



**HAL**  
open science

# LMI conditions for contraction, integral action, and output feedback stabilization for a class of nonlinear systems

Mattia Giaccagli, Vincent Andrieu, Sophie Tarbouriech, Daniele Astolfi

## ► To cite this version:

Mattia Giaccagli, Vincent Andrieu, Sophie Tarbouriech, Daniele Astolfi. LMI conditions for contraction, integral action, and output feedback stabilization for a class of nonlinear systems. *Automatica*, 2023, 154, pp.111106. 10.1016/j.automatica.2023.111106 . hal-04093784

**HAL Id: hal-04093784**

**<https://hal.science/hal-04093784>**

Submitted on 10 May 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LMI conditions for contraction, integral action, and output feedback stabilization for a class of nonlinear systems

M. Giaccagli<sup>a</sup>, V. Andrieu<sup>a</sup>, S. Tarbouriech<sup>b</sup>, D. Astolfi<sup>a</sup>

<sup>a</sup>Université Lyon 1, Villeurbanne, France – CNRS, UMR 5007, LAGEP, France. (e-mail: giaccagli.mattia@gmail.com, danielle.astolfi@univ-lyon1.fr, vincent.andrieu@univ-lyon1.fr)

<sup>b</sup>LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France (e-mail: tarbour@laas.fr)

---

## Abstract

This article focuses on the class of linear systems coupled with a nonlinearity satisfying a sector bound or a monotonic condition. First, we present a set of sufficient conditions based on Linear Matrix Inequalities (LMIs) to establish if such a system defines a contraction. Then, we propose a contractive state-feedback control law designed via the use of LMIs which may include nonlinear terms. Furthermore, the design is extended to the case in which an integral action is included in order to achieve constant output regulation. Following a similar approach, the design of contractive observers for the aforementioned class of systems is also considered. Finally, based on previous conditions, we establish a new nonlinear separation principle in which the feedback and the observer gains may be designed independently. Throughout the paper, we illustrate our results by means of practical examples.

*Keywords:* Contraction, incremental stability, nonlinear systems, LMIs, sector bound, monotonic, integral action, observers, separation principle.

---

## 1. Introduction

In many control applications, it is of primary importance to guarantee some convergence properties of any two trajectories. This is the case, for instance, of observers design [1–3], synchronization problems [4–7], or output regulation [8–10]. Such a property has been defined and characterized in the literature in several different manners: incremental stability [3, 11–16], convergence [9, 17] or quadratic stability [18].

In this work, we consider the notion of incremental stability, also usually referred as contraction [19]. The problem of designing a control feedback achieving incremental stability for the closed loop is not new in literature. We can divide the existing techniques in three classes. The first group considers design tools that are specific for a particular class of systems. Among them, we recall, for instance, a backstepping control design proposed in [20] for systems in strict feedback form, a piece-wise approximated controller for Lur’e systems [21] and a forwarding control design in [8] for systems in feed-forward form. Controllers within this first class, usually provide sufficient tractable conditions to achieve closed-loop incremental stability properties. However, they can be applied only to specific classes of systems. The second class of controllers is the one that considers different notions than the one of incremental stability, such as

control designs achieving convergence [9, 17] or quadratic stability [18]. These notions are similar but different to the one of incremental stability, see [17]. Therefore the existing designs may be applied to have incremental properties, but each case has to be carefully addressed. The last class of control designs makes use of optimization-based tools to compute the solution, see e.g. the control contraction metric approach in [22]. These designs are applicable to more general classes of systems, with the drawback that they usually require the solution to an on-line optimization problem. Recently, this disadvantage has been handled by means of Neural Network-based tools, see [23]. If this allows to reduce the quantity of on-line computations needed, it usually reflects in a loss of stability guarantees that an analytic solution may provide.

In this paper, we focus on the design of feedback laws for the class of nonlinear systems that can be described by a combination of linear dynamics with a nonlinear term satisfying a sector bound or monotonic condition. In other words, the results presented in this work belong to the first group of solutions considered above. This class of systems is of relevant practical interest since several real-life engineering systems can be modeled in a such way, see [24–28] and references therein. Differently from the Lyapunov based approaches in [11, 12] or the logarithmic norm approach in [14, 16], in order to establish the desired contractivity properties we follow the framework based on metric analysis [13, 29]. In particular, following [3, 13], we look for the existence of a Riemannian metric for which the distance between different trajectories of

---

\*This work was supported in part by the ANR Labex CIMI (grant ANR-11-LABX- 0040) within the French State Programme “Investissement d’Avenir” and ANR project HANDY, number ANR-18-CE40-0010.

the closed loop system is monotonically decreasing in forward time. By supposing that such a Riemannian metric is Euclidean, sufficient conditions based on linear matrix inequalities (LMIs) are then derived. This allows to obtain easy-to-check and easy-to-compute conditions that apply to a class of systems whose state-space model description is frequent in many practical cases.

With respect to the existing literature, we provide the following results. First, we present sufficient conditions based on LMIs to state if an autonomous system defines a contraction and a state-feedback control design achieving incremental stability for the closed loop. This first set of results sums up our preliminary work [4]. Then, we apply the proposed conditions to the design of a contractive integral action law, that is able to achieve global output regulation in presence of constant perturbations and references, see, e.g., [8, 10]. We then focus on the problem of contractive observer design. To this end, we first show how to design a contractive observer following the structure proposed in [2]. The gains of the observer are obtained via some LMI conditions. Finally, we establish a nonlinear separation principle. Under some mild conditions, if both contractive state-feedback and observer gains are designed, we can derive an output-feedback law by simply replacing the state with its estimate. Such an output-feedback law guarantees the origin to be globally exponentially stable. It is important to emphasize that the design of the feedback and of the observer can be made independently and doesn't need to satisfy any time-scale separation, contrarily to most of results in nonlinear systems (see, e.g., [30–32]).

The paper is structured as follows. Section 2 provides some sufficient conditions in terms of LMIs to ensure that the flow of an autonomous system defines a contraction. In Section 3, such LMI conditions are employed to design a state-feedback control law ensuring contractivity of the closed-loop system dynamics. In Section 4, we apply such conditions to design a contractive integral action. Section 5 is devoted to design of contractive observers. The design of output feedback control laws and a new nonlinear separation principle are presented in Section 6. Conclusions are drawn in Section 7. Examples are presented at end of each section in order to illustrate the main results of the paper and the feasibility of the presented LMIs.

**Notation.** The identity matrix is denoted by  $I$ . For any matrix  $A$ ,  $A^\top$  denotes its transpose matrix and, for square matrices,  $\text{He}\{A\} := A + A^\top$ . For symmetric positive matrices,  $\star$  denotes the symmetric component of the matrix. Given two square matrices  $A, B$ , we indicate with  $\text{blkdiag}\{A, B\}$  the block-diagonal matrix which has  $A$  and  $B$  in the main diagonal and 0 everywhere else. Given a matrix  $\Omega \in \mathbb{R}^{n \times m}$ , we denote with  $\Omega_{ij}$  the scalar element in row  $i$  and column  $j$  of  $\Omega$ . We write that a matrix  $P$  satisfies  $P > 0$  ( $\geq 0$ ) if  $P$  is symmetric and strictly positive definite (semi-positive definite).

