)$ then x and ω satisfy:

$$\begin{cases} e^{2(x-s_0)} \sin(2\omega) = 0 \\ e^{-2(x-s_0)} \cos(2\omega) = 1. \end{cases}$$

From the first equation one gets that $\omega = \frac{k}{2}\pi$ with $k \in \mathbb{Z}^*$ and the second equation yields $x = s_0$, which concludes the proof. \square

Now, let us consider again the quasipolynomial (7.74) where $\alpha \neq |\gamma|$ and $|\beta| \neq 1$, *i.e.*, the case where the quasipolynomial's degree is equal to three.

Theorem 7.2. *Consider the quasipolynomial $Q(\cdot, \tau)$ given by (7.74) and let $\tau \neq 2$.*

A given real number s_0 is a triple root of (7.74) if, and only if,

$$(7.79) \quad \begin{cases} \alpha = \frac{(\tau-2)e^{2s_0} - (\tau+2)}{(\tau+2)e^{2s_0} - (\tau-2)} e^{\tau s_0}, \\ \beta = \frac{(\tau+2)e^{2s_0} + (\tau-2)}{(\tau+2)e^{2s_0} - (\tau-2)}, \\ \gamma = \frac{(\tau-2)e^{2s_0} + (\tau+2)}{(\tau+2)e^{2s_0} - (\tau-2)} e^{\tau s_0}. \end{cases}$$

If (7.79) is satisfied and τ is an even integer strictly greater than 2, then the GMID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot, 2k)$ given by (7.74).

To prove the above Theorem, one needs the following two results: the first one which is a classical Theorem of A. Cohn [36] and the second is due to Eneström-Kakeya [51, 43]; see also [41, 61, 55, 34] for further insights on polynomials with all zeros on the unit circle.

Theorem 7.3 ([36]). *All zeros of the polynomial*

$$P(s) = \sum_{k=0}^n a_k s^k \in \mathbb{C}[s]$$

of degree n lie on the unit circle if, and only if,

i) *P is self-inversive, *i.e.**

$$(7.80) \quad a_{n-k} = \epsilon \bar{a}_k \quad (k = 0, \dots, n) \quad \text{where } \epsilon \in \mathbb{C}, \quad |\epsilon| = 1;$$

ii) *all zeros of P' lie inside or on this circle.*

If the coefficients a_k ($k = 0, \dots, n$) are real, then in equalities (7.80), the coefficient ϵ should also be real, thus these polynomials are self-inversive if, and only if,

j) *either $\epsilon = 1$ and $a_{n-k} = a_k$ ($k = 0, \dots, n$) *i.e.* P is reciprocal,*

jj) *or $\epsilon = -1$ and $a_{n-k} = -a_k$ ($k = 0, \dots, n$) *i.e.* P is anti-reciprocal.*

Theorem 7.4 ([51, 43]). *The absolute value of zeros of the polynomial*

$$P(s) = \sum_{k=0}^n a_k s^k \in \mathbb{R}[s]$$

whose coefficients are positive, lie between the greatest and the least of the n quotients

$$\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_{n-1}}, \dots, \frac{a_1}{a_2}, \frac{a_0}{a_1}.$$

Proof. The first claim is shown in the same manner as the proof of Theorem 7.1. Next, let us introduce the change of variable $e^{-2(s-s_0)} = X$. One can show that for any $\tau = 2k$ where k is an integer greater than 2, one has $Q(s, 2k) = R_k(X)$ with:

$$R_k(X) = \sum_{l=0}^{k+1} a_k X^k = -X^{k+1} + \frac{k+1}{k-1} X^k - \frac{k+1}{k-1} X + 1.$$

Interestingly, the R_k coefficients satisfy: $a_{k+1-l} = -a_l$ for $l = 0, \dots, k+1$, which entails that R_k is *self-inversive*, anti-reciprocal and skew-adjoint polynomial since $R_k(X) = -X^{k+1} R_k(\frac{1}{X})$. Notice that $X = 1$ is a triple root of R_k .

Furthermore, the first derivative of R_k can be written:

$$R'_k(X) = -X^k(k+1) + \frac{(k+1)X^{k-1}k}{k-1} - \frac{k+1}{k-1},$$

for which $X = 1$ is a double root. Let us investigate the zeros of R'_k or equivalently the following polynomial:

$$\frac{1-k}{k+1} R'_k(X) = (k-1)X^k - kX^{k-1} + 1$$

One easily shows by recurrence that:

$$\frac{1-k}{k+1} R'_k(X) = (X-1)^2 \sum_{l=0}^{k-2} (l+1) X^l.$$

Hence, Theorem 7.4 immediately incurs that beyond the double root at $X = 1$ the roots of R'_k satisfy

$$\frac{1}{2} \leq |X| \leq \frac{k-2}{k-1} < 1.$$

Finally, since R_k is self-inversive and all zeros of R'_k lie inside or on the unit circle, then Theorem 7.3 asserts that all roots of R_k lie on the unit circle, which concludes the proof. \square

Remark 7.5. *It should be pointed out that apart from delay values $\tau = 2k$ the GMID property holds for $\tau = 1$ as it will be shown in the sequel. As a matter of fact, if (7.79) is satisfied and $\tau = 1$, then one is in the following particular case:*

$$Q(s, 1) = -\left(e^{-(s-s_0)} - 1\right)^3,$$

showing that the GMID obviously holds, i.e. s_0 corresponds to the spectral abscissa of the above quasipolynomial.

Remark 7.6. *From a control theory viewpoint, the MID property can be exploited by tuning the control parameters as emphasized above after prescribing a negative number s_0 which corresponds to the closed-loop system solution's decay rate.*

When the MID property fails, one can prescribe a lower bound for the decay rate as will be discussed in the next section.

