Almost-fuchsian structures on disk bundles over a surface

Samuel Bronstein

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ABSTRACT. Considering an integer $d > 0$, we show the existence of convex-cocompact representations of surface groups into $\mathrm{SO}(4,1)$ admitting an embedded minimal map with curvatures in $(-1,1)$ and whose associated hyperbolic 4-manifolds are disk bundles of degree d over the surface, provided the genus g of the surface is large enough. We also show that we can realize these representations as complex variation of Hodge structures. This gives examples of quasicircles in $\mathbb{S}^3$ bounding superminimal disks in $\mathbb{H}^4$ of arbitrarily small second fundamental form. Those are examples of generalized almost-Fuchsian representations which are not deformations of Fuchsian representations.

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1. Introduction

1.1. **Context.**

1.1.1. *Minimal surfaces and convex-cocompact representations.* In this paper, we consider representations of a closed surface group into $\mathrm{SO}(4,1) \approx \mathrm{Isom}(\mathbb{H}^4)$ and minimal surfaces in the hyperbolic 4-space equivariant under the group action. We prove the existence of "exotic" minimal surfaces of small curvature, that is minimal surfaces with arbitrarily small second fundamental form while the quotient of the hyperbolic space by the group action is a nontrivial disc bundle over the surface considered.

Minimal surfaces are surfaces which are locally critical points of the area functional. Considering an immersion from a Riemann surface Σ to $\mathbb{H}^4$, the image of an immersion is a minimal surface if the immersion is conformal and harmonic.

The study of minimal surfaces in hyperbolic manifolds is a rich and diverse topic. Schoen and Yau [SY79] and Sacks–Uhlenbeck [SU82] showed that in any isotopy class of an incompressible surface in a compact manifold, there is a weakly branched immersion whose image is minimal. See [MIY82] for the case when the target manifold has mean convex boundary. In the case of a hyperbolic three-dimensional manifold with convex boundary, there is an immersion, who will be an embedding if we're in the isotopy class of an incompressible surface.

Considering a faithful and discrete representation ρ in $\mathrm{Isom}(\mathbb{H}^n)$, it is then natural to ask whether the quotient manifold $\rho\backslash\mathbb{H}^n$ contains a non-empty compact convex set. When that is the case, we say that ρ is *convex-cocompact*. In particular, for a convex-cocompact representation ρ from a closed surface group in $\mathrm{Isom}(\mathbb{H}^n)$, there is always an equivariant weak immersion $\mathbb{D}^2 \to \mathbb{H}^4$ whose image is minimal (but has branch points in general).

While branch points can be ruled out for least area minimal surfaces in hyperbolic 3-manifolds with convex boundary, it is not true from the dimension 4 on. In this regard, our theorem gives examples of minimal maps into a disc bundle over a surface which are embeddings, so without branch points and self-intersections.

1.1.2. *Hyperbolic structures on disc bundles.* Among the family of hyperbolic 4-manifolds whose fundamental group is a surface group, one family is of specific interest to us: disc bundles over a Riemann surface. Considering a Fuchsian representation of a surface group into $\mathrm{SO}(4,1)$, it preserves a totally geodesic disc in $\mathbb{H}^4$ and the quotient manifold identifies with the normal bundle to the quotient of the disc. Topologically the quotient manifold is the product of a disc with the Riemann surface. However, these are not the only examples of hyperbolic disc bundles. Gromov–Lawson–Thurston [GLJT88] were the first to give examples of hyperbolic structures on nontrivial disc bundles over a surface, considering equivariant piecewise-linear embeddings of the disc into $\mathbb{H}^4$. Kuiper [Kui88] also gave examples of those, using tessellations of the hyperbolic 4-space.

Topologically, disc bundles over a Riemann surface are classified by the Euler characteristic of the surface and the Euler class of the bundle. The precise list of disc bundles over a closed surface admitting a hyperbolic structure is still an open problem. Gromov–Lawson–Thurston conjecture that a necessary condition to admit a hyperbolic structure would be that the degree d is smaller in absolute value than the Euler characteristic $2g-2$. Kapovich [Kap89] showed that for $0 \leq d \leq g - 1/11$, there is indeed a hyperbolic structure on the corresponding disc bundle.

Our result gives a new method of finding some of those hyperbolic structures, with more geometric properties, although we don't have an explicit condition on how small the degree has to be with regard to g. Check Ho [Ho14] for the latest results concerning Gromov–Lawson–Thurston's conjecture.

1.1.3. *Almost-Fuchsian representations.* Another important question regarding minimal surfaces is the potential uniqueness in a fixed isotopy class. Already in dimension 3, there are examples of hyperbolic 3-manifolds containing arbitrarily many isotopic minimal surfaces, see Huang and Wang [HW15]. However, we may get uniqueness if we add some conditions on the manifold: In 1983, Uhlenbeck [Uhl83] introduced the notion of *almost-fuchsian representation* which are faithful and discrete representations of a surface group into $\mathrm{PSL}(2,\mathbb{C})$ admitting an equivariant immersion $\mathbb{D}^2 \to \mathbb{H}^3$ minimal and with principal curvatures in $(-1,1)$.

Remark that asking the principal curvatures to be zero is equivalent to ask that our immersion is totally geodesic. Here as we relaxed this closed condition into asking the principal curvatures to be in $(-1,1)$, the notion is open in the character variety.

This differential geometric assumption has several consequences: the representation is then convex-cocompact, the map f is the only minimal map in its isotopy class and it is an embedding.

Since then, the notion of almost-fuchsian representation has been extended to almost-fuchsian immersions [KS07]. Seppi [Sep16] studied almost-fuchsian discs in $\mathbb{H}^3$, Donaldson [Don03], Hodge [Hod05] and Trautwein [Tra19] considered the hyperkähler structure on the moduli space of almost-fuchsian representations extending the hyperkähler structure on Teichmüller space. With Smith [BS23], we have exhibited a parametrization of the set of almost-fuchsian discs in $\mathbb{H}^3$ by a convex set of bounded holomorphic quadratic differentials on the disc. There are important questions related to foliations by surfaces of almost-fuchsian manifolds, see [GHW10] and [CMS23].

Jiang [Jia21] considered almost-fuchsian discs in $\mathbb{H}^n$. For faithful and discrete representations into $\mathrm{SO}(n,1)$, the existence of an equivariant minimally immersed disc whose scalar second fundamental forms have curvatures in $(-1,1)$ actually implies the uniqueness of the minimal surface in its isotopy class, its embeddedness, and the convex-cocompactness of the representation.

Recently, Davalo [Dav23] extended the notion of almost-fuchsian immersion into nearly-geodesic immersion into a higher rank symmetric space of noncompact type. The only known examples of nearly-geodesic immersions are deformations of totally geodesic immersions. While in $\mathrm{PSL}(2,\mathbb{C})$ those are indeed the only examples, we bring here examples of such immersions which cannot be deformed to totally geodesic ones. In particular, the limit set of our representations is a quasi-circle, there is a quasiconformal homeomorphism of $\mathbb{S}^3$ sending our limit set to a round curve, but this homeomorphism cannot be made equivariant.

1.1.4. *Superminimal maps and complex variations of Hodge structures.* The construction we will exhibit will naturally give examples of complex variations of Hodge structures in $\mathrm{SO}(4,1)$. The notion of complex variation of Hodge structures for representations of surface groups come from the Nonabelian Hodge Correspondence, see for instance this survey by Wentworth [FMS+16]. The Nonabelian Hodge Correspondence is a 1 to 1 correspondence between the moduli space of representations of surface group with the moduli space of an object called Higgs bundle, on which there is an action of $\mathbb{C}^*$. The fixed points of this action are called complex variations of Hodge structures. Loftin–McIntosh [LM19] give an extensive description of those representations in their article about equivariant minimal surfaces in $\mathbb{H}^4$. For a representation ρ with an equivariant minimal immersion f, asking that ρ is a complex variation of Hodge structure is equivalent to asking that f is superminimal, i.e. that it has circular ellipse of curvature, see Definition 2.5.

The term superminimal was coined by Kommerell in [Kom97]. In particular, it implies the minimality of the image of the map. Regarding the study of superminimal surfaces in the sphere $\mathbb{S}^4$, there are some impressive results: see for instance Bryant [Bry82], who exhibited a duality between superminimal surfaces in $\mathbb{S}^4$ and holomorphic legendrian curves in $\mathbb{C}P^3$. Check Forstnerič [For21] for the analogous statement in $\mathbb{H}^4$.

More precisely, Loftin–McIntosh show that for any pair of integers d and $g \geq 2$ such that $|d| < 2g-2$, there are representations from $\pi_1\Sigma_g$ into $\mathrm{SO}(4,1)$ with an equivariant superminimal immersion whose normal bundle is of degree d over Σ_g. Our result states that for d very small before g, there are such pairs (ρ, f) where f is actually embedded and ρ is convex-cocompact. We also have, as the normal bundle to f will identify to the quotient manifold $\rho\backslash\mathbb{H}^4$, hyperbolic structures on degree d disc bundles over Σ_g.

1.1.5. *Solving "mixed" Toda systems.* Finally, the method used to build our examples is to solve a non-linear PDE system which we call a mixed Toda system. Toda systems, and Toda systems with opposite signs are defined by Guest–Lin [GL14] and designate PDE systems where the unknown is (w_i) and satisfy an equation of the type

$$\Delta w_i = e^{w_i - w_{i-1}} - e^{w_i - w_{i+1}} \tag{1.1}$$

While our system definitely cannot be written in this form, it belongs to a class of equations arising frequently when dealing with flat curvature equations.

This kind of system appears quite easily in the theory of flat bundles. Li–Mochizuki [LM20] encounter Toda systems (of the opposite type) when dealing with cyclic Higgs bundles associated to representations in rank 2 Lie groups. Malchiodi–Ruiz [MR11] and Battaglia–Jevnikar–Malchiodi–Ruiz [BJMR15] made use of the Moser–Trudinger inequality to solve some Toda systems motivated by the non-abelian Chern-Simons theory. Here in rank 1 with the complex Hodge variation structure assumption, we get something alike, but with tweaked signs:

$$\begin{cases} \Delta u &= -1 + e^{2u} + e^{-2u}e^{2v}|\alpha|^2 \\ \Delta v &= \frac{1}{2g-2} - e^{-2u}e^{2v}|\alpha|^2 \end{cases} \tag{1.2}$$

We think the method used in this article to solve this system could be applied to a broader class of PDE systems, yet to be described.

1.2. **Statement of the results.** First, we state our result in dimension 4 concerning degree 1-disc bundles over Riemann surfaces.

Theorem 1.1. *Fix $r \in (0,1)$. There is $g_0 \geq 2$ such that, for any $g \geq g_0$ and Σ a closed Riemann surface of genus g, there is a representation $\rho : \pi_1\Sigma \to \mathrm{Isom}(\mathbb{H}^4)$ and an immersion $f : \mathbb{S}^2 \to \mathbb{H}^4$ satisfying:*

(a) *ρ is faithful, discrete, convex-cocompact, almost-fuchsian*
(b) *f is a ρ-equivariant embedding, superminimal*
(c) *The second fundamental form of f satisfies $|\mathit{II}_f| < r$ everywhere.*
(d) *The quotient hyperbolic 4-manifold $\rho\backslash\mathbb{H}^4$ is a degree 1 disc bundle over Σ.*

As a corollary, we have the existence of almost-fuchsian representations uniformizing degree d disc bundles over Riemann surfaces of high genus.

