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# ON THE MOSER-TRUDINGER INEQUALITY FOR SURFACES

SAMUEL BRONSTEIN

ABSTRACT. This paper is devoted to the Moser-Trudinger inequality on smooth riemannian surfaces. We establish that the constants involved can be chosen to depend on only 3 parameters, which are the systole, isoperimetric constant and curvature of the surface. We have two analogous statements, considering respectively infinite-volume surfaces and closed surfaces. As an application, we show that there are sequences of coverings of a hyperbolic closed surface which admit a uniform Moser-Trudinger inequality.

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## 1. INTRODUCTION

In this paper we consider  $(\Sigma, \sigma)$  a Riemannian surface, and we look at the famous Sobolev embeddings  $W^{1,2}(\Sigma) \hookrightarrow L^p(\Sigma)$  for any  $p \geq 2$ .

When  $\Sigma$  is of infinite volume, we prove the following version of the celebrated Moser-Trudinger inequality:

**Theorem 1.1.** *Let  $(\Sigma, \sigma)$  be an infinite volume complete surface. Assume that the curvature of  $\Sigma$  is upper bounded by  $K$ . Assume that the systole  $\delta$  and the Cheeger isoperimetric constant  $h$  are both nonzero. Then there is a constant  $C$  depending only on  $(\delta, h, K)$  such that, for any  $u \in W^{1,2}(\Sigma)$ :*

$$(1.1) \quad \int_{\Sigma} |\nabla u|^2 d\sigma \leq 1 \Rightarrow \int_{\Sigma} \exp(4\pi u^2) - 1 d\sigma \leq C(\delta, h, K)$$

We refer to section 2 for precise definitions of systole and isoperimetric constant for non-compact surfaces.

For compact surfaces, this statement cannot be true because of the existence of nonzero harmonic maps. The standard way to deal with this problem is to add the assumption that our maps have zero average, restricting effectively to the orthogonal of harmonic maps in  $W^{1,2}$ .

In this setting, we prove the following inequality:

**Theorem 1.2.** *Let  $a > 0$  and  $K \in \mathbb{R}$ . There is a constant  $C(a, \delta, h, K)$  such that, for any closed surface  $(\Sigma, \sigma)$  with systole  $\delta$ , Cheeger constant  $h$ , curvature upper bounded by  $K$  and volume bigger than  $a$ , for any  $u \in W^{1,2}(\Sigma)$  satisfying:*

$$(1.2) \quad \int_{\Sigma} u d\sigma = 0, \quad \int_{\Sigma} |\nabla u|^2 d\sigma \leq 1$$

Then:

$$(1.3) \quad \int_{\Sigma} \exp(4\pi u^2) - 1 d\sigma \leq C(a, \delta, h, K)$$

When the considered surface is the sphere  $\mathbb{S}^2$  with its round metric, our theorem is exactly the Moser-Trudinger inequality, first proved by Trudinger [Tru67] and then generalized by Moser to the higher-dimensional spheres [Mos71].

When considering the hyperbolic plane  $\mathbb{H}^2$  which satisfies the condition of the abovementioned theorem, we get back a result of Mancini and Sandeep [MS10].

The constant  $4\pi$  is the biggest one for which the statement is true, statement proven by Li [Li01] for any compact Riemannian surface.

We insist that the main novelty is the dependency of the constant  $C$  involved in the parameters, and particularly that it is roughly independent of the volume of the surface. There are other ways to prove the Trudinger-Moser inequality for compact surfaces, by a concentration argument, as Li did [Li01]. We would expect other more recent methods to work to, for instance the fractional Sobolev inequalities have recently been used to prove an analogous inequality in the case of the  $n$ -dimensional sphere, see Xiong [Xio18].

The same kind of dependency is also true for the constants involved in the Sobolev embeddings, see Ilias's work [Ili83]. See also Faget [Fag06], for a generalization of the exponential decay of Sobolev's constants in higher dimensional manifolds.

This reinforces the feeling that Moser-Trudinger is a nice way of understanding the asymptotic behavior of the embeddings  $W^{1,2}(\Sigma) \hookrightarrow L^p(\Sigma)$ .

Generally, we have no idea of the sharp constant  $C$  involved in the theorem. This question was asked in a paper of Dolbeault-Esteban–Jankowiak [DEJ14]. Our paper gives a partial answer as it claims that the constant won't explode if we control the right parameters, but we think the exhibition of the sharp constant should involve more complicated techniques.

Our proof mainly relies on a symmetrisation argument, which naturally involves the isoperimetric inequality. That's why we preferred to write the dependency in terms of the isoperimetric constant rather than the spectral gap, even if they are closely linked by the Cheeger and Buser inequalities [Che70, Bus82].

Trudinger-Moser inequality has had tremendous consequences regarding the prescribed curvature problem on  $\mathbb{S}^2$ . Famously, Moser used it to solve the problem of prescribed curvature on the projective plane [Mos73]. Mancini–Sandeep [MS10] use their version of Moser-Trudinger for the prescribed curvature problem on subsets of  $\mathbb{R}^2$ . It has also been used by Jost–Wang [JW01] to solve some Toda systems on closed surfaces. Malchiodi–Ruiz [MR11] and [BJMR15] deduced existence theorems for mean-field equations from the Trudinger-Moser inequality.