7.2. Beyond the MID property in output feedback stabilization of the wave equation in (3.10) . By substituting the expressions of α, β and γ given in system (7.79) into the expression of Q in expression (7.74) and by introducing the variable change

$$(7.81) \quad p := s - s_0,$$

it comes

$$(7.82) \quad \tilde{Q}(p, \tau) := 1 - \frac{\tau+2}{\tau-2} e^{-2p} + \frac{\tau+2}{\tau-2} e^{-\tau p} - e^{-(\tau+2)p},$$

and we have now to examine the roots of \tilde{Q} with respect to τ . It is well-known that zeros of quasipolynomials with real coefficients are symmetric with respect to the real axis, that is, zeros are either real or appear in complex conjugate pairs. The following lemma underlines an additional symmetry structuring the distribution of zeros of \tilde{Q} with respect to the imaginary axis, that is a central symmetry.

Lemma 7.7. *Let $p \in \mathbb{C}$ be a zero of \tilde{Q} defined by expression (7.82). Then, $-p$ is also a zero of \tilde{Q} .*

Proof. Let us evaluate \tilde{Q} at $-p$ where p is a zero of \tilde{Q} .

$$(7.83) \quad \tilde{Q}(-p, \tau) = 1 - \frac{\tau+2}{\tau-2} e^{2p} + \frac{\tau+2}{\tau-2} e^{\tau p} - e^{(\tau+2)p}$$

Multiplying both sides by $e^{-(\tau+2)p}$, one gets

$$(7.84) \quad e^{-(\tau+2)p} \tilde{Q}(-p, \tau) = e^{-(\tau+2)p} - \frac{\tau+2}{\tau-2} e^{-\tau p} + \frac{\tau+2}{\tau-2} e^{-2p} - 1$$

which yields $e^{-(\tau+2)p} \tilde{Q}(-p, \tau) = -\tilde{Q}(p, \tau)$. Since for all $p \in \mathbb{C}$ one has $e^{-(\tau+2)p} \neq 0$, then necessarily $\tilde{Q}(-p, \tau) = 0$. \square

In order to locate the zeros of (7.82), we require the following settings and results from [48], see also [14] for further refinements. Consider the quasipolynomial

$$(7.85) \quad \Theta(p, \kappa, h) := \sum_{k=0}^N \kappa_k e^{-p \chi_k \cdot h}$$

where $\kappa = (\kappa_1, \dots, \kappa_N)^T \in \mathbb{R}^N$, $h = (h_1, \dots, h_M)^T \in \mathbb{R}_+^M$, $\chi_j = (\chi_{j,1}, \dots, \chi_{j,M})$, $\chi_{j,k} \in \mathbb{N}^*$ ($j \in \llbracket 1, N \rrbracket$, $k \in \llbracket 1, M \rrbracket$) and $\chi_j \cdot h = \sum_{k=1}^M \chi_{j,k} h_k$. We also adopt the notations $\kappa_0 = 1$ and $\chi_0 = (0, \dots, 0)$. Define $Z_\Theta(\kappa, h) := \{\Re(p) : \Theta(p, \kappa, h) = 0\}$ and denote its closure by $\bar{Z}_\Theta(\kappa, h)$.

Lemma 7.8 ([48]). *If the equation $\Theta(x + i\omega, \kappa, h) = 0$ is satisfied for some reals x and ω , then the lengths $\{|\kappa_j| e^{-x \chi_j \cdot h}, j \in \llbracket 0, N \rrbracket\}$ can form a closed polygon; that is, none of these lengths is larger than the sum of the others: $|\kappa_j| e^{-x \chi_j \cdot h} \leq \sum_{k \neq j} |\kappa_k| e^{-x \chi_k \cdot h}$ for $j \in \llbracket 0, N \rrbracket$.*

Also, following Henry [48], let us define $\rho_j = \rho_j(\kappa, h)$ ($j \in \llbracket 0, N \rrbracket$), if they exist, by the relation

$$(7.86) \quad |\kappa_j| e^{-\rho_j \chi_j \cdot h} = \sum_{k \neq j} |\kappa_k| e^{-\rho_j \chi_k \cdot h} \quad \text{for } j \in \llbracket 0, N \rrbracket.$$

If $\chi_N \cdot h \geq \chi_j \cdot h > 0$ for $j \in \llbracket 1, N-1 \rrbracket$, then ρ_N and ρ_0 are uniquely defined and $\rho_N < \rho_0$ for $N \geq 2$.

Lemma 7.9 ([48]). *If $\chi_N.h \geq \chi_{N-1}.h > \dots > \chi_1.h > 0$, then*

$$(7.87) \quad \bar{Z}_\Theta(\kappa, h) \subseteq [\rho_N, \rho_0].$$

The following lemma provides a vertical strip in the complex plane, which is symmetric with respect to the imaginary axis and contains the set of zeros of \bar{Q} :

Lemma 7.10.

$$(7.88) \quad \bar{Z}_{\bar{Q}}(\kappa, h) \subseteq [-\rho^*, \rho^*],$$

where ρ^* is the unique positive zero of

$$(7.89) \quad \hat{Q}(\rho, \tau) := 1 - \left| \frac{\tau+2}{\tau-2} \right| e^{-2\rho} - \left| \frac{\tau+2}{\tau-2} \right| e^{-\tau\rho} - e^{-(\tau+2)\rho}.$$