## 2. Sufficient conditions for contraction

Consider the following nonlinear continuous-time system:

$$\dot{x} = Ax + G\varphi(\zeta), \quad \zeta = Hx \quad (1)$$

where  $x$  and  $\zeta$  are vectors in  $\mathbb{R}^{n_x}$  and  $\mathbb{R}^{n_\zeta}$ , respectively, the function  $\varphi : \mathbb{R}^{n_\zeta} \rightarrow \mathbb{R}^{n_\varphi}$  is  $C^1$ , and  $A, G$  and  $H$  are constant matrices of appropriate dimensions. Given  $x_0 \in \mathbb{R}^{n_x}$ , we denote  $X(x_0, t)$  the solution to (1) initialized from  $x_0$  evaluated at time  $t$  where  $t$  is taken on the time domain of existence of the solution.

Assuming some sector bound condition on  $\varphi$ , it is possible to give some sufficient conditions ensuring global<sup>1</sup> asymptotic (or exponential) stability of the origin for all the nonlinear mappings which satisfy the same sector bound condition. See, for instance, [21, 26]. One may however be interested in another different (and stronger) property that the one of asymptotic stability of an equilibrium point. In particular, in many contexts it is interesting to study the contractivity properties of system (1), according to the following definition.

**Definition 1.** *System (1) is incrementally exponentially stable (or exponentially contractive) if there exist positive real numbers  $k, \lambda > 0$  such that for all  $(x_1, x_2) \in \mathbb{R}^{2n_x}$ , the following holds*

$$|X(x_1, t) - X(x_2, t)| \leq k \exp(-\lambda t) |x_1 - x_2| ,$$

for all  $t \geq 0$  in the time domain of existence of solutions.

The characterization of contraction properties for autonomous systems has been studied thoroughly, for example, in [11]. All these studies are based on the notion of contracting flows, which has been widely studied in the literature [12, 13, 29]. These flows generate trajectories for which an appropriately defined distance is monotonically decreasing forward in time. As shown for instance in [3], the dynamical system (1) defines a uniform exponential contraction if there exists a  $C^1$  matrix function  $\mathcal{P} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_x}$ , taking symmetric positive definite values, such that<sup>2</sup>

$$L_f \mathcal{P}(x) < -\nu I, \quad \underline{p}I \leq \mathcal{P}(x) \leq \bar{p}I, \quad \forall x \in \mathbb{R}^{n_x}, \quad (2)$$

for some positive real numbers  $\nu, \underline{p}, \bar{p}$  and where we indicated, with a compact notation,  $\bar{f}(x) = Ax + G\varphi(Hx)$ . When the system is autonomous, sufficiently smooth and

<sup>1</sup>Local characterizations can be given in the case in which the sector bound condition holds only locally. In this paper we will consider only the global case.

<sup>2</sup>Given a  $C^1$  matrix function  $\mathcal{P} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_x}$ , and a  $C^1$  vector field  $f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ ,  $L_f \mathcal{P}$  denotes the Lie derivative in the direction of  $f$  of the quadratic form  $\mathcal{P}$ , i.e.  $L_f \mathcal{P}(x) = \lim_{h \rightarrow 0} \frac{(I+h \frac{\partial f}{\partial x}(x))^\top \mathcal{P}(x+h f(x,t))(I+h \frac{\partial f}{\partial x}(x)) - \mathcal{P}(x)}{h}$ , with coordinates  $(L_f \mathcal{P}(x))_{i,j} = \sum_k \left[ 2\mathcal{P}_{ik} \frac{\partial f_k}{\partial x_j}(x) + \frac{\partial \mathcal{P}_{ij}}{\partial x_k}(x) f_k(x) \right]$ .

globally Lipschitz, it is shown in [3, 33] that conditions (2) are also necessary. Note however that equation (2) is a first order partial differential equation (PDE) in  $\mathcal{P}$ , and an explicit solution may be difficult to compute, especially for high-dimensional systems with significant nonlinearities. In the particular case in which the function  $\mathcal{P}$  is constant, i.e.  $\mathcal{P}(x) = P = P^\top > 0$  for all  $x \in \mathbb{R}^{n_x}$ , the former equation, applied to system (1), reads as

$$\text{He} \left( P \left[ A + M \frac{\partial \varphi}{\partial \zeta}(\zeta) H \right] \right) \leq -\nu I, \quad (3)$$

for all  $\zeta \in \mathbb{R}^{n_\zeta}$ . We recognize in this case the contractivity condition given in [13]. However, even in this particular context of constant metric  $\mathcal{P} = P$ , we still need to solve an infinite dimensional equation (which is linear in  $P$  but has to be solved for all  $\zeta$  in  $\mathbb{R}^{n_\zeta}$ ). By restricting the class of nonlinearities under consideration, one can however exhibit tractable conditions for contraction. This is the purpose of this paper. In particular, we will restrict to the case in which the metric  $\mathcal{P}$  is a symmetric positive constant matrix  $P = P^\top > 0$  and we will consider the class of systems (1) with  $\varphi$  satisfying a sector bound or a monotonic condition. These constraints reduce the set of solutions, but, as it will be shown, allow for the use of LMI-tools to provide sufficient easy-to-compute conditions. In light of this, we recall the following result.

**Theorem 1.** *If there exists a symmetric positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$  such that (3) is satisfied for all  $\zeta \in \mathbb{R}^{n_\zeta}$ , then system (1) is exponentially contractive.*

A proof of Theorem 1 can be found for instance in [13]. In the next section we introduce the sector bound and monotonic conditions characterizing the nonlinearity  $\varphi$ .

### 2.1. Sector bound condition

We define now the notion of *incremental sector bound condition* (see also [5]).

**Assumption 1 (Incremental Sector Bound).** *The function  $\varphi : \mathbb{R}^{n_\zeta} \rightarrow \mathbb{R}^{n_\varphi}$  satisfies*

$$\text{He} \left\{ \frac{\partial \varphi}{\partial \zeta}(\zeta)^\top S \left[ \frac{\partial \varphi}{\partial \zeta}(\zeta) + \Omega \right] \right\} \leq 0, \quad \forall \zeta \in \mathbb{R}^{n_\zeta}, \quad (4)$$

for some matrix  $\Omega \in \mathbb{R}^{n_\varphi \times n_\zeta}$  and a symmetric positive definite matrix  $S \in \mathbb{R}^{n_\varphi \times n_\varphi}$ .

Note that Assumption 1 implies that  $\varphi$  is a globally Lipschitz function. By using Assumption 1, the following sufficient condition for contraction for system (1) can be stated.

**Proposition 1.** *Consider system (1) and suppose that  $\varphi$  satisfies Assumption 1. If there exist a symmetric positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$  and a positive real number  $\eta$  such that the following inequality holds*

$$\begin{bmatrix} A^\top P + PA + \eta I & PG - H^\top \Omega^\top S \\ \star & -2S \end{bmatrix} \leq 0, \quad (5)$$

then system (1) is exponentially contractive.

