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Asymptotic confidence intervals for extreme quantiles in a maximum domain of attraction

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Abstract

Extreme quantiles are commonly used to assess the risk of extreme events such as large floods, extreme temperature, financial crisis among many others. When the underlying distribution belongs to a maximum domain of attraction, extreme quantile estimation has been widely studied in the literature but, surprisingly, little work has been dedicated to the construction of an asymptotic confidence interval. This is precisely the question addressed in this paper where a confidence interval for extreme quantile is proposed. This confidence interval can be used whatever the maximum domain of attraction of the distribution. The convergence of its coverage probability to the nominal one is established and its finite sample performance is investigated through a simulation study. An application on lifespan of French supercentenarians is done.

Keywords: Extreme quantile; confidence interval; maximum domain of attraction.

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1 Introduction

Let X be a random variable defined on an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $F_X(\cdot) := \mathbb{P}(X \leq \cdot)$ its cumulative distribution function and by $\bar{F}_X = 1 - F_X$ its survival function. Throughout this paper, we assume that F_X is a continuous and strictly increasing function and that the support \mathcal{S} of the distribution is included in $[0, \infty)$. The associated quantile of level $\beta \in (0, 1)$ is given by

$$Q_X(\beta) = \bar{F}_X^{\leftarrow}(\beta) = \inf\{x \in \mathcal{S} \mid \bar{F}_X(x) \leq \beta\}.$$

In various fields of application, large quantiles are commonly used to assess the risk of extreme events such as large floods, extreme temperature, financial crisis among many others. More details can be found in the monographs of Pisarenko and Rodkin [14] and Embrechts et al. [8]. If one observes n independent realizations of X , a quantile $Q_X(\beta_n)$ is said to be large if $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. When β_n does not converge too fast to 0, i.e., $n\beta_n \rightarrow \infty$, the quantile can be simply estimated by an order statistic. A more challenging question arises when $n\beta_n \rightarrow c > 0$ i.e., when $Q_X(\beta_n)$ is an extreme quantile. In this situation a rough knowledge of the shape of the tail distribution is required. In this paper, we assume that F_X belongs to a maximum domain of attraction (MDA) that is to say there exist sequences $(a_n) > 0$ and (b_n) and a real ξ such that

$$\mathbb{P}\left(\frac{\max(X_1, \dots, X_n) - b_n}{a_n} \leq x\right) = \exp\left(-(1 + \xi x)^{-1/\xi}\right),$$

for all x satisfying $1 + \xi x > 0$. This convergence in distribution can be reformulated in terms of the quantile function Q_X . More specifically, F_X belongs to a MDA if and only if there exist a parameter $\xi \in \mathbb{R}$ and an auxiliary function $a : [1, \infty) \rightarrow (0, \infty)$ such that for all $t > 0$,

$$\lim_{\alpha \rightarrow 0} \frac{Q_X(t\alpha) - Q_X(\alpha)}{a(\alpha^{-1})} = K_\xi(t^{-1}) := \int_1^{t^{-1}} s^{\xi-1} ds. \quad (1)$$

The parameter ξ , called the extreme value index, plays an important role in modeling extreme events since it controls the tail heaviness of the distribution. More precisely, if $\xi < 0$, the right endpoint $Q_X(0)$ is finite and

\bar{F}_X decreases at a polynomial rate. When $\xi = 0$, the survival function decreases at an exponential rate and when $\xi > 0$, the distribution is said to be heavy-tailed that is to say, for all $t > 0$,

$$\lim_{\alpha \rightarrow 0} \frac{Q_X(t\alpha)}{Q_X(\alpha)} = t^{-\xi}.$$

In other words, the quantile function Q_X is heavy-tailed if and only if the function $Q_X(1/\cdot)$ is a regularly varying function with index ξ or equivalently, $Q_X(1/x) = x^{-\xi}L(x)$ where L is a slowly varying function that is to say, for all $t > 0$, the ratio $L(tx)/L(x)$ converges to 1 as $x \rightarrow \infty$.

While the estimation of extreme quantiles has been widely studied in the literature (see for instance the references [16, 7, 5]), little work has been dedicated to the construction of an asymptotic confidence interval. This is precisely the question addressed in this paper. More specifically, for a preselected level $\gamma \in (0, 1)$, we are searching for positive random variables $L_{n,\gamma}$ and $R_{n,\gamma}$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(L_{n,\gamma} \leq Q_X(\beta_n)) = \lim_{n \rightarrow \infty} \mathbb{P}(R_{n,\gamma} \geq Q_X(\beta_n)) = \frac{1 + \gamma}{2}.$$

The random interval $[L_{n,\gamma}, R_{n,\gamma}]$ is then an asymptotic two-sided confidence interval of level γ for $Q_X(\beta_n)$.

In the recent literature, one can mention the existence of three papers dedicated to the construction of an asymptotic interval for extreme quantiles. In the first one (Gardes [9]), the construction is addressed only for quantiles of order β_n with $n\beta_n \rightarrow \infty$. The construction of an asymptotic confidence interval for extreme quantiles ($n\beta_n \rightarrow c > 0$) has been tackled in Gardes and Maistre [10] and Buitendag [4]. In both works, the underlying distribution is assumed to be heavy-tailed and the proposed confidence intervals are not suitable when $\xi \leq 0$. The novelty brought by this work is the proposition of an asymptotic confidence interval for extreme quantiles that can be used whatever the sign of the extreme value index.

The paper is organized as follows. The proposed confidence interval is introduced in Section 2. Roughly speaking, it is obtained by extrapolating two order statistics over the range of the data. The estimation of these extrapolation factors is considered in Section 3. A general estimation procedure is presented in Section 3.1 where sufficient conditions for the convergence of the coverage probability to the nominal one are given. An application to a classical extreme quantile estimator is given in Section 3.2. The finite sample behavior of our procedure is investigated on simulated data in Section 4 and on a real dataset in Section 5. All the proofs are gathered in Section 6.

2 Construction of the confidence interval

Let us consider a random variable X such that (1) holds for the quantile function Q_X . Let X_1, \dots, X_n be a sample of n independent replications of X and let $X_{1,n} \leq \dots \leq X_{n,n}$ be the sample arranged in ascending order. Let us briefly explain the procedure used to find the right bound of the confidence interval, namely the random variable $R_{n,\gamma}$. The same procedure can be applied to find the left bound $L_{n,\gamma}$. Since the probability space is atomless, there exist independent standard uniform random variables U_1, \dots, U_n such that the random vector $(X_{n-j,n}, j = 0, \dots, n-1)$ is distributed as the random vector $(Q_X(U_{j+1,n}), j = 0, \dots, n-1)$. Hence, for any integers $j_R \in \mathbb{N}$, it is readily seen that

$$\mathbb{P}(X_{n-j_R,n} \geq Q_X(\beta_n)) = \mathbb{P}(U_{j_R+1,n} \leq \beta_n) = F_{\text{Beta}}^{(j_R+1, n-j_R)}(\beta_n),$$

where $F_{\text{Beta}}^{(r,s)}$ is the cumulative distribution function of a Beta distribution with parameters $r > 0$ and $s > 0$. If $n\beta_n \rightarrow c > 0$, we have that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_{n-j_R,n} \geq Q_X(\beta_n)) = \lim_{n \rightarrow \infty} F_{\text{Beta}}^{(j_R+1, n-j_R)}(\beta_n) = F_{\text{Gamma}}^{(j_R+1, 1)}(c),$$

where $F_{\text{Gamma}}^{(r,s)}$ is the cumulative distribution function of a Gamma distribution with parameters $r > 0$ and $s > 0$, see Lemma 1 for a proof. As a consequence, for a given level β_n and a given probability γ , there is no guarantee of finding an integer $j_R \in \mathbb{N}$ such that $\mathbb{P}(X_{n-j_R,n} \geq Q_X(\beta_n)) \rightarrow (1+\gamma)/2$. As an example, if $\beta_n = 1/(2n)$, we have for all $j \in \mathbb{N}$ that $F_{\text{Gamma}}^{(j+1, 1)}(1/2) \leq$