Proof. The symmetry of $\bar{Z}_{\bar{Q}}(\kappa, h)$ with respect to the imaginary axis is a direct consequence of Lemma 7.9 and Lemma 7.7. Furthermore, Lemma 7.9 asserts that, if it exists, ρ^* is a positive zero of \hat{Q} given by (7.89). Finally, independently from the sign of $\tau-2$, using simple real analysis arguments, one easily checks that $\hat{Q}(0, \tau) < 0$ and $\lim_{\rho \rightarrow \infty} \hat{Q}(\rho, \tau) = 1$ for all $\tau \in \mathbb{R}_+ \setminus \{2\}$ and shows by computing \hat{Q}_ρ , the first derivative of \hat{Q} with respect to ρ , that \hat{Q} is strictly increasing. Hence, ρ^* exists as a function of τ and it is unique. \square

Now, ρ^* is a root of (7.89) if, and only if,

$$(7.90) \quad e^{-(\tau+2)\rho^*} = 1 - \left| \frac{\tau+2}{\tau-2} \right| e^{-2\rho^*} - \left| \frac{\tau+2}{\tau-2} \right| e^{-\tau\rho^*}$$

Substituting the above expression into \hat{Q}_ρ , the first derivative of \hat{Q} with respect to ρ , and evaluating the obtained expression at ρ^* entails:

$$\hat{Q}_\rho(\rho^*, \tau) = - \frac{(\tau+2)(e^{-2\rho^*} \tau - |\tau-2| + 2e^{-\tau\rho^*})}{|\tau-2|}.$$

which never vanish for any $\tau \in \mathbb{R}_+ \setminus \{2\}$. Indeed, for $\tau < 2$, \hat{Q}_ρ is of constant sign and strictly decreasing with respect to ρ^* . In addition, if we assume that \hat{Q}_ρ vanishes at ρ^* for $\tau > 2$, we get

$$(7.91) \quad \tau = \frac{-2e^{-\tau\rho^*} - 2}{e^{-2\rho^*} - 1}.$$

However, by eliminating $e^{-\tau\rho^*}$ from the above expression of \hat{Q}_ρ and substituting it into the expression of \hat{Q} , we obtain

$$\hat{Q}(\rho^*, \tau) = \frac{(\tau^2 - 4)e^{-2\rho^*}}{2|\tau-2|} - e^{-(\tau+2)\rho^*} - \frac{\tau}{2} = 0,$$

that is,

$$(7.92) \quad \tau = \frac{2e^{-(\tau+2)\rho^*} - 2e^{-2\rho^*}}{e^{-2\rho^*} - 1},$$

which is inconsistent with (7.91) and the fact that $\rho^* > 0$. Consequently, the *Implicit Function Theorem* is then applicable to (7.89) and asserts that $\rho^* = \rho^*(\tau)$ with

$$\rho^{*'}(\tau) = \frac{(4 + (-\tau^2 + 4)\rho^*(\tau))e^{-2\rho^*(\tau)} + 4e^{-\tau\rho^*(\tau)} + \rho^*(\tau)|\tau - 2|(\tau - 2)}{(e^{-2\rho^*(\tau)}\tau - |\tau - 2| + 2e^{-\tau\rho^*(\tau)})(\tau^2 - 4)}.$$

Lemma 7.11. *Consider the quasipolynomial \hat{Q} given by (7.89) with $\tau \neq 2$. Then the spectral abscissa σ of \hat{Q} is lower-bounded by $s_0 + \hat{\rho}(\tau)$ where $\hat{\rho}$ is given by*

$$(7.93) \quad \hat{\rho}(\tau) := \frac{1}{\min\{\tau, 2\}} \ln \left(1 + 2 \frac{\tau + 2}{|\tau - 2|} \right).$$

Proof. It is easy to observe that for $\rho > 0$ and $\tau \neq 2$

$$\hat{Q}(\rho, \tau) \geq 1 - \left(1 + 2 \frac{\tau + 2}{|\tau - 2|} \right) e^{-\tau\rho}.$$

We remark that the right-hand side of this last inequality admits a single root at $\hat{\rho}$ given by (7.93). In conclusion, for any $\tau \neq 2$, one has $\hat{Q}(\hat{\rho}(\tau), \tau) > 0$, which asserts that $\rho^*(\tau) \leq \hat{\rho}(\tau)$ from the *Intermediate Value Theorem*. \square

Thanks to the above results and lemmas, the proof of the following theorem, which gives a certified decay rate's lower-bound for the closed-loop system's solution, is immediate.

Theorem 7.12. *Consider the output feedback stabilization of the wave equation in (3.10) with an arbitrary positive delay $\tau > 0$ then the following assertions hold:*

- *If $\tau = 2$, then the control parameter tuning prescribed in system (7.78) allows to assign the solution's exponential decay rate at an arbitrary $-s_0$;*
- *If $\tau = 2k$ where k is an integer greater than one, then the control parameter tuning prescribed in system (7.79) allows to assign the solution's exponential decay rate at an arbitrary $-s_0$;*
- *If $\tau \neq 2k$, then the control parameter tuning prescribed in system (7.79) allows a closed-loop solution decaying exponentially faster than $-s_0 - \hat{\rho}(\tau)$, where $\hat{\rho}$ is defined by expression (7.93).*

Proof. The D'Alembert transformation of the wave equation (3.10) turns it into the difference equation (6.73). Next, applying the Laplace transform in the frequency-domain to the latter, yields the characteristic quasipolynomial function (7.74). Finally, using the normalization (7.81), we end up with expression (7.82). The first assertion is a direct consequence of Theorem 7.1. The second assertion is a direct consequence of Theorem 7.2. The third assertion follows directly from Lemma 7.11. \square

Remark 7.13. *Figure 2 shows the locus of $\hat{\rho}$ given by (7.93) (the proposed upper-bound on the real parts of the zeros of the quasipolynomial \hat{Q}) as a function of the delay τ . Thanks to the linear change of variables (7.81), this enables the selection of an appropriate pair (s_0, τ) in the filled gray region, providing an upper-bound on the spectral abscissa of the quasipolynomial Q given in (7.74). As asserted in Theorem 7.12, the desired decay rate towards the steady state equilibrium is greater than*

