Peculiarities of Space Dependent Conservation Laws: Inverse Design and Asymptotics

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Abstract Recently, results regarding the Inverse Design problem for Conservation Laws and Hamilton-Jacobi equations with x-dependent convex fluxes were obtained in [6, 7]. More precisely, characterizations of attainable sets and the set of initial data evolving at a prescribed time into a prescribed profile were obtained. Here, we present an explicit example that underlines deep differences between the x-dependent and x-independent cases. Moreover, we add a detailed analysis of the time asymptotic solution of this example, again underlining differences with the x-independent case.

1 Introduction

Consider the scalar one dimensional conservation law

$$\begin{cases} \partial_t u + \partial_x (H(x,u)) = 0 & (t,x) \in]0,+\infty[\times \mathbb{R} \\ u(0,x) = u_o(x) & x \in \mathbb{R}\,. \end{cases} \tag{CL}$$

Denote by

$$S^{CL} \colon \mathbb{R}^+ \times \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \tag{1}$$

the semigroup whose orbits are the entropy solutions to (**CL**). For any positive T and for any assigned profile $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ the inverse design is the set of initial data

Rinaldo M. Colombo
Unità INdAM & Dipartimento di Ingegneria dell'Informazione, Università di Brescia
e-mail: rinaldo.colombo@unibs.it

Vincent Perrollaz
Institut Denis Poisson, Université de Tours, CNRS UMR 7013, Université d'Orléans
e-mail: Vincent.Perrollaz@univ-tours.fr

Abraham Sylla
Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca
e-mail: abraham.sylla@unimib.it

evolving into this profile at time T, i.e.,

$$I_T^{CL}(w) := \left\{u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}) \colon S_T^{CL} u_o = w\right\}. \tag{2}$$

In the homogeneous case, a general characterization of $I_T^{CL}(w)$ is given in [5]. Other more specific results in this setting are [12], devoted to Burgers' equation; [3], specific to the attainable set for boundary value problems arising in the modeling of vehicular traffic. However, the homogeneous case significantly differs from the present non-homogeneous one and less results in the literature are available. A first step in this direction, limited to the study of the attainable set, is [2], see also the related preprint [1], where H in (**CL**) consists of an expression for $x > 0$ and another expression for $x < 0$. A specific inverse design problem related to conservation laws, e.g., the minimization of a sonic boom, is considered in [10], while [9] is devoted also to Hamilton-Jacobi equation.

In [7], the characterizations obtained in [5] is extended to the non-homogeneous case. The analytic techniques developed in [7] take advantage of the connection $u = \partial_x U$ between (**CL**) and the Hamilton-Jacobi equation

$$\begin{cases} \partial_t U + H(x, \partial_x U) = 0 & (t,x) \in]0,+\infty[\times \mathbb{R} \\ U(0,x) = U_o(x) & x \in \mathbb{R}. \end{cases} \tag{**HJ**}$$

We know, on the basis of [6], that both Cauchy problems for (**CL**) and (**HJ**) are globally well posed under the same set of assumptions, namely:

$$\textbf{Smoothness}: \quad H = H(x,u) \in \mathbf{C}^3(\mathbb{R}^2,\mathbb{R}). \tag{**C3**}$$

$$\textbf{Compact NonHomogeneity}: \quad \begin{array}{l} \exists X > 0,\ \forall (x,u) \in \mathbb{R}^2 \\ |x| > X \implies \partial_x H(x,u) = 0. \end{array} \tag{**CNH**}$$

$$\textbf{Strong Convexity}: \quad \begin{array}{l} \forall x \in \mathbb{R},\ u \mapsto \partial_u H(x,u) \text{ is an} \\ \text{increasing } \mathbf{C}^1\text{-diffeomorphism} \\ \text{of } \mathbb{R} \text{ onto itself.} \end{array} \tag{**CVX**}$$

Assumption (**CVX**) implies that H is strictly convex with respect to the second variable. Clearly, the strongly concave case is entirely analogous. Let us also mention that for well-posedness, (**CVX**) is relaxed to a uniform coercivity assumption coupled to a genuine nonlinearity assumption in [6, **(UC)-(WGNL)**]. Rather than tackling directly the characterization of the inverse design for (**CL**), [7] deals first with (**HJ**) and then uses the correspondence to get back to (**CL**).