$0.39 < (1 + \gamma)/2$ for all $\gamma \in (0, 1)$. It thus appears that for a level β_n close to 0, an extreme order statistic is not large enough to be considered as a bound of an asymptotic confidence interval for $Q_X(\beta_n)$. We thus need to find a positive extrapolation factor $t_{n,R}$ (depending on j_R) in order to have $\mathbb{P}(t_{n,R}X_{n-j_R,n} \geq Q_X(\beta_n)) \approx (1 + \gamma)/2$. To find such a sequence, we use the methodology introduced in [10]. More specifically, using the same arguments as above, we get that

$$\mathbb{P}(t_{n,R}X_{n-j_R,n} \geq Q_X(\beta_n)) = F_{\text{Beta}}^{(j_R+1, n-j_R)}(\overline{F}_X(t_{n,R}^{-1}Q_X(\beta_n))).$$

A good candidate is the sequence $t_{n,R} = Q_X(\beta_n)/Q_X(\mathbf{a}_{n,R}(\gamma))$ where, denoting by $q_{\text{Beta}}^{(r,s)}$ the inverse of $F_{\text{Beta}}^{(r,s)}$,

$$\mathbf{a}_{n,R}(\gamma) := q_{\text{Beta}}^{(j_R+1, n-j_R)}\left(\frac{1+\gamma}{2}\right). \quad (2)$$

Of course, the factor $t_{n,R}$ is unknown and need to be estimated by any consistent estimator $\widehat{t}_{n,R}$. An example of an estimator is given later. Finally, we propose to take for the right bound of the asymptotic confidence interval for $Q_X(\beta_n)$, the random variable $R_{n,\gamma} = \widehat{t}_{n,R}X_{n-j_R,n}$. Similarly, we propose to take $L_{n,\gamma} = \widehat{t}_{n,L}X_{n-j_L,n}$ where $\widehat{t}_{n,L}$ is any consistent estimator of $t_{n,L} := Q_X(\beta_n)/Q_X(\mathbf{a}_{n,L}(\gamma))$ where

$$\mathbf{a}_{n,L}(\gamma) := q_{\text{Beta}}^{(j_L+1, n-j_L)}\left(\frac{1-\gamma}{2}\right). \quad (3)$$

To sum up, we give below the definition of the two-sided asymptotic confidence interval for $Q_X(\beta_n)$ that is considered in all what follows.

Definition 1 *Let X be a positive random variable such that (1) holds for the quantile function Q_X . Let X_1, \dots, X_n be n independent replications of X . For a preselected probability $\gamma \in (0, 1)$, a sequence β_n such that $n\beta_n \rightarrow c > 0$ and $(j_L, j_R) \in \mathbb{N}^2$, the two-sided confidence interval for the extreme quantile $Q_X(\beta_n)$ is*

$$\text{CI}(\gamma, \beta_n) := \left[\widehat{t}_{n,L}X_{n-j_L,n}, \widehat{t}_{n,R}X_{n-j_R,n} \right],$$

where $\widehat{t}_{n,L}$ and $\widehat{t}_{n,R}$ are any consistent estimators of the ratios $t_{n,L}$ and $t_{n,R}$ with

$$t_{n,\bullet} := \frac{Q_X(\beta_n)}{Q_X(\mathbf{a}_{n,\bullet}(\gamma))},$$

the symbol \bullet being either L or R .

The confidence interval $\text{CI}(\gamma, \beta_n)$ is thus a rescaled version of the random interval $[X_{n-j_L, n}, X_{n-j_R, n}]$. The choice of the two integers j_L and j_R is left to the user. A proposition for the couple (j_L, j_R) is given in the simulation study.

In the next result, we provide sufficient conditions on the estimators $\widehat{t_{n,L}}$ and $\widehat{t_{n,R}}$ ensuring that the asymptotic coverage probability of $\text{CI}(\gamma, \beta_n)$ is equal to γ . Before that, let us remind that a sequence of real random variables Z_n is $\mathcal{O}_{\mathbb{P}}(1)$ if for all $\epsilon > 0$, there exist $N_\epsilon \in \mathbb{N}$ and $c_\epsilon > 0$ such that for all $n \geq N_\epsilon$, $\mathbb{P}(Z_n \in [-c_\epsilon, c_\epsilon]) > 1 - \epsilon$. In particular, if Z_n converges in distribution, then $Z_n = \mathcal{O}_{\mathbb{P}}(1)$.

Theorem 1 *Assume that (1) holds for Q_X and let $(\beta_n) \in (0, 1)$ be a sequence such that $n\beta_n \rightarrow c > 0$. If the extrapolation factors $\widehat{t_{n,L}}$ and $\widehat{t_{n,R}}$ in Definition 1 are such that $\sigma_n^{-1}(t_{n,\bullet}/\widehat{t_{n,\bullet}} - 1) = \mathcal{O}_{\mathbb{P}}(1)$ where (σ_n) is a positive sequence satisfying*

$$\lim_{n \rightarrow \infty} \sigma_n \frac{Q_X(1/n)}{a(n)} = 0, \quad (4)$$

then the probabilities

$$\mathbb{P}\left(\widehat{t_{n,L}} X_{n-j_L, n} \leq Q_X(\beta_n)\right) \quad \text{and} \quad \mathbb{P}\left(\widehat{t_{n,R}} X_{n-j_R, n} \geq Q_X(\beta_n)\right),$$

both converge to $(1 + \gamma)/2$ as $n \rightarrow \infty$.

From de Haan and Ferreira [11, Lemma 1.2.9], we have

$$\lim_{n \rightarrow \infty} \frac{a(n)}{Q_X(1/n)} = \max(0, \xi).$$

Hence, condition (4) entails that the sequence σ_n converges to 0 and thus that $\widehat{t_{n,\bullet}}$ is a consistent estimator of $t_{n,\bullet}$ in the sense that the ratio $\widehat{t_{n,\bullet}}/t_{n,\bullet}$ converges in probability to 1. More specifically, if $\xi > 0$, condition (4) reduces to $\sigma_n \rightarrow 0$. When $\xi < 0$, we have that $Q_X(1/n) \rightarrow Q_X(0) =: x^* < \infty$ and, according to [11, Lemma 1.2.9 and Corollary 1.2.10], $a(n) \sim -\xi\{x^* - Q_X(1/n)\} = n^\xi L(n)$, where L is a slowly varying function. So, when

$\xi < 0$, condition (4) takes the form $\sigma_n n^{-\xi}/L(n) \rightarrow 0$. In particular, when $\xi < -1/2$, the ratio $\widehat{t_{n,\bullet}}/t_{n,\bullet}$ must converges to 1 faster than the classical parametric rate of convergence $n^{1/2}$. Constructing an asymptotic confidence interval for $Q_X(\beta_n)$ in this situation seems to be a quite difficult task.

3 Estimation of the extrapolation factors

Starting from a sample X_1, \dots, X_n of independent copies of X , this section is dedicated to the estimation of the extrapolation factors $t_{n,L}$ and $t_{n,R}$. Recall that, for some preselected probability γ , these extrapolation factors are given by

$$t_{n,\bullet} := \frac{Q_X(\beta_n)}{Q_X(\mathbf{a}_{n,\bullet}(\gamma))}.$$

As before, we assume that the quantile function Q_X satisfies (1).

3.1 General estimation procedure

The natural way to estimate $t_{n,\bullet}$ is to plug any consistent estimator $\widehat{Q}_{X,n}$ of $Q_X(\cdot)$ in the expression of $t_{n,\bullet}$ leading to the class of estimators

$$\widehat{t}_{n,\bullet}^{(*)} := \frac{\widehat{Q}_{X,n}(\beta_n)}{\widehat{Q}_{X,n}(\mathbf{a}_{n,\bullet}(\gamma))}. \quad (5)$$

For this class of estimators, we have the following version of Theorem 1.