Our focus here is rather on hyperbolic surfaces of high genus. A corollary of our theorem is the possibility of taking high degree covers of a fixed hyperbolic surface keeping a uniform Sobolev embedding

**Corollary 1.3.** *Let  $\Sigma$  be a closed hyperbolic surface. There are sequence  $(\Sigma_n)$  of Riemannian covers of  $\Sigma$  of degree  $n$  and  $C(\Sigma) > 0$  such that, for any  $n \geq 1$ ,  $u \in W^{1,2}(\Sigma_n)$  satisfying*

$$(1.4) \quad \int_{\Sigma_n} u = 0, \quad \int_{\Sigma_n} |\nabla u|^2 \leq 1$$

then

$$(1.5) \quad \int_{\Sigma_n} e^{4\pi u^2} - 1 \leq C(\Sigma)$$

Indeed, thanks to Magee–Naud–Puder [MNP22] we know that for any  $\varepsilon > 0$ , with probability going to one when the index of the covering increases, a random cover of  $X$  has relative spectral gap bigger than  $\frac{3}{16} - \varepsilon$ . This means that there are sequences of coverings of a closed surface with spectral gap uniformly lower bounded, hence for such a sequence  $(X_n)$  the Cheeger constant is lower bounded. The systole and volume are bigger than the systole and volume of  $X$ , and the curvature is  $-1$ , hence the constant in Moser-Trudinger is uniform.

The author is very thankful to Nicolas Tholozan and Paul Laurain for their time and discussion on these topics.

## 2. NOTATIONS AND ISOPERIMETRIC COMPARISON

Throughout this section  $(\Sigma, \sigma)$  will be a Riemannian surface. Recall the definitions of systole and Cheeger constant:

**Definition 2.1** (Systole). The systole of  $(\Sigma, \sigma)$  is the length of the shortest noncontractible closed curve in  $\Sigma$ .

**Definition 2.2** (Cheeger constant). Denote  $l$  the length measure associated to  $\sigma$ . The Cheeger constant of  $(\Sigma, \sigma)$  is:

$$(2.1) \quad h := \inf_{A \subset \Sigma, 2\text{Vol}(A) < \text{Vol}(M)} \frac{l(\partial A)}{\text{Vol}(A)}$$

If  $\text{Vol}(\Sigma) = \infty$ , we require  $\bar{A}$  to be compact.

The Cheeger constant is heavily linked with the spectral gap of the surface.

**Definition 2.3.** For a closed surface, the spectral gap  $\Lambda$  of  $(\Sigma, \sigma)$  is the smallest nonzero eigenvalue of  $-\Delta$ .

It is a standard fact that the spectrum of the Laplacian on a compact surface is discrete, hence the spectral gap of a closed surface is always strictly positive. Remark that there is a way to generalize the spectral gap for asymptotically negatively curved surfaces, using the resolvent of the laplacian. See for instance Borthwick [Bor07] for the spectral gap of infinite area hyperbolic surfaces. On the hyperbolic plane, the spectral gap will be  $\frac{1}{4}$ , which is a reformulation of Hardy's inequality. The spectral gap and Cheeger's isoperimetric constant are closely linked thanks to two relations. Recall the first one, proved by Cheeger in [Che70]:

**Theorem 2.4** (Cheeger). *Let  $\Sigma$  be a complete surface. Denote  $h$  and  $\Lambda$  its Cheeger constant and spectral gap. Then*

$$(2.2) \quad 4\Lambda \geq h^2.$$

Remark that Cheeger states it for closed surfaces, but it has since been extended for complete noncompact surfaces, see [BPP<sup>+</sup>93]. Buser [Bus82], theorem 7.1., showed that for surfaces with curvature bounded below, a converse holds:

**Theorem 2.5** (Buser). *There is a universal constant  $C$ , such that: Let  $(\Sigma, \sigma)$  be a complete Riemannian surface with curvature bounded below by  $-\delta^2$ ,  $\delta \geq 0$ . Denote  $h$  and  $\Lambda$  its Cheeger constant and spectral gap. Then*

$$(2.3) \quad \Lambda \leq C(\delta h + h^2)$$

We will need a stronger insight on the isoperimetric behavior of a surface, hence we introduce the *isoperimetric profile*, as considered by Grimaldi–Nardulli–Pansu [GNP09]:

**Definition 2.6** (Isoperimetric profile). Let  $(\Sigma, \sigma)$  be a Riemannian surface. The isoperimetric profile is the following map:

$$(2.4) \quad \forall t > 0, \varphi(t) = \inf_{A \subset \Sigma, \text{Vol}(A)=t} l(\partial A)$$

Remark that this notation differs from the isoperimetric profile for revolution manifolds considered by Gallot in [Gal88], but it is easy to go from one definition to another.

**Lemma 2.7.** *Let  $\Sigma$  be a complete surface with nonzero systole  $\delta$  and curvature bounded above by  $K$ . Denote  $\varphi$  its isoperimetric profile. Denote  $V_0$  the following:*

$$(2.5) \quad V_0 = \frac{\delta^2}{2\pi + \sqrt{4\pi^2 - K\delta^2}}$$

*Consider  $t_0 = \min\{V_0, \frac{\text{Vol}(\Sigma)}{2}\}$  if  $K \leq 0$  and  $t_0 = \min\{V_0, \frac{\text{Vol}(\Sigma)}{2}, \frac{4\pi}{1+K}\}$  if  $K > 0$  Then for  $t \leq t_0$ , we have:*

$$(2.6) \quad \varphi(t) \geq \sqrt{t(4\pi - Kt)}$$

*Proof.* Assume first  $V_0 \leq \frac{\text{Vol}(\Sigma)}{2}$  Let  $A \subset \Sigma$  be a domain of area  $t \leq V_0$ .  $V_0$  was chosen so that:

$$(2.7) \quad \sqrt{V_0(4\pi - KV_0)} \leq \delta$$

Hence the wanted inequality is obviously verified if  $l(\partial A) \geq \delta$ .