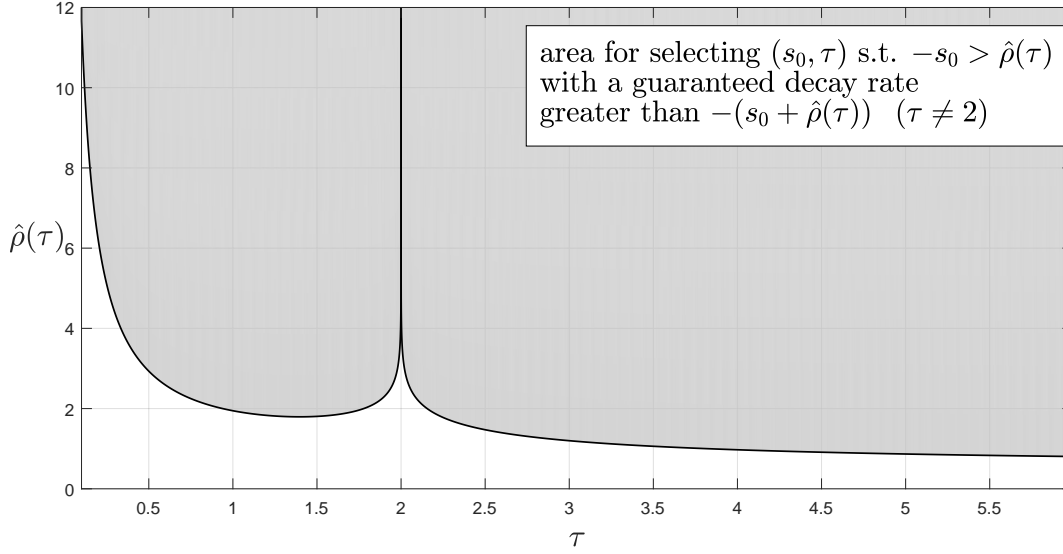


FIGURE 2. The behavior of $\hat{\rho}$ given by expression (7.93) as a function of the delay τ .

$|s_0 + \hat{\rho}(\tau)| = -(s_0 + \hat{\rho}(\tau))$, since we have proven that any root $s_i \in \mathbb{C}$ of equation (7.74) ($i \in \mathbb{N}$) is such that $\Re(s_i) < s_0 + \rho^* < s_0 + \hat{\rho}(\tau)$. By prescribing a minimal decay rate $\sigma > 0$, s_0 is chosen such that $s_0 + \hat{\rho}(\tau) < -\sigma < 0$ to ensure the asymptotic stability, i.e., $s_0 < -\sigma - \hat{\rho}(\tau)$.

8. Application: Longitudinal vibration control of a long drill pipe transmission line

Drilling is an important operation for the production of oil and gas. It is a key process in this production and exploration industry. According to us, a phenomenon that can cause poor operational performances, and even develop into the catastrophic failure of the well, is the presence of vibrations in the drill-string, which is mainly composed of long pipes, due to torsional, axial and also lateral vibrations. This is a spatially distributed system that is modeled fundamentally by PDEs. Such a system represents a very challenging modeling and control problem, see for instance [62, 77] and references therein.

In this section, we propose to illustrate the main result of this work through the example of the longitudinal vibrations of a long steel drill pipe, for which we aim to reduce the vibrations thanks to the QPB controller set as above.

8.1. Description of the problem. Figure 3 shows a sketch of the considered drill pipe, clamped at one end and controlled by an axial force at the free-end. Such a situation can occur when the end of the drill-string is blocked at the bottom of the well. The drill pipe is submitted to an axial force $d(t)$ located at position l , which disturbs it and provokes axial vibrations.

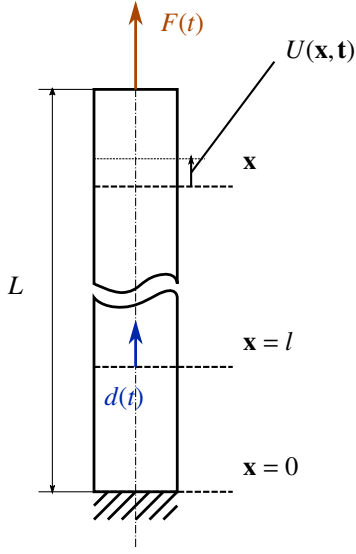


FIGURE 3

Young modulus (N/m^2)	E	$200 \cdot 10^9$
Steel density (kg/m^3)	ρ	7800
Length (m)	L	3000
Cross-sectional area - cylindrical pipe (m^2) (with outer & inner diameters: $D_o = 0.127$ (m), $D_i = 0.108$ (m))	A	$3.5068 \cdot 10^{-3}$
Wave propagation speed (m/s)	c	5063.7
Time scaling factor (s)	d	0.59245
Disturbance location (m)	l	1000

TABLE 1. Material and geometrical parameters of the pipe.