Corollary 1.2. *Let $d > 0$. For any $g_0 > 0$, there is $g \geq g_0$, Σ a surface of genus g ; $\rho : \pi_1\Sigma \to \mathrm{Isom}(\mathbb{H}^4)$ and $f : \mathbb{D}^2 \to \mathbb{H}^4$ satisfying:*

(a) *ρ is faithful, discrete, convex-cocompact, almost-fuchsian*
(b) *f is a ρ-equivariant embedding, superminimal.*
(c) *The second fundamental form of f satisfies $I\!I_f < r$ everywhere.*
(d) *The quotient manifold $\rho\backslash\mathbb{H}^4$ is a degree d disc bundle over Σ.*

1.3. **Scheme of proof.**

1.3.1. *Writing the Flat curvature equations.* In order to build a hyperbolic structure on a degree d disc bundle, we first show the existence of a hyperbolic structure on a degree 1 disc bundle over a surface with the desired properties. Once this example is built, we can take a cover of degree d of this bundle in order to obtain a hyperbolic structure on a degree d disc bundle over a surface (of genus large enough).

Consider a Riemann surface Σ of genus g. Our proof will rely on solving a PDE system of the following kind:

$$\begin{cases} \Delta u & = -1 + e^{2u} + e^{-2u}e^{2v}|\alpha|^2 \\ \Delta v & = \frac{1}{2g-2} - e^{2v}e^{-2u}|\alpha|^2 \end{cases} \tag{1.3}$$

where α has to be a holomorphic section of a degree $4g-3$ line bundle over Σ. The Laplacian is always taken with respect to the Poincaré volume form on the surface, and every line bundle will be considered endowed with a metric whose curvature form is proportional to the Poincaré volume form, hence defining the norm $|\alpha|$.

The second section will be devoted to showing how solving this system yields a minimal disc in $\mathbb{H}^4$ equivariant under a representation ρ. With the good pointwise control on $e^{2v}e^{-2u}|\alpha|^2$, the representation ρ will be almost-fuchsian and the quotient manifold $\rho\backslash\mathbb{H}^4$ will be a degree 1 disc bundle over the Riemann surface.

The associated minimal map will have a holomorphic second fundamental form (see Definition 2.2.) $\alpha \oplus 0$, hence be superminimal.

1.3.2. *Controlling the geometric data.* The third section will be devoted to finding families (Σ_n) of surfaces with genus going to infinity, with data (α_n) associated, while controlling the behavior of solutions of PDEs on (Σ_n) depending on (α_n). To do so, we need to control the systole and spectral gap of (Σ_n), and the ratio $\frac{\int |\alpha_n|^2}{|\alpha_n|^2_\infty}$.

Controlling the systole is the easiest part, as the systole is nondecreasing when taking the Riemannian cover of a surface. Controlling the spectral gap is trickier, but can be done by taking (Σ_n) a random cover of a genus 2 surface thanks to a result of Magee–Naud–Petri [MNP22]: considering a cover $\Sigma' \to \Sigma$ of closed Riemann surfaces, the relative spectrum of Σ' is defined to be the part of the spectrum of Σ' which is not in the spectrum of Σ. Magee–Naud–Petri states that for a random cover of large degree of Σ, the probability that the relative spectral gap is lower than $\frac{3}{16} - \varepsilon$ goes to zero when the degree goes to infinity. Using this result, we are able to find families of coverings of a Riemann surface whose spectral gap is uniformly lower bounded.

Finally, we need to build α_n such that the ratios $\frac{|\alpha_n|^2_\infty}{\int |\alpha_n|^2}$ are uniformly bounded. This property will be called *balanced* family of sections. We will show that multiplying a cover of a section with a section of a line bundle of degree 1 yields a balanced family of sections of the desired type.

1.3.3. *Building the maps* Φ *and* Ψ. Once this is done, we need to solve our PDE system. For this, we will consider each line of the system and build two maps. Fixing v and solving the first line in u, we will build a map $\Phi : C^{0,\alpha}(\Sigma) \to C^{2,\alpha}(\Sigma)$ which will be well-defined on the set $\mathcal{K}_v$, where:

$$\mathcal{K}_v := \{v \in C^{0,\alpha}(\Sigma) : e^{2v}|\alpha|^2 \leq \frac{1}{2}\} \tag{1.4}$$

And it happens that the image of $\mathcal{K}_v$ is in the set $\mathcal{K}_u$, where

$$\mathcal{K}_u := \{u \in C^{0,\alpha}(\Sigma) : -\frac{\ln 2}{2} \leq u \leq 0\} \tag{1.5}$$

We would like to do the same work for the second equation, fixing v, solving in u to get a map Ψ. However, the work to build Ψ is considerably harder. The third section will be devoted to the study of Ψ. The method used to solve it is heavily inspired from the analysis of the prescribed curvature equation in the projective plane $\mathbb{R}P^2$: for an even function on the sphere which is nonnegative on an open set, it is the curvature of a metric conformal to the round metric on $\mathbb{S}^2$ (and even). See Chang–Yang [CY03] for a precise explanation of this. With the same tools, we are able to solve the second line and to get the following estimate: There is a constant $C > 0$ such that the solution v satisfies:

$$e^{2v}e^{-2u}|\alpha|^2 \leq \frac{C}{2g-2} \tag{1.6}$$

The involved constant C will depend on three parameters: δ the systole of the surface, Λ its spectral gap and the ratio between the L^∞-norm of α and its L^2-average. Hence thanks to the third section it can be bounded on surfaces of arbitrarily large genus.

1.3.4. *Piecing everything together.* Considering a family of covers (Σ_n) with a balanced family of sections of sections (α_n) and controlled systole and spectral gap, we are able, provided n is large enough, to build a map $\Psi : \mathcal{K}_u \to \mathcal{K}_v$, which will be continuous.

With Ψ constructed and well-defined, We will show that the composed map $\Phi \circ \Psi$ defined on $\mathcal{K}_u$ has image in a compact convex subset of $\mathcal{K}_u$. Hence by Schauder's fixed-point theorem, $C^{0,\alpha}(\Sigma)$ being a Banach space, $\Phi \circ \Psi$ has a fixed point $u \in \mathcal{K}_u$. Denote $v = \Psi(u)$.

It is clear that (u, v) is a solution of the PDE system considered, and so it corresponds to an almost-fuchsian representation ρ whose quotient manifold is a degree 1 disc bundle over the Riemann surface.

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2. The curvature equations

2.1. **Minimal equivariant immersions in $\mathbb{H}^4$.** Let Σ be a closed oriented surface of genus $g \geq 2$.

Consider a representation $\rho : \pi_1\Sigma \to \mathrm{SO}_0(4,1) \approx \mathrm{Isom}_+(\mathbb{H}^4)$. We will always assume that ρ is faithful and discrete. Let $f : \mathbb{D}^2 = \tilde{\Sigma} \to \mathbb{H}^4$ be an equivariant immersion with minimal image.

Endowing Σ with the induced metric by f, we get a conformal and harmonic map.

Associated to f, there is a flat $\mathbb{R}^{4,1}$ vector bundle E, which decomposes as follows:

Denote $\mathrm{I\!I}$ the second fundamental form of f, and B its shape operator. The vector bundle (E, ∇_E) splits as $E = O_{\mathbb{R}} \oplus T_{\mathbb{R}} \oplus N_{\mathbb{R}}$, where $O_{\mathbb{R}}$ is the trivial vector bundle of rank 1, $T_{\mathbb{R}}$ the tangent bundle is a rank 2 vector bundle, and $N_{\mathbb{R}}$ is the normal bundle to f, again of rank 2. In this splitting, the connection has the following expression:

$$\nabla_E = \begin{pmatrix} \nabla & a^\dagger & 0 \\ a & \nabla_T & B \\ 0 & \mathrm{I\!I} & \nabla_N \end{pmatrix} \tag{2.1}$$

Denote q the quadratic form in $\mathbb{R}^{4,1}$ given by

$$q(x_1, \ldots, x_5) := x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 \tag{2.2}$$

As the map f is valued in the quadric $q = -1$ in $\mathbb{R}^{4,1}$, the image of f gives a reduction of E into a $S(O(1) \times O(4))$ bundle. The differential of f being valued, by definition, in the Tangent bundle, we denote $a = df$ the nonvanishing section of $T^*\Sigma \otimes \mathrm{Hom}(O_{\mathbb{R}}, T)$. The definition of $\mathrm{I\!I}$ and of B gives the desired decomposition of ∇_E.

Remark 2.1. This splitting is orthogonal with respect to b the signature $(4,1)$ bilinear form, with $O_{\mathbb{R}}$ being a negative line. As the surface is oriented and our representation is into orientation-preserving isometries $\mathrm{SO}_0(4,1) \approx \mathrm{Isom}_+(\mathbb{H}^4)$, $T_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are endowed with an $SO(2)$ structure, corresponding to the induced metrics on the tangent and normal bundle to f. The nondegenerate metric b also implies that $\mathrm{I\!I}$ and B are antiadjoint with respect to b:

$$b(B(w)u, v) + b(\mathrm{I\!I}(u,v), w) = 0 \quad \forall (u,v) \in T_{\mathbb{R}},\, w \in N_{\mathbb{R}}\,. \tag{2.3}$$

We denote $B = -\mathrm{I\!I}^\dagger$, where $\dagger$ denotes the transposition, identifying $\mathrm{Hom}(E,F)$ with $\mathrm{Hom}(F^*, E^*)$.

Complexify now the bundles $T_{\mathbb{R}}$ and $N_{\mathbb{R}}$. As each of these are equipped with an SO(2)-structure, their complexification splits into the sum of two line bundles, the restriction of B giving a nondegenerate pairing between those. Now for each line subbundle of E coming into play, we consider ∇_L the induced connection of ∇_E and we take its $(0,1)$-part to get $\overline{\partial}_L$ a holomorphic structure on L. Remark that the $(0,1)$-part is well-defined because the induced metric endows the surface with a Riemann surface structure.

The diagonalization of the $SO(2)$ structure with regard to the metrics yields:

Proposition 2.1. *Viewing $(\mathit{I\!I}_{\mathbb{C}})$ as a 1-form valued in $\mathrm{Hom}(K^{-1}, N)$, the complexification of the second fundamental form can be written:*

$$(T_{\mathbb{R}})_{\mathbb{C}} = \overline{K} \oplus K = K^{-1} \oplus K \tag{2.4}$$

$$(N_{\mathbb{R}})_{\mathbb{C}} = N \oplus \overline{N} = N \oplus N^{-1} \tag{2.5}$$

$$(\mathit{I\!I})_{\mathbb{C}} = \begin{pmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{pmatrix} \tag{2.6}$$

with α and β $(1,0)$-forms.

Proof. The splitting of the complexification of an SO(2)-bundle is given by the eigenspaces of J the fiberwise rotation of angle $\frac{\pi}{2}$. Remark that this splitting diagonalizes ∇_T and ∇_N. In this splitting, write down the second fundamental form:

$$(\mathrm{I\!I})_{\mathbb{C}} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \tag{2.7}$$

As $\mathrm{I\!I}$ is real-valued, we have:

$$\gamma = \overline{\beta} = \beta^* \tag{2.8}$$

$$\delta = \overline{\alpha} = \alpha^* \tag{2.9}$$

Finally, as f is minimal, $\mathrm{I\!I}$ is traceless, and so the $(1,1)$-part of the second fundamental form is zero: This implies that both $\alpha \in \Gamma(K \otimes \mathrm{Hom}(K^{-1}, N))$ and $\beta \in \Gamma(K \otimes \mathrm{Hom}(K^{-1}, N^{-1}))$ are $(1,0)$-forms. □

Remark 2.2. In the process, we endowed the bundle $E \otimes \mathbb{C}$ with the following holomorphic structure:

$$E \otimes \mathbb{C} = \mathcal{O} \oplus K^{-1} \oplus K \oplus N \oplus N^{-1} \tag{2.10}$$

$$\overline{\partial}_E = \overline{\partial}_{\mathcal{O}} \oplus \overline{\partial}_{K^{-1}} \oplus \overline{\partial}_K \oplus \overline{\partial}_N \oplus \overline{\partial}_{N^{-1}} \tag{2.11}$$

It is clear that the sections α, β characterize wholly the second fundamental form. Hence we define the *Holomorphic second fundamental form*:

Definition 2.2. Let $f : \mathbb{D}^2 \hookrightarrow \mathbb{H}^4$ be a minimal immersion. The *Holomorphic second fundamental form* of f is the $(2,0)$-part of the complexification of the second fundamental form:

$$\mathrm{I\!I}^{2,0} = \alpha \oplus \beta \in \Gamma(K^2(N \oplus N^{-1})) \tag{2.12}$$

2.2. **Writing the flat curvature equations as a scalar PDE system.** In order to write this system as a scalar PDE system, we need to write down each metric as a conformal change of a preferred one. For the metric on K, we denote $e^{2u}\sigma$ the induced metric, where σ is the unique complete hyperbolic metric in the conformal class.