Corollary 1 *Assume that (1) holds for Q_X and let $(\beta_n) \in (0, 1)$ be a sequence such that $n\beta_n \rightarrow c > 0$. If there exists a positive sequence (ϑ_n) with $\vartheta_n/a(n) \rightarrow 0$ and*

$$\vartheta_n^{-1} \left(\widehat{Q}_{X,n}(u_n) - Q_X(u_n) \right) = \mathcal{O}_{\mathbb{P}}(1),$$

for any sequence (u_n) satisfying $nu_n \rightarrow d > 0$, then, the conclusion of Theorem 1 holds.

3.2 Application: Moment Estimator

A classical estimator of the extreme quantile $Q_X(u_n)$ where the sequence $(u_n) \in (0, 1)$ satisfies $nu_n \rightarrow c > 0$ as $n \rightarrow \infty$ is provided by the Moment Estimator $\widehat{Q}_{X,n}^{(M)}$ (see for instance de Haan and Ferreira [11, Section 4.3.2]) which is based on the moment statistics

$$M_n^{(j)} := \frac{1}{k_n} \sum_{i=1}^{k_n} (\ln X_{n-i+1,n} - \ln X_{n-k_n,n})^j, \quad j \in \{1, 2\}.$$

More specifically, for a sequence $(k_n) \in \{1, \dots, n\}$, its expression is given by

$$\widehat{Q}_{X,n}^{(M)}(u_n) := X_{n-k_n,n} + \widehat{a}_n^{(M)} \left(\frac{n}{k_n} \right) K_{\widehat{\xi}_n^{(M)}} \left(\frac{k_n}{nu_n} \right),$$

where,

$$\widehat{\xi}_n^{(M)} := M_n^{(1)} + 1 - \frac{1}{2} \left(1 - \frac{\{M_n^{(1)}\}^2}{M_n^{(2)}} \right)^{-1},$$

is the moment estimator of the extreme value index ξ introduced in Dekkers et al. [6] and

$$\widehat{a}_n^{(M)} \left(\frac{n}{k_n} \right) := X_{n-k_n,n} M_n^{(1)} \left(1 - \widehat{\xi}_n^{(M)} + M_n^{(1)} \right),$$

is the moment estimator of the auxiliary function $a(n/k_n)$.

To deal with the asymptotic behavior of the Moment Estimator, the rate of convergence in (1) needs to be specified. The following second-order condition is then introduced.

(SOC) There exist $\rho < 0$, $\xi \neq \rho$ and functions $a > 0$ and A with $A(x) \rightarrow 0$ as $x \rightarrow \infty$ such that for all $t > 0$,

$$\lim_{\alpha \rightarrow 0} \frac{1}{A(\alpha^{-1})} \left(\frac{Q_X(t\alpha) - Q_X(\alpha)}{a(\alpha^{-1})} - K_\xi(t^{-1}) \right) = H_{\xi,\rho}(t^{-1}),$$

where $H_{\xi,\rho}(s) := \{K_{\xi+\rho}(s) - K_\xi(s)\}/\rho$.

A detailed discussion on this condition can be found in the monographs of Beirlant et al. [2, Section 3.3] and de Haan and Ferreira [11, Section 2.3]. The second-order parameter $\rho < 0$, controls the rate of convergence in (1): the larger the value of ρ , the slower the convergence. Note that when there exists an auxiliary function a such that

$$\frac{Q_X(t\alpha) - Q_X(\alpha)}{a(\alpha^{-1})} = K_\xi(t^{-1}),$$

the second-order parameter ρ is by convention equal to $-\infty$. From [11, Lemma B.3.16], if Q_X satisfies the second-order condition (SOC) then, there exist $\rho' \in [0, \rho]$ and a function B with $B(x) \rightarrow 0$ as $x \rightarrow \infty$ such that

$$\lim_{\alpha \rightarrow 0} \frac{1}{B(\alpha^{-1})} \left(\frac{\ln Q_X(t\alpha) - \ln Q_X(\alpha)}{a(\alpha^{-1})/Q_X(\alpha)} - K_{\xi_-}(t^{-1}) \right) = H_{\xi_-, \rho'}(t^{-1}), \quad (6)$$

where $\xi_- := \min(0, \xi)$ as before. Note that the function B is regularly varying with index ρ' and, as a consequence, $A(x) = \mathcal{O}(B(x))$.

The asymptotic behavior of $\widehat{Q}_{X,n}^{(M)}(u_n)$ has been established in de Haan and Ferreira [11, Section 4.3.2]. Let us recall the obtained result.

Proposition 1 *Assume that the second-order condition (SOC) holds. For any intermediate sequence (k_n) satisfying $k_n^{1/2} B(n/k_n) \rightarrow \lambda \in \mathbb{R}$ where B satisfies (6), we have, for any sequence (u_n) with $nu_n \rightarrow c > 0$,*

$$\{\vartheta_n^{(M)}\}^{-1} \left(\widehat{Q}_{X,n}^{(M)}(u_n) - Q_X(u_n) \right) = \mathcal{O}_{\mathbb{P}}(1),$$

where

$$\vartheta_n^{(M)} = k_n^{-1/2} a \left(\frac{n}{k_n} \right) \int_1^{k_n} s^{\xi-1} \ln(s) ds.$$

The corresponding asymptotic confidence interval as defined in Definition 1 is given by

$$\text{CI}^{(M)}(\gamma, \beta_n) := \left[\widehat{t}_{n,L}^{(M)} X_{n-j_L,n}, \widehat{t}_{n,R}^{(M)} X_{n-j_R,n} \right], \quad (7)$$

with

$$\widehat{t}_{n,\bullet}^{(M)} = \frac{\widehat{Q}_{X,n}^{(M)}(\beta_n)}{\widehat{Q}_{X,n}^{(M)}(\mathbf{a}_{n,\bullet}(\gamma))}, \quad \bullet \in \{L, R\}.$$

According to Corollary 1, the random interval $\text{CI}^{(M)}(\gamma, \beta_n)$ is an asymptotic two-sided confidence interval for $Q_X(\beta_n)$ if the condition

$$\lim_{n \rightarrow \infty} \frac{\vartheta_n^{(M)}}{a(n)} = \lim_{n \rightarrow \infty} \frac{a(n/k_n)}{a(n)} \frac{1}{k_n^{1/2}} \int_1^{k_n} s^{\xi-1} \ln(s) ds = 0, \quad (8)$$

holds. We have the following result.

Corollary 2 *Assume that the second-order condition (SOC) holds for Q_X and let (β_n) be a sequence such that $n\beta_n \rightarrow c > 0$.*

- i) If $\xi > -1/2$, conditions of Proposition 1 hold by taking for (k_n) any sequence asymptotically proportional to n^δ with $\delta = -2(\eta + \rho')/(1 - 2\rho')$ where $\eta \in (0, -\rho')$ can be taken as close as we want to 0.*
- ii) If $\xi < -1/2$, there does not exist a sequence (k_n) for which condition (8) is satisfied.*

A clear conclusion cannot be given in the situation where $\xi = -1/2$. It depends on the asymptotic behavior of the auxiliary function a .

When $\xi < -1/2$, the moment estimator cannot be used to construct a valid confidence interval. As mentioned before, in this case, we need to use an estimator of $t_{n,\bullet}$ that converges faster than the parametric rate $n^{1/2}$. Up to our knowledge, such an estimator is not available in the literature.