Otherwise, if  $l(\partial A) \leq \delta$ , because of completeness one can assume that  $A$  is a disjoint union of disks. But for a disk, as the curvature is lower than  $K$ , the length of its boundary is bigger than the length of a disk of same area in constant curvature  $K$  model space. Hence, we get the inequality desired. □

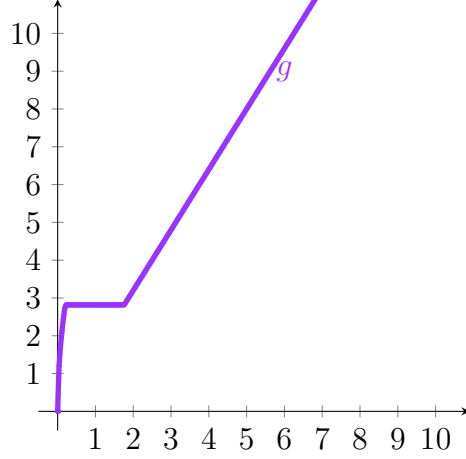
We will compare the isoperimetric profile of a surface with the behavior of a radial metric on the disk.

**Lemma 2.8.** *Consider  $\varepsilon > 0$ ,  $h > 0$  and  $K \in \mathbb{R}$ . Let  $g$  be defined as the following map:*

$$(2.8) \quad \forall t > 0, g(t) := \begin{cases} \sqrt{t(4\pi - Kt)} & \text{if } t \leq \varepsilon \\ g(\varepsilon) & \text{if } \varepsilon < t \leq \frac{g(\varepsilon)}{h} \\ h \cdot t & \text{if } t > \frac{g(\varepsilon)}{h} \end{cases}$$

*Then there exists a  $C^1$  map  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that the disk endowed with radial metric  $dr^2 + f'(r)^2 d\theta^2$  satisfies:*

- (1) The perimeter of a centered disk of radius  $A$  is  $g(A)$
- (2) there is a constant  $C(g) > 0$  such that the metric  $g_{\mathbb{D}} = dr^2 + f'(r)^2 d\theta^2$  is uniformly conformal to a disk with constant negative curvature.


 FIGURE 2.1. graph of  $g$  for  $(K, h, \varepsilon) = (-1, 1.6, 0.2)$ 

*Proof.* The assumption (1) is verified if and only if  $f$  satisfies the functional equation:

$$(2.9) \quad 2\pi f'(r) = g(2\pi f(r)) \quad \forall r > 0$$

This equation can be explicitly solved on  $(0, \varepsilon)$  with:

$$(2.10) \quad f(r) = \begin{cases} \frac{-2}{K} \sinh\left(\frac{\sqrt{-K}r}{2}\right) & \text{if } K < 0 \\ \frac{r^2}{2} & \text{if } K = 0 \\ \frac{2}{K} \sin\left(\frac{\sqrt{K}r}{2}\right) & \text{if } K > 0 \end{cases}$$

On  $(\varepsilon, \infty)$  there is a nondecreasing solution on  $\mathbb{R}$ , which has at least linear growth, as  $g$  is lower bounded. The solution is defined on all of  $\mathbb{R}$  because  $g$  is sublinear. Actually, we have the following explicit description of  $f$ :

$$(2.11) \quad \forall \varepsilon < t \leq \frac{g(\varepsilon)}{h}, f(t) = f(\varepsilon) + (t - \varepsilon) \frac{g(\varepsilon)}{2\pi}$$

$$(2.12) \quad \forall t > \frac{g(\varepsilon)}{h}, f(t) = f\left(\frac{g(\varepsilon)}{h}\right) \exp\left(h\left(t - \frac{g(\varepsilon)}{h}\right)\right)$$

As for  $t > \frac{g(\varepsilon)}{h}$ , we have:

$$(2.13) \quad \frac{f'(t)}{\sinh(ht)} = 2hf\left(\frac{g(\varepsilon)}{h}\right)e^{-g(\varepsilon)} \frac{1}{1 - e^{-2ht}}$$

And  $\frac{f'(t)}{\sinh(ht)}$  has a nonzero limit when  $t \rightarrow 0$ . We deduce that there is a constants  $C(K, \varepsilon, h)$  so that

$$(2.14) \quad \frac{1}{C(K, \varepsilon, h)} \leq \frac{f'(t)}{\sinh(ht)} \leq C(K, \varepsilon, h)$$

so  $f'$  is uniformly comparable to  $\sinh(hr)$ , which means that our disk is uniformly quasisconformal to the complete disk with constant sectional curvature  $-h$ . As this disk is

quasiconformal to the hyperbolic plane, we get the statement (2). We point out that the quasiconformal factor depends only on the map  $g$ , so in the parameters  $(K, \varepsilon, h)$ .  $\square$

### 3. THE MOSER-TRUDINGER INEQUALITY FOR INFINITE-VOLUME SURFACES

It is known that a Moser-Trudinger type inequality can't be true for all noncompact surfaces, as it implies the existence of a Poincaré inequality. We will show here that for a class of surfaces which at infinity, look like they are negatively curved, a Moser-Trudinger inequality is true. We will also show that the relevant geometric data are the systole and Cheeger constant. Throughout this section  $(\Sigma, \sigma)$  will be a Riemannian surface, noncompact. We will show that the following geometric assumptions are enough to get a Moser-Trudinger inequality:

**Theorem 3.1.** *Let  $(\Sigma, \sigma)$  be an infinite volume complete surface. Assume that the curvature of  $\Sigma$  is upper bounded by  $K$ . Assume that the systole  $\delta$  and the Cheeger isoperimetric constant  $h$  are both nonzero. Then there is a constant  $C$  depending only on  $\delta, h, K$  such that, for any  $u \in W^{1,2}(\Sigma)$ :*

$$(3.1) \quad \int_{\Sigma} |\nabla u|^2 d\sigma \leq 1 \Rightarrow \int_{\Sigma} \exp(4\pi u^2) - 1 d\sigma \leq C(\delta, h, K)$$

Note that the Moser-Trudinger with exponent  $4\pi$  was already shown by Mancini–Sandeep [MS10] in the case of disks dominated by the hyperbolic metric. Our proof relies on a radial rearrangement adapted to the surface  $\Sigma$ , the same used by Ilias [Ili83] to estimate the constants involved in the Sobolev embeddings. The following formulas are well-known measure-theoretic tools:

**Lemma 3.2.** *Let  $u$  be a positive  $C^2$  Morse function. Assume it is  $W^{1,2}(\Sigma)$ . Denote  $\nu$  the length measure associated to  $\sigma$ . Consider the functions  $l(t)$  and  $A(t)$ :*

$$(3.2) \quad A(t) := \sigma\{u \geq t\}$$

$$(3.3) \quad l(t) := \nu\{u = t\}$$

Then, for any measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  :

$$(3.4) \quad \int_{\Sigma} f(u) = \int_{\mathbb{R}} f(t) A(t) dt$$

$$(3.5) \quad \int_{\Sigma} |\nabla u|^2 \geq \int_{\mathbb{R}} \frac{l^2(t)}{A'(t)} dt$$

*Proof.* The first term is a standard formula in differential geometry, it is proven via a change of variables. For the second inequality, we use the following formula for the derivative of  $A$ :

$$(3.6) \quad A'(t) = \int_{\{u=t\}} \frac{d\nu}{|\nabla u|}$$

Now the co-area formula yields (see [Kes06], theorem 2.2.2)

$$(3.7) \quad \int_{\Sigma} |\nabla u|^2 = \int_{\mathbb{R}} \left( \int_{\{u=s\}} |\nabla u| d\nu \right) ds$$

And finally, applying Cauchy-Schwarz:

$$(3.8) \quad \int_{\{u=s\}} |\nabla u| d\nu \int_{\{u=s\}} \frac{d\nu}{|\nabla u|} \geq l^2(s)$$

which finishes the proof.  $\square$

**Theorem 3.3.** *Let  $(\Sigma, \sigma)$  be an infinite volume complete surface, with nonzero systole, Cheeger constant and upper bounded curvature  $\delta, h, K$ . There is an nondecreasing function  $f$  depending only on  $(\delta, h, K)$ , a disk uniformly conformal to the complete hyperbolic plane, with radial metric  $dr^2 + f'(r)^2 d\theta^2$ . Moreover, the radial rearrangement mapping on that disk:*

$$(3.9) \quad \begin{cases} W^{1,2}(\Sigma) & \rightarrow W^{1,2}(\mathbb{D}) \\ u & \mapsto u^* \end{cases}$$

defined so that  $u^*$  is nonincreasing, radial, and:

$$(3.10) \quad A(t) = |\{u^* \geq t\}| \quad \forall t \in \mathbb{R}$$

satisfies:

$$(3.11) \quad \int_{\Sigma} f(u) d\sigma = \int_{\mathbb{D}} f(u^*)$$

$$(3.12) \quad \int_{\Sigma} |\nabla u|^2 \geq \int_{\mathbb{D}} |\nabla u^*|^2$$

*Proof.* Denote  $\varphi$  the isoperimetric profile of  $\Sigma, \sigma$  By lemma 2.7, there is  $V_0(K, \delta) > 0$  such that, on  $(0, V_0)$  the isoperimetric profile satisfies:

$$(3.13) \quad \varphi(t) \geq \sqrt{t(4\pi - Kt)}$$

Consider the map  $g$  defined in lemma 2.8 with  $\varepsilon = V_0$ . Then we claim:

$$(3.14) \quad \forall t \in \mathbb{R}, \varphi(t) \geq g(t)$$

Indeed, for small values it is the lemma 2.7, for big values it is the definition of the Cheeger constant and in between it comes from the fact that  $g(V_0) \leq \delta$ .

Hence define  $f$  by the lemma 2.8. For  $u \in C^2$ , Morse, nonnegative and  $W^{1,2}$  on  $\Sigma$  define  $u^*$  its radial rearrangement on  $\mathbb{D}, dr^2 + f'(r)^2 d\theta^2$ . By construction, it is clear that

$$(3.15) \quad \int_{\Sigma} f(u) = \int_{\mathbb{D}} f(u^*)$$

for any measurable  $f$ . Also, denote  $l$  and  $l^*$  the length of level set functions:

$$(3.16) \quad l(t) = \nu\{u = t\}$$

$$(3.17) \quad l^*(t) = |\{u^* = t\}|$$

Because  $g \leq \varphi$  everywhere, it is clear that  $l \geq l^*$ . Thanks to lemma 3.2, we deduce

$$(3.18) \quad \int_{\Sigma} |\nabla u|^2 \geq \int_{\mathbb{D}} |\nabla u^*|^2$$

$\square$

*Proof of theorem 3.1.* Consider  $\Sigma$  such a surface. Consider  $u \in W^{1,2}(\Sigma)$ . First we will prove the inequality for Morse surfaces, then by density it will be true on  $W^{1,2}(\Sigma)$ .