The parameters of the considered pipe are given in Table 1. The pipe is assumed to be cylindrical, steel based, and homogeneous, *i.e.*, the density ρ and Young modulus E are identically distributed along the pipe's length. Let us denote by $U(\mathbf{x}, \mathbf{t})$ the longitudinal displacement of a point (cross-section) of the pipe located at position \mathbf{x} and given at time \mathbf{t} . The wave equation that describes the longitudinal vibrations of the pipe is given by

$$(8.94) \quad U_{\mathbf{t}\mathbf{t}}(\mathbf{x}, \mathbf{t}) = c^2 U_{\mathbf{x}\mathbf{x}}(\mathbf{x}, \mathbf{t}),$$

for $\mathbf{x} \in (0, L)$, where c is the wave propagation speed given by the relation $c := \sqrt{\frac{E}{\rho}}$. The long pipe is fixed at one end, which is expressed by the boundary condition

$$(8.95) \quad U(0, \mathbf{t}) = 0.$$

The control is made via the free end by exerting a force $F(t)$ that is related to the strain $U_x(L, \mathbf{t})$ by the relation $F(t) = EA U_x(L, \mathbf{t})$. For $x \in (0, L)$, the initial conditions are $U(\mathbf{x}, 0) = 0$, $U_{\mathbf{t}}(\mathbf{x}, 0) = 0$. Finally, for $\mathbf{x} > 0$, the denormalized control law in equation (3.10) is

$$(8.96) \quad U_x(L, \mathbf{t}) = -\alpha U_x(L, \mathbf{t} - d\tau) + \frac{\beta}{c} U_{\mathbf{t}}(L, \mathbf{t}) + \frac{\gamma}{c} U_{\mathbf{t}}(L, \mathbf{t} - d\tau),$$

where $d := \frac{L}{c}$ is the time scaling factor weighting the normalized time variable \mathbf{t} to give the true time variable \mathbf{t} . This control law is assumed to be initiated by $U_{\mathbf{t}}(L, \mathbf{t} - d\tau) = 0$, $U_x(L, \mathbf{t} - d\tau) = 0$ for $\mathbf{t} \in (0, d\tau)$.

Remark 8.1. *The wave equation problem modeled by equation (3.10) has used some normalized (i.e., dimensionless) time t and position x variables, in such a way that their corresponding relations to the considered true variables are $\mathbf{x} = Lx$ and $\mathbf{t} = dt$. In this case, the normalized longitudinal displacement*

is related to the true one by the relation

$$(8.97) \quad \begin{aligned} u(x, t) &:= U(\mathbf{x}, \mathbf{t}), \\ &= U(Lx, dt). \end{aligned}$$

To perform numerical simulations of the closed-loop system (3.10), we propose to adopt the mechanical engineering approach which consists in handling numerically the system's dynamics separately from the control law's dynamics. Both of them are then combined once the numerical schemes are established. Concerning the wave equation, there are usually two numerical methods: the finite difference and the finite element methods. In this section, to address this issue, we propose to use the Finite Element Method in order to approximate the wave equation's PDE, because this approach is nowadays very well mastered by scientists; several books are devoted to this method [78, 50, 52]. Moreover, this approach requires a lower amount of computations than the one required by the finite difference approach for a given computation precision.

8.2. Finite element modeling.

The long pipe of Figure 3 is divided into $N \in \mathbb{N}^*$ basic parts called *elements*, as in Figure 4. This step is called the meshing of the structure. Each element connects two consecutive points of the meshing that are the *nodes*, with a total of $N + 1$ nodes. A basic element e for the wave equation case is well described by a *truss* element, which is assumed to carry only tension or compression loads, resulting in axial displacements only. In this case, the displacement field within an element e is given, in the element's local coordinate $\tilde{x} \in [0, l_{e_i}]$, by

$$U_i(\tilde{x}, t) = \begin{pmatrix} 1 - \xi_i(\tilde{x}) & \xi_i(\tilde{x}) \end{pmatrix} \begin{pmatrix} U_{i_1}(t) \\ U_{i_2}(t) \end{pmatrix} =: S_i(\tilde{x}) q_i^e(t),$$

where $\xi_i(\tilde{x}) := \tilde{x}/l_{e_i}$ is the normalized space variable inside the element i (connecting the nodes n_i and n_{i+1}), $S_i(\tilde{x})$ is the consistent interpolating (shape) function, $q_i^e(t) \in \mathbb{R}^2$ is the nodal displacement vector of the i^{th} element and $U_{i_1}(t), U_{i_2}(t) \in \mathbb{R}$ are the i^{th} element's nodal coordinates corresponding to its nodes' displacements.

The next step consists in computing the kinetic T_i and strain U_i energies at time t within a given element i , which are defined by:

$$\begin{aligned} T_i(t) &:= \frac{1}{2} \int_0^{l_{e_i}} \rho (\partial_t U_i(\tilde{x}, t))^T (\partial_t U_i(\tilde{x}, t)) A d\tilde{x}, \\ U_i(t) &:= \frac{1}{2} \int_0^{l_{e_i}} E (\partial_{\tilde{x}} U_i(\tilde{x}, t))^T (\partial_{\tilde{x}} U_i(\tilde{x}, t)) A d\tilde{x}. \end{aligned}$$

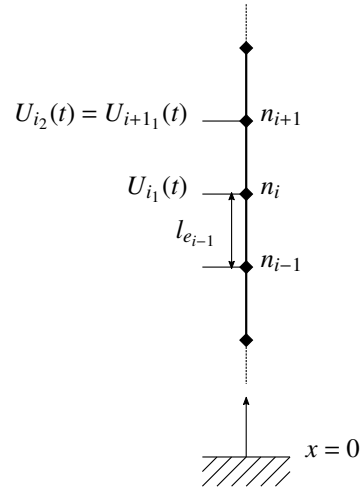


FIGURE 4

Once computed, this yields

$$\begin{aligned} T_i(t) &= \frac{1}{2} \dot{q}_i^e(t)^T M_i \dot{q}_i^e(t), \\ U_i(t) &= \frac{1}{2} q_i^e(t)^T K_i q_i^e(t), \end{aligned}$$

where $\dot{q}_i^e(t)$ is the time derivative of $q_i^e(t)$, M_i is the element mass matrix given by