We conclude this section by a focus on the situation where Q_X is heavy-tailed (i.e., $\xi > 0$). It is well known that for any intermediate sequence (k_n) (i.e., $k_n \rightarrow \infty$ and $n/k_n \rightarrow \infty$ as $n \rightarrow \infty$) and any sequence (u_n) with $nu_n \rightarrow c > 0$, the Weissman estimator [16] given by

$$\widehat{Q}_{X,n}^{(W)}(u_n) := X_{n-k_n,n} \left(\frac{nu_n}{k_n} \right)^{-\widehat{\xi}_n},$$

where $\widehat{\xi}_n$ is the classical Hill estimator [12], is a consistent estimator of $Q_X(\beta_n)$ (see de Haan and Ferreira [11, Theorem 4.3.8]). Replacing in (5), $\widehat{Q}_{X,n}$ by the Weissman estimator leads to the asymptotic confidence interval for $Q_X(\beta_n)$ given by

$$\text{CI}^{(W)}(\gamma, \beta_n) := \left[\left(\frac{\beta_n}{\mathbf{a}_{n,L}(\gamma)} \right)^{-\widehat{\xi}_n} X_{n-j_{L,n}}, \left(\frac{\beta_n}{\mathbf{a}_{n,R}(\gamma)} \right)^{-\widehat{\xi}_n} X_{n-j_{R,n}} \right]. \quad (9)$$

This random interval is slightly different from the confidence interval introduced in Gardes and Maistre [10]. The difference lies in the choice of j_L and j_R . In (9), j_L and j_R are assumed to be fixed (independent of the sample size n) while in [10], $j_L = j_R = j_n$ where (j_n) is an intermediate sequence. Of course, the confidence interval (9) is only adapted to heavy-tailed quantiles.

4 Simulation study

In this section, we are interested in the behavior of the confidence interval $\text{CI}^{(M)}(\gamma, \beta_n)$ for $Q_X(\beta_n)$ when a sample of size $n \in \{200; 500; 1,000; 2,000\}$ is observed. For this study, we take $\gamma = .95$ and $\beta_n = 1/n$. Several distributions are considered. The two distributions below correspond to the case $\xi > 0$ i.e., to heavy-tailed quantiles.

- **Absolute student distribution** – The quantile function is given for all $u \in (0, 1)$ by $Q_X(u) = \text{qt}(1 - u/2; \mu)$, where $\text{qt}(\cdot; \mu)$ is the quantile function of a Student distribution with $\mu > 0$ degree of freedom. The extreme-value index is $\xi = 1/\mu$ and the second-order parameter is $\rho = -1$.
- **Burr distribution** – For $\kappa > 0$ and $c > 0$, the quantile function is given for all $u \in (0, 1)$ by $Q_X(u) = \{u^{-1/\kappa} - 1\}^{1/c}$. For this distribution, $\xi = 1/(c\kappa)$ and $\rho = -1/\kappa$.

The extreme value index of the next three distributions is $\xi = 0$. In this case, the distribution is said to be light-tailed.

- **Exponential distribution** – For all values of the rate parameter $\lambda > 0$, the second-order parameter is $\rho = -\infty$.
- **Normal distribution** – The second-order parameter is $\rho = 0$ whatever the mean and the standard deviation of the distribution.
- **Log-Normal distribution** – For $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, the quantile function is given for all $u \in (0, 1)$ by $Q_X(u) = \exp\{\text{qN}(u; \mu, \sigma^2)\}$, where $\text{qN}(\cdot; \mu, \sigma^2)$ is the quantile function of a normal distribution with mean μ and variance σ^2 . The second-order parameter is $\rho = 0$.

Finally, the following distributions belong to a MDA with a negative extreme value index.

- **Beta distribution** – For $\alpha > 0$ and $\beta > 0$, the probability density function is proportional to $x^{\alpha-1}(1-x)^{\beta-1}$. The extreme value index is $\xi = -1/\beta$ and the second-order parameter $\rho = -\infty$.
- **Reverse Burr distribution** – For $\kappa > 0$ and $c > 0$, the quantile function is given for all $u \in (0, 1)$ by $Q_X(u) = 1 - \{u^{-1/\kappa} - 1\}^{-1/c}$. The extreme value index and the second-order parameter are $\xi = -1/(c\kappa)$ and $\rho = -1/\kappa$.

As noticed in the introduction, few works are dedicated to the construction of a confidence interval for an extreme quantile. We can however mention the papers of Buitendag et al. [4] and Gardes and Maistre [10] where this question has been tackled in the case where the underlying distribution is heavy-tailed. As mentioned at the end of Section 3.2, the interval proposed in [10] is a slightly different version of the interval $\text{CI}^{(W)}(\gamma, \beta_n)$ given in (9) which is obtained by using the Weissman estimator instead of the Moment Estimator $\widehat{Q}_{X,n}^{(M)}$. Without going further into details, Gardes and Maistre [10] proposed to use the same intermediate sequence (j_n) instead of the integers j_L and j_R in (9). In other words, the left and right bounds in [10] are extrapolations of the intermediate order statistic $X_{n-j_n,n}$ while in $\text{CI}^{(W)}(\gamma, \beta_n)$, the left (resp. right) bound in an extrapolation of the extreme order statistic $X_{n-j_L,n}$ (resp. $X_{n-j_R,n}$). Moreover, Gardes and Maistre [10] introduced a bias-reduced version of their interval (denoted hereafter $\text{CI}^{(BR)}(\gamma, \beta_n)$) whose performance has been compared to the one of the interval introduced in Buitendag et al. [4]. It appears that the bias-reduced confidence interval in Gardes and Maistre [10] is most of the time associated to a better (or at least equivalent) coverage probability. For the two heavy-tailed distributions in our simulation study, we thus compare the new confidence interval $\text{CI}^{(M)}(.95, 1/n)$ to the bias-reduced interval $\text{CI}^{(BR)}(.95, 1/n)$ in [10]. The accuracy of the confidence interval $\text{CI}^{(M)}(.95, 1/n)$ is evaluated by

$$E^{(M)}(k_n) := \frac{1}{2} \left\{ |\gamma_L(k_n) - .975| + |\gamma_R(k_n) - .975| \right\},$$

where

$$\gamma_L(k_n) := \mathbb{P} \left(\widehat{t}_{n,L}^{(M)} X_{n-j_L,n} \leq Q_X(1/n) \right),$$

and

$$\gamma_R(k_n) := \mathbb{P} \left(\widehat{t}_{n,R}^{(M)} X_{n-j_R,n} \geq Q_X(1/n) \right).$$

For of $\text{CI}^{(M)}(.95, 1/n)$, a good performance in terms of coverage probability corresponds to a value of $E^{(M)}(k_n)$ close to 0. Of course, the true value of $E^{(M)}(k_n)$ is unknown but it can be approximated with a desired precision by means of a Monte Carlo (MC) procedure. In this study, the MC procedure is done by generating $N = 25,000$ independent samples. When the distribution is heavy-tailed, the same quantity is computed for the bias-reduced interval $\text{CI}^{(BR)}(.95, 1/n)$ and it is denoted $E^{(BR)}(k_n)$.

Selection of the hyper-parameters

First, for a heavy-tailed distribution, the interval $\text{CI}^{(BR)}(.95, 1/n)$ depends on the choice of two intermediate sequences: a sequence (k_n) used to compute the Weissman estimator and a sequence (j_n) used in the intermediate order statistic $X_{n-j_n,n}$. The authors suggest to take $j_n = \max\{3, \lfloor \ln^{.85}(k_n) \rfloor\}$. The same choice is adopted in this simulation study. The sequence (k_n) is selected by using the procedure described in [10]. The obtained sequence is denoted $(\widehat{k}_n^{(BR)})$.