As the functional  $e^{4\pi x^2}$  is even and the absolute-value is 1-lipschitz, up to replacing  $u$  by  $|u|$ , we can assume that  $u$  is nonnegative and

$$(3.19) \quad \int_{\Sigma} |\nabla u|^2 \leq 1.$$



Consider  $(\mathbb{D}, dr^2 + f'(r)^2 d\theta^2)$  the disk defined in theorem 3.3 and  $u^*$  the radial rearrangement. By construction,

$$(3.20) \quad \int_{\mathbb{D}} \exp(4\pi(u^*)^2) - 1 = \int_{\Sigma} \exp(4\pi u^2) - 1$$

$$(3.21) \quad \int_{\mathbb{D}} |\nabla u^*|^2 \leq 1$$

As  $\int_{\mathbb{D}} |\nabla u^*|^2$  is a conformal invariant, consider  $C$  the conformal distance between  $dr^2 + f'(r)^2 d\theta^2$  and the metric of the Poincaré disk. Naturally, with  $C$  being the conformal distortion, we control:

$$(3.22) \quad \int_0^\infty (\exp(4\pi(u^*)^2) - 1) f'(r) dr \leq C \int_0^\infty (\exp(4\pi(u^*)^2) - 1) \sinh(r) dr$$

Now, cf [MS10], this is upperbounded by some bound depending only on  $h$ . Finally, as  $C$  depends only on  $(\delta, h, K)$  we get a bound depending on those parameters.  $\square$

We finish our discussion about infinite volume surfaces with 3 examples of noncompact surfaces which do not satisfy the Moser-Trudinger inequality, showing that none of the 3 assumptions could have been dropped.

**Theorem 3.4.** *Let  $\Sigma$  be a (noncompact) surface with Cheeger constant 0, and lower bounded curvature. Then the Moser-Trudinger inequality is false.*

*Proof.* Thanks to the work of Buser [Bus82], we know that such a surface has no spectral gap, hence it cannot satisfy a Moser-Trudinger inequality.  $\square$

*Remark 3.1.* There are examples of surfaces from Buser [Bus79] of surfaces with Cheeger constant arbitrarily small with regard to the spectral gap, but these examples involve small systoles and very negative curvature. It is not known to the author if with those examples one could show a noncompact surface satisfying the Moser-Trudinger inequality, but with Cheeger constant zero.

**Theorem 3.5.** *Let  $\Sigma, \sigma$  be a (noncompact) surface with curvature  $\kappa$  satisfying:*

$$(3.23) \quad \sup \kappa = \infty$$

*Then the Moser-Trudinger inequality is false.*

*Proof.* The main tool for this is to show that the constant of Moser-Trudinger explodes in high positive curvature. Fix  $M > 0$  Let  $\Omega$  be a small disk on which  $\kappa \geq M$  Consider  $C_{\mathbb{S}}$  the Moser-Trudinger constant for the sphere. Thanks to works of Chang and Yang, there is a bubbling sequence  $u_n$  of functions on  $\mathbb{S}^2$ , weakly converging towards a Dirac distribution, so that:

$$(3.24) \quad \int_{\mathbb{S}^2} |\nabla u_n|^2 \leq 1$$

$$(3.25) \quad \int_{\mathbb{S}^2} \exp(4\pi u_n^2) - 1 \rightarrow C_{\mathbb{S}}$$

Because of the weak convergence, one can assume that  $\text{Supp}u_n$  is in small neighborhood of a point. Hence for any  $\varepsilon > 0$  we easily get a function  $v$  supported in  $\Omega$  satisfying:

$$(3.26) \quad \int_{\Sigma} |\nabla v|^2 \leq 1$$

$$(3.27) \quad \int_{\Sigma} \exp(4\pi v^2) - 1 \geq MC_{\mathbb{S}} - \varepsilon$$

So the Moser-Trudinger inequality is false.  $\square$

When the systole goes to zero, there are plenty known examples where the inequality is false.

**Theorem 3.6.** *Let  $\Sigma$  be the hyperbolic cusp:  $\mathbb{R} \times \mathbb{S}^1$  equipped with the metric  $dt^2 + e^{2t}d\theta^2$ . Then it does not satisfy the Moser-Trudinger inequality.*

*Proof.* Fix  $b \in \mathbb{R}$ , and consider the function

$$(3.28) \quad f(t, \theta) = (e^t - e^b)\mathbf{1}_{t \leq b}$$

One can make the following computations:

$$(3.29) \quad |\nabla f|_2^2 = \frac{2\pi}{3}e^{3b}$$

$$(3.30) \quad |f|_2^2 = \frac{2\pi}{3}e^{3b}$$

$$(3.31) \quad |f|_4^4 = \frac{2\pi}{5}e^{5b}$$

hence we see that, when  $b \rightarrow -\infty$

$$(3.32) \quad \frac{|f|_4^4}{|\nabla f|_2^4} \rightarrow +\infty$$

which contradicts the Moser-Trudinger inequality.  $\square$

#### 4. A MOSER-TRUDINGER INEQUALITY FOR CLOSED SURFACES

In this section, we need to adapt our proof with more subtlety in order to get the same theorem for closed surface. We point out that Li [Li01] already proved that all compact surfaces satisfy the Moser-Trudinger inequality with exponent  $4\pi$ , however his proof does not show the dependency of the upper bound in the systole, curvature and Cheeger constant.