$$M_i = \frac{m_{e_i}}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

where $m_{e_i} := \rho A l_{e_i}$ is the element mass, and K_i is the element stiffness matrix given by

$$K_i = k_i \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where $k_i := EA/l_{e_i}$ is the element stiffness. Lastly, the virtual work of an axial force $F_i(t)$ acting on the i^{th} element at a point located at the element's local coordinate \tilde{x} is given by $W_i := F_i(t) U_i(\tilde{x}, t) = F_i(t) S_i(\tilde{x}) q_i^e(t)$. The whole structure's meshing is now considered by computing the global kinetic and strain energies, as well as the global virtual work (the assembling step):

$$(8.98) \quad T := \sum_{i=1}^N T_i, \quad U := \sum_{i=1}^N U_i, \quad W := \sum_{i=1}^N W_i,$$

where N is the number of elements. Let us define now the *Generalized Coordinates* displacement vector $q(t) := \left(q_{N+1}(t) \quad q_N(t) \quad \cdots \quad q_2(t) \quad q_1(t) \right)^T \in \mathbb{R}^{N+1}$, where $q_i(t)$ denotes the i^{th} node's axial displacement which is, according to the notations of Figure 4, equal to $U_{i-1_2}(t) \equiv U_{i_1}(t)$. This last relation transcribes the connectivity of elements $i-1$ and i . By establishing the relation between $q_i^e(t)$ and $q(t)$, $q_i^e(t) = B_i q(t)$ where B_i is an appropriate matrix derived from the previously defined element connectivity relations ($B_i \in \mathbb{R}^{2 \times (N+1)}$), one can deduce from relations (8.98) the matrix motion equation given by

$$(8.99) \quad M \ddot{q}(t) + K q(t) = \mathcal{F}(t),$$

where $M := \sum_{i=1}^N B_i^T M_i B_i$ ($M \in \mathbb{R}^{(N+1) \times (N+1)}$) is the structure's mass matrix, $K := \sum_{i=1}^N B_i^T K_i B_i$ ($K \in \mathbb{R}^{(N+1) \times (N+1)}$) is the structure's stiffness matrix, and $\mathcal{F}(t) \in \mathbb{R}^{(N+1) \times (N)}$ is the Generalized Forces' vector given by $\mathcal{F}(t) := \sum_{i=1}^N B_i^T S_i(\tilde{x})^T F_i(t)$, assuming only one axial force $F_i(t)$ applied to the i^{th} element at the local position \tilde{x} . In our case, the boundary conditions are taken into account by letting first $q_1(t) \equiv 0$ (the fixed end at $\mathbf{x} = 0$). Moreover, the control force $F(t)$ applied at $\mathbf{x} = L$, *i.e.*, the N^{th} element, is described by the generalized force vector $e_1 F(t)$, where e_{N+1} is the first column vector of the identity matrix of size $N+1$. The disturbance force $d(t)$ can be addressed in the same way once the exact drill pipe meshing is performed.

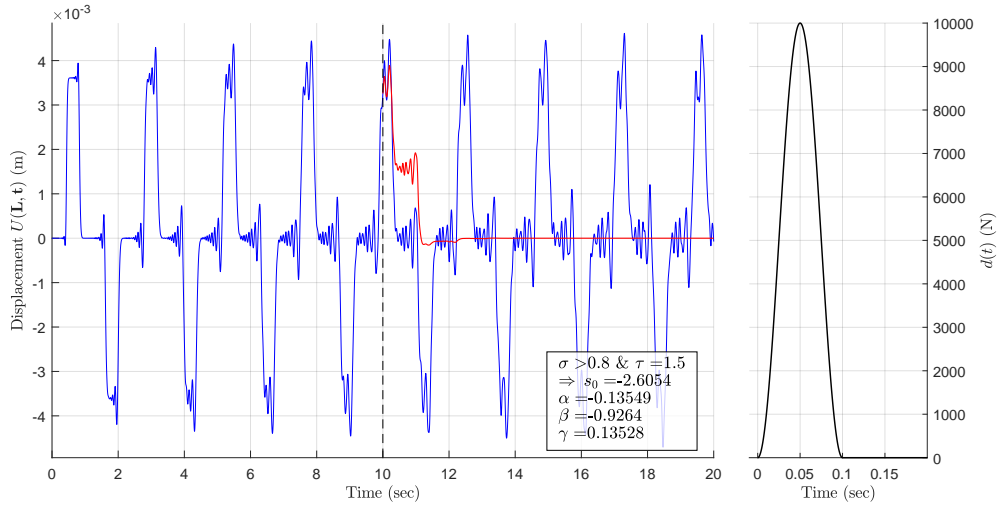
8.3. Numerical developments and physical intuitive interpretations. The considered drill pipe described by Figure 3 and Table 1 has been meshed with $N = 60$ elements of equal lengths, $l_{e_i} = 50$ (m), for $i = 1$ to N . Even if this number may seem insufficient to correctly discretize the wave length of the fastest wave, it appropriately describes its low frequencies' behavior, which is what we aim to analyze

with a low-frequency spectrum for the considered disturbance. The time simulation of the motion equation (8.99) is performed with a fixed time-step of 10^{-4} (sec) thanks to an ODE 3 numerical method (the Runge-Kutta of order 3 proposed by Bogacki and Shampine [79]). This time-step's choice is made to cope with the delay's numerical value considered in the QPB's control law.