For the interval $\text{CI}^{(M)}(.95, 1/n)$, which can be used whatever the value of the extreme value index, the choice of two integers j_L and j_R and of an intermediate sequence (k_n) are required. The integers j_L and j_R are selected in order to minimize the extrapolation factors $\widehat{t}_{n,L}^{(M)}$ and $\widehat{t}_{n,R}^{(M)}$ in $\text{CI}^{(M)}(\gamma, \beta_n)$. In other words, we take j_L and j_R so that $\mathbf{a}_{n,L}(.95) = q_{\text{Beta}}^{(j_L+1, n-j_L)}(.025)$ and $\mathbf{a}_{n,R}(.95) q_{\text{Beta}}^{(j_R+1, n-j_R)}(.975)$ are as near as possible to $\beta_n = 1/n$ that is to say

$$j_L = \arg \min_{j \in \mathbb{N}} \left| q_{\text{Beta}}^{(j+1, n-j)}(.025) - \frac{1}{n} \right| \quad \text{and} \quad j_R = \arg \min_{j \in \mathbb{N}} \left| q_{\text{Beta}}^{(j+1, n-j)}(.975) - \frac{1}{n} \right|.$$

For all $n \in \{200; 500; 1,000; 2,000\}$, we have $j_L = 3$ and $j_R = 0$.

To select the number of observations k_n used for the Moment Estima-

tor, we consider a method based on the excesses over a random threshold that has been proposed by Hüsler et al. [13]. Let us briefly describe this selection procedure. For a given intermediate sequence (k_n) , let $\{Z_i = X_{n-i+1,n} - X_{n-k_n,n}, i = 1, \dots, k_n\}$ be the k_n excesses over the random threshold $X_{n-k_n,n}$. From [11, Lemma 3.4.1], given $X_{n-k_n} = t$, the random vector (Z_1, \dots, Z_{k_n}) is distributed as the random vector of order statistics $(Z_{k_n,k_n}^*, \dots, Z_{1,k_n}^*)$ obtained from independent and identically distributed random variables $Z_1^*, \dots, Z_{k_n}^*$. Moreover, under (1), for t large enough, the common distribution of the Z_i^* can be approximated by a generalized Pareto (GP) distribution with cumulative distribution function given by

$$H_\xi \left(\frac{x}{f(t)} \right) := 1 - \left(1 + \xi \frac{x}{f(t)} \right)^{-1/\xi}$$

for all $x > 0$ such that $1 + \xi x/f(t) > 0$, where $f(t) = a(1/(1 - F_X(t)))$. Hence, since the intermediate order statistic $X_{n-k_n,n}$ is a consistent estimator of $Q_X(k_n/n)$ we can reasonably assume that, for n large enough and for a well chosen sequence (k_n) , the random variables $Z_1^*, \dots, Z_{k_n}^*$ are independent with common distribution function $H_\xi(\cdot/a(n/k_n))$ and consequently that the random vector $(H_\xi(Z_1/a(n/k_n)), \dots, H_\xi(Z_{k_n}/a(n/k_n)))$ is distributed as the order statistics $(U_{k_n,k_n}, \dots, U_{1,k_n})$ of a standard uniform distribution. Replacing ξ and $a(n/k_n)$ by their moment estimators $\widehat{\xi}_n^{(M)}$ and $\widehat{a}_n^{(M)}(n/k_n)$ yields to the discrepancy measure defined by

$$D(k_n) := \frac{1}{k_n} \sum_{i=1}^{k_n} \left| H_{\widehat{\xi}_n^{(M)}} \left(\frac{Z_i}{\widehat{a}_n^{(H)}(n/k_n)} \right) - \frac{k_n - i + 1}{k_n + 1} \right|.$$

In view of the previous discussion, if (k_n) is well chosen, the discrepancy measure $D(k_n)$ should be close to 0. The selected value of k_n is thus given by

$$\widehat{k}_n^{(M)} := \arg \min_{k \in \{k_1, \dots, k_2\}} D(k),$$

where $\{k_1, \dots, k_2\} \subset \{1, \dots, n\}$. In this simulation study, we take $k_1 = \lfloor .02n \rfloor$ and $k_2 = \lfloor .8n \rfloor$.

The results are gathered in Tables 1 to 5. Let us first emphasize that for all the considered distributions and for all sample sizes, the choice $k_n = \widehat{k}_n^{(M)}$

for the confidence interval $\text{CI}^{(M)}(.95, 1/n)$ leads to better results in terms of coverage probability, than the arbitrary choice $k_n = \lfloor n/5 \rfloor$. Note also that, as expected, the accuracy measure $E^{(M)}(\widehat{k}_n^{(M)})$ decreases as the sample size increases. Next, even if $\text{CI}^{(M)}(.95, 1/n)$ is not specifically dedicated for heavy-tailed distributions (while $\text{CI}^{(BR)}(.95, 1/n)$ is), its accuracy is better for the student distribution with $\mu = 2$ and the Burr distribution as soon as $n \geq 500$.

5 Real data study

Jeanne Calment was a French supercentenarian (i.e., a person who has reached the age of 110 years) who was born on 21 February 1875 and died on 4 August 1997 at the age of 122 years and 164 days. This exceptional longevity makes her the longest-living human at least among those whose birth and death dates could be verified. To get an idea of this exceptional lifespan, the second longest human lifespan observed in France (and in Europe) was 118 years and 341 days.

We would like to check whether Jeanne Calment's lifespan is an outlier or not. For this purpose, we compare her with French supercentenarians born during a similar period. Our study is based on the International Database on Longevity (IDL) which is the product of an international collaboration of several statistical offices and a large number of experts. This database contains birth and death dates of supercentenarians for 13 countries, and semi-supercentenarians (persons with a lifespan of between 105 and 110 years old) for 10 countries.

The population chosen for our comparison is the French supercentenarians and semi-supercentenarians born between 1890 and 1899. The number of observations per year over this period is comparable ranging from 162 for the year 1890 to 310 for the year 1898. We denote by x_1, \dots, x_n the lifespans in years of the $n = 2400$ individuals in the study. We assume that these observations are realizations of n independent replications X_1, \dots, X_n of a random variable X . In terms of the years studied and the period covered by the IDL (1978-2018), none of the persons concerned is still alive. This allows us to state that the variable X is not right-censored because of omissions of

supercentenarians not included in the sample due to their longevity. We can consider that the lifespan of Jeanne Calment is not an outlier if it is close to the $1/2400$ -quantile which is the lifespan that is expected to be observed every 2400 people over the age of 105. We thus check if Jeanne Calment's lifespan belongs to the observed 95% confidence interval for $Q_X(1/2400)$.

We proceed with the confidence interval based on the Moment Estimator (see Section 3.2). The different parameters are taken following the procedure described in section 4. The obtained parameters are $j_L = 3$, $j_R = 0$ and $\widehat{k}_n^{(M)} = 1388$. The extreme value index is estimated by $\widehat{\xi}_X = -0.059$ and the observed confidence interval is $[113.8; 116.1]$. Clearly, the value 122.5 is much larger than the upper bound. It thus seems that Jeanne Calment's lifespan can be considered as an outlier.

To go further, for each year, the 95% confidence intervals for the $1/250$ -th quantile are represented in Figure 1. As expected, the lifespan of Jeanne Calment is clearly larger than all the upper bounds. We also note that the estimates of the extreme value index are always close to 0. This is in line with several studies such as the one of Alvarez et al. [1] where it is shown that the risk of dying appears constant over the age of 105. This surprising phenomenon is known as the "mortality plateau". In other words, it seems that X follows an exponential law (constant hazard rate).

6 Proofs

6.1 Preliminary results

Lemma 1 *For all integer j and for all $\delta \in (0, 1)$,*

$$\lim_{n \rightarrow \infty} n q_{\text{Beta}}^{(j+1, n-j)}(\delta) = q_{\text{Gam}}^{(j+1, 1)}(\delta),$$

where $q_{\text{Gam}}^{(r, s)}$ is the quantile function of a Gamma distribution with parameters $r > 0$ and $s > 0$.

Proof – Let us first give some notations. Let E_1, \dots be independent standard exponential random variables and, for $k \in \mathbb{N} \setminus \{0\}$, let $T_k := E_1 + \dots + E_k$. The distribution of T_k is then a Gamma distribution with parameters k and 1.