**Theorem 4.1.** *Let  $\Sigma, \sigma$  be a closed surface with nonzero systole  $\delta$ , Cheeger constant  $h$  and upper bounded curvature  $K$ . Then there is a constant  $C(\delta, h, K, \text{Vol}(\Sigma))$  such that, for any  $u \in W^{1,2}(\Sigma)$  with zero average and satisfying:*

$$(4.1) \quad \int_{\Sigma} |\nabla u|^2 d\sigma \leq 1$$

Then

$$(4.2) \quad \int_{\Sigma} \exp(4\pi u^2) - 1 d\sigma \leq C(\delta, h, K).$$

The constant  $C$  can be chosen to be nonincreasing in  $\text{Vol}(\Sigma)$ .

*Remark 4.1.* In the case of hyperbolic surfaces, an important feature is that the bound is actually independent of the volume, as long as the systole and spectral gap are controlled.

In the case of tori, it actually depends on the volume, as big volume flat tori must have small Cheeger constant.

When dealing with zero average function, we cannot assume, of course, that they are nonnegative. Hence we must be more careful when considering radial rearrangements:

**Theorem 4.2.** *Let  $(\Sigma, \sigma)$  be a closed surface. Denote  $\delta, h, K$  its systole, Cheeger constant and supremum of curvature. Then there is a disk of radius  $R = \frac{\text{Vol}(\Sigma)}{2}$ , with radial metric  $dr^2 + f'(r)^2 d\theta^2$ , so that for any  $u$  Morse function on  $\Sigma$  one can define  $u_-$  and  $u_+$  its lower and upper radial rearrangements satisfying:*

- (1)  $u_+$  is nonincreasing radial on  $\mathbb{D}$ .
- (2)  $u_-$  is nondecreasing radial on  $\mathbb{D}$
- (3)

$$(4.3) \quad \forall t \text{ such that } \sigma(u \geq t) < \frac{\text{Vol}(\Sigma)}{2}, \sigma(u \geq t) = |\{u_+ \geq t\}|$$

$$(4.4) \quad \forall t \text{ such that } \sigma(u \leq t) < \frac{\text{Vol}(\Sigma)}{2}, \sigma(u \leq t) = |\{u_- \leq t\}|$$

- (4) For any measurable  $f$ ,

$$(4.5) \quad \int_{\Sigma} f(u) d\sigma = \int_{\mathbb{D}} f(u_-) + f(u_+)$$

$$(4.6) \quad \int_{\Sigma} |\nabla u|^2 \geq |\nabla u_+|_2^2 + |\nabla u_-|_2^2$$

*Proof.* Denote  $\varphi$  the isoperimetric profile of  $\Sigma$ . Thanks to lemma 2.7, there is  $V_0 > 0$  so that, on  $(0, V_0)$ :

$$(4.7) \quad \varphi(t) \geq \sqrt{t(4\pi - Kt)}$$

Also, on  $(0, \frac{\text{Vol}(\Sigma)}{2})$ , by definition of the Cheeger constant:

$$(4.8) \quad \varphi(t) \geq h \cdot t$$

Consider the map  $g$  defined in lemma 2.8 with  $\varepsilon = V_0$ . We check

$$(4.9) \quad \forall t \leq \frac{\text{Vol}(\Sigma)}{2}, \varphi(t) \geq g(t)$$

Hence define  $f$  by the lemma 2.8, and consider  $u_+$  the nonincreasing radial rearrangement of the part of  $u$  above its median  $m$ . In the same way, define  $m - u_-$  to be the nonincreasing radial rearrangement of the part of  $m - u$  above  $u$ . Thanks to the formulas 3.2, we deduce easily, for any measurable  $f$ :

$$(4.10) \quad \int_{\Sigma} f(u) = \int_{\mathbb{D}} f(u_+) + f(u_-)$$

And because  $g$  is lower than the isoperimetric profile on  $(0, \frac{\text{Vol}(\Sigma)}{2})$ , we deduce:

$$(4.11) \quad \int_{\Sigma} |\nabla u|^2 \geq |\nabla u_+|_2^2 + |\nabla u_-|_2^2$$

□

As a direct consequence, one can mimic the proof of the noncompact case and get:

**Corollary 4.3.** *Let  $(\Sigma, \sigma)$  be a compact surface. Denote  $\delta$ ,  $h$  and  $K$  its systole, Cheeger constant and supremum of curvature. There is a constant  $C$  depending only on  $\delta, h, K$  such that, for any  $u \in W^{1,2}(\Sigma)$ , satisfying:*

$$(4.12) \quad \int_{\Sigma} |\nabla u|^2 \leq 1$$

$$(4.13) \quad \sigma(u \geq 0) = \sigma(u \leq 0)$$

we have

$$(4.14) \quad \int_{\Sigma} \exp(4\pi u^2) - 1 \leq C$$

*Proof.* Consider  $u$ , and its radial rearrangements  $u_+, u_-$  defined by theorem 4.2. As the disk is conformal to a disk of curvature  $-h$  with conformal bound depending only on  $r_0, h, K$ , one can assume  $\mathbb{D}$  is a disk with constant curvature  $-h$ . We can then extend by zero the maps  $u_+ - m$  and  $m - u_-$  to get functions in  $W^{1,2}$  on the space of constant curvature  $-h$ . Then, by [MS10], we have  $C$  depending on  $\delta, h, K$  such that:

$$(4.15) \quad \int_{\mathbb{D}} \exp(4\pi(u_+ - m)^2) - 1 \leq C$$

$$(4.16) \quad \int_{\mathbb{D}} \exp(4\pi(u_- - m)^2) - 1 \leq C$$

The properties of the radial rearrangements are then enough to get the desired result:

$$(4.17) \quad \int_{\Sigma} \exp(4\pi(u - m)^2) - 1 \leq C$$

□

In order to get the same statement for zero average functions, we will need two technical lemmas.