Denote ω the Poincaré $(1,1)$-form associated to σ, which is locally $dz \wedge d\overline{z}$. Also, we denote by Δ the Laplace-Beltrami operator on (Σ, σ). Note that Δ is defined to be locally $\partial_x^2 + \partial_y^2$, so it is a nonpositive operator satisfying:

$$\forall (f,g) \in C(\Sigma), \quad \int_\Sigma (\Delta f) g \, \omega + \int_\Sigma \nabla f \cdot \nabla g \, \omega = 0 \tag{2.13}$$

Generally, on a holomorphic line bundle L of degree d, there is a hermitian metric h_L whose curvature form is the following:

$$\frac{i}{2\pi} F_{L,h_L} = \frac{d}{2g-2}\omega \tag{2.14}$$

The metric h_L is uniquely defined up to multiplicative constant. From now on, every line bundle will be endowed with a metric of this kind, and a given fiberwise norm $|\cdot|_L$.

Let's write down the flat curvature condition as a scalar PDE system:

Theorem 2.3. *Let $f : \Sigma \hookrightarrow M$ be a conformal harmonic immersion into a complete hyperbolic 4-manifold. Consider $\alpha \oplus \beta$ its holomorphic second fundamental form. Denote $e^{2u}\sigma$ the induced metric on Σ, $e^{2v}h_N$ the induced metric on the normal bundle. Then the flat curvature equations are equivalent to the system:*

$$\begin{cases} \overline{\partial}\alpha &= 0 \\ \overline{\partial}\beta &= 0 \\ \Delta u &= e^{2u} - 1 + e^{-2u}(e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2 \\ \Delta v &= c - e^{-2u}(e^{2v}|\alpha|^2 - e^{-2v}|\beta|^2) \end{cases} \tag{2.15}$$

Proof. First, as $B = -\mathrm{II}^\dagger$, we have the explicit description of B

$$B = \begin{pmatrix} -\alpha^{\dagger,*} & -\beta^{\dagger,*} \\ -\beta^\dagger & -\alpha^\dagger \end{pmatrix} \tag{2.16}$$

Now the flat curvature equation is $F_{\nabla_E} = 0$, so coordinatewise:

$$\left\{ \begin{aligned} \nabla\alpha &= 0 \\ \nabla\beta &= 0 \\ F_{K^{-1}} + aa^* - \alpha^{*,\dagger}\alpha - \beta^{*,\dagger}\beta &= 0 \\ F_N - \alpha\alpha^{*,\dagger} - \beta^*\beta^\dagger &= 0 \end{aligned} \right. \tag{2.17}$$

where we wrote $\alpha\alpha^*$ for the wedge product $\alpha \wedge \alpha^*$. Also, the notation $\nabla\alpha = 0$ means that α is parallel for the connection on $\mathrm{Hom}(K^{-1}, N)$, namely $\nabla_N \wedge \alpha + \alpha \wedge \nabla_{K^{-1}} = 0$.

The standard formula for the curvature under a conformal change of metric is

$$\left\{ \begin{aligned} iF_{K^{-1}} &= e^{-2u}(-\Delta u - 1)\omega \\ iF_N &= e^{-2u}(-\Delta v + c)\omega \end{aligned} \right. \tag{2.18}$$

Now recall that as α is a $(1,0)$-form, and locally $dz \wedge d\overline{z} = -2i dx \wedge dy$, hence

$$\left\{ \begin{aligned} iaa^* &= \omega \\ i\alpha\alpha^* &= e^{-4u}e^{2v}|\alpha|^2\omega \\ i\beta\beta^* &= e^{-4u}e^{-2v}|\beta|^2\omega \end{aligned} \right. \tag{2.19}$$

So the flat curvature equation is equivalent to:

$$\left\{ \begin{aligned} e^{-2u}(\Delta u + 1) &= 1 + e^{-4u}(e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2) \\ e^{-2u}(\Delta v - c) &= e^{-4u}(-e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2) \end{aligned} \right. \tag{2.20}$$

Also, as α is of type $(1,0)$, necessarily $\partial\alpha = 0$. Hence $\nabla\alpha = 0$ rewrites as $\overline{\partial}\alpha = 0$, and the same applies to β. Finally, rearranging the right and left hand sides, we get the desired scalar PDE system:

$$\left\{ \begin{aligned} \overline{\partial}\alpha &= 0 \\ \overline{\partial}\beta &= 0 \\ \Delta u &= -1 + e^{2u} + e^{-2u}(e^{2v}|\alpha|^2 + e^{-2v}|\beta|^2) \\ \Delta v &= c - e^{-2u}(e^{2v}|\alpha|^2 - e^{-2v}|\beta|^2) \end{aligned} \right. \tag{2.21}$$

□

Remark 2.3. The Higgs bundle associated to f can be read with our notation, it will be the following: $\mathcal{E} = \mathcal{O} \oplus K^{-1} \oplus K \oplus N \oplus N^{-1}$, and the holomorphic structure with the Higgs field have the explicit description:

$$\overline{\partial}_{\mathcal{E}} = \begin{pmatrix} \overline{\partial} & 0 & 0 & 0 & 0 \\ 0 & \overline{\partial} & 0 & -\alpha^* & -\beta^* \\ 0 & 0 & \overline{\partial} & 0 & 0 \\ 0 & 0 & \beta^* & \overline{\partial} & 0 \\ 0 & 0 & \alpha^* & 0 & \overline{\partial} \end{pmatrix}, \quad \Phi_{\mathcal{E}} = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.22}$$

Recall that we chose N so that $\deg N \geq 0$. Observing that $\mathcal{O} \oplus K^{-1} \oplus N$ is an invariant subbundle, the stability condition implies:

$$0 \leq \deg N \leq 2g - 2\,. \tag{2.23}$$

Proposition 2.4. *Let f be a minimal map with $\alpha = 0$. Then $\deg N = 0$ and f is totally geodesic.*

Proof. As $\alpha = 0$, N is an invariant subbundle of the corresponding Higgs bundle to f. Then by semi-stability, $\deg N = 0$. Now if $\beta \neq 0$, v is solution of:

$$\Delta v = e^{-2u} e^{-2v} |\beta|^2 \tag{2.24}$$

Integrating this identity implies

$$\int_\Sigma e^{-2u} e^{-2v} |\beta|^2 = 0 \tag{2.25}$$

Hence β is the zero section and f is totally geodesic, as asserted. □

We will be interested in the specific case when the second fundamental form, viewed as a map $T_\mathbb{R} \to (T_\mathbb{R})^*$, has *circular ellipse of curvature*, i.e. the image of the unit circle in $T_\mathbb{R}$ by the map $X \mapsto \mathrm{I\!I}(X, X)$ is a circle. This corresponds to asking the $(2, 0)$ and $(0, 2)$ part of $\mathrm{I\!I}_\mathbb{C}$ to be orthogonal.

Following Loftin–McIntosh [LM19]:

Definition 2.5. We say that f is superminimal if f is harmonic, conformal and its Holomorphic second fundamental form $\alpha \oplus \beta$ satisfies:

$$\alpha \cdot \beta = 0 \in H^0(K^4) \tag{2.26}$$

Remark 2.4. The notion of superminimal maps goes back to 1897 with Kommerell [Kom97] who studied superminimal immersions in $\mathbb{S}^4$. See Forstnerič [For21] for a recent review of the topic. Another interpretation of the superminimality is that it is equivalent to asking the normal Gauss map, valued in the Grassmannian of geodesic disks in $\mathbb{H}^4$, is conformal.

Depending on the degree of N superminimal maps behave differently:

Proposition 2.6. *Let f be a superminimal map with $\deg N = 0$, equivariant under a faithful and discrete representation $\rho : \pi_1\Sigma \to SO(4, 1)$, Σ being a closed surface. Then f is totally geodesic and ρ is a fuchsian representation.*

Proof. While this proposition can be found in [LM19], we reproduce it here as it can be seen as a consequence of our PDE system: As the normal bundle is trivial, we have $c = 0$ and the superminimality implies $\alpha = 0$ or $\beta = 0$. For convenience's sake, assume $\beta = 0$. Then (u, v) must be a solution of the system

$$\begin{cases} \Delta u &= -1 + e^{2u} + e^{-2u} e^{2v} |\alpha|^2 \\ \Delta v &= -e^{2u} e^{2v} |\alpha|^2 \end{cases} \tag{2.27}$$

Fix u, and observe that, on a closed Riemann surface, we must have $\int \Delta v = 0$. Hence $\alpha = 0$, the immersion f is totally geodesic and ρ is fuchsian. □

2.3. Almost-fuchsian disks in $\mathbb{H}^4$.

First introduced by Uhlenbeck for immersions into $\mathbb{H}^3$ [Uhl83], the notion of almost-fuchsian immersion naturally generalizes to immersions into $\mathbb{H}^4$.

Definition 2.7. Let $f : \mathbb{D} \hookrightarrow \mathbb{H}^4$ be a proper immersion. f is said to be *almost-fuchsian* if it is minimal and there is $\delta > 0$ such that the second fundamental form satisifes:

$$\forall u \in T_f,\ |\mathrm{I\!I}(u, u)| \leq (1 - \delta)|u|^2 \tag{2.28}$$

Proposition 2.8. *Let $f : \mathbb{D} \to \mathbb{H}^4$ be a proper almost-fuchsian immersion. Then the exponential map $N_f \to \mathbb{H}^4$ is a diffeomorphism*

As this constitutes, with the following result, the lemma 2.2. of [Jia21], we won't reproduce a proof here. Note that Jiang's convention for the second fundamental form differs from ours by a factor one half, hence the difference in the condition. More specifically, almost-fuchsian disks in $\mathbb{H}^4$ are embeddings, bound a quasi-circle at infinity, and if they are ρ-equivariant for some $\rho : \Gamma \to \mathrm{SO}(4,1)$ discrete and faithful, ρ is convex-cocompact. In particular,

Proposition 2.9. *Let Σ be a closed hyperbolic surface, and $\rho : \pi_1\Sigma \to SO(4,1)$ a faithful and discrete representation such that there exists $f : \mathbb{D} \hookrightarrow \mathbb{H}^4$ a ρ-equivariant proper almost-fuchsian immersion. Then ρ is convex-cocompact, f is an embedding and the hyperbolic 4-manifold $M = \rho\backslash\mathbb{H}^4$ is diffeomorphic via $\exp_f$ to a disk bundle over Σ.*

Also f is the unique ρ-equivariant minimal immersion.

Definition 2.10. A representation $\rho : \Gamma \to \mathrm{SO}(4,1)$. is said to be *almost-fuchsian* if it is discrete, faithful and it admits a proper almost-fuchsian ρ-equivariant immersion $f : \widetilde{\Sigma} \to \mathbb{H}^4$.