Below, we construct an explicit example showing that when H depends on x, the inverse design $I_T^{CL}(w)$ may have properties, in a sense, opposite to those that hold in the homogeneous case, according to [5]. Indeed, in the x-independent case, the presence of a shock in w implies that $I_T^{CL}(w)$ is a cone with infinite dimensional extremal faces. In our x-dependent example, in spite of w displaying a shock, $I_T^{CL}(w)$ is a singleton. On the contrary, in the homogeneous case, $I_T^{CL}(w)$ is a singleton if and only if w is continuous.

Furthermore, in this example, the time asymptotic solution is neither constant nor a rescaling of a solution to a Riemann Problem. On the contrary, as $t \to +\infty$, the solution converges locally uniformly to a compactly supported profile displaying a single singularity, which is a stationary entropic shock wave. This suggests various questions about the characterization of time asymptotic profiles to non homogeneous conservation laws and, in particular, about their stability properties.

2 Notations and Results

Recall the classical definition of entropy solution [11, Definition 1], as tweaked in [6].

Definition 1 Fix $u_o \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. A bounded function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$ is a *solution* to (**CL**) if for all test functions $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R};\mathbb{R}^+)$ and for all scalar $k \in \mathbb{R}$:

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}} |u(t,x)-k|\,\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}(u(t,x)-k)\ \left(H(x,u(t,x))-H(x,k)\right)\ \partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &-\int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}(u(t,x)-k)\ \partial_x H(x,k)\,\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_{\mathbb{R}} |u_o(x)-k|\,\varphi(0,x)\,\mathrm{d}x \geq 0. \end{aligned} \tag{3}$$

Definition 1, taken from by [6, Definition 2.1] is, apparently, weaker than the classical Kružkov definition since it does not require the *"trace at* 0 *condition"* [11, Formula (2.2)]. Nevertheless, under Assumption (**C3**), Definition 1 ensures uniqueness and uniform $\mathbf{L}^1_{\mathbf{loc}}$-continuity in time of the solution, as proved in [6, Theorem 2.6].

As usual, in connection with (**CL**) we use the system of ordinary differential equations

$$\begin{cases} \dot q = \partial_p H(q,p) \\ \dot p = -\partial_q H(q,p), \end{cases} \tag{**HS**}$$

which we consider equipped either with initial or with final conditions. For all $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that $I_T^{CL}(w) \neq \emptyset$, define

$$\pi_w\colon \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & q(0) \end{array} \text{ where } (q,p) \text{ solves (\textbf{HS}) with } \begin{cases} q(T)=x \\ p(T)=w(x)\,. \end{cases} \tag{4}$$

We also introduce the set

$$\mathcal{R}_T := \left\{ q \in \mathbf{C}^1([0,T];\mathbb{R})\colon \exists\, p \in \mathbf{C}^1([0,T];\mathbb{R}) \text{ such that } (q,p) \text{ solves (\textbf{HS})} \right\}.$$

whose elements we call Hamiltonian rays. The map π_w assigns to $x \in \mathbb{R}$ the intersection of the minimal backward characteristics emanating from (T,x), see [8, Defi-

nition 3.1, Theorems 3.2-3.3], with the axis $t = 0$. Remark that in the x-independent case, all Hamiltonian rays are straight lines, as also any extremal characteristics, a key simplification exploited in [5, Formula (2.3)].

Theorem 1 [7, Corollary 3.5] *Let H satisfy* (**C3**)*,* (**CNH**) *and* (**CVX**)*. Fix $T > 0$ and $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$. Then, $I_T^{CL}(w) \neq \emptyset$ if and only if π_w admits a non decreasing representative. In this case, $I_T^{CL}(w)$ is a closed convex cone with a unique vertex u_o^*.*

A characterization of the attainable set for (**CL**) can be obtained by that for (**HJ**) provided in [7, Theorem 3.2]. The latter theorem extends to the x-dependent case some of the properties known to hold in the x-independent case, see [5]. However, the extension to the x-dependent case can not be merely reduced to the rise of technical difficulties. Indeed, some properties are irremediably lost and new phenomena arise.