**Lemma 4.4.** *Let  $(\Sigma, \sigma)$  be a finite volume surface. Denote  $\Lambda$  its spectral gap. Let  $u \in W^{1,2}(\Sigma)$ . Denote  $\bar{u}$  its average and  $m$  its median. Then*

$$(4.18) \quad |m - \bar{u}| \leq \sqrt{\frac{2\Lambda}{\text{Vol}(\Sigma)}} |\nabla u|_2$$

*Proof.* Recall the Markov inequality, for  $\varepsilon > 0$ :

$$(4.19) \quad \sigma(\{|u - \bar{u}| \geq \varepsilon\}) \leq \varepsilon^{-2} |u - \bar{u}|_2^2 \leq \varepsilon^{-2} \Lambda |\nabla u|_2^2$$

We apply it to

$$(4.20) \quad \varepsilon^2 = \frac{2\Lambda |\nabla u|_2^2}{\text{Vol}(\Sigma)}$$

in order to get

$$(4.21) \quad \sigma(\{|u - \bar{u}| \geq \varepsilon\}) \leq \frac{\text{Vol}(\Sigma)}{2}.$$

By definition of the median, necessarily

$$(4.22) \quad |\bar{u} - m| \leq \varepsilon$$

as asserted. □

**Lemma 4.5.** *Let  $\mathbb{D}$  be a disk of area  $A$ , and spectral gap  $\Lambda$ . Let  $m > 0$ . Let  $v$  be a radial nondecreasing map satisfying:*

$$(4.23) \quad v(\partial\mathbb{D}) = m$$

$$(4.24) \quad \int_{\mathbb{D}} v \leq -mA.$$

Then

$$(4.25) \quad |\nabla v|_2^2 \geq 4\Lambda m^2 A$$

*Proof.* First, we use Cauchy-Schwarz for the estimate:

$$(4.26) \quad A^2 m^2 \leq A|v|_2^2$$

We deduce that

$$(4.27) \quad 4A^2 m^2 \leq |v - m|_2^2.$$

Hence, by definition of the spectral gap:

$$(4.28) \quad |\nabla v|_2^2 \geq 4\Lambda A^2 m^2$$

as asserted. □

*Proof of theorem 4.1.* In order to get the result for zero average functions, we have to be more careful. Let  $\Sigma$  be a compact surface, and consider  $u$  a Morse function on it satisfying:

$$(4.29) \quad \int_{\Sigma} u = 0$$

$$(4.30) \quad \int_{\Sigma} |\nabla u|^2 \leq 1$$

Up to replacing  $u$  by  $-u$ , we can assume its median satisfies:

$$(4.31) \quad m > 0$$

Consider its radial rearrangements  $u_-, u_+$  on the disk  $\mathbb{D}$ , defined by theorem 4.2. As the disk is uniformly conformal to a hyperbolic disk, it has a spectral gap  $\Lambda(\delta, h, K)$ . As  $u$  is of zero average,  $u_-$  satisfies the assumptions of lemma 4.5, and we have:

$$(4.32) \quad |\nabla u_-|_2^2 \geq 4\Lambda m^2 A$$

where  $A$  denotes the area of our disk. Of course, this implies

$$(4.33) \quad |\nabla u_+|_2^2 \leq 1 - 4\Lambda m^2 A$$

We can now apply the Moser-Trudinger inequality to  $\frac{1}{\sqrt{1-m^2A}}(u_+ - m)$  to get

$$(4.34) \quad \int_{\mathbb{D}} \exp\left(\frac{4\pi}{1-4\Lambda m^2 A}(u_+ - m)^2\right) - 1 \leq C(\delta, h, K)$$

We compute

$$(4.35) \quad \frac{(u - m)^2}{1 - 4\Lambda m^2 A} - u^2 = \frac{4\Lambda m^2 A}{1 - 4\Lambda m^2 A} \left[ u^2 - \frac{2u}{4\Lambda m A} + \frac{1}{4\Lambda A} \right]$$

$$(4.36) \quad \geq -\frac{1}{4\Lambda A}$$

and we deduce that

$$(4.37) \quad \int_{\mathbb{D}} \exp(4\pi u_+^2) - 1 \leq e^{\frac{1}{4\Lambda A}} C + 4\Lambda A (e^{\frac{1}{4\Lambda A}} - 1)$$

By construction,

$$(4.38) \quad 2A = \text{Vol}(\Sigma)$$

Hence we get a bound  $C'$  depending on  $(\delta, h, K)$  only so that:

$$(4.39) \quad \int_{\mathbb{D}} \exp(4\pi u_+^2) - 1 \leq C' \exp\left(\frac{1}{2\Lambda \text{Vol}(\Sigma)}\right) + 2\Lambda \text{Vol}(\Sigma) \left(\exp\frac{1}{2\Lambda \text{Vol}(\Sigma)} - 1\right)$$