It is not known which disk bundles can be uniformized by almost-fuchsian representations. Fuchsian representations uniformize the trivial bundle, and we will show in this paper that for a surface of large enough genus, there are almost-fuchsian representations uniformizing vector bundles with positive degree over the surface.

Proposition 2.11. *Let Σ be a closed hyperbolic surface of genus g. Denote w its volume form. Consider N a line bundle of degree $d \geq 0$, endowed with a metric h of curvature form $c\omega = \frac{d}{2g-2}\omega$. Let $\alpha \in H^0(K^2N)$ be such that there exist smooth functions u, v on Σ satisfying:*

$$\Delta u = -1 + e^{2u} + e^{-2u}e^{2v}|\alpha|^2 \tag{2.29}$$

$$\Delta v = c - e^{-2u}e^{2v}|\alpha|^2 \tag{2.30}$$

$$\sup e^{-4u}e^{2v}|\alpha|^2 < 1 \tag{2.31}$$

Then there is a convex-cocompact representation $\rho : \pi_1\Sigma \to SO(4,1)$ and a minimal, superminimal, ρ-equivariant and almost-fuchsian immersion $f : \mathbb{D} \to \mathbb{H}^4$ such that:

1. *The induced metric by f is $e^{2u}w$*
2. *The hyperbolic manifold $\rho\backslash\mathbb{H}^4$ identifies with the normal bundle to f and has degree d*
3. *The metric induced on the normal bundle is $e^{2v}h$*
4. *The Holomorphic second fundamental form of f is α.*

Proof. With the given data, consider the bundle $\mathcal{E} = \mathcal{O} \oplus K^{-1} \oplus K \oplus N \oplus N^{-1}$, with the metric gotten by product of the metrics on each line. Adding α and its h-dual α^*, the 2 equations are exactly the flat curvature equation, corresponding to a representation ρ and a map $f : \mathbb{D} \to \mathbb{H}^4$ whose Holomorphic second fundamental form satisfies:

$$(\mathrm{II}_f)_{\mathbb{C}} = \alpha \oplus 0 \in \Gamma(K^2(N \oplus N^{-1})) \tag{2.32}$$

It is clear that such an f is minimal and superminimal. The last hypothesis

$$\sup e^{-4u}e^{2v}|\alpha|^2 < 1 \tag{2.33}$$

ensures that f and ρ are almost-fuchsian, as asserted. $\square$

3. Balanced holomorphic sections of positive line bundles

In this section, we show that there are families of holomorphic sections of line bundles of degree $4g-3$ which behave roughly like covers in the following sense:

Definition 3.1 (Balanced family)**.** Let (Σ_g) be a family of genus g Riemann surfaces, equipped with their hyperbolic metric. Let N_g be a family of line bundles of degree $n_g \geq 0$ Let (α_g) be a family of holomorphic sections of N_g.

On each Σ_g, denote by ω_g the Poincaré volume form. On each N_g, denote h_g the metric on N_g with curvature form $\frac{n_g}{2g-2}\omega_g$. We say that (α_g, N_g) is *balanced* if there is $C > 0$ such that:

$$\sup_g \frac{\sup_{x\in X_g} |\alpha_g|^2_{\omega_g,h_g}}{⨍_{X_g} |\alpha_g|^2_{h_g} d\omega_g} \leq C \tag{3.1}$$

The main idea behing this control is that when solving elliptic PDEs depending on α_g, as α_g is coarsely periodic, then the solutions should be too, and we will show that the solutions satisfy estimates which are independent of the genus g.

We will make extensive use of the following notations:

Notation 3.2. In this section, Σ denotes a closed hyperbolic surface of genus g. We denote respectively by δ and Λ the systole and spectral gap of Σ. Any N positive line bundle over Σ is endowed with a metric of curvature form proportional to the Poincaré volume form ω.

Remark 3.1. It is easy to exhibit examples of balanced sections for n_g proportional to $g-1$: Indeed, if $n_g = a(g-1)$, consider N_2 a degree a line bundle over Σ_2, and α a holomorphic section of N_2. and consider covers $\Sigma_g \to \Sigma_2$. Then the family of lifts of N_2 and α yields a balanced family.

Here we will prove the existence of balanced families for line bundles of degree $4g-3$:

Theorem 3.3. *Let Σ be a closed hyperbolic surface of genus $g \geq 2$ Consider (Σ_n) a family of covers of Σ of degree n. Then there is a balanced family (α_n, N_n), with N_n a line bundle of degree $(4g-4)n+1$ over Σ_n.*

The main idea will be to consider covers to get a balanced section of a degree $4n(g-1)$ line bundle, and then to tensorize with a well-chosen degree 1 line bundle to get the desired family of sections. This requires a description of the oscillatory behavior of a degree 1 bundle section, which we provide here.

Before all, we will make use of the continuous embedding $W^{2,2} \hookrightarrow L^\infty$.

Lemma 3.4. *Let Σ be a hyperbolic surface with systole δ. Denote M the constant of continuity of the embedding $W^{2,2}(\mathbb{H}^2) \hookrightarrow L^\infty(\mathbb{H}^2)$. Then, for any $u \in W^{2,2}(\Sigma)$*

$$|u|_\infty \leq \frac{C}{\delta}|u|_{2,2}\,. \tag{3.2}$$

Proof. As smooth maps are dense in $W^{2,2}(\Sigma)$, consider $u \in W^{2,2}(\Sigma)$ smooth. Let $z_0 \in \Sigma$. By definition of the systole, there is an isometric embedding from a hyperbolic disk of radius δ around z_0. Fix χ a smooth radial map on $D(z_0,\delta)$,$\frac{1}{\delta}$ Lipschitz, between 0 and 1, such that $\chi(z_0) = 1$ and $\chi(\partial D(z_0,\delta)) = 0$. Choose for χ a rescaling of a cutoff map on $D(0,1) \subset \mathbb{H}^2$, so that

$$\sup D\chi = \sup D^2\chi = \frac{1}{\delta} \tag{3.3}$$

Then we can extend by zero to consider $u\chi \in W^{2,2}(\mathbb{H}^2)$. Using our cutoff function, we get

$$|u(z_0)| = |u\chi(z_0)| \leq |u\chi|_\infty \leq M|u\chi|_{2,2} \leq \frac{M}{\delta}|u|_{2,2} \tag{3.4}$$

, as asserted. □

Proposition 3.5. *Let $z_0 \in \Sigma$ Denote $D = D(z_0, \delta)$ the disk of radius δ around z_0. Denote by a the constant $a = \frac{2\pi}{Vol(D)}$. Consider $g : \Sigma \to \mathbb{R}$ the C^1 function satisfying:*

$$\begin{cases} \int_\Sigma g &= 0 \\ \Delta g &= \frac{1}{2g-2} - a\mathbf{1}_D \end{cases} \tag{3.5}$$

Then there is a bound $C(\delta, \Lambda)$ such that:

$$|\sup g - \inf g| \leq C(\delta, \Lambda) \tag{3.6}$$

Proof. as $a = \frac{2\pi}{\mathrm{Vol}D}$,

$$\int_\Sigma \frac{1}{2g-2} - a\mathbf{1}_D = 0 \tag{3.7}$$

hence g is well-defined.

First, we have $\int g\Delta g = -|\nabla g|_2^2$, so

$$|\nabla g|_2^2 = a\int_D g \leq 2\pi \sup g \tag{3.8}$$

Recall Bochner's identity

$$\int |\nabla^2 g|^2 = \int (\Delta g)^2 + \int |\nabla g|^2 \tag{3.9}$$

which translates to give the estimate

$$|\nabla^2 g|_2^2 \leq \frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{2\pi}{2g-2} + 2\pi \sup g \tag{3.10}$$

As $\int g = 0$, by definition of the spectral gap

$$|g|_2^2 \leq \Lambda |\nabla g|_2^2 \leq 2\pi\Lambda \sup g\,. \tag{3.11}$$

Combining all this, we bound the $W^{2,2}$ norm

$$|g|_{2,2}^2 = |\nabla^2 g|_2^2 + |\nabla g|_2^2 + |g|_2^2 \leq \frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{2\pi}{2g-2} + 2\pi(2+\Lambda)\sup g \tag{3.12}$$

By lemma 3.4, this implies

$$|\sup g - \inf g| \leq \frac{2M}{\delta}\left(\frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{2\pi}{2g-2} + 2\pi(2+\Lambda)\sup g\right)^{\frac{1}{2}}. \tag{3.13}$$

Because $\int g = 0$, we know that $\inf g \leq 0$ and $\sup g \geq 0$. Thus

$$0 \leq (\sup g)^2 \leq \frac{4M^2}{\delta^2}\left(\frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{\pi}{g-1} + 2\pi(2+\Lambda)\sup g\right) \tag{3.14}$$

Going to the squares, this is a control of the type $X^2 \leq AX + B$, with $A, B > 0$. Explicitely, A and B are the following

$$\begin{cases} A &= \frac{8\pi(2+\Lambda)M^2}{\delta^2} \\ B &= \frac{4M^2}{\delta^1}\left(\frac{4\pi^2}{\mathrm{Vol}(D)} - \frac{\pi}{g-1}\right) \end{cases} \tag{3.15}$$

so it implies the existence of $C(\delta, \Lambda) = \frac{A+\sqrt{A+4B}}{2}$ such that

$$\sup g \leq C(\delta, \Lambda) \tag{3.16}$$

Replugging this into the Inequation 3.13, we get the existence of $\widetilde{C}(\delta, \Lambda)$ satisfying

$$|\sup g - \inf g| \leq \widetilde{C}(\delta, \Lambda) \tag{3.17}$$

as asserted. □

It happens that the oscillations of g away from $D(z_0, r)$ are actually the same as those of the log of the section s associated to the line bundle of divisor $\{z_0\}$.

Proposition 3.6. *Let N be a degree 1 line bundle over Σ and $s \in H^0(N)$ section, with a unique zero z_0. Denote $D = D(z_0, \delta)$ the disk of radius δ around z_0. Denote $a = \frac{2\pi}{Vol(D)}$. Consider $g : \Sigma \to \mathbb{R}$ the function associated to z_0 defined in Proposition 3.5. Denote by f the function $f = -\frac{1}{2}\log|s|^2$. Then $f - g$ is bounded and constant on $\Sigma - D(z_0, \delta)$.*

Proof. Let r be the radial geodesic coordinate on $D(z_0, \delta)$. With the notations above, define the map $h : D \to \mathbb{R}$:

$$h(r) = -2a \ln\cosh(\frac{r}{2}) + B + \ln(\tanh\frac{r}{2})\,. \tag{3.18}$$

As h is radial, we can explicitly compute the laplacian of h:

$$\Delta h = \partial_r^2 h + \frac{1}{\tanh r}\partial_r h = -a \tag{3.19}$$

As we also have

$$|z - z_0| = \tanh(\frac{r}{2}) \tag{3.20}$$

Because z_0 is a simple zero of s, locally:

$$|s(z)|^2 \approx K|z - z_0|^2, \quad f(z) = -\ln\tanh\frac{r}{2} + \mathcal{O}(1). \tag{3.21}$$

And so $f + h$ is bounded.