The most apparent difference between the two situations is described in Figure 1, with reference to extremal backward generalized characteristics (thicker curves), whose behaviors in the two cases are quite different. In the x-independent case, extremal backward characteristics define a *non uniqueness gap*. On the contrary, in the x-dependent case, extremal backward characteristics may well intersect at the initial time, so that the non uniqueness gap disappears. Furthermore, in the x-independent case, an isentropic solution, see [6, Theorem 3.1], can be constructed filling the non uniqueness gap with Hamiltonian rays, that is solutions to (**HS**), emanating from $q(T) = 0$, $p(T) = (1 - \lambda)w(0-) + \lambda w(0+)$, for $\lambda \in [0, 1]$. On the contrary, the same procedure might not work in the x-dependent case, see also [4] for a multi-d study. The numerical integrations in Figure 1, right, referred to (**HS**) with Hamiltonian (5), show that extremal backward characteristics still do not intersect in $]0, T[\times \mathbb{R}$, but the intermediate Hamiltonian rays may well cross each other and even exit the region bounded by the extremal characteristics.

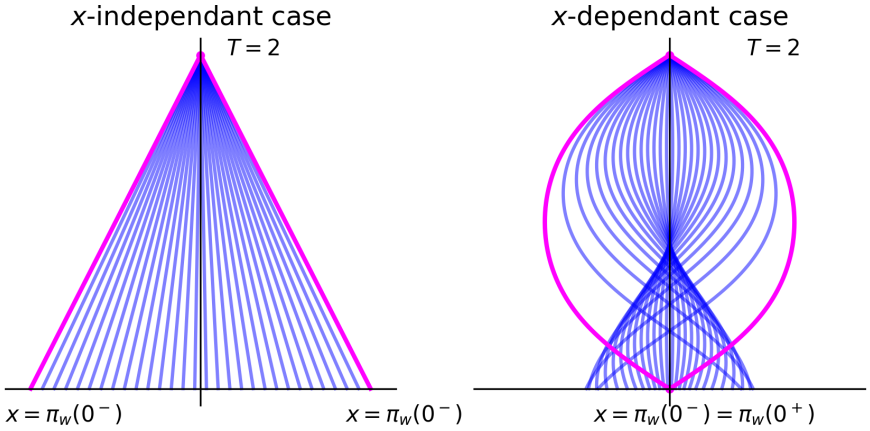

Fig. 1 Left, in the x-independent case, the Hamiltonian rays fill the non uniqueness gap. Right, in the x-dependent case defined by the Hamiltonian (5), extremal characteristics still do not intersect, but Hamiltonian rays do and may well exit the non uniqueness gap or also intersect.

When H does not depend on x, u_o^* defined in Theorem 1 is characterized by [5, (G2) in Proposition 5.2]. Then, [5, (R1) in Lemma 7.2] ensures not only that u_o^* is one sided Lipschitz continuous, but also that the solution $\tilde{u}$ to (**CL**) with initial datum u_o^* evolving into w is Lipschitz continuous on any compact subset of $]0,T[\times\mathbb{R}$. Thus, $\tilde{u}$ satisfies the inequality in Definition 1 with an equality, *i.e.*, it is an *isentropic* solution and also reversible in time. All this is no longer true in the heterogeneous case, as highlighted by the following theorem.

Theorem 2 [7, Theorem 4.1] *Define*

$$H(x,p) \coloneqq \frac{p^2}{2} + g(x) \quad \textit{where} \quad g(x) \coloneqq \begin{cases} 1-(1-x^2)^4 & \textit{if } |x| \le 1 \\ 1 & \textit{otherwise.} \end{cases} \tag{5}$$

Then, (**C3**)*,* (**CNH**) *and* (**CVX**) *hold. Moreover, there exists* $w \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ *such that*

(**i**) *For all* $T>0$*,* $I_T^{CL}(w)$ *is a singleton.*

(**ii**) *For all* $T > \pi/(2\sqrt{2})$ *the solution* u *to* (**CL**) *with* $u(0)$ *being the vertex of* $I_T^{CL}(w)$ *displays an entropic shock sited at* $x=0$*, whose size grows in time, that arises at time* $t=\pi/(2\sqrt{2})$.

Elements of proof are presented in Section 3. As we mentioned before, the contrary to **(ii)** holds when H does not depend on x. In the homogeneous case, the contrary to **(i)** also holds true, see [5, (G1) in Proposition 5.2]. The evolution of the numerical solution to (**CL**)-(5) with initial data in $I_T^{CL}(w)$ (which is a singleton) computed with a standard finite volume scheme, is represented in Figure 2, see also Figure 4.

Therefore, there exist an x-dependent Hamiltonian H, a profile w and a time $T>0$ such that $I_T^{CL}(w)\neq\emptyset$ but in any solution evolving from an initial datum in $I_T^{CL}(w)$, shocks arise before time T, so that no reversible solution is possible, see Figure 4. In other words, the profile w can be reached exclusively producing a sufficient amount of entropy and no isentropic solution evolves into w. Each of these facts requires H to depend on x and is false in an x-independent setting. As a consequence, no direct definition of u_o^* is available, as it was in the x-independent case, explaining why in [7] the authors had to resort to (**HJ**) for its construction.