From Rényi's representation of ordered standard uniform random variables, the random vector $(U_{j+1,n}, j = 0, \dots, n-1)$ is distributed as the random vector $(T_{j+1}/T_{n+1}, j = 0, \dots, n-1)$. Since $U_{j+1,n}$ is distributed according to a Beta distribution with parameters $j+1$ and $n-j$, we have

$$\mathbb{P}\left(\frac{nT_{j+1}}{T_{n+1}} \leq nq_{\text{Beta}}^{(j+1, n-j)}(\delta)\right) = \delta.$$

Moreover, from the law of large numbers, n/T_{n+1} converges almost surely to 1 and thus nT_{j+1}/T_{n+1} converges in distribution to a Gamma distribution with parameters k and 1. As a consequence, since the cumulative distribution function $F_{\text{Gam}}^{(j+1,1)}$ of a Gamma distribution is continuous, we have

$$\left|F_{\text{Gam}}^{(j+1,1)}\left(nq_{\text{Beta}}^{(j+1, n-j)}(\delta)\right) - \delta\right| \leq \sup_{t \in \mathbb{R}} \left|F_{\text{Gam}}^{(j+1,1)}(t) - \mathbb{P}\left(\frac{nT_{j+1}}{T_{n+1}} \leq t\right)\right| \rightarrow 0,$$

as $n \rightarrow \infty$. We conclude the proof by using the continuity of $q_{\text{Beta}}^{(j+1, n-j)}$. \square

Lemma 2 *Let $f_{\text{Beta}}^{(r,s)}$ be the density function of a Beta distribution with parameters $r > 0$ and $s > 0$. For any sequence $(z_n) \in (0, 1)$ such that $nz_n \rightarrow c > 0$ as $n \rightarrow \infty$ and for any positive integer j , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} f_{\text{Beta}}^{(j+1, n-1)}(z_n) = \frac{c^j e^{-c}}{j!}.$$

Proof – The density of a Beta distribution with parameters $j+1$ and $n-j$ is given by

$$f_{\text{Beta}}^{(j+1, n-j)}(z_n) = \frac{n!}{j!(n-j-1)!} z_n^j (1-z_n)^{n-j-1}.$$

Since $nz_n \rightarrow c > 0$, it is readily seen that $z_n^j \sim c^j n^{-j}$ and $(1-z_n)^{n-j-1} \rightarrow e^{-c}$ as $n \rightarrow \infty$. Hence,

$$f_{\text{Beta}}^{(j+1, n-j)}(z_n) \sim \frac{c^j e^{-c}}{j!} \frac{n!}{n^j (n-j-1)!}.$$

Stirling's formula ensures that, as $n \rightarrow \infty$, we have $n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}$. Replacing in the above expression $n!$ and $(n-j-1)!$ by their equivalents leads to

$$\frac{n!}{n^j (n-j-1)!} \sim n,$$

and the proof is complete. \square

From [11, Theorem B.2.21], condition (1) on the quantile function Q_X is equivalent to

$$\lim_{n \rightarrow \infty} \frac{\overline{F}_X(x_n + tg(x_n))}{\overline{F}_X(x_n)} = \frac{1}{K_\xi^\leftarrow(t)}, \quad (10)$$

where (x_n) is a sequence converging to the right endpoint $Q_X(0)$ and $g(x) := a(1/\overline{F}_X(x))$. A generalization of (10) when t is replaced by a sequence $t_n \rightarrow t_0$ was proved in [5, Lemma 2].

Lemma 3 *Assume that (1) holds and let (x_n) be a sequence converging to the right endpoint $Q_X(0)$. For all sequence (t_n) such that $t_n \rightarrow t_0$, if there exists $\eta > 0$ for which $1 + \xi t_0 \geq \eta$ then,*

$$\lim_{n \rightarrow \infty} \frac{\overline{F}_X(x_n + t_n g(x_n))}{\overline{F}_X(x_n)} = \frac{1}{K_\xi^\leftarrow(t_0)}.$$

6.2 Proofs of main results

6.2.1 Proof of Theorem 1

We need to prove that

$$\mathbb{P} \left(\widehat{t_{n,L}} X_{n-j_L,n} \leq Q_X(\beta_n) \right) \rightarrow \frac{1+\gamma}{2}, \quad (11)$$

and

$$\mathbb{P} \left(\widehat{t_{n,R}} X_{n-j_R,n} \geq Q_X(\beta_n) \right) \rightarrow \frac{1+\gamma}{2}. \quad (12)$$

We only prove (11), the proof of (12) being similar. Let $\epsilon > 0$. By assumption, there exist $c_\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$,

$$\mathbb{P} \left(A_{n,\epsilon} := \left\{ \sigma_n^{-1} \left(\frac{\widehat{t_{n,L}}}{t_{n,L}} - 1 \right) \in [-c_\epsilon, c_\epsilon] \right\} \right) \geq 1 - \epsilon,$$

where we recall that $t_{n,L} := Q_X(\beta_n)/Q_X(\mathbf{a}_{n,L}(\gamma))$. We have the decomposition

$$\begin{aligned} \mathbb{P}\left(\widehat{t_{n,L}}X_{n-j_L,n} \leq Q_X(\beta_n)\right) &= \mathbb{P}\left(\left\{\widehat{t_{n,L}}X_{n-j_L,n} \leq Q_X(\beta_n)\right\} \cap A_{n,\epsilon}\right) \\ &\quad + \mathbb{P}\left(\left\{\widehat{t_{n,L}}X_{n-j_L,n} \leq Q_X(\beta_n)\right\} \cap \Omega \setminus A_{n,\epsilon}\right) \\ &=: T_{1,n} + T_{2,n}. \end{aligned}$$

Obviously,

$$0 \leq T_{2,n} \leq \epsilon, \quad (13)$$

and thus $T_{2,n} \rightarrow 0$ as $n \rightarrow \infty$. Let us focus on the term $T_{1,n}$. We have

$$T_{1,n} = \mathbb{P}\left(\left\{\overline{F}_X(X_{n-j_L,n}) \geq \overline{F}_X\left(\frac{Q_X(\beta_n)}{\widehat{t_{n,L}}}\right)\right\} \cap A_{n,\epsilon}\right).$$

Let $x_{n,L} := Q_X(\mathbf{a}_{n,L}(\gamma))$. When the event $A_{n,\epsilon}$ occurs, we have for all $n \geq N_\epsilon$,

$$\overline{F}_X(x_{n,L} + s_{n,\epsilon}g(x_{n,L})) \leq \overline{F}_X\left(\frac{Q_X(\beta_n)}{\widehat{t_{n,L}}}\right) \leq \overline{F}_X(x_{n,L} - s_{n,\epsilon}g(x_{n,L})),$$

where $g(y) = a(1/\overline{F}_X(y))$ and $s_{n,\epsilon} := c_\epsilon \sigma_n x_{n,L}/g(x_{n,L})$. Lemma 1 entails that $\mathbf{a}_{n,L}(\gamma)$ is asymptotically proportional to $1/n$. Using the fact that Q_X and a are regularly varying functions, respectively with indices $\max(0, \xi)$ and ξ , we get that $s_{n,\epsilon}$ is asymptotically proportional to $Q_X(1/n)/a(n)$. Hence, by assumption, $s_{n,\epsilon} \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, $x_{n,L}$ converges to the right endpoint $Q_X(0)$ since $\mathbf{a}_{n,L}(\gamma)$ converges to 0. We can thus apply Lemma 3 leading to

$$\overline{F}_X(x_{n,L} \pm s_{n,\epsilon}g(x_{n,L})) \sim \overline{F}_X(x_{n,L}) = \mathbf{a}_{n,L}(\gamma).$$