Also, we compute the radial derivative at the boundary of the disk of h:

$$\partial_r h(\delta) = (\frac{1}{2} - a)\tanh\frac{\delta}{2} + \frac{1}{2\tanh\frac{\delta}{2}} \tag{3.22}$$

But by definition of a,

$$a = \frac{2\pi}{\mathrm{Vol}(D)} = \frac{2\pi}{4\pi\sinh^2(\frac{r}{2})} = \frac{1}{2\sinh^2\frac{r}{2}} \tag{3.23}$$

which implies

$$\partial_r h(\delta) = 0\,. \tag{3.24}$$

Hence with the appropriate choice of B we can extend h by zero to get a C^1 function on Σ such that $f+h$ is bounded and satisfies

$$\Delta(f+h) = \frac{1}{2g-2} - a\mathbf{1}_D \tag{3.25}$$

This implies that $f+h = g+A$ for some constant $: wA$. As h is constant out of D, this means that $f-g$ is constant on $\Sigma - D$, as asserted. ☐

Corollary 3.7. *Let Σ be a hyperbolic surface with systole δ and spectral gap Λ. Let N be a degree 1 line bundle over Σ and $\alpha \in H^0(N)$ section, with a unique zero z_0. Denote $D = D(z_0, \delta)$ the disk of radius δ around z_0. Then there is some $C(\delta, \Lambda)$ and $\lambda > 0$ such that*

$$\forall z \in \Sigma - D(z_0,\delta), \quad \frac{1}{C(\delta,\Lambda)} \leq \lambda|\alpha|^2 \leq C(\delta,\Lambda) \tag{3.26}$$

Proof. With the g defined in proposition 3.5, we know

$$|\sup g - \inf g| \leq C(\delta,\Lambda) \tag{3.27}$$

Denote $f = -\frac{1}{2}\ln|\alpha|^2$. By proposition 3.6, $f-g$ is constant on $\Sigma - D$. Thus

$$|\sup_{\Sigma-D} f - \inf_{\Sigma-D} f| = |\sup_{\Sigma-D} g - \inf_{\Sigma-D} g| \leq C(\delta,\Lambda) \tag{3.28}$$

which directly gives, on $\Sigma - D$:

$$\frac{\sup|\alpha|^2}{\inf|\alpha|^2} \leq e^{2C(\delta,\Lambda)} \tag{3.29}$$

Multiply by λ such that $(\sup\lambda|\alpha|^2)(\inf\lambda|\alpha|^2) = 1$ to get the desired result:

$$\forall z \in \Sigma - D, \quad e^{-C(\delta,\Lambda)} \leq \lambda|\alpha|^2(z) \leq e^{C(\delta,\Lambda)} \tag{3.30}$$

☐

It remains to check that we can also have an upper bound on D. We can do this by some version of the Schwarz lemma adapted to sections of positive line bundles:

Proposition 3.8. *Let Σ be a hyperbolic surface with systole δ and spectral gap Λ. Let N be a degree 1 line bundle over Σ and $\alpha \in H^0(N)$ section, with a unique zero z_0. Let $D = D(z_0, \delta)$ the disk of radius δ around z_0. Then there is $C(\delta) > 0$ satisfying:*

$$\forall z \in D, \quad |\alpha|^2(z) \leq C(\delta)|z|^2 \sup_{\partial D}|\alpha|^2 . \tag{3.31}$$

Proof. Denote $f = -\frac{1}{2}\ln|\alpha|^2$. f satisfies

$$\Delta f = \frac{1}{2g-2} \tag{3.32}$$

Denote r the radial geodesic coordinate on D, and consider the map h

$$h = f + \ln\tanh\frac{r}{2} - \frac{2}{2g-2}\ln\cosh\frac{r}{2} \tag{3.33}$$

Then h is bounded on D and satisfies

$$\Delta h = 0 \tag{3.34}$$

By the minimum principle,

$$\inf_D h = \inf_{\partial D} h = \inf_{\partial D} f + \ln\tanh\frac{\delta}{2} - \frac{1}{g-1}\ln\cosh\frac{\delta}{2}. \tag{3.35}$$

Exponentiating, we get the upper bound on $|\alpha|$:

$$\forall z \in D, \quad \frac{|\alpha|^2}{|z|^2}\cosh(\frac{r}{2})^{\frac{2}{g-1}} \leq \sup_{\partial D}|\alpha|^2 \frac{(\cosh\frac{\delta}{2})^{\frac{2}{g-1}}}{(\tanh\frac{\delta}{2})^2} \tag{3.36}$$

Because $g > 1$ and $\cosh(t) \geq 1$, this implies

$$|\alpha|^2(z) \leq |z|^2 \sup_{\partial D}|\alpha|^2 \frac{(\cosh\frac{\delta}{2})^2}{(\tanh\frac{\delta}{2})^2} \tag{3.37}$$

□

Finally, we combine these results to show our estimate of oscillations of sections of degree 1 line bundles:

Proposition 3.9. *Let Σ be a hyperbolic surface. Denote δ its systole, Λ its spectral gap and consider $r < \frac{\delta}{2}$. There are constants $C_i(\delta, \Lambda, r)$ such that, for any holomorphic section s of a degree 1 line bundle with a unique zero $z_0 \in \Sigma$, there is a $\lambda > 0$ such that:*

$$|\lambda s|^2 \leq C_2 \text{ on } B(z_0, r) \tag{3.38}$$

and

$$\frac{1}{C_1} \leq |\lambda s|^2 \leq C_1 \text{ on } \Sigma - B(z_0, r) \tag{3.39}$$

Remark 3.2. Up to renormalizing our section, one can always assume the first condition to be satisfied on the ball of radius r, and then the results states that the oscillations away from the zero of the section are controlled, essentially by the spectral gap and the systole of the surface.

Proof. Let Σ be a hyperbolic surface with systole δ and spectral gap Λ. Let N be a degree 1 line bundle over Σ, with $\alpha \in H^0(N)$ a holomorphic section having one unique zero z_0. Consider $D = D(z_0, \delta)$. Thanks to corollary 3.7, up to renormalizing α, there is $C(\delta, \Lambda) > 0$ such that

$$\forall z \in \Sigma - D, \quad \frac{1}{C(\delta, \Lambda)} \leq |\alpha|^2(z) \leq C(\delta, \Lambda) \tag{3.40}$$

Now because of proposition 3.8, there is $C(\delta)$ such that

$$\forall z \in D, \quad |\alpha|^2 \leq C(\delta)C(\delta, \Lambda) \tag{3.41}$$

as asserted. □

Multiplying those sections with covers of a section of a degree $4(g-1)$ line bundle will result in a balanced section of a bundle of desired degree $4n(g-1)+1$.

proof of theorem 3.3. Let Σ be a closed hyperbolic surface of genus $g \geq 2$. Consider Σ_n a family of degree n covers of Σ such that the spectral gap of Σ_n remains lower bounded by $\Lambda > 0$. This kind of cover exists, for instance we can use the result of [MNP22], which states that for a random cover of Σ, the probability that the relative spectral gap (i.e. the part of the spectral gap which is not in Σ) of the cover is bigger than $\frac{3}{16} - \varepsilon$ goes to 1 when the degree is large. Hence for any $\eta < \min\{\Lambda, \frac{3}{16}\}$, we can choose a family of covers whose spectral gap is larger than η.

Of course, the systole is nondecreasing by taking a finite cover, hence we can consider δ the systole of Σ.

Consider s a 2-form on Σ, hence a section of K^2, which is of degree $4g-4$. Denote s_n the lift of s to Σ_n, which is a section of a degree $4n(g-1)$ line bundle. Obviously, we have

$$\int |s_n|^2 = n\int |s|^2, \quad |s_n|_\infty = |s|_\infty \tag{3.42}$$

Also, consider (z_n) a sequence of points, $z_n \in \Sigma_n$. Define $L_n = \mathcal{O}(z_n)$ the degree 1 line bundle associated to the divisor $\{z_n\}$.

Thanks to Proposition 3.9, there are sections τ_n of L_n, and constants $C_i(\delta, \Lambda)$ such that

$$\begin{cases} \frac{1}{C_1(\delta,\Lambda)} \leq |\tau_n|^2 \leq C_1(\delta,\Lambda) & \text{out of} \quad D(z_n,\delta) \\ \qquad |\tau_n|^2 \leq C_2(\delta,\Lambda) & \text{on} \quad D(z_n,\delta) \end{cases} \tag{3.43}$$

Consider now $N_n = K^2 L_n$ and $\alpha_n = s_n\tau_n$ section of N_n line bundle of degree $4n(g-1)+1$. By construction, we have the estimates

$$\frac{1}{C_1(\delta,\Lambda)}|s|_\infty^2 \leq |\alpha_n|_\infty^2 \leq C_2(\delta,\Lambda)|s|_\infty^2 \tag{3.44}$$

$$\frac{n-1}{C_1(\delta,\Lambda)}\int |s|^2 \leq \int |\alpha_n|^2 \leq C_1(n-1)\int |s|^2 + C_2\int |s|^2 \tag{3.45}$$

Eventually,

$$\frac{|\alpha_n|_\infty^2}{\fint |\alpha_n|^2} \leq \frac{n}{n-1}\frac{|s|_\infty^2}{\int |s|^2} C_1(\delta,\Lambda)C_2(\delta,\Lambda)\mathrm{Vol}(\Sigma) \tag{3.46}$$

which is bounded in n, so the family (N_n, α_n) is balanced. □

Remark 3.3. In a broader setting, given a section s of a degree d line bundle, we expect the ratio between the L^2 average and the L^∞ norm to be an quantitative indicator of how much does s look like a lift of a section on a smaller surface. With this in mind, we expect this ratio to be controlled by the lowest distance between the zeroes of s. This would have the advantage of being an open property, and that when it is actually possible, the minimal ratio is gotten by a lift.

Remark 3.4. The same proof can be adapted to show that for any $a, d \geq 0$, we can build a balanced family $(\Sigma_g, N_g, \alpha_g)$ with N_g of degree $a(g-1)+d$.

4. Estimates on the solutions to the flat curvature equations

Throughout this section, Σ will be a closed hyperbolic surface of genus $g \geq 2$. Unless otherwise precised, N will be a degree 1 line bundle over Σ and α will be a holomorphic 2-form valued in N, i.e. $\alpha \in H^0(K^2N)$. Fix $c = \frac{1}{2g-2}$, and consider the following PDE system:

$$\begin{cases} \Delta u = & e^{2u} - 1 + e^{-2u}e^{2v}|\alpha|^2 \\ \Delta v = & c - e^{-2u}e^{2v}|\alpha|^2 \end{cases} \tag{4.1}$$

In this part, we consider each equation separately, considering the first as giving u depending on v, and vice-versa for the second.

4.1. **Dealing with Gauss'equation.** Fixing a map u, the first equation is the famous Gauss equation, and is well-behaved. As the data v and α are fixed, we denote $f = e^{2v}|\alpha|^2$, and our problem boils down to solve

$$\Delta u = -1 + e^{2u} + e^{-2u} f \tag{4.2}$$

Denote by $C^{0,\alpha}(\Sigma)$ the set of continuous α-Hölder functions on Σ. Looking at the results of [BS23], we can apply the same sub-supersolution method to get:

Theorem 4.1. *Let $\eta \in (0,1]$. Consider the set*

$$\mathcal{C} := \{f \in C^{0,\alpha}(\Sigma) : 0 \leq f \leq \frac{\eta}{(1+\eta)^2}\} \tag{4.3}$$

Then for any $f \in \mathcal{C}$, there is a unique $u \in C^{2,\alpha}(\Sigma)$ satisfying:

$$-\frac{\ln 2}{2} \leq u \leq 0 \tag{4.4}$$

$$\Delta u = e^{2u} - 1 + e^{-2u} f \tag{4.5}$$

Moreover, u satisfies the following upper bound:

$$e^{-2u} f \leq \eta\,. \tag{4.6}$$

Proof. We will proceed with the sub-supersolution method. A subsolution v_- to the equation $\Delta u = F(u)$ is a function v_- satisfying $\Delta v_- \geq F(v_-)$. Analogously, a supersolution is a function v_+ satisfying $\Delta v_+ \leq F(v_+)$.