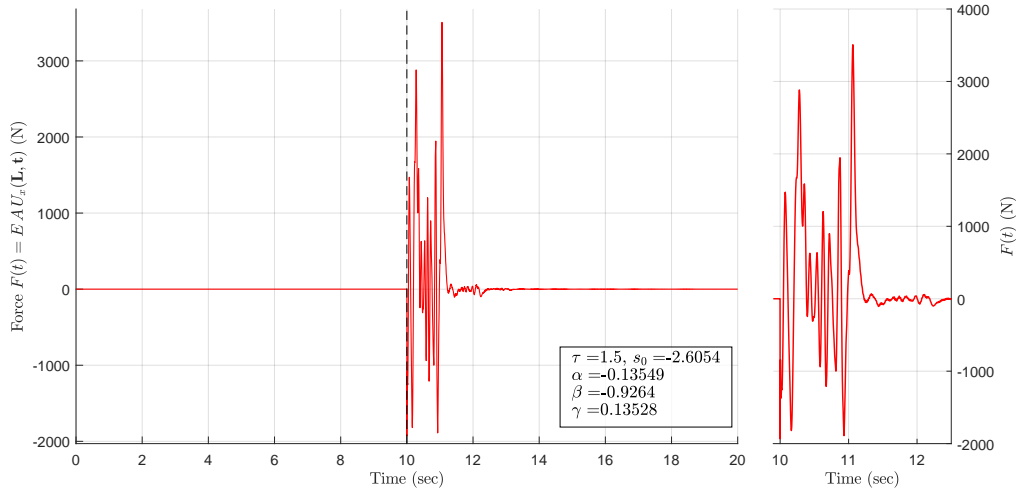
The left side of Figure 5a shows a simulation of the drill pipe elongation, supposed to be at rest at the initial time, which is executed to illustrate the QPB controller's performances for the active vibration damping of the drill pipe. The pipe is enduring an impact happening at $\mathbf{x} = l$ modeled by a force $d(t)$ that corresponds to half a period of a sinus function of magnitude 10000 (N) with an impulse width of 0.1 second displayed to the right. The time propagation of the shock-wave to the free-end can be seen on this figure and computed thanks to the wave propagation speed c , by $t_p := (L - l)/c \approx 0.395$ seconds. Since the drill pipe model of (8.94) does not include any dissipating term, the pipe begins to vibrate persistently until the output feedback control applies. The latter is designed to implement the following specifications. A nominal decay rate of $\sigma = 0.8$ has been fixed in the normalized framework. By using (7.79) combined with Theorem 7.12, a pair (s_0, τ) is selected according the chart given in Fig. 2. The resulting controller's coefficients are reported within Figure 5a in the normalized form. One can observe that the dissipativity conditions in (4.27) of Section 4 are satisfied. The first ten seconds show the natural behavior in blue of the undamped pipe. At $t = 10$ seconds, the output feedback control is applied and one can compare the closed-loop behavior in red with the natural one in blue. From the closing loop time $t = 10$, the settling time to zero within an interval of $\pm 5\%$ of the value $U(L, 10)$, can be observed roughly equal to 2.2 seconds. It corresponds to three times the time constant computed with the prescribed decay rate ($3/(-s_0 - \sigma) \approx 2.2$). Moreover, whatever the choice of (s_0, τ) is made, an incompressible duration of roughly 1.18 second is observed between the closing loop time and the moment the controlled damping is effective. It corresponds to $2d$, where 2 is the normalized delay due to the forward-backward duration of the wave propagation.

Remark 8.2. *It is worth mentioning that, with the QPB controller's four degrees of freedom and the results gathered in Theorem 7.12, the pair (s_0, τ) can be arbitrarily selected. This enables the control designer to make a choice for the performances that is robust to numerical approximations, e.g., the delay parameter. Figure 6a illustrates this through time simulations with the same configuration as those in Figures 5a & 5b.*

8.4. P3 δ Software. P3 δ , whose name stands for *Partial pole placement via delay action*, is a Python software with a friendly user interface for the design of parametric stabilizing feedback laws with time delays, thanks to properties of the distribution of the quasipolynomials' zeros. It exploits mainly the MID/GMID property as well as the coexisting real roots-induced-dominancy, or CRRID for short, established in [25, 17]. Initially, P3 δ has been established for the control design of retarded differential equations. Based on the results of this work as well as the references [65, 59], the software has been updated and is now able to treat linear neutral functional differential equations as well.



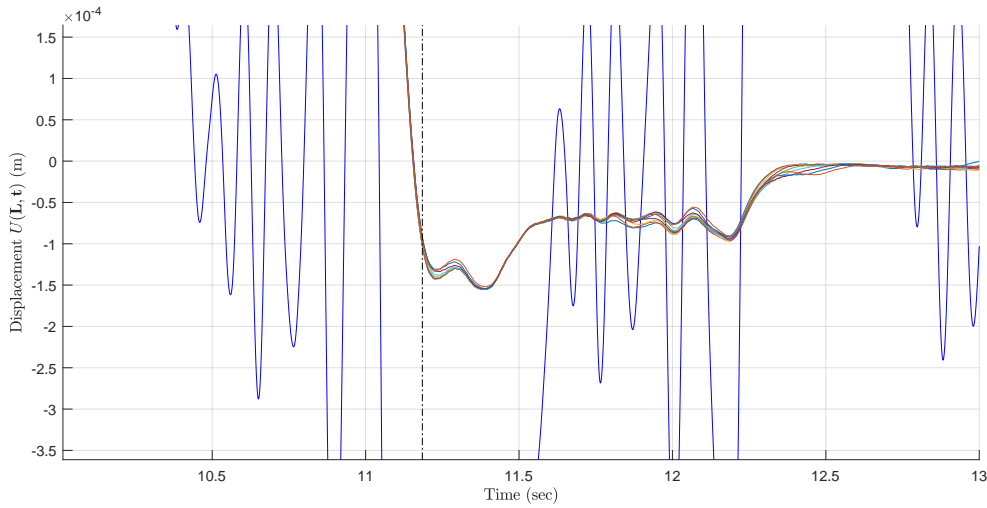
(A) Free-end's time response to a shock-like force $d(t)$ - with (red) & without (blue) feedback control. The feedback control loop is closed at $t = 10$ (sec).