**Proof.** Let us define

$$\mathcal{L} := \text{He} \left\{ P \left[ A + G \frac{\partial \varphi}{\partial \zeta}(\zeta) H \right] \right\} \quad (6)$$

for all  $\zeta \in \mathbb{R}^{n_\zeta}$ . In view of Assumption 1, it yields, by pre-multiplying (resp. post-multiplying) inequality (4) by  $H^\top$  (resp.  $H$ ):

$$\begin{aligned} \mathcal{L} \leq \mathcal{L} - H^\top \frac{\partial \varphi}{\partial \zeta}(\zeta)^\top S \left[ \frac{\partial \varphi}{\partial \zeta}(\zeta) + \Omega \right] H \\ - H^\top \left[ \frac{\partial \varphi}{\partial \zeta}(\zeta) + \Omega \right]^\top S \frac{\partial \varphi}{\partial \zeta}(\zeta) H. \end{aligned} \quad (7)$$

By defining

$$\mathcal{Q} = \begin{bmatrix} A^\top P + PA + \eta I & PG - H^\top \Omega^\top S \\ \star & -2S \end{bmatrix},$$

and using the definition of  $\mathcal{L}$ , inequality (7) gives

$$\mathcal{L} \leq -\eta I + \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial \zeta}(\zeta) H \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial \zeta}(\zeta) H \end{bmatrix}.$$

Hence, if relation (5) is satisfied,  $\mathcal{Q} \leq 0$  and one can conclude from previous inequality  $\mathcal{L} \leq -\eta I$ . In other words, relation (3) holds and then, according to Theorem 1, system (1) is a contraction. This concludes the proof.  $\square$

### 2.2. Monotonic nonlinearities

Instead of using the previous incremental sector bound condition, one may assume that  $\varphi$  is monotonic. More precisely, we consider the case in which  $n_\varphi = n_\zeta$  (see also [34, Equation (1)]) and we state the following.

**Assumption 2 (Monotonic).** *The function  $\varphi : \mathbb{R}^{n_\varphi} \rightarrow \mathbb{R}^{n_\varphi}$  satisfies*

$$\frac{\partial \varphi}{\partial \zeta}(\zeta) = \frac{\partial \varphi^\top}{\partial \zeta}(\zeta), \quad 0 \leq \text{He} \left\{ \frac{\partial \varphi}{\partial \zeta}(\zeta) \right\} \leq \Gamma \quad (8)$$

for all  $\zeta \in \mathbb{R}^{n_\varphi}$ , where  $\Gamma \in \mathbb{R}^{n_\varphi \times n_\varphi}$  is a symmetric positive definite matrix.

Under the previous assumption, the following sufficient condition for contraction can be provided.

**Proposition 2.** *Consider system (1) and assume that  $\varphi$  satisfies Assumption 2. If there exist a symmetric positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$  and a positive real number  $\eta > 0$  such that*

$$\begin{bmatrix} A^\top P + PA + \eta I & PG + H^\top \\ \star & -4\Gamma^{-1} \end{bmatrix} \leq 0. \quad (9)$$

then system (1) is exponentially contractive.

**Proof.** The function  $\mathcal{L}$  defined in (6) can be rewritten as:

$$\mathcal{L} := PA + A^\top P + w^\top \frac{\partial \varphi}{\partial \zeta}(\zeta)v + v^\top \frac{\partial \varphi}{\partial \zeta}(\zeta)^\top w,$$

with the notation  $v = H$  and  $w = G^\top P$ . The latter gives

$$\begin{aligned} \mathcal{L} = \text{He}\{PA\} - \frac{1}{4}(v-w)^\top \text{He} \left\{ \frac{\partial \varphi}{\partial \zeta}(\zeta) \right\} (v-w) \\ + \frac{1}{4}(v+w)^\top \text{He} \left\{ \frac{\partial \varphi}{\partial \zeta}(\zeta) \right\} (v+w). \end{aligned}$$

With Assumption 2, this implies

$$\mathcal{L} \leq -\eta I + \text{He}\{PA\} + \eta I + \frac{1}{4}(v+w)^\top \Gamma(v+w). \quad (10)$$

If relation (9) holds then, by using the Schur's complement of the matrix in the left-hand term of (9), we get

$$PA + A^\top P + \eta I + \frac{1}{4}(H^\top + PG)\Gamma(H + G^\top P) \leq 0,$$

corresponding to the right-hand term of (10). In other words, we obtain  $\mathcal{L} \leq -\eta I$ . Then, one can conclude that if relation (9) holds then system (1) is exponentially contractive in view of Theorem 1.  $\square$

**Remark 1.** In the particular case in which  $n_\varphi = n_\zeta = 1$  (in other words,  $\varphi$  is a mapping from  $\mathbb{R}$  to  $\mathbb{R}$ ), Assumptions 1 and 2 are equivalent. Moreover, when  $n_\varphi = n_\zeta$ , inequality (9) is recovered from inequality (5) by selecting  $S = 2\Gamma^{-1}$  and  $\Omega = \Omega^\top = -\frac{\Gamma}{2}$ . In other words, Assumption 1 implies Assumption 2. For larger dimensions, the two assumptions are generically different.

An interesting aspect of the approach developed from Assumption 2 is that the condition (8) can be relaxed as

$$0 \leq \frac{\partial \varphi}{\partial \zeta}(\zeta) + \frac{\partial \varphi}{\partial \zeta}(\zeta)^\top, \quad \forall \zeta \in \mathbb{R}^{n_\zeta}. \quad (11)$$

In this case, no globally Lipschitz assumption is imposed on the mapping  $\varphi$ . In this case, the matrix inequality (9) reads

$$\begin{bmatrix} A^\top P + PA + \eta I & PG + H^\top \\ \star & 0 \end{bmatrix} \leq 0. \quad (12)$$

This implies that  $P = P^\top > 0$  has to satisfy the following constraints

$$A^\top P + PA + \eta I < 0, \quad H^\top = -PG.$$

Hence, no restriction on the slope of the nonlinearity has to be imposed provided that a part of the LMI is replaced by an equality constraint.

**Remark 2.** All the results given in this section still hold when  $\varphi$  is an explicit function of time  $\varphi(\zeta, t)$  (see [4]).

### 3. Making a system contractive with a state-feedback

The framework proposed in Section 2 can be applied for the design of a contractive control law for controlled systems. Consider the following nonlinear continuous-time system:

$$\dot{x} = Ax + G\varphi(Hx) + Bu \quad (13)$$

where  $u$  is a control input in  $\mathbb{R}^{n_u}$  and as in system (1), the state  $x$  is in  $\mathbb{R}^{n_x}$ ,  $\varphi : \mathbb{R}^{n_\zeta} \rightarrow \mathbb{R}^{n_\varphi}$  and  $A, G, B, H$  are constant matrices of appropriate dimensions. We are now interested in designing a feedback law  $\alpha : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  such that the system (13) in closed loop with  $u = \alpha(x)$  is an exponential contraction. In our framework, we restrict ourselves to a specific feedback of the form

$$\alpha(x) = Kx + N\varphi(Hx) \quad (14)$$

leading to the following closed-loop system dynamics

$$\dot{x} = (A + BK)x + (G + BN)\varphi(Hx). \quad (15)$$

Based on the condition (3), we look for gains  $K \in \mathbb{R}^{n_u \times n_x}$  and  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a constant metric  $P = P^\top > 0$  satisfying

$$\text{He} \left\{ P \left[ A + BK + (G + BN) \frac{\partial \varphi}{\partial \zeta}(\zeta)H \right] \right\} \leq -\eta I \quad (16)$$

for all  $\zeta \in \mathbb{R}^{n_\zeta}$ , and for some  $\eta > 0$ . Under condition (16) we can therefore establish the exponential contractivity properties of the closed-loop system in light of Theorem 1. Based on Assumptions 1 and 2, we propose, in the next sections, two different sets of constructive conditions.