The sub-supersolution principle states that the existence of a subsolution v_- and a supersolution v_+ satisfying $v_- \leq v_+$ ensures the existence of a solution v satisfying $v_- \leq v \leq v_+$.

Freely, 0 is a supersolution to the equation 4.5. As $f \in \mathcal{C}$, the constant map $-\frac{\ln(1+\eta)}{2}$ is a subsolution. Hence by the sub-supersolution principle, there is a solution u satisfying

$$-\frac{\ln 2}{2} \leq u \leq 0\,. \tag{4.7}$$

To prove uniqueness, assume u_1 and u_2 are both solutions in $(-\frac{\ln 2}{2}, 0)$ of

$$\Delta u_i = e^{2u_i} - 1 + e^{-2u_i} f \tag{4.8}$$

Then the difference $w = u_1 - u_2$ satisfies

$$\Delta w = (e^{2u_2} - e^{-2u_2} f)(e^{2w} - 1) + f e^{-2u_2}(e^{2w} + e^{-2w} - 2)\,. \tag{4.9}$$

We observe that the second term is nonnegative, thus

$$\Delta w \geq (e^{2u_2} - e^{-2u_2} f)(e^{2w} - 1) \tag{4.10}$$

but as $f \in \mathcal{C}$ and $u_2 \geq \frac{-\ln 2}{2}$, we get

$$e^{-4u_2} f \leq 1 \tag{4.11}$$

which implies

$$e^{2u_2} - e^{-2u_2} f| \geq 0 \tag{4.12}$$

hence, by the maximum principle, at a maximum of w,

$$(e^{2u_2} - e^{-2u_2} f)(e^{2w} - 1) \leq 0 \tag{4.13}$$

Hence

$$w \leq 0\,. \tag{4.14}$$

Exchanging the roles of u_1 and u_2 shows

$$u_1 = u_2\,. \tag{4.15}$$

So there is a unique solution in $[-\frac{\ln 2}{2}, 0]$.

Finally, because $-\frac{\ln(1+\eta)}{2} \le u \le 0$, we deduce

$$e^{-2u} f \le \frac{\eta}{1+\eta} \le \eta\,. \tag{4.16}$$

As asserted. ☐

Proposition 4.2. *The map $\Phi : C^{0,\alpha}(\Sigma, [0, \frac{1}{2}]) \to C^{2,\alpha}(\Sigma)$ which to f associates the unique u solution of*

$$\Delta u = e^{2u} - 1 + e^{-2u} f \tag{4.17}$$

satisfying $\frac{-\ln 2}{2} \le u \le 0$ is continuous.

Proof. Thanks to Theorem 4.1, Φ is well-defined. For $f \in C^{0,\alpha}(\Sigma, [0, \frac{1}{2}])$ given, consider the linearized equation

$$\Delta \dot{u} = 2(e^{2u} - e^{-2u} f)\dot{u} \tag{4.18}$$

The assumptions on u imply

$$e^{2u} - e^{-2u} f \ge 0 \tag{4.19}$$

By a maximum principle, the only solution of the linearized equation is 0, hence our solutions are stable and the defined map is continuous. ☐

4.2. **Dealing with Ricci's equation.** Considering the second equation, we will fix v and express a solution v as the maximizer of the following functional:

Notation 4.3. For $u \in C^{1,\alpha}(\Sigma)$, consider $W_0^{1,2}(\Sigma)$ the Sobolev space of zero average functions that are square integrable with square integrable gradient. Define the functional $J_u : W_0^{1,2}(\Sigma) \to \mathbb{R}$ as

$$J_u(w) := \log\Big(\fint e^{-2u}|\alpha|^2 e^{2w}\Big) - \frac{1}{c\mathrm{Vol}\Sigma}\int |\nabla w|^2 \tag{4.20}$$

with $c = \frac{1}{2g-2}$. The notation $\fint$ is used for the average of the considered function.

We will also use the notation $|w|_2^2$ for the squared L^2-norm:

$$|w|_2^2 = \int_\Sigma |w|^2 \tag{4.21}$$

When no confusion is possible, we will write J for J_u. We will prove that J attains its maximum at a function w which will be, up to an addittive constant, the solution v that we are looking for. This proof is heavily inspired from the method used to solve the prescribed curvature equation on the sphere when the prescribed data is even. Chang–Yang [CY03] use this method in this precise case. Keep in mind that our equation is not a prescribed curvature equation on the surface, but rather a prescribed curvature equation on a disk bundle on it.

Lemma 4.4. *J is well-defined and weakly continuous on $W_0^{1,2}(\Sigma)$.*

Proof. First, we need to check that J is well-defined. This is true as Σ is compact, and the Sobolev constants in the injections $W^{1,2} \to L^p$ are exponentially integrable (see Ilias [Ili83])

$$\exists C > 0 : \forall w \in W_0^{1,2},\ \text{with} \int |\nabla w|^2 \leq 1 : \int e^{2w} \leq C \tag{4.22}$$

As $e^{-2u}|\alpha|^2$ is bounded on Σ, we deduce that $J(w)$ is finite. Hence J is well-defined. It is standard that the L^2 norm of the gradient is weakly continuous on $W^{1,2}(\Sigma)$. To be sure that J is weakly continous, it remains to prove that the map

$$H : w \mapsto \log\big(\fint e^{-2u}|\alpha|^2 e^{2w}\big) \tag{4.23}$$

is. A simple computation shows that H is actually convex, hence it is weakly continuous if and only if it is strongly continuous. It is strongly continuous because of the aforementioned property, equation (4.22). $\square$

In order to study explicitly the behavior of J, we will need a more precise control on the Sobolev constants of the functions involved, namely we will use a precise Moser-Trudinger inequality obtained in [Bro23].

Theorem 4.5 (Moser–Trudinger, B.)**.** *Denote respectively by δ and Λ the systole and spectral gap of Σ. There is a constant $C(\delta, \Lambda)$ depending only on δ, Λ and continuously in those parameters such that:*

$$\forall u \in W_0^{1,2}(\Sigma), \int |\nabla u|^2 \leq 1 \Rightarrow \int (e^{4\pi u^2} - 1) \leq C(\delta, \Lambda) \tag{4.24}$$

Corollary 4.6. *The functional J is upper bounded on $W_0^{1,2}(\Sigma)$, and we have the following estimate:*

$$\log\big(\fint e^{-2u}|\alpha|^2\big) \leq \sup J \leq \log\max e^{-2u}|\alpha|^2 + \log\big(1 + \frac{C(\delta,\Lambda)}{Vol(\Sigma)}\big)\,. \tag{4.25}$$

Also, J is proper on $W_0^{1,2}(\Sigma)$, as the following holds

$$\forall w \in W_0^{1,2}(\Sigma),\ J(w) + \frac{1}{4\pi}|\nabla w|_2^2 \leq \log\max e^{-2u}|\alpha|^2 + \log\big(1 + \frac{C(\delta,\Lambda)}{Vol(\Sigma)}\big)\,. \tag{4.26}$$

Proof. Let $w \in W_0^{1,2}(\Sigma)$. Consider that

$$2w \leq 4\pi\frac{w^2}{|\nabla w|_2^2} + \frac{|\nabla w|_2^2}{4\pi} \tag{4.27}$$

Exponentiating this inequality, we get

$$e^{2w} \leq e^{\frac{|\nabla w|_2^2}{4\pi}} e^{4\pi\left(\frac{w}{|\nabla w|_2}\right)^2} \tag{4.28}$$

Using theorem 4.5 to bound the integral of the right-hand side,

$$\int e^{2w} \leq (C(\delta,\Lambda) + \mathrm{Vol}(\Sigma))e^{\frac{|\nabla w|_2^2}{4\pi}} \tag{4.29}$$

We deduce

$$J(w) \leq \log\max e^{-2u}|\alpha|^2 + \big(\frac{1}{4\pi} - \frac{1}{c\mathrm{Vol}\Sigma}\big)|\nabla w|_2^2 + \log\big(1 + \frac{C(\delta,\Lambda)}{\mathrm{Vol}\Sigma}\big) \tag{4.30}$$

As $\mathrm{Vol}(\Sigma) = 4\pi(g-1)$,and the bundle is of degree 1, necessarily

$$\frac{1}{4\pi} - \frac{1}{c\mathrm{Vol}(\Sigma)} = -\frac{1}{4\pi} \tag{4.31}$$

Inserting it on the left-hand side, we get

$$J(w) + \frac{1}{4\pi}|\nabla w|_2^2 \leq \log \max e^{-2u}|\alpha|^2 + \log\left(1 + \frac{C(\delta,\Lambda)}{\mathrm{Vol}(\Sigma)}\right). \tag{4.32}$$

So J is indeed proper and upper-bounded with the desired explicit upper bound.
The lower estimate for $\sup J$ is gotten by expliciting $J(0) \leq \sup J$. □

Proposition 4.7. *The functional J attains its maximum at $w \in W_0^{1,2}(\Sigma)$ satisfying*

$$\frac{1}{4\pi}|\nabla w|_2^2 \leq \log \frac{\max e^{-2u}|\alpha|^2}{\fint e^{-2u}|\alpha|^2} + \log\left(1 + \frac{C(\delta,\Lambda)}{Vol\Sigma}\right). \tag{4.33}$$

Proof. Thanks to Corollary 4.6, we know that J is upper bounded. Consider a maximizing sequence (w_n) of J is $W_0^{1,2}(\Sigma)$. Writing down our estimate

$$J(w_n) + \frac{1}{4\pi}|\nabla w_n|_2^2 \leq \log \max e^{-2u}|\alpha|^2 + \log\left(1 + \frac{C(\delta,\Lambda)}{\mathrm{Vol}\Sigma}\right) \tag{4.34}$$

As $(J(w_n))$ is upper bounded by $\sup J$, so is $(|\nabla w_n|_2^2$, We deduce that $(|\nabla w_n|_2^2)$ is bounded. The explicit bound, assuming $J(w_n) \geq J(0)$, is

$$\frac{1}{4\pi}|\nabla w_n|_2^2 \leq \log \frac{\max e^{-2u}|\alpha|^2}{\fint e^{-2u}|\alpha|^2} + \log\left(1 + \frac{C(\delta,\Lambda)}{\mathrm{Vol}\Sigma}\right). \tag{4.35}$$

As (w_n) is bounded in $W_0^{1,2}(\Sigma)$, extracting a subsequence we can assume it converges weakly towards some $w \in W_0^{1,2}(\Sigma)$. By lemma 4.4, J is weakly continuous, so w is a maximizer of J. As $|\nabla w|_2^2 \leq \liminf |\nabla w_n|_2^2$, we get the desired bound on w. □

Theorem 4.8. *Let N be of degree 1, $\alpha \in H^0(K^2N)$, $c = \frac{1}{2g-2}$ and $u \in C^{0,\alpha}(\Sigma, [\frac{-\ln 2}{2}, 0])$. Then there is a solution v to*

$$\Delta v = c - e^{2v}e^{-2u}|\alpha|^2 \tag{4.36}$$

satisfying

$$\frac{1}{4\pi}|\nabla v|_2^2 \leq \log \frac{\max e^{-2u}|\alpha|^2}{\fint e^{-2u}|\alpha|^2} + \log\left(1 + \frac{C(\delta,\Lambda)}{Vol\Sigma}\right). \tag{4.37}$$

Proof. Thanks to proposition 4.7, there exist $w \in W_0^{1,2}(\Sigma)$ a maximizer of the functional J. For any $s \in W_0^{1,2}$, $D_w J \cdot s = 0$ and so

$$\frac{2}{\mathrm{Vol}(\Sigma)} \frac{\int e^{-2u}|\alpha|^2 e^{2w} s}{\fint e^{-2u}|\alpha|^2 e^{2w}} - \frac{2}{c\mathrm{Vol}(\Sigma)} \int \nabla w \cdot \nabla s = 0 \tag{4.38}$$

Weakly, this means

$$\frac{1}{c}\Delta w + \frac{e^{-2u}|\alpha|^2 e^{2w}}{\fint e^{-2u}|\alpha|^2 e^{2w}} \in (L_0^2)^\perp \tag{4.39}$$

As the orthogonal of the zero average functions are the constant maps, there is a constant A such that

$$\Delta w = A - c\frac{e^{-2u}|\alpha|^2 e^{2w}}{\fint e^{-2u}|\alpha|^2 e^{2w}} \tag{4.40}$$

By the elliptic regularity principle, as u is in $C^{0,\alpha}(\Sigma)$, w belongs to $C^{2,\alpha}(\Sigma)$ and satisfies the above equation strongly.