A further difference between the homogeneous and the non homogeneous cases is in the long time behavior.

Theorem 3 *Let* H *be as in* (5)*, choose*

$$u_o(x) \coloneqq \begin{cases} -2 \textit{ if } x<0 \\ 2 \quad \textit{if } x>0\,, \end{cases} \tag{6}$$

and consider the corresponding entropy solution u *to* (**CL**)*. Then, for all* $x\in\mathbb{R}\backslash\{0\}$*,*

$$\lim_{t\to+\infty} u(t,x) = -\operatorname{sgn} x\,\sqrt{2(1-g(x))} = \begin{cases} -\sqrt{2}\operatorname{sgn} x\,(1-x^2)^2 & |x|\le 1\,, \\ 0 & |x|>1\,. \end{cases} \tag{7}$$

The proof is sketched in Section 4, refer to Figure 2, bottom right, for the graph of (7).

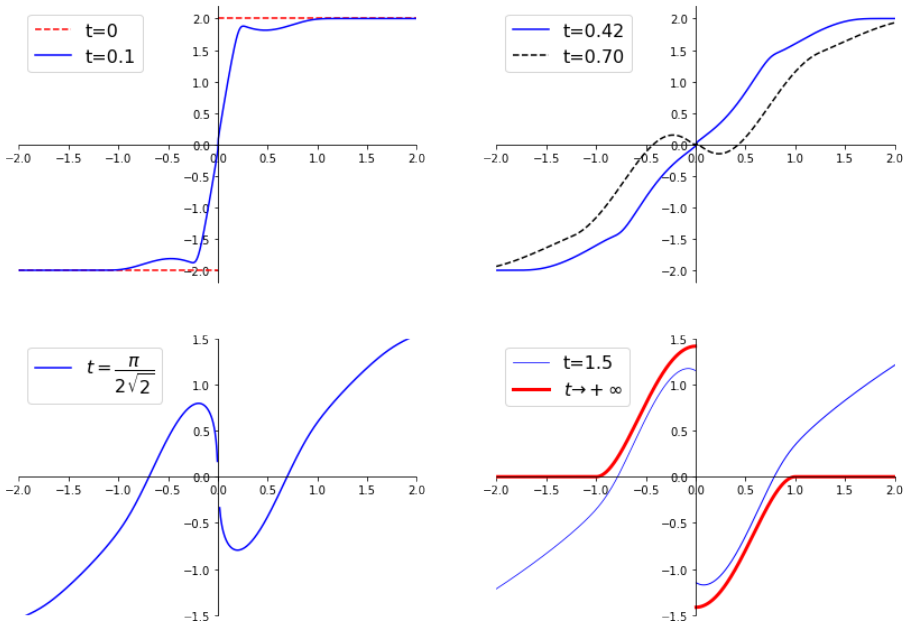

Fig. 2 Numerically computed solution to (**CL**) with H as in (5) and initial datum (6), as described in Theorem 2. The Riemann initial datum (6) first evolves similarly to a rarefaction (top left); then gets non monotone (top right); at time $t = \pi/(2\sqrt{2})$ a shock is formed (bottom left) and in subsequent times the size of the shock grows (bottom right). The latter graph also portrays the asymptotic profile (7).

In particular, note that a stationary shock is present at $x = 0$ in the time asymptotic limit. Outside this shock, the solution is classical.

3 Theorem 2 – Sketch of the Proof

Note first that H, as defined in (5), satisfies (**C3**), (**CNH**) with $X = 1$ and (**CVX**).

Here follows a sequence of lemmas describing the behaviors of the solutions to (**HS**)– (5). The proofs, relying on standard ODE arguments, can be found in [7]. We refer to Figure 3 for an illustration of these behaviors.