Finally, we check that

$$(4.40) \quad u_-^2 \leq (u_- - m)^2 + m^2$$

And by lemma 4.4,

$$(4.41) \quad m \leq \sqrt{\frac{2\Lambda}{\text{Vol}(\Sigma)}}.$$

Hence we can bound

$$(4.42) \quad \int_{\mathbb{D}} \exp(4\pi u_-^2) - 1 \leq C(\delta, h, K) e^{m^2} + \frac{\text{Vol}(\Sigma)}{2} (e^{m^2} - 1)$$

$$(4.43) \quad \leq C(\delta, h, K) e^{\frac{2\Lambda}{\text{Vol}(\Sigma)}} + \frac{\text{Vol}(\Sigma)}{2} (e^{\frac{2\Lambda}{\text{Vol}(\Sigma)}} - 1)$$

All in all, we get nonnegative constants  $C''$ ,  $a, b$  depending on  $\delta, h, K$  so that:

$$(4.44) \quad \int_{\mathbb{D}} \exp(4\pi u^2) - 1 \leq C'' \left(\exp\frac{a}{\text{Vol}(\Sigma)}\right) + \text{Vol}(\Sigma) \left(\exp\frac{b}{\text{Vol}(\Sigma)} - 1\right)$$

The map  $t \mapsto t(\exp\frac{b}{t} - 1)$  is nonincreasing on  $\mathbb{R}_+ - \{0\}$ , hence we get an upper bound which is nonincreasing in  $\text{Vol}(\Sigma)$ , as asserted.  $\square$

We can see that these dependencies are actually sharp for the case of hyperbolic closed surfaces:

**Proposition 4.6.** *Let  $\varepsilon_n \rightarrow 0$  and let  $(X_n)$  be a family of hyperbolic closed surfaces with systole  $\varepsilon_n$ . Then there is a family of functions  $f_n \in W^{1,2}(X_n)$  satisfying:*

$$(4.45) \quad \int_{X_n} f_n = 0, \quad \int_{X_n} |\nabla f_n|^2 \leq 1$$

And

$$(4.46) \quad \int_{X_n} |f_n|^4 \rightarrow +\infty$$

*Proof.* Fix  $R_0 > 0$ . Thanks to the collar lemma (see [Bus10], theorem 4.1.1), if the systole goes to zero, for  $\varepsilon$  small enough  $X_n$  contains an isometric collar  $(-R_0, R_0) \times \mathbb{S}^1$  with metric  $dt^2 + \varepsilon_n^2 ch(t)^2 d\theta^2$ . Consider the map:

$$(4.47) \quad g_n(t, \theta) = \mathbf{1}_{(-R_0, R_0)}(\cosh(t) - \cosh(R_0))$$

It belongs to  $W^{1,2}(X_n)$ , and satisfies:

$$(4.48) \quad \int_{X_n} |\nabla g_n|^2 = \frac{2}{3} \varepsilon_n \cosh(R_0)$$

$$(4.49) \quad \int_{X_n} g_n = 2\varepsilon_n \left( \frac{-\sinh(2R_0)}{4} + \frac{R_0}{2} \right)$$

$$(4.50) \quad \int_{X_n} |g_n|^2 = 2\varepsilon_n \left( \frac{\sinh(R_0)^3}{3} + \sinh(R_0) - R_0 \cosh(R_0) \right)$$

A thorough computation gives, when  $\varepsilon_n \rightarrow 0$  and  $R_0$  too, with  $\varepsilon_n$  negligible with regard to  $R_0$ :

$$(4.51) \quad \int_{X_n} \left| g_n - \frac{1}{\text{Vol}(\Sigma)} \int_{X_n} f \right|^4 \sim \frac{16\varepsilon_n R_0^9}{315}$$

All in all, fix  $R_0 = \varepsilon^{\frac{1}{10}}$  small enough and you get, as  $n \rightarrow +\infty$ :

$$(4.52) \quad \frac{|g_n - \overline{g_n}|_4^4}{|\nabla g_n|_2^4} \sim \frac{4}{35} \varepsilon^{-\frac{1}{10}} \rightarrow +\infty$$

as asserted. □

In particular, if we consider a family with lower bounded volume and systole, and uniformly bounded curvature and Cheeger constant, we get a Moser-Trudinger bound independent of the surface considered in the family.

**Corollary 4.7.** *Let  $X$  be a closed hyperbolic surface, and  $\varepsilon > 0$ . Consider  $\mathcal{X}$  the collection of Riemannian covers of  $X$  with Cheeger constant bigger than  $\varepsilon$ . Then there is a constant  $C(\varepsilon)$  such that, for any  $Y \in \mathcal{X}$ , for any  $u \in W^{1,2}(Y)$  satisfying:*

$$(4.53) \quad \int_Y u = 0 \quad \int_Y |\nabla u|^2 \leq 1$$

we have

$$(4.54) \quad \int_Y (e^{4\pi u^2} - 1) \leq C(\varepsilon)$$

*Remark 4.2.* Now depending on  $\varepsilon$  this family has different characteristics: For  $\varepsilon > h_1(X)$ , this family is empty. The result of [MNP22] combined with Buser's control tells us that for  $\varepsilon$  close enough to 0, the family is infinite, and even "big", in the sense that the probability that a random cover of degree  $d$  doesn't belong to  $\mathcal{X}$  goes to zero as  $d$  is large.

It is not known to the author if a random tower of covers of  $X$  has a nonzero probability to belong to some  $\mathcal{X}(\varepsilon)$ , for  $\varepsilon > 0$ .

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