#### 3.1. Sector bounded condition

Let us consider Assumption 1. The following result provides a solution to the problem of designing a contractive law.

**Lemma 1.** Assume that  $\varphi$  satisfies Assumption 1. If there exist a symmetric positive definite matrix  $W \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $Z \in \mathbb{R}^{n_u \times n_x}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a positive real number  $\nu$  such that the following LMI holds

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ G^\top + N^\top B^\top - S\Omega HW & -S & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \leq 0, \quad (17)$$

then the closed-loop system (13)-(14) is exponentially contractive with  $K = ZW^{-1}$  and such a  $N$ .

**Proof.** By applying the Schur complement to relation (17), and by pre- and post-multiplying the resulting one by  $\text{blkdiag}\{P, I\}$  where  $P = W^{-1}$  and  $K = ZW^{-1}$ , we obtain

$$\begin{bmatrix} \text{He}\{P(A + BK)\} + \nu^{-1}I & P(G + BN) - H^\top \Omega^\top S \\ \star & -2S \end{bmatrix} \leq 0.$$

By denoting  $\nu^{-1} = \eta$ , one retrieves relation (5) of Proposition 1. Hence, by applying Theorem 1 to the closed-loop system (15), we conclude that system (15) is contractive.  $\square$

### 3.2. Monotonic nonlinear mapping

The following result provides a design of the feedback law (14) for nonlinearities satisfying Assumption 2 with  $n_\varphi = n_\zeta$ .

**Lemma 2.** *Assume that  $\varphi$  satisfies Assumption 2. If there exist a symmetric positive definite matrix  $W \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $Z \in \mathbb{R}^{n_u \times n_x}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a positive real number  $\nu$  such that the following LMI holds*

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ HW + (G + BN)^\top & -2\Gamma^{-1} & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \leq 0, \quad (18)$$

then the closed-loop system (13)-(14) is exponentially contractive with  $K = ZW^{-1}$  and such a  $N$ .

**Proof.** By applying the Schur complement to relation (18), and by pre- and post-multiplying the resulting one by  $\text{blkdiag}\{P, I\}$  with  $P = W^{-1}$  and  $K = ZW^{-1}$ , we obtain

$$\left[ \begin{array}{cc} \text{He}\{P(A + BK)\} + \nu^{-1}I & P(G + BN) + H^\top \\ \star & -4\Gamma^{-1} \end{array} \right] \leq 0.$$

By denoting  $\nu^{-1} = \eta$ , one retrieves relation (9) of Proposition 2. Theorem 1 implies that system (15) is contractive.  $\square$

Similarly to the considerations at the end of Section 2, if the function  $\varphi$  satisfies the inequality (11), the LMI condition (18) is transformed into an equality constraint of the form

$$\left[ \begin{array}{cc} \text{He}\{AW + BZ\} & W \\ \star & -\nu I \end{array} \right] \leq 0, \quad W > 0, \quad (19)$$

$$WH^\top = -(G + BN). \quad (20)$$

Finally, note that, as in Section 2, the same conditions hold for time-varying functions  $\varphi$ . Furthermore, similar results can be obtained in a semi-global setting by restricting the analysis to desired bounded compact sets.

### 3.3. An example: the surge system

Consider the surge subsystem of an axial compressor (see for instance [35, Eq. 2]), with system dynamics described by

$$\begin{cases} \dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 &= x_1 - u, \end{cases} \quad (21)$$

with  $x = (x_1, x_2) \in \mathbb{R}^2$ . Following [2], we can rewrite this system as

$$\begin{cases} \dot{x}_1 &= -x_2 + \ell x_1 - \varphi_\ell(x_1) \\ \dot{x}_2 &= v \end{cases}, \quad (22)$$

where

$$\varphi_\ell(\zeta) = \ell\zeta + \frac{3}{2}\zeta^2 + \frac{1}{2}\zeta^3 \quad (23)$$

and where  $\ell \in \mathbb{R}$  is a free parameter,  $u = x_1 - v$  and  $v$  is an additional input. It is shown in [2] that  $\varphi_\ell$  satisfies the sector condition  $\varphi_\ell(\zeta)\zeta \geq 0$  if  $\ell \geq \frac{9}{8}$ . If  $\ell \geq 6$ , it yields

$$\frac{\partial \varphi_\ell}{\partial \zeta}(\zeta) = \ell + 3\zeta + \frac{3}{2}\zeta^2 \geq \ell - 6. \quad (24)$$

Hence, equation (11) is satisfied and we may try to apply the former approach. In particular, for all  $\ell \geq 6$ , there exist  $K$  in  $\mathbb{R}^{1 \times 2}$  and  $N$  in  $\mathbb{R}$ , such that the feedback law

$$v = Kx + N\varphi_\ell(x_1), \quad (25)$$

makes the system (22) exponentially contractive. Note that it can be shown that the conditions (19)-(20) are always feasible. Indeed, the system (21) is in the form (13) with

$$A = \begin{bmatrix} \ell & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad H = [1 \quad 0]. \quad (26)$$

Then, letting

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}, \quad Z = [z_1 \quad z_2],$$

the equation (20) is solved with  $w_{11} = 1$  and  $w_{12} = -N$ . Now, select  $w_{12} = -N$ , and  $w_{22}$  sufficiently large ( $w_{22} - N^2 > 0$ ) such that  $W$  is positive definite. Then, by ignoring the term  $\nu$  and selecting  $w_{11} = 1$ , inequality (19) becomes

$$\text{He}\{AW + BZ\} = \text{He} \left\{ \begin{bmatrix} \ell + N & -N\ell - w_{22} \\ z_1 & z_2 \end{bmatrix} \right\}. \quad (27)$$

Since  $N < -\ell < 0$ , this matrix can be always made negative definite provided  $z_2$  is taken sufficiently negative. For example, one can choose  $z_1 = w_{22}$  and  $z_2 \leq -\frac{\ell^2 N^2}{4|l+N|}$ . Applying Lemma 2, the result follows. The introduction of the nonlinearity explicitly appearing in the controller in this case plays an important role. Note indeed that in case  $N = 0$ , there is no solution, as in such a case, the top-left term of (27) reduces to  $2\ell$  and therefore is always positive definite.

## 4. Contraction and integral action

A very important application of contraction theory is in output regulation problems, see for instance [8-10] and

references therein. Consider in particular the following nonlinear dynamical system

$$\begin{aligned} \dot{z} &= A_z z + G_z \varphi(H_z z) + B_z u + \delta \\ e &= C_z z + D_z \varphi(H_z z) - r \end{aligned} \quad (28)$$

where  $z \in \mathbb{R}^{n_z}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control input,  $e \in \mathbb{R}^{n_e}$  is the regulated output,  $\delta \in \mathbb{R}^{n_z}$  is a constant perturbation and  $r \in \mathbb{R}^{n_e}$  is a constant reference. The objective of this section is to design a feedback control law able to solve a classical output regulation problem, that is to asymptotically regulate the output  $e$  to zero, see, e.g. [8].