Lemma 1 [7, Lemma 5.11] *Let H be as in* (5) *and u_o be as in* (6). *Fix $q_o \geq 0$. Denote by (q, p) the solution to* (**HS**)–(5) *with initial datum $(q_o, u_o(q_o+)) = (q_o, 2)$. Then, q is increasing on $[0, +\infty[$ and $q(t) \underset{t\to+\infty}{\longrightarrow} +\infty$.*

Lemma 2 [7, Lemma 5.12] *Let H be as in* (5) *and u_o be as in* (6). *Fix $0 \leq q_o < \widetilde{q_o}$ and denote by (q, p), respectively $(\widetilde{q}, \widetilde{p})$, the global solution to* (**HS**)–(5) *with initial*

datum $(q_o, u_o(q_o+)) = (q_o, 2)$*, respectively* $(\widetilde{q_o}, u_o(\widetilde{q_o}+)) = (\widetilde{q_o}, 2)$*. Then,* $q(t) < \widetilde{q}(t)$*, for all* $t \geq 0$*.*

Lemma 3 [7, Lemmas 5.13 – 5.15] *Let H be as in* (5) *and* u_o *be as in* (6)*. Fix* $p_o \in]0, 2[$*. Denote by* (q, p) *the global solution to* (**HS**)–(5) *with initial datum* $(0, p_o)$*.*

(i) If $p_o \in]\sqrt{2}, 2[$*, then* q *is increasing on* $[0, +\infty[$ *and* $q(t) \underset{t\to+\infty}{\longrightarrow} +\infty$*.*

(ii) If $p_o = \sqrt{2}$*, then* q *is increasing on* $[0, +\infty[$ *and* $q(t) \underset{t\to+\infty}{\longrightarrow} 1$*.*

(iii) If $p_o \in]0, \sqrt{2}[$*, then* q *is periodic and the map*

$$\begin{array}{rccc} \mathcal{T}\colon &]0,\sqrt{2}[& \longrightarrow &]0,+\infty[\\ & p_o & \longmapsto & \text{the smallest period of } q \end{array} \tag{8}$$

is increasing, continuous and $\lim_{p_o\to\sqrt{2}} \mathcal{T}(p_o) = +\infty$*. Moreover,* $q(t) > 0$ *for* $t \in]0, \mathcal{T}(p_o)/2[$ *and* $q(t) < 0$ *for* $t \in]\mathcal{T}(p_o)/2, \mathcal{T}(p_o)[$*.*

Lemma 4 [7, Lemma 5.16] *Let H be as in* (5) *and* u_o *be as in* (6)*. Fix* $0 < p_o < \widetilde{p_o} < 2$ *and denote by* (q, p)*, respectively* $(\widetilde{q}, \widetilde{p})$*, the global solution to* (**HS**)–(5) *with initial datum* $(0, p_o)$*, respectively* $(0, \widetilde{p_o})$*. Then,*

$$\begin{aligned} &(i)\ \ p_o \in]0,\sqrt{2}[\implies \forall t \in]0, \mathcal{T}(p_o)/2],\ q(t) < \widetilde{q}(t) \\ &(ii)\ \ p_o \in [\sqrt{2}, 2[\implies \forall t \in]0, +\infty[,\ q(t) < \widetilde{q}(t). \end{aligned}$$

Recall that (**C3**)-(**CNH**)-(**CVX**) ensure that for all $(q_o, p_o) \in \mathbb{R}^2$, calling (q, p) the solution to (**HS**)–(5) with datum (q_o, p_o) at time 0, the map

$$\begin{array}{rccc} \mathcal{F}\colon & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (t, q_o, p_o) & \longmapsto & (q(t), p(t)) \end{array} \tag{9}$$

and its two projections $\mathcal{F}_q, \mathcal{F}_p$ are of class $\mathbf{C}^2$, see [7, Lemma 5.2].

Lemma 5 [7, Lemma 5.17] *Let H be as in* (5) *and* u_o *be as in* (6)*. Then, there exists a unique map*

$$\begin{array}{rccc} \Delta\colon &]0,+\infty[^2 & \longrightarrow & ([0,+\infty[\times \{2\}) \cup (\{0\} \times]0, 2]) \\ & (t, x) & \longmapsto & (q_o, p_o) \end{array} \tag{10}$$

such that

$$\mathcal{F}_q(t, q_o, p_o) = x \quad \text{and} \quad \forall\, s \in]0, t[,\ \mathcal{F}_q(s, q_o, p_o) > 0\,. \tag{11}$$

Moreover,

(i) Δ *is continuous.*

(ii) Δ *is monotone, in the sense that setting* $\Delta(t_o, x_o) = (0, p_o)$ *and* $\Delta(t_o, x_o') =$