Writing $\int \Delta w = 0$, we deduce

$$A = c\,. \tag{4.41}$$

Consider the element of $W^{1,2}(\Sigma)$:

$$v = w + \frac{1}{2}\ln\big(\frac{c}{\fint e^{-2u}|\alpha|^2 e^{2w}}\big) \tag{4.42}$$

As $\Delta v = \Delta w$, we get

$$\Delta v = c - e^{-2u}|\alpha|^2 e^{2v} \tag{4.43}$$

As $|\nabla w|_2^2 = |\nabla v|_2^2$, the estimate of Proposition 4.7 ensures the desired bound on $|\nabla v|_2^2$. $\square$

Remark 4.1. In contrast with the Gauss equation, we cannot in this case establish the uniqueness of the solution v. We will address this problem by showing stability of the solutions under some conditions after stating some precise estimates on the solution w and v.

4.3. **Estimating the regularity of the solution.** In order to apply later a fixed point theorem, we need to understand the behavior of the solutions v. Particularly, we need an L^∞ bound on $e^{2v}|\alpha|^2$. Because of the sign difference with Gauss'equation, applying a maximum principle won't yield bounds on v. We get our upper bound via a $W^{2,2}$-estimate on the maximizer w of J_u.

Proposition 4.9. *Assume $u \in C^{0,\alpha}(\Sigma)$. Then a maximizer w of the functional J_u satisfies*

$$|\nabla^2 w|_2^2 \leq c^2\Bigg(\big(Vol\Sigma + C(\delta,\Lambda)\big)\big(\frac{\max|\alpha|^2}{\fint |\alpha|^2}\big)^2 - Vol\Sigma\Bigg) + 4\pi\Bigg(\log\frac{\max e^{-2u}|\alpha|^2}{\fint e^{-2u}|\alpha|^2} + \log\big(1+\frac{C(\delta,\Lambda)}{Vol\Sigma}\big)\Bigg) \tag{4.44}$$

Proof. Let w be a maximizer of J. First, we use the Bochner identity to estimate the Hessian of w

$$|\nabla^2 w|_2^2 = \int (\Delta w)^2 + \int |\nabla w|^2\,. \tag{4.45}$$

Since Δw satisfies

$$\Delta w = c - \frac{c|\alpha|^2 e^{-2u} e^{2w}}{\fint e^{-2u}|\alpha|^2 e^{2w}}\,. \tag{4.46}$$

We deduce that

$$\int (\Delta w)^2 = -c^2\mathrm{Vol}\Sigma + c^2(\mathrm{Vol}\Sigma)^2\frac{\int e^{-4u}|\alpha|^4 e^{4w}}{(\int e^{-2u}|\alpha|^2 e^{2w})^2} \tag{4.47}$$

To control the right hand side, we estimate $4w$

$$4w \leq \frac{4\pi w^2}{|\nabla w|_2^2} + \frac{|\nabla w|_2^2}{\pi} \tag{4.48}$$

and apply the Moser–Trudinger inequality of Theorem 4.5 to get

$$\int e^{4w} \leq (\mathrm{Vol}\Sigma + C(\delta,\Lambda)) e^{\frac{|\nabla w|_2^2}{\pi}} \tag{4.49}$$

This ensures

$$\int |\alpha|^4 e^{-4u} e^{4w} \leq (\mathrm{Vol}\Sigma + C(\delta,\Lambda)) \max e^{-4u}|\alpha|^4 \exp\big(\frac{|\nabla w|_2^2}{\pi}\big)\,. \tag{4.50}$$

It remains to estimate the denominator of the right hand side. As $c\mathrm{Vol}\Sigma = 2\pi$ and by maximality $J(w) \geq J(0)$, we have

$$\log \fint |\alpha|^2 e^{-2u} \leq \log \fint e^{-2u}|\alpha|^2 e^{2w} - \frac{1}{2\pi}|\nabla w|_2^2\,. \tag{4.51}$$

Exponentiating this,

$$\int e^{-2u}|\alpha|^2 \exp\big(\frac{1}{2\pi}|\nabla w|_2^2\big) \leq \int e^{-2u}|\alpha|^2 e^{2w} \tag{4.52}$$

Combining this with the upper bound on the numerator, we get

$$\frac{\int e^{-4u}|\alpha|^4 e^{4w}}{\big(\int e^{-2u}|\alpha|^2 e^{2w}\big)^2} \leq \frac{(\mathrm{Vol}\Sigma + C(\delta,\Lambda)) \max e^{-4u}|\alpha|^4 \exp(\frac{|\nabla w|_2^2}{\pi})}{\big(\int e^{-2u}|\alpha|^2 \exp(\frac{1}{2\pi}|\nabla w|_2^2)\big)^2} \tag{4.53}$$

It happens that the left-hand side simplifies to give

$$\int (\Delta w)^2 \leq -c^2\mathrm{Vol}\Sigma + c^2(\mathrm{Vol}\Sigma + C(\delta,\Lambda)) \frac{\big(\max e^{-2u}|\alpha|^2\big)^2}{(\fint e^{-2u}|\alpha|^2)^2} \tag{4.54}$$

Combining with the bound on $|\nabla w|_2^2$ from Proposition 4.7, we get the desired result. □

We sum up with the local upper bound on the solution, which will be key to building solutions of the PDE system:

Proposition 4.10. *Let $u : \Sigma \to \mathbb{R}$ be a map in $C^{0,\alpha}(\Sigma)$ such that*

$$1 \leq e^{-2u} \leq 2\,. \tag{4.55}$$

Then there is a constant $C_1 = C_1(\delta, \Lambda, \frac{|\alpha|_\infty^2}{\fint |\alpha|^2})$, and $v \in C^{2,\alpha}(\Sigma)$ a solution of

$$\Delta v = c - e^{-2u}e^{2v}|\alpha|^2 \tag{4.56}$$

satisfying

$$\sup e^{2v}|\alpha|^2 \leq \frac{C_1(\delta,\Lambda,\frac{|\alpha|_\infty^2}{\fint |\alpha|^2})}{2g-2} \tag{4.57}$$

The constant C_1 is continuous in the parameters δ, Λ and $\frac{|\alpha|_\infty^2}{\fint |\alpha|^2}$.

Proof. Once again, consider $w \in W_0^{1,2}$ a maximizer of J And build the solution v as a translated of w so that

$$w = v - \overline{v}\,. \tag{4.58}$$

Thanks to Proposition 4.9, there is a bound $C_1(\delta, \Lambda, \frac{|\alpha|_\infty^2}{\fint |\alpha|^2})$ such that

$$|w|_{2,2} \leq C_1(\delta,\Lambda,\frac{|\alpha|_\infty^2}{\fint |\alpha|^2}) \tag{4.59}$$

As the continuous embedding $W^{2,2} \hookrightarrow L^\infty$ depends only on the systole δ, this implies that we have $C_2 > 0$ with the same dependencies such that

$$|w|_\infty \leq C_2(\delta, \Lambda, \frac{|\alpha|_\infty^2}{\fint |\alpha|^2}) \,. \tag{4.60}$$

We compute the average of $e^{2v}|\alpha|^2$

$$c = \fint e^{-2u}e^{2v}|\alpha|^2 = e^{2\overline{v}} \fint e^{-2u}e^{2w}|\alpha|^2 \tag{4.61}$$

from which we get the estimate:

$$e^{2\overline{v}}e^{-2C_2} \fint |\alpha|^2 \leq \frac{1}{2g-2} \leq 2e^{2\overline{v}}e^{2C_2} \fint |\alpha|^2 \tag{4.62}$$

Combining this, we get

$$e^{2v}|\alpha|^2 = e^{2\overline{v}}e^{2w}|\alpha|^2 \leq \frac{e^{4C_2}}{2g-2}\frac{|\alpha|_\infty^2}{\fint |\alpha|^2} \tag{4.63}$$

as asserted. As the constant C_1 from Proposition 4.9 is continuous in the parameters, so is the exhibited constant C_2. □

It remains to show the stability of the solutions under suitable perturbations.

Lemma 4.11. *Consider $f \geq 0$ a function in $C^{0,\alpha}(\Sigma)$, such that there is v solution of*

$$\Delta v = c - e^{2v} f \tag{4.64}$$

satisfying $2e^{2v}f < \Lambda$. Consider the operator $H = \Delta + 2e^{2v}f : W^{2,2}(\Sigma) \to L^2(\Sigma)$. Then H is invertible, and we have the upper bound on its inverse:

$$||H^{-1}|| \leq \max\{(\Lambda - 2|e^{2v}f|_\infty)^{-1}, c^{-1}\} \tag{4.65}$$

Proof. H is a self-adjoint operator between Hilbert spaces. Moreover, for λ large enough, we have the estimate:

$$|(\lambda I - H)w|_2^2 |(\lambda - e^{2v}f)w - \Delta w|_2^2 \tag{4.66}$$

$$\leq (\lambda - \frac{\Lambda}{2})^2 \int w^2 + \int (\Delta w)^2 - 2\int (\lambda - e^{2v}f)w\Delta w \tag{4.67}$$

$$\leq C|w|_{2,2}^2 \tag{4.68}$$

Hence for λ large enough, $(\lambda I - H)$ has compact inverse, so it is a Fredholm operator, and H has discrete spectrum.

To conclude the lemma, it remains to check that H has no eigenvalue in the range $(-\Lambda + |2e^{2v}f|_\infty, 2c)$.