Similarly to [8–10], we follow the classical linear paradigm of extending the system with an integral action processing the regulation error  $e$  and looking for a feedback controller stabilizing the extended system. In the case in which the external reference  $r$  and perturbation  $\delta$  are allowed to have any amplitude, a sufficient condition to address the output regulation objective is to design a globally exponentially contractive feedback (uniform in the signals  $r, \delta$ ), see [8]. Following such a strategy and relying on the conditions given in Section 3, we look for a dynamical regulator of the form

$$\begin{aligned} \dot{\xi} &= C_z z + D_z \varphi(H_z z) - r, \\ u &= K_z z + K_\xi \xi + K_\varphi \varphi(H_z z), \end{aligned} \quad (29)$$

with  $\xi \in \mathbb{R}^{n_e}$ , and we look for matrices  $K_z$ ,  $K_\xi$  and  $K_\varphi$  such that the closed-loop system (28)-(29) is an exponential contraction with respect to a constant metric. Indeed, if such a condition is satisfied, it is possible to show that the closed-loop system admits a unique equilibrium point toward which all the trajectories converge and in which (thanks to the integral action) the regulated error is zero, for each initial condition  $(z_0, \xi_0) \in \mathbb{R}^{n_z+n_e}$  and for each  $(\delta, r) \in \mathbb{R}^{n_z+n_e}$ , see [8, Theorem 1] or [10].

To this end, we rely on the results developed in Section 3. In particular, rewriting the extended closed-loop system (28)-(29) in the form (15) with  $x = (z, \xi) \in \mathbb{R}^{n_z+n_e}$  and

$$\begin{aligned} A &= \begin{bmatrix} A_z & 0 \\ C_z & 0 \end{bmatrix}, & B &= \begin{bmatrix} B_z \\ 0 \end{bmatrix}, & G &= \begin{bmatrix} G_z \\ D_z \end{bmatrix}, \\ H &= \begin{bmatrix} H_z & 0 \end{bmatrix}, & K &= \begin{bmatrix} K_z & K_\xi \end{bmatrix}, & N &= K_\varphi, \end{aligned}$$

we can state the following results for the design of the gains  $K_x, K_\xi, K_\varphi$ , according to the particular assumption on the nonlinearity  $\varphi$  at hand. In particular, Corollary 1, respectively Corollary 2, applies for sector bounded nonlinearities, respectively monotonic, and is directly derived from Lemma 1, respectively from Lemma 2. Proofs are omitted for compactness and can be easily derived by combining Theorem 1 with [8, Theorem 1].

**Corollary 1.** *Assume that  $\varphi$  satisfies Assumption 1. If there exist six matrices  $W_1 = W_1^\top > 0, W_3 = W_3^\top > 0,$*

*$W_2, Z_1, Z_2$  and  $N$  of appropriate dimensions and a scalar  $\nu > 0$  such that the following LMI*

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ \Upsilon & -\Lambda & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \leq 0 \quad (30)$$

*holds with  $\Lambda = S,$*

$$\Upsilon = [(G_z + B_z N)^\top - S\Omega H_z W_1 \quad D_z^\top - S\Omega H_z W_2]$$

*and*

$$Z = [Z_1 \quad Z_2], \quad W = \begin{bmatrix} W_1 & W_2 \\ W_2^\top & W_3 \end{bmatrix} > 0, \quad (31)$$

*then the closed-loop system (28), (29) with  $K = [K_z \quad K_\xi] = [Z_1 \quad Z_2] W^{-1}$  and  $K_\varphi = N$  is exponentially contractive and the regulated output  $e$  is exponentially converging to zero for any  $(r, \delta) \in \mathbb{R}^{n_e+n_z}.$*

**Corollary 2.** *Assume that  $\varphi$  satisfies Assumption 2. If there exist six matrices  $W_1 = W_1^\top > 0, W_3 = W_3^\top > 0, W_2, Z_1, Z_2$  and  $N$  of appropriate dimensions and a scalar  $\nu > 0$  such that the LMIs (30), with  $\Lambda = 2\Gamma^{-1},$*

$$\Upsilon = [H_z W_1 + (G_z + B_z N)^\top \quad D_z^\top + H_z W_2]$$

*and (31) hold. Then the closed-loop system (28), (29) with  $K = [K_z \quad K_\xi] = [Z_1 \quad Z_2] W^{-1}$  and  $K_\varphi = N$  is exponentially contractive and the regulated output  $e$  is exponentially converging to zero for any  $(r, \delta) \in \mathbb{R}^{n_e+n_z}.$*

#### 4.1. An example: the mechanical ventilation system

Consider the example of a mechanical ventilation system presented in [27, 36] and whose system dynamics is modeled as (28) with

$$\begin{aligned} A &= \begin{bmatrix} -2\omega_n & -\omega_n^2 & 0 \\ 1 & 0 & 0 \\ 0 & b_p \omega_n^2 \ell & -b_p c_p \ell + a_p \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ 0 \\ b_p \end{bmatrix}, \\ H &= [0 \quad -\omega_n^2 \quad +c_p], \quad C = [0 \quad -d_p \ell \omega_n^2 \quad c_p(\ell d_p + 1)], \\ D &= -d_p, \quad \varphi(\zeta) = \text{sign}(\zeta) \frac{a - \sqrt{a^2 + 2\rho|\zeta|}}{\rho} + \ell \zeta, \end{aligned}$$

where  $\zeta \in \mathbb{R}$ ,  $a_p, b_p, c_p, d_p, \omega_n, a, \rho$  are constant plant parameters (see [27] for a physical interpretation) and  $\ell \in \mathbb{R}$  is a free parameter. Note that  $\varphi$  is  $C^1$  and if  $\ell > \frac{1}{a}$  with  $a > 0$ , Assumption 2 is satisfied with  $\Gamma = -\frac{2}{a} + 2\ell$ . We then apply Corollary 2 with  $w_n = 188.4, a_p = 37.45 \cdot 10^{-3}, b_p = 31.65 \cdot 10^{-3}, c_p = d_p = 3.165 \cdot 10^{-3}, a = 4.865 \cdot 10^{-3}, \rho = 1.6 \cdot 10^{-6}$  and  $\ell = 10^{-3}$ . The LMIs are feasible, and the controller (29) with

$$\begin{aligned} K_z &= [-4.76 \cdot 10^3 \quad -4.37 \cdot 10^7 \quad 505.32], \\ K_\xi &= 153.18, \quad K_\varphi = -7.56 \cdot 10^6, \end{aligned}$$

makes the closed-loop system exponentially contractive in the whole state space. Therefore, for any  $(\delta, r)$  the closed-loop trajectories exponentially converge to an equilibrium point where the tracking error is zero.