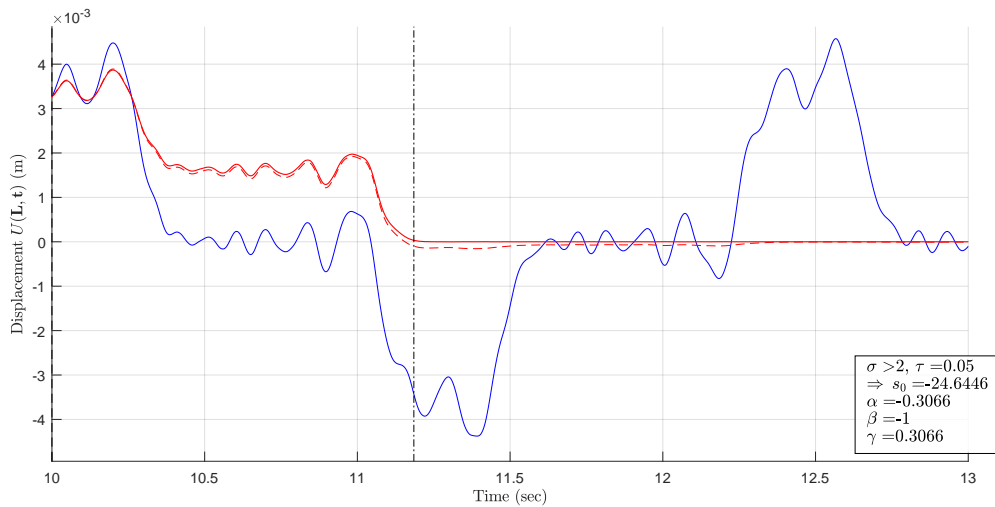
(B) Time response of the controlled force $F(t)$ to a shock-like force $d(t)$ happening at the initial time $t = 0$. The feedback control loop is closed at $t = 10$ (sec). On the right, a zoom on the time interval where the closed-loop control signal is activated.

FIGURE 5. Time responses without & with output feedback control.

P3 δ is freely available for download on <https://cutt.ly/p3delta>, where installation instructions, video demonstrations, and the user's guide are also available. Interested readers may also directly contact one of the authors of the paper. Since its inception, P3 δ is committed to be available to the greatest number and on all available platforms. The current version of the software exists as a local executable version as well as an online version ready to use in one click. The online version of P3 δ is hosted on servers thanks to the *Binder* service [74]. *Binder* allows to create instances of personalized computing



(A) Free-end's time response to a shock-like force $d(t)$ - with (various colours) & without (blue) feedback control. The feedback control loop is closed at $t = 10$ (sec). The wave propagation time is represented by the vertical dot-dashed line, at $t \approx 11.18$ seconds. Different delay parameters τ have been implemented around its nominal computed value within an interval of $\pm 20\%$ (with a 5% increment), without changing the nominal values for the other parameters α, β and γ . This emphasizes the robustness of the QPB controller with respect to the implemented delay.



(B) Free-end's time response to a shock-like force $d(t)$ - with (red) & without (blue) feedback control with another choice for the pair (s_0, τ) (see the normalized values within the figure). The same time response of the configuration depicted in Figure 5a is superimposed (dashed red curve) on this new one (red), emphasizing a better exponential decay rate.

FIGURE 6. Free-end's time responses through various situations.

environment directly from a *GitHub* repository that can be employed and shared by users. The *Binder* service is free to use and is powered by *BinderHub*, an open-source tool that deploys the service in the cloud. The online version of P3 δ is written in Python and structured as a *Jupyter Notebook*, an open document format which can contain live code, equations, visualizations, and text. The Jupyter Notebook is completed by a friendly user interface built using interactive widgets from Python's *ipywidgets* module. Future development of P3 δ will also provide a certified upper-bound of the decay rate when the MID property fails.

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LR ANALYSE ET CONTRÔLE DES EDP, LR 22ES03, DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES DE MONASTIR, UNIVERSITÉ DE MONASTIR, 5019 MONASTIR, TUNISIE

E-mail address: `kais.ammari@fsm.rnu.tn`

UNIVERSITÉ PARIS-SACLAY, CNRS, CENTRALESUPÉLEC, LABORATOIRE DES SIGNAUX ET SYSTÈMES, INRIA SACLAY ÎLE-DE-FRANCE, DISCO TEAM, 91190, GIF-SUR-YVETTE, FRANCE & INSTITUT POLYTECHNIQUE DES SCIENCES AVANCÉES (IPSA), 63 BOULEVARD DE BRANDEBOURG, 94200 IVRY-SUR-SEINE, FRANCE

E-mail address: `islam.boussaada@l2s.centralesupelec.fr`

LABORATOIRE DES SIGNAUX ET SYSTÈMES, UNIVERSITÉ PARIS-SACLAY–CNRS–CENTRALESUPÉLEC, INRIA, 3 RUE JOLIOT-CURIE, 91192 GIF-SUR-YVETTE CEDEX, FRANCE

E-mail address: `Silviu.Niculescu@l2s.centralesupelec.fr`

LABORATOIRE DES SIGNAUX ET SYSTÈMES, UNIVERSITÉ PARIS-SACLAY–CNRS–CENTRALESUPÉLEC, 3 RUE JOLIOT-CURIE, 91192 GIF-SUR-YVETTE CEDEX, FRANCE

E-mail address: `sami.tliba@l2s.centralesupelec.fr`