As a consequence, there exists $N_\epsilon^{(*)} \in \mathbb{N}$ such that for all $n \geq \max(N_\epsilon, N_\epsilon^{(*)})$,

$$\mathbf{a}_{n,L}(\gamma)(1 - \epsilon) \leq \overline{F}_X\left(\frac{Q_X(\beta_n)}{\widehat{t_{n,L}}}\right) \leq \mathbf{a}_{n,L}(\gamma)(1 + \epsilon).$$

Note that $\overline{F}_X(X_{n-j_L,n})$ is distributed as a Beta distribution with parameters $j_L + 1$ and $n - j_L$. We obtain for $n \geq \max(N_\epsilon, N_\epsilon^{(*)})$,

$$\begin{aligned} T_{1,n} &\leq \mathbb{P}\left(\left\{\overline{F}_X(X_{n-j_L,n}) \geq \mathbf{a}_{n,L}(\gamma)(1 - \epsilon)\right\} \cap A_{n,\epsilon}\right) \\ &\leq \mathbb{P}\left(\overline{F}_X(X_{n-j_L,n}) \geq \mathbf{a}_{n,L}(\gamma)(1 - \epsilon)\right) = \overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 - \epsilon)). \end{aligned}$$

Moreover, still for $n \geq \max(N_\epsilon, N_\epsilon^{(*)})$,

$$\begin{aligned}
T_{1,n} &\geq \mathbb{P} \left(\left\{ \overline{F}_X(X_{n-j_L, n}) \geq \mathbf{a}_{n,L}(\gamma)(1 + \epsilon) \right\} \cap A_{n,\epsilon} \right) \\
&= \mathbb{P} \left(\overline{F}_X(X_{n-j_L, n}) \geq \mathbf{a}_{n,L}(\gamma)(1 + \epsilon) \right) \\
&\quad - \mathbb{P} \left(\left\{ \overline{F}_X(X_{n-j_L, n}) \geq \mathbf{a}_{n,L}(\gamma)(1 + \epsilon) \right\} \cap \Omega \setminus A_{n,\epsilon} \right) \\
&\geq \overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 + \epsilon)) - \epsilon.
\end{aligned}$$

To sum up, we have proved that for all $\epsilon > 0$ and for all $n \geq \max(N_\epsilon, N_\epsilon^{(*)})$,

$$\overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 + \epsilon)) - \epsilon \leq T_{1,n} \leq \overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 - \epsilon)).$$

To prove that $T_{1,n} \rightarrow 0$ and to conclude the proof, it remains to show that

$$\lim_{n \rightarrow \infty} \overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 \pm \epsilon)) = \frac{1 + \gamma}{2}. \quad (14)$$

Recall that $\overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)) = (1 + \gamma)/2$. The mean value theorem entails that there exists $\kappa \in (0, 1)$ such that

$$\overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 \pm \epsilon)) = \frac{1 + \gamma}{2} \mp \epsilon \mathbf{a}_{n,L}(\gamma) f_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 \pm \kappa \epsilon)).$$

From Lemma 1, there exists $c_1 = c_1(\gamma, j_L) > 0$ such that $\mathbf{a}_{n,L}(\gamma) \sim c_1/n$. Hence, from Lemma 2

$$f_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 \pm \kappa \epsilon)) \sim \frac{n}{j_L!} [c_1(1 \pm \kappa \epsilon)]^{j_L} \exp[-c_1(1 \pm \kappa \epsilon)] =: nc_2.$$

It is then readily seen that there exists positive constants c_3 and c_4 such that for all $\epsilon > 0$ and for n large enough,

$$\frac{1 + \gamma}{2} - c_3 \epsilon \leq \overline{F}_{\text{Beta}}^{(j_L+1, n-j_L)}(\mathbf{a}_{n,L}(\gamma)(1 \pm \epsilon)) \leq \frac{1 + \gamma}{2} + c_4 \epsilon,$$

proving (14). \square

6.3 Proof of Corollary 1

By assumption, $n\beta_n \rightarrow c > 0$ and, from Lemma 1, there exists $c^* > 0$ such that $n\mathbf{a}_{n,\bullet}(\gamma) \rightarrow c^*$. Hence, using the assumption on the asymptotic behavior of the extreme quantile estimator $\widehat{Q}_{X,n}$, we get

$$\widehat{t}_{n,\bullet}^{(*)} = t_{n,\bullet} \left(\frac{1 + \vartheta_n/Q_X(\beta_n)Z_n}{1 + \vartheta_n/Q_X(\mathbf{a}_{n,\bullet}(\gamma))Z_n^*} \right),$$

where $Z_n = \mathcal{O}_{\mathbb{P}}(1)$ and $Z_n^* = \mathcal{O}_{\mathbb{P}}(1)$. Let $\xi_+ := \max(0, \xi)$. From [11, Lemma 1.2.9 and Corollary 1.2.10], we have as $n \rightarrow \infty$ that $Q_X(\beta_n) \sim c^{-\xi_+} Q_X(1/n)$ and $Q_X(\mathbf{a}_{n,\bullet}(\gamma)) \sim \{c^*\}^{-\xi_+} Q_X(1/n)$. Moreover, since $a(n)/Q_X(1/n) \rightarrow \xi_+$, it is readily seen that assumption $\vartheta_n/a(n) \rightarrow 0$ entails that $\vartheta_n/Q_X(1/n) \rightarrow 0$. We deduce from these results that

$$\frac{\tilde{t}_{n,\bullet}^{(*)}}{t_{n,\bullet}} - 1 = \mathcal{O}_{\mathbb{P}}\left(\frac{\vartheta_n}{Q_X(1/n)}\right).$$

An application of Theorem 1 with $\sigma_n = \vartheta_n/Q_X(1/n)$ concludes the proof. \square

6.4 Proof of Corollary 2

Proof of part i): First, assume that $\xi > 0$. From [11, Remark 4.3.3] and since $k_n \rightarrow \infty$, we have as $n \rightarrow \infty$

$$\int_1^{k_n} s^{\xi-1} \ln(s) ds \sim \frac{1}{\xi} k_n^{\xi} \ln(k_n).$$

Moreover, the auxiliary function a is regularly varying with index $\xi > 0$ (see [11, Lemma 1.2.9 and Corollary 1.2.10]). Hence, there exists a slowly varying function L such that

$$\frac{a(n/k_n)}{a(n)} = k_n^{-\xi} \frac{L(n/k_n)}{L(n)}.$$

We then get that as $n \rightarrow \infty$,

$$\frac{\vartheta_n^{(H)}}{a(n)} \sim \frac{1}{\xi} k_n^{-1/2} \frac{L(n/k_n)}{L(n)} \ln(k_n).$$

Taking k_n asymptotically equivalent to n^{δ} with $\delta > 0$ is sufficient to ensure that $\vartheta_n^{(H)}/a(n) \rightarrow 0$ and it is readily seen that conditions $k_n \rightarrow \infty$, $n/k_n \rightarrow \infty$ and $k_n^{1/2} B(n/k_n) \rightarrow \lambda \in \mathbb{R}$ are checked if $\delta = -2(\eta + \rho')/(1 - 2\rho')$ where $\eta \in (0, -\rho')$ can be taken as close as we want to 0.

Assume now that $\xi = 0$. We have from [11, Remark 4.3.3] that, as $n \rightarrow \infty$,

$$\int_1^{k_n} s^{-1} \ln(s) ds \sim \frac{1}{2} \ln^2(k_n).$$

Since a is in this case a slowly varying function, it is easy to check that $\vartheta_n^{(H)}/a(n) \rightarrow 0$ for any sequence (k_n) which is asymptotically equivalent to n^δ with $\delta > 0$. We next conclude as in the case $\xi > 0$.

Finally, if $\xi \in (-1/2, 0)$, we have from [11, Remark 4.3.3] that,

$$\lim_{n \rightarrow \infty} \int_1^{k_n} s^{\xi-1} \ln(s) ds = \frac{1}{\xi^2}.$$

and thus, as $n \rightarrow \infty$,

$$\frac{\vartheta_n^{(M)}}{a(n)} \sim \frac{1}{\xi^2} k_n^{-(\xi+1/2)} \frac{a(n/k_n)}{a(n)}. \quad (15)$$

Since a is a slowly varying function, we conclude as in the case $\xi = 0$.