## 5. Contractive observer design

In this section, we consider the case in which the dynamical system (13) is complemented with a measured output and reads

$$\dot{x} = Ax + Bu + G\varphi(Hx), \quad y = Cx, \quad (32)$$

where  $x \in \mathbb{R}^{n_x}$  is the system state,  $u \in \mathbb{R}^{n_u}$  is the control input,  $y \in \mathbb{R}^{n_y}$  the measured output and  $A, B, G, H, C$  are matrices of appropriate dimensions. Contraction properties have been deeply investigated for the design of observers. See, e.g., the recent survey [1] and more in particular [3, 37]. In our context, following [25], we consider an observer of the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) + G\varphi(\hat{\zeta}), \\ \hat{\zeta} &= H\hat{x} - E(y - C\hat{x}), \end{aligned} \quad (33)$$

where  $E, L$  are matrices in  $\mathbb{R}^{n_y \times n_y}$  and  $\mathbb{R}^{n_x \times n_y}$  respectively. In particular, differently from many other standard design (such as Luenberger observers, Kalman like observers, high-gain observers and so on), the observer (33) is characterized by the presence of a correction term inside the nonlinear term. A sufficient condition to guarantee the convergence of the observer, namely, that the observer trajectories  $\hat{X}(\hat{x}, t)$  converge to the plant's trajectory  $X(x, t)$ , is to select the matrices  $E$  and  $L$  so that to ensure that the observer defines a uniform (with respect to  $y$ ) contraction. For systems of the form (32) with an observer selected as (33), this is achieved for instance if there exists a positive definite matrix  $Q$  in  $\mathbb{R}^{n_x \times n_x}$  such that

$$\text{He} \left\{ Q \left[ A - LC + G \frac{\partial \varphi}{\partial \zeta}(\zeta)(H - EC) \right] \right\} \leq -\nu I \quad (34)$$

for all  $\zeta$ , for some strictly positive real number  $\nu > 0$ . Indeed, under the condition (34), it is easy to show<sup>3</sup> the existence of positive real numbers  $(k, \lambda)$  such that for all  $(x, \hat{x})$  in  $\mathbb{R}^{2n_x}$ ,

$$\left| X(x, t) - \hat{X}(\hat{x}, x, t) \right| \leq k \exp(-\lambda t) |x - \hat{x}|,$$

for all  $t \geq 0$  in the time domain of existence of solutions.

Similar to the results presented in the former sections, it is possible to give sufficient conditions in the form of LMI in order to obtain constructive conditions for the design of  $L$  and  $E$ . For instance, based on the sector bound condition in Assumption 1, we obtain the following result.

**Corollary 3.** *Assume that  $\varphi$  satisfies Assumption 1. If there exist a symmetric positive definite matrix  $Q \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $R$  and  $E$  of appropriate dimensions and a positive real number  $q$  such that*

$$\text{He} \left\{ \begin{bmatrix} QA - RC + \frac{q}{2}I & 0 \\ G^\top Q - S\Omega(H - EC) & -S \end{bmatrix} \right\} < 0, \quad (35)$$

<sup>3</sup>This can be done, for instance, by applying Theorem 1 to the error-coordinates dynamics  $e := x - \hat{x}$ .

then (33) is an exponentially contractive observer with  $L = Q^{-1}R$  and  $E$ .

Based on monotonic nonlinearities, the same result has been obtained in [25]. In particular, under Assumption 2, system (33) is an exponential observer for system (32) if there exists a solution to the following LMI

$$\text{He} \left\{ \begin{bmatrix} QA - RC + \frac{q}{2}I & 0 \\ H - EC + G^\top Q & -2\Gamma^{-1} \end{bmatrix} \right\} < 0, \quad (36)$$

with  $L = Q^{-1}R$ . In the following we specialize such a result for a nonlinearity satisfying (11).

**Corollary 4.** *Assume that  $\varphi$  satisfies Assumption 2. If there exist a symmetric positive definite matrix  $Q \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $R$  and  $E$  of appropriate dimensions and a positive real number  $q$  such that*

$$\text{He} \{ QA - RC + \frac{q}{2}I \} \leq 0, \quad (H - EC)^\top = -QG. \quad (37)$$

then (33) is an exponentially contractive observer for system (32) with  $L = Q^{-1}R$  and  $E$ .

Proofs are omitted for compactness.

### 5.1. An example: the surge system

Consider again the surge subsystem studied in Section 3.3. Recall that the nonlinearity satisfies relation (11). The output matrix  $C$  is given as  $C = [c_1, c_2]$ . Depending on the values of  $c_1$  and  $c_2$  different cases can be considered. In the following, we consider the case in which  $c_1 = 0$  and  $c_2 = 1$ . It can be shown that for all  $\ell \geq 6$ , there exists  $(L_1, L_2, E)$  in  $\mathbb{R}^3$  such that the system

$$\begin{cases} \dot{\hat{x}}_1 &= -\hat{x}_2 + \ell \hat{x}_1 - \varphi_\ell(\hat{x}_1 + E(y - \hat{x}_2)) + L_1(y - \hat{x}_2) \\ \dot{\hat{x}}_2 &= \hat{x}_1 - u + L_2(y - \hat{x}_2) \end{cases}$$

with  $\varphi_\ell$  defined in (23) is an exponentially contractive observer. As shown in (24), the function  $\varphi_\ell$  satisfies inequality (11) if  $\ell \geq 6$ . Moreover, the system (21) is in the form of (32) with matrices  $A, B, G, H$  as in (26) and  $C = [0 \ 1]$ . Let

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

Then, the equality constraint in (37) reads  $q_{11} = 1$  and  $q_{12} = -E$ . To satisfy  $Q > 0$  one has to satisfy  $q_{22} - E^2 > 0$ . From inequality (37), one also obtain

$$\text{He}\{QA - RC\} = \text{He} \left\{ \begin{bmatrix} \ell - E & -1 - R_1 \\ -E\ell + q_{22} & E - R_2 \end{bmatrix} \right\},$$

which can be made definite negative selecting

$$R_1 = -E\ell + q_{22} - 1, \quad E - \ell > 0, \quad R_2 - E > 0.$$

Hence, (37) is satisfied for sufficiently small  $q$ .



## 6. A nonlinear separation principle

One of the motivations of observer design is surely the design of output feedback control laws. In this section, we establish a separation principle by showing that a globally stabilizing output feedback law can be obtained by first designing a contractive state-feedback law, and then replacing the state by an estimate provided by a contractive observer. In contrast with most of nonlinear separation principle, with the proposed conditions one may recover standard results of linear systems in which the design of gains of the state-feedback law and the observer output injection are independent. As a matter of fact, most of the existing results about the separation principle for nonlinear systems, rely on time-separation scale conditions, see, e.g. [30, 31, 38]. In other words, the typical nonlinear approach consists in first designing a state-feedback law, and then replacing the state by an estimate which converges to the true state trajectory sufficiently fast. Such a goal is typically achieved by means of high-gain observers [30–32, 38]. This need of time-separation scale however leads to well-known drawbacks due to the employment of such a class of observers, such as the peaking phenomenon and the poor sensitivity to measurement noise properties [39]. In our case, such a time-separation scale condition is not anymore needed thanks to the contractivity properties. As a result, the observer's convergence may be tuned independently of the feedback design. For instance, the observer dynamics can be selected slower than the convergence of the state-feedback law, so that to improve the overall performances in the presence of measurement noise. We recover in this sense, the standard linear separation principle result.

### 6.1. Main result

In order to state the main result of this section, consider system (32) coupled with an output feedback control law of the form

$$u = K\hat{x} + N\varphi(H\hat{x}) \quad (38)$$

in which the estimate  $\hat{x}$  is provided by an observer of the form (33). For space reasons, in this section we consider only the case of functions  $\varphi$  satisfying the monotonic condition in Assumption 2, but similar results can be extended to the case of sector bound conditions of Assumption 1. Now, following the framework in [38] based on the notion of input-to-state stability (ISS), we recall that two different paths may be followed for the design of an output feedback law:

**Direct approach.** An ISS property is imposed on measurement error in the state feedback law to cope with the mismatch between  $x$  and  $\hat{x}$ .