Let t be in this range. If $\dot{v}$ is an eigenvector for the value t, we have

$$\Delta \dot{v} + (2e^{2v}f - t)\dot{v} = 0 \tag{4.69}$$

Decompose $\dot{v} = m(\dot{v}) + \check{v}$ with $m(\dot{v})$ being the average of $\dot{v}$. Then it comes

$$m(\dot{v}) \int 2e^{2v}f - t = -\int (2e^{2v}f - t)\check{v} \tag{4.70}$$

$$\int |\nabla \check{v}|^2 - (2e^{2v}f - t)\check{v}^2 = m(\dot{v}) \int (2e^{2v}f - t)\check{v} \tag{4.71}$$

Combining these two equations with $\int e^{2v} f = c$, we get

$$\int |\nabla \check{v}|^2 - (2e^{2v} f - t)\check{v}^2 = -2m(v)^2(2c-t)\mathrm{Vol}\Sigma \tag{4.72}$$

As $t < 2c$, the right-hand side is nonpositive, while because $-\Lambda + 2e^{2v} f < t$, the left-hand side is nonnegative. So both must be zero, and $\dot{v} = 0$, as desired. $\square$

A consequence of this lemma is that we locally able to lift solutions to $\Delta v = c - e^{2v} f$. But we will need to be able to lift to a continuous branch for all f obtained from solving the Gauss-Codazzi equation. To this aim, we give a more precise description of the open set on which we are able to lift:

Theorem 4.12. *Let Σ be a closed hyperbolic surface. We will denote by (C_i) constants depending on the systole, the spectral gap, and the ratio $\frac{|\alpha|^2_\infty}{f\,|\alpha|^2}$. Consider v_0 a solution of*

$$\Delta v_0 = c - e^{2v_0}|\alpha|^2 \tag{4.73}$$

satisfying

$$e^{2v}|\alpha|^2 \leq \frac{C_1}{2g-2} < \Lambda\,. \tag{4.74}$$

Fix $\varepsilon > 0$. Consider the functional $\mathcal{H} : C^{0,\alpha}(\Sigma) \times W_{2,2}(\Sigma) \to L^2(\Sigma)$:

$$\mathcal{H}(f, v) = \Delta v - c + e^{2v} f|\alpha|^2 \tag{4.75}$$

Then there is $C_2 > 0$, such that for $|f-1|_\infty \leq \frac{C_2}{\sqrt{2g-2}}$, there is a unique $v \in W^{2,2}$ satisfying:

$$\begin{cases} |v - v_0|_{2,2} & \leq \varepsilon \\ \Delta v & = c - e^{2v} f|\alpha|^2 \end{cases} \tag{4.76}$$

In particular, this v also satisfies:

$$e^{2v}|\alpha|^2 \leq \frac{C_3}{2g-2} \tag{4.77}$$

Proof. This explicit lift comes from an estimation of the bounds appearing in the implicit function theorem. We will use the version in [Hol70]. With the notations of the theorem, we need to compute $k_1 = |(D_v\mathcal{H}(f_0, v_0))^{-1}|$. The lemma 4.11 shows that, provided $\frac{1}{2(2g-2)} \leq \Lambda$,

$$k_1 \leq \frac{1}{2}(2g-2)\,. \tag{4.78}$$

Now we need to find $\delta > 0$ and g_1 such that, for $s \in [0, \delta]$, $v \in [0, \varepsilon]$

$$|f-1| \leq \delta,\ |v - v_0|_{2,2} \leq \varepsilon \Rightarrow ||D_v\mathcal{H}(f, v) - D_v\mathcal{H}(1, v_0)|| \leq g_1(|f-1|_\infty, |v-v_0|_{2,2}) \tag{4.79}$$

But, as

$$|D_v\mathcal{H}(f,v)\dot{v} - D_v\mathcal{H}(1,v_0)\dot{v}|_2^2 = \int (2(e^{2v}f - e^{2v_0})|\alpha|^2\dot{v})^2 \leq \frac{4C_1^2}{(2g-2)^2} e^{4|v-v_0|_\infty}|1-f|_\infty|\dot{v}|_2^2 \tag{4.80}$$

from which we deduce

$$||D_v\mathcal{H}(f,v) - D_v\mathcal{H}(1,v_0)|| \leq \frac{C_4}{2g-2} e^{2C_5|v-v_0|_{2,2}}|1-f|_\infty \tag{4.81}$$

Finally, we must find g_2 such that

$$|f-1|_\infty \leq \delta \Rightarrow ||\mathcal{H}(f,v_0)|| \leq g_2(|f-1|_\infty) \tag{4.82}$$

Explicitly,

$$|\mathcal{H}(f,v_0)|_2^2 = \int (e^{2v_0}|\alpha|^2(f-1)^2)^2 \leq \frac{C_6}{2g-2} \tag{4.83}$$

Of which we get

$$g_2(s) = \frac{C_7 s}{\sqrt{2g-2}} \tag{4.84}$$

Now we can lift a continuous branch satisfying $\mathcal{H}(f,\Psi(f)) = 0$ provided δ satisfies

$$k_1 g_1(\delta,\varepsilon) \leq \eta < 1 \text{ and } k_1 g_2(\delta) \leq \varepsilon(1-\eta) \tag{4.85}$$

This is verified for δ of the kind:

$$\delta = \frac{C_8}{\sqrt{2g-2}} \tag{4.86}$$

and the solutions $v = \Psi(f)$ satisfy

$$|v - v_0|_{2,2} \leq \varepsilon \tag{4.87}$$

Of which we deduce

$$e^{2v}|\alpha|^2 \leq e^{2C_9|v-v_0|_{2,2}} e^{2v_0}|\alpha|^2 \leq \frac{C_{10}}{2g-2} \tag{4.88}$$

as desired. □

5. Proof of the main theorem

In this section we will prove the geometric meaning of all this study:

Theorem 5.1. *Let Σ be a closed hyperbolic surface of genus g. Fix $\eta \in (0,1)$ and consider a family $(\Sigma_n, N_n, \alpha_n)_{n\geq 2}$ satisfying:*

(a) *Σ_n is a degree n cover of Σ, equipped with the lift of the hyperbolic metric on Σ.*
(b) *N_n is a degree $4n(g-1)+1$ line bundle over Σ_n, equipped with a metric of constant sectional curvature.*
(c) *$\alpha_n \in H^0(N_n)$ is a section of N_n.*
(d) *There is $\Lambda_0 > 0$ such that for all n the spectral gap of Σ_n is bigger than Λ_0.*
(e) *The family of sections is balanced, i.e.*

$$\sup_{n\geq 2} \frac{|\alpha_n|_\infty^2}{⨍ |\alpha_n|^2} < \infty\,. \tag{5.1}$$

Then, for n large enough, there are $u, v : \Sigma_n \to \mathbb{R}$ such that

$$\sup_{z\in\Sigma_n} e^{-4u} e^{2v}|\alpha|^2(z) < \eta \tag{5.2}$$

$$\begin{cases} \Delta u &= e^{2u} - 1 + e^{-2u}e^{2v}|\alpha_n|^2 \\ \Delta v &= \frac{1}{n(2g-2)} - e^{-2u}e^{2v}|\alpha_n|^2 \end{cases} \tag{5.3}$$

As a corollary, we get the desired existence theorem for almost-fuchsian representations with nontrivial normal bundle.

Corollary 5.2. *There is a genus g_0 such that for any $g \geq g_0$, there exists a representation $\rho : \pi_1\Sigma_g \to SO(4,1)$ which is almost-fuchsian, a complex variation of Hodge structure and the hyperbolic 4-manifold $\rho\backslash\mathbb{H}^4$ is a degree 1 disk bundle over Σ_g.*

Proof. Assuming Theorem 5.1, let Σ be a closed hyperbolic surface of genus 2. Thanks to the result of Magee–Naud–Petri [MNP22], we know the existence of a family of covers (Σ_n) satisfying condition (d) of Theorem 5.1. Thanks to Theorem 3.3, there is a balanced sequence (α_n, N_n) satisfying conditions (b), (c) and (e). Thus for n large enough we have (u, v) satisfying

$$\sup_{z\in\Sigma_n} e^{-4u}e^{2v}|\alpha|^2(z) < 1 \tag{5.4}$$

and

$$\left\{\begin{array}{ll} \Delta u & = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha_n|^2 \\ \Delta v & = \frac{1}{n(2g-2)} - e^{-2u}e^{2v}|\alpha_n|^2 \end{array}\right. \tag{5.5}$$

Thanks to Theorem 2.11, this corresponds to an almost-fuchsian representation and complex variation of Hodge structure, with holomorphic data α, and the hyperbolic 4-manifold is a degree 1 disk bundle over Σ_n. □

The main argument of Theorem 5.1 will be a fixed point theorem, namely Schauder's fixed point theorem: a continuous map which preserves a compact convex set has a fixed point.

Proof of Theorem 5.1 . Denote $\mathcal{K}_u^n$ the following convex set:

$$\mathcal{K}_u^n := \{u \in C^{0,\alpha}(\Sigma_n) : -\frac{\ln(1+\eta)}{2} \leq u \leq 0\} \tag{5.6}$$

Thanks to Theorem 4.8, for $u = 0$, there is a function v_0 solution of

$$\Delta v_0 = \frac{1}{|\chi(\Sigma_n)|} - e^{2v_0}|\alpha_n|^2 \tag{5.7}$$

and by Proposition 4.10, there is $C_0 > 0$ depending continuously on δ, Λ and the ratio $\frac{|\alpha_n|^2_\infty}{\fint |\alpha_n|^2}$ such that

$$|e^{2v_0}|\alpha_n|^2 \leq \frac{C_0}{2n(g-1)} \tag{5.8}$$

The assumptions (a), (d) and (e) ensure that the constant C can be chosen to be independent of n, as it must be for our proof. Recall Λ_0 defined in (d) as a common lower bound to the spectral gaps of (Σ_n). Consider now n large enough so that:

$$\frac{C_0}{2n(g-1)} < \min\{\Lambda_0, \frac{1}{2}\} \tag{5.9}$$

Thanks to proposition 4.12, there are constants C_1, C_2, C_3 independent of n, and a continuous map

$$\Psi : \{u \in C^{0,\alpha} || e^{2u} - 1|_\infty \leq \frac{C_1}{\sqrt{2n(g-1)}}\} \to \{v || v - v_0|_\infty \leq \frac{C_2}{2n(g-1)}\} \tag{5.10}$$

such that $v = \Psi(u)$ is a solution of

$$\Delta v = \frac{1}{|\chi(\Sigma_n)|} - e^{2v}e^{-2u}|\alpha_n|^2 \tag{5.11}$$

And $v = \Psi(u)$ satisfies

$$|e^{2v}|\alpha_n|^2 \leq \frac{C_3}{2n(g-1)} \tag{5.12}$$

Provided $\frac{C_3}{2n(g-1} \leq \frac{\eta}{(1+\eta)^2}$, theorem 4.1 gives us a map Φ and an n-independent constant C_4

$$\Phi : \{v|e^{2v}|\alpha_n|^2 \leq \frac{C_3}{2n(g-1)}\} \to \{u||e^{2u}-1|_\infty \leq \frac{C_4}{2n(g-1)}\} \tag{5.13}$$

such that $u = \Phi(v)$ is a solution of

$$\Delta u = e^{2u} - 1 + e^{-2u}e^{2v}|\alpha_n|^2 \tag{5.14}$$

and u satisfies

$$\sup e^{-4u}e^{2v}|\alpha_n|^2 < \eta \tag{5.15}$$

Finally for n sufficiently large,

$$\frac{C_4}{2n(g-1)} \leq \frac{C_1}{\sqrt{2n(g-1)}} \tag{5.16}$$

and $\Phi \circ \Psi$ is a well-defined self-map of $\mathcal{K} = \{u||e^{2u}-1| \leq \frac{C_4}{2n(g-1)}\} \subset \mathcal{K}_n^u$. We also point out that ,by ellipticity, every element in the image of Φ is in $C^{2,\alpha}$ and satisfies

$$\begin{cases} -\frac{\ln 2}{2} & \leq u \leq 0 \\ -1 & \leq \Delta u \leq 1 \end{cases} \tag{5.17}$$

And so the image of Φ is in the following set :

$$\mathcal{Q} = \{u \in C^{2,\alpha}(\Sigma_n) : -1 \leq \Delta u \leq 1 \text{ and } -\frac{\ln 2}{2} \leq u \leq 0\} \tag{5.18}$$

whose closure in $C^{0,\alpha}(\Sigma_n)$ is a convex compact set.

Finally, $\Phi \circ \Psi$ is a continuous self-map of $\mathcal{K}$, which is convex, and its image is contained in a compact convex set. Hence by Schauder fixed point theorem, it has a fixed point u. Denote $v = \Psi(u)$, then (u, v) is solution of the desired PDE system, as asserted. By construction, it satisfies the estimate

$$e^{-4u}e^{2v}|\alpha_n|^2 \leq \eta\,. \tag{5.19}$$

□

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