Proof of part ii): We start with (15). It appears that

$$\frac{\vartheta_n^{(M)}}{a(n)} \sim \frac{U(n/k_n)}{U(n)},$$

where U is a regularly varying function with index $\xi + 1/2 < 0$. Using [3, Theorem 1.5.2], since $k_n \rightarrow \infty$ and $n/k_n \rightarrow \infty$, we have for any $a > 0$ and for n large enough,

$$\left| \frac{U(n)}{U(n/k_n)} - k_n^{\xi+1/2} \right| \leq \sup_{\lambda \geq a} \left| \frac{U(\lambda n/k_n)}{U(n/k_n)} - \lambda^{\xi+1/2} \right| \rightarrow 0.$$

Hence, since $k_n^{\xi+1/2} \rightarrow 0$, we have $U(n)/U(n/k_n) \rightarrow 0$ proving that, for any sequence (k_n) satisfying $k_n \rightarrow \infty$ and $n/k_n \rightarrow \infty$, $\vartheta_n^{(M)}/a(n) \rightarrow \infty$. \square

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	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
$E^{(M)}(\widehat{k}_n^{(M)})$	2.046 %	1.45 %	0.970 %	0.614 %
$E^{(BR)}(\widehat{k}_n^{(BR)})$	0.447 %	1.747 %	2.338 %	2.835 %
$E^{(M)}(\lfloor n/5 \rfloor)$	3.274 %	2.194 %	1.398 %	0.830 %
$E^{(BR)}(\lfloor n/5 \rfloor)$	0.350 %	0.954 %	1.912 %	1.828 %

Table 1: Student distribution with $\mu = 2$. Value of the accuracy measures $E^{(M)}(k_n)$ and $E^{(BR)}(k_n)$ for different choice of the intermediate sequence (k_n) .

	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
$E^{(M)}(\widehat{k}_n^{(M)})$	1.526 %	0.932 %	0.622 %	0.292 %
$E^{(BR)}(\widehat{k}_n^{(BR)})$	1.420 %	0.585 %	0.261 %	0.482 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.948 %	1.732 %	1.020 %	0.52 %
$E^{(BR)}(\lfloor n/5 \rfloor)$	2.31 %	0.970 %	0.709 %	0.268 %

Table 2: Student distribution with $\mu = 1$. Value of the accuracy measures $E^{(M)}(k_n)$ and $E^{(BR)}(k_n)$ for different choice of the intermediate sequence (k_n) .

	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
$E^{(M)}(\widehat{k}_n^{(M)})$	2.052 %	1.374 %	0.920 %	0.544 %
$E^{(BR)}(\widehat{k}_n^{(BR)})$	0.125 %	1.364 %	1.911 %	2.102 %
$E^{(M)}(\lfloor n/5 \rfloor)$	3.352 %	2.248 %	1.434 %	0.852 %
$E^{(BR)}(\lfloor n/5 \rfloor)$	0.620 %	0.554 %	1.382 %	1.400 %

Table 3: Burr distribution with $\kappa = 1$ and $c = 1/2$. Value of the accuracy measures $E^{(M)}(k_n)$ and $E^{(BR)}(k_n)$ for different choice of the intermediate sequence (k_n) .

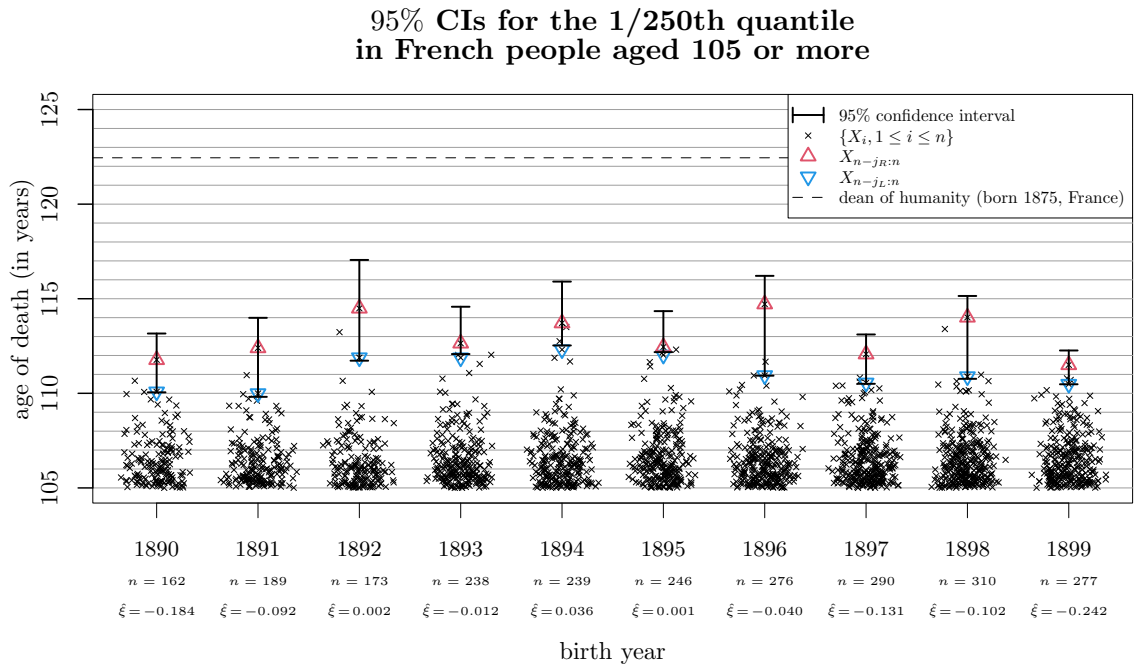


Figure 1: Confidence intervals of level 95% for the lifespan of French supercentenarians for the years 1890 to 1899.

	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
Exponential distribution with $\lambda = 1/2$				
$E^{(M)}(\widehat{k}_n^{(M)})$	1.206 %	0.892 %	0.452 %	0.146 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.106 %	1.310 %	0.500 %	0.146 %
Normal distribution with $\mu = 0$ and $\sigma^2 = 1$				
$E^{(M)}(\widehat{k}_n^{(M)})$	1.776 %	1.782 %	1.410 %	1.260 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.334 %	2.116 %	1.898 %	1.812 %
Log-Normal distribution with $\mu = 0$ and $\sigma^2 = 1$				
$E^{(M)}(\widehat{k}_n^{(M)})$	0.886 %	0.418 %	0.334 %	0.582 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.316 %	1.174 %	0.284 %	0.486 %

Table 4: Gumbel MDA. Value of the accuracy measure $E^{(M)}(k_n)$ for different choice of the intermediate sequence (k_n) .

	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
Beta distribution with $\alpha = 4$ and $\beta = 4$				
$E^{(M)}(\widehat{k}_n^{(M)})$	2.136 %	2.168 %	2.174 %	1.930 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.442 %	2.256 %	2.226 %	2.206 %
Reverse Burr distribution with $\kappa = 1$ and $c = 1/4$				
$E^{(M)}(\widehat{k}_n^{(M)})$	1.478 %	1.462 %	1.138 %	0.890 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.190 %	2.078 %	1.918 %	1.774 %
Reverse Burr distribution with $\kappa = 1$ and $c = 1/2$				
$E^{(M)}(\widehat{k}_n^{(M)})$	0.736 %	0.848 %	0.752 %	0.596 %
$E^{(M)}(\lfloor n/5 \rfloor)$	2.082 %	2.508 %	2.886 %	3.112 %

Table 5: Weibull MDA. Value of the accuracy measure $E^{(M)}(k_n)$ for different choice of the intermediate sequence (k_n) .