**Indirect approach.** An ISS property is imposed to cope with the observer correction term.

In our context, both cases may be pursued under an extra assumption, as shown in the following.

**Theorem 2.** *Consider the system (32) and suppose Assumption 2 holds. Consider the feedback output law given by the observer (33) and control law (38). Suppose that the LMIs (18) and (36) are feasible for some matrices  $Q, R, E, W, Z, N$  of appropriate dimensions and any positive scalar  $\nu, q > 0$ . Assume moreover that one of the following properties hold:*

- the nonlinear term in the feedback (38) is zero ( $N = 0$ );
- the injection term inside the nonlinearity in (33) is zero ( $E = 0$ ).

*Then, the origin of the closed-loop system (32), (33), (38) is globally exponentially stable with  $K = ZW^{-1}$ ,  $L = Q^{-1}R$ , and  $N = 0$  or  $E = 0$ .*

**Proof. Direct approach ( $N = 0$ ):** With the change of coordinates  $\hat{x} \mapsto e := x - \hat{x}$ , the closed-loop dynamics  $(x, e)$  reads

$$\begin{aligned} \dot{x} &= (A + BK)x - BKe + G\varphi(Hx) \\ \dot{e} &= (A - LC)e + G[\varphi(Hx) - \varphi(Hx - (H + EC)e)]. \end{aligned}$$

Now, consider the Lyapunov function  $V(x) = x^\top Px$  where  $P = W^{-1}$  with  $W$  defined as solution to Lemma 2. We have that

$$\begin{aligned} \dot{V} &\leq 2x^\top P \left[ A + BK + G \int_0^1 \frac{\partial \varphi}{\partial \zeta}(s\zeta) ds H \right] x \\ &\quad + 2x^\top PBKe \\ &\leq -2\nu|x|^2 + 2x^\top PBKe \end{aligned}$$

where the last inequality comes from Lemma 2. Therefore, the  $x$ -dynamics is ISS with respect to the input  $e$ . Since (36) holds,  $\lim_{t \rightarrow \infty} e(t) = 0$  (see [2]). Hence the closed loop can be seen as a cascade of a system having an equilibrium which is globally asymptotically stable and an ISS system. Hence,  $(x, e) = (0, 0)$  is globally asymptotically stable for the closed loop system.

**Indirect approach ( $E = 0$ ):** With the change of coordinates  $x \mapsto e := x - \hat{x}$ , the closed-loop dynamics  $(\hat{x}, e)$  reads

$$\begin{aligned} \dot{\hat{x}} &= (A + BK)\hat{x} + (BN + G)\varphi(H\hat{x}) + LCe \\ \dot{e} &= (A - LC)e + G\varphi(\hat{x} + e) - G\varphi(\hat{x}) \end{aligned}$$

Consider the Lyapunov function  $V(\hat{x}) = \hat{x}^\top P\hat{x}$  where  $P = W^{-1}$  with  $W$  defined as solution to Lemma 2. We have that

$$\begin{aligned} \dot{V} &\leq 2\hat{x}^\top P \left[ A + BK + (BN + G) \int_0^1 \frac{\partial \varphi}{\partial s}(sH\hat{x}) ds H \right] \hat{x} \\ &\quad + 2\hat{x}^\top PLCe \\ &\leq -\nu|\hat{x}|^2 + 2\hat{x}^\top PLCe. \end{aligned}$$

Again, we have obtained a cascade of a contractive system and an ISS system. Therefore, the origin of the closed loop

is globally exponentially stable.  $\square$

**Remark 3.** *In the general case in which  $E \neq 0$  and  $N \neq 0$  no stability results can be stated a priori. However, under additional assumptions, one might still be able to claim stability of the closed loop by addressing the problem through a small-gain analysis. In this case, the designs of the feedback and the observer cannot be made disjointed and typically a time-scale separation between these dynamics is needed.*

**Remark 4.** *The output feedback law (33), (38) doesn't ensure in general any contractivity property for the closed-loop system (32), (33), (38). As a consequence, if one aims at obtaining a contractive output feedback law, a more general dynamic output feedback of the form*

$$\begin{aligned} u &= K_1 x_c + K_2 y + \sum_{j=1}^m N_j \varphi(H J_j x_c + E_j y), \\ \dot{x}_c &= A_c x_c + \sum_{j=1}^m M_j \varphi(H J_j x_c + E_j y) \end{aligned} \quad (39)$$

*needs to be considered. By rewriting the closed-loop system in the form (1), LMI conditions similar to those derived in Section 2 can be established in order to show contractivity of the closed-loop system in the sense of Theorem 1.*

## 6.2. Example: a link manipulator

Consider the example of a flexible link manipulator presented in [28] with system (13) defined by matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I_m} & -\frac{C_{vf}}{I_m} & \frac{k}{I_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_l} & 0 & -\frac{k}{I_l} - b \frac{mgh}{I_m} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ k_T \\ \frac{I_m}{I_m} \\ 0 \\ 0 \end{bmatrix},$$

$$G^\top = \begin{bmatrix} 0 & 0 & 0 & \frac{mgh}{I_m} \end{bmatrix}, H = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

and nonlinearity  $\varphi$  satisfying Assumption 2 with  $\Gamma = 2(b-1)$  where  $b > 1$  is a free parameter. The system is coupled with a linear output  $C = [1 \ 0 \ 0 \ 0]$ . For such a system with plant parameters selected as  $k = 1.8$ ,  $I_m = 3.7$ ,  $I_l = 9.3$ ,  $C_{vf} = 4.6$ ,  $k_T = 8$ ,  $m = 2.1$ ,  $g = 9.81$ ,  $h = 1$  and  $b = 1.1$ , the assumptions of Theorem 2 hold with a zero nonlinear injection term  $E = 0$  and the system is output-feedback stabilizable by means of the dynamical controller (33), (38) with gains selected as  $K = [-43.2 \ -5.4 \ 203.8 \ -599.9]$ ,  $N = 16.9$  and  $L = [-9.4 \ -87.1 \ -858.7 \ -1177.5]$ , and with positive scalars  $\nu = 2$  and  $q = 1$ . In this case, the convergence of the observer is slower than the guaranteed convergence of the state-feedback law (differently from most of output feedback results for nonlinear systems).

## 7. Conclusion

In this paper we have introduced some LMI sufficient conditions to analyze the contraction properties of nonlinear systems composed of a linear part to which is added an incremental sector bounded or monotonic nonlinearity. These sufficient conditions can be translated when dealing with state-feedback, integral action based controllers, observers and observed-based output-feedback designs. Throughout the paper, some practical examples are presented as illustrations. The proposed results pave the way for future studies. In particular, it could be interesting to address the design of general dynamical output feedback controllers, for example as defined in (39) ensuring contractivity properties of the overall closed-loop dynamics. Furthermore, it should be interesting to consider other kinds of nonlinearity  $\varphi$ , in particular, those affecting the input  $u$  of the system.

## References

- [1] P. Bernard, V. Andrieu, D. Astolfi, Observer design for continuous-time dynamical systems, *Annual Reviews in Control* 53 (2022) 224–248.
- [2] M. Arcak, M. Larsen, P. Kokotović, Circle and Popov criteria as tools for nonlinear feedback design, *Automatica* 39 (4) (2003) 643–650.
- [3] V. Andrieu, B. Jayawardhana, L. Praly, Characterizations of global transversal exponential stability, *IEEE Transaction on Automatic Control* 66 (8) (2020) 3682–3694.
- [4] V. Andrieu, S. Tarbouriech, LMI conditions for contraction and synchronization, *IFAC-PapersOnLine* 52 (16) (2019) 616–621.
- [5] F. Zhang, H. L. Trentelman, J. Scherpen, Fully distributed robust synchronization of networked Lur'e systems with incremental nonlinearities, *Automatica* 50 (10) (2014) 2515–2526.
- [6] V. Andrieu, B. Jayawardhana, S. Tarbouriech, Some results on exponential synchronization of nonlinear systems, *IEEE Transaction on Automatic Control* 63 (4) (2018) 1213–1219.
- [7] M. Giaccagli, V. Andrieu, D. Astolfi, G. Casadei, Sufficient metric conditions for synchronization of leader-connected homogeneous nonlinear multi-agent systems, *IFAC-PapersOnLine* 54 (14) (2021) 412–417.
- [8] M. Giaccagli, D. Astolfi, V. Andrieu, L. Marconi, Sufficient conditions for global integral action via incremental forwarding for input-affine nonlinear systems, *IEEE Transaction on Automatic Control* 67 (12) (2022) 6537–6551.
- [9] A. Pavlov, N. Van De Wouw, H. Nijmeijer, Uniform output regulation of nonlinear systems: a convergent dynamics approach, Springer Science & Business Media, 2006.
- [10] M. Giaccagli, D. Astolfi, V. Andrieu, L. Marconi, Sufficient conditions for output reference tracking for nonlinear systems: a contractive approach, in: *59th IEEE Conference on Decision and Control*, 2020, pp. 4580–4585.
- [11] D. Angeli, Further results on incremental input-to-state stability, *IEEE Transaction on Automatic Control* 54 (6) (2009) 1386–1391.
- [12] F. Forni, R. Sepulchre, A differential Lyapunov framework for contraction analysis, *IEEE Transaction on Automatic Control* 59 (3) (2014) 614–628.
- [13] W. Lohmiller, J. Slotine, On contraction analysis for non-linear systems, *Automatica* 34 (6) (1998) 683–696.
- [14] E. Sontag, Contractive systems with inputs, in: *Perspectives in mathematical system theory, control, and signal processing*, Springer, 2010, pp. 217–228.
- [15] M. Giaccagli, V. Andrieu, S. Tarbouriech, D. Astolfi, Infinite gain margin, contraction and optimality: an LMI-based design, *European Journal of Control* 68 (2022) 100685.

- [16] F. Bullo, *Contraction Theory for Dynamical Systems*, 1.1 Edition, Kindle Direct Publishing, 2023.  
URL <https://fbullo.github.io/ctds>
- [17] B. S. Rüffer, N. van de Wouw, M. Mueller, Convergent systems vs. incremental stability, *Systems & Control Letters* 62 (3) (2013) 277–285.
- [18] L. D’Alto, M. Corless, Incremental quadratic stability, *Numerical Algebra, Control & Optimization* 3 (1) (2013) 175.
- [19] J. Jouffroy, T. Fossen, A tutorial on incremental stability analysis using contraction theory, *MIC Journal* 31 (3) (2010) 93–106.
- [20] M. Zamani, N. van de Wouw, R. Majumdar, Backstepping controller synthesis and characterizations of incremental stability, *Systems & Control Letters* 62 (10) (2013) 949–962.
- [21] S. Waitman, L. Bako, P. Massioni, G. Scorletti, V. Fromion, Incremental stability of Lur’e systems through piecewise-affine approximations, *IFAC-PapersOnLine* 50 (1) (2017) 1673–1679.
- [22] I. R. Manchester, J. E. Slotine, Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design, *IEEE Transaction Automatic Control* 62 (6) (2017) 3046–3053.
- [23] H. Tsukamoto, S. Chung, Neural contraction metrics for robust estimation and control: A convex optimization approach, *IEEE Control Systems Letters* 5 (1) (2020) 211–216.
- [24] G. Zames, P. Falb, Stability conditions for systems with monotone and slope-restricted nonlinearities, *SIAM Journal on Control* 6 (1) (1968) 89–108.
- [25] M. Arcak, P. Kokotović, Nonlinear observers: a circle criterion design and robustness analysis, *Automatica* 37 (12) (2001) 1923–1930.
- [26] A. Zemouche, M. Boutayeb, On LMI conditions to design observers for Lipschitz nonlinear systems, *Automatica* 49 (2) (2013) 585–591.
- [27] J. Reinders, B. Hunnekens, F. Heck, T. Oomen, N. van de Wouw, Accurate pressure tracking to support mechanically ventilated patients using an estimated nonlinear hose model and delay compensation, *Control Engineering Practice* 106 (2021) 104660.
- [28] R. Wu, W. Zhang, F. Song, Z. Wu, W. Guo, Observer-based stabilization of one-sided Lipschitz systems with application to flexible link manipulator, *Advances in Mechanical Engineering* 7 (12) (2015) 1687814015619555.
- [29] V. Andrieu, B. Jayawardhana, L. Praly, Transverse exponential stability and applications, *IEEE Transaction on Automatic Control* 61 (11) (2016) 3396–3411.
- [30] A. Atassi, H. Khalil, A separation principle for the stabilization of a class of nonlinear systems, *IEEE Transaction on Automatic Control* 44 (9) (1999) 1672–1687.
- [31] A. Teel, L. Praly, Global stabilizability and observability imply semi-global stabilizability by output feedback, *Systems & Control Letters* 22 (5) (1994) 313–325.
- [32] H. Khalil, L. Praly, High-gain observers in nonlinear feedback control, *International Journal of Robust and Nonlinear Control* 24 (6) (2014) 993–1015.
- [33] V. Andrieu, Lyapunov functions obtained from first order approximations, in: *Feedback Stabilization of Controlled Dynamical Systems*, Springer, 2017, pp. 3–28.
- [34] X. Fan, M. Arcak, Observer design for systems with multivariable monotone nonlinearities, *Systems & Control Letters* 50 (4) (2003) 319–330.
- [35] A. Andersson, A. Robertsson, A. Shiriaev, L. Freidovich, R. Johansson, Robustness of the Moore-Greitzer compressor model’s surge subsystem with new dynamic output feedback controllers, *IFAC Proceedings Volumes* 47 (3) (2014) 3690–3695.
- [36] J. Reinders, M. Giaccagli, B. Hunnekens, D. Astolfi, T. Oomen, N. van De Wouw, Repetitive control for Lur’e-type systems: application to mechanical ventilation, *IEEE Transactions on Control Systems Technology* (2023).
- [37] R. Sanfelice, L. Praly, Convergence of nonlinear observers on  $\mathbb{R}^n$  with a Riemannian metric (part I), *IEEE Transaction on Automatic Control* 57 (7) (2011) 1709–1722.
- [38] V. Andrieu, L. Praly, A unifying point of view on output feedback designs for global asymptotic stabilization, *Automatica* 45 (8) (2009) 1789–1798.
- [39] D. Astolfi, L. Marconi, A high-gain nonlinear observer with limited gain power, *IEEE Transaction on Automatic Control* 60 (11) (2015) 3059–3064.