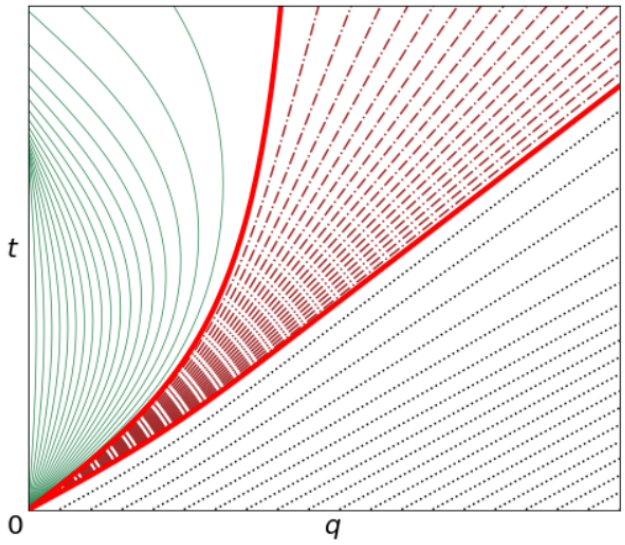

Fig. 3 On the horizontal axis, the q component of solutions to (**HS**)–(5), while time t is on the vertical axis. Dotted curves are those considered in Lemma 1 and in Lemma 2; dashed-dotted curves are those considered in *(i)* of Lemma 3; solid curves refer to *(iii)* of Lemma 3. The two thicker curves depict solutions corresponding to the initial data $(0, \sqrt{2})$ and $(0, 2)$.

$(0, p'_o)$, *if* $0 < x_o < x'_o$, *then* $p_o < p'_o$.

(iii) For all $x \in]0, +\infty[$, $\lim_{t \to 0+} \Delta(t, x) = (x, 2)$.

The construction of the solution to (**CL**)–(6) now follows from the next proposition.

Proposition 1 [7, Proposition 5.18] *Let H be as in* (5) *and u_o as in* (6). *Then,*

$$u \colon \begin{array}{ccc}]0, +\infty[\times (\mathbb{R} \setminus \{0\}) & \longrightarrow & \mathbb{R} \\ (t, x) & \longmapsto & \begin{cases} \mathcal{F}_p\,(t, \Delta(t, x)) & \text{if } x > 0 \\ -\mathcal{F}_p\,(t, \Delta(t, -x)) & \text{if } x < 0. \end{cases} \end{array} \tag{12}$$

is the solution to (**CL**) *in the sense of Definition 1 and it is a classical solution outside* $x = 0$.

We refer to [7, § 5.4] for the concluding remarks proving the properties of u.

4 Proof of Theorem 3

Since u is odd in the space variable, we consider only the case $x > 0$. Using the notation (9), introduce the maps

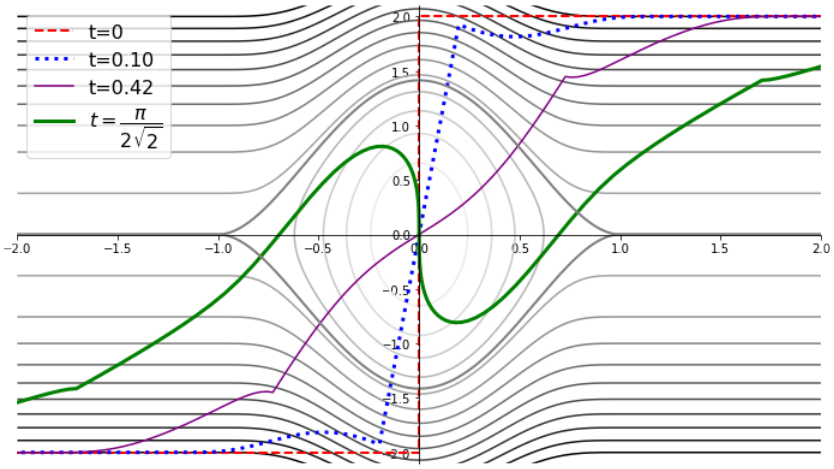

Fig. 4 Illustration of Proposition 1: evolution in time of the solution to (**CL**) in connection with the orbits of (**HS**)–(5).

$$q^\flat(t) = \mathcal{F}_q(t, 0, \sqrt{2}) \quad \text{and} \quad q^\sharp(t) = \mathcal{F}_q(t, 0, 2) \tag{13}$$

whose graphs are thicker in Figure 3. By Lemma 1 and Lemma 3, $\lim_{t\to+\infty} q^\flat(t) = 1$ and $\lim_{t\to+\infty} q^\sharp(t) = +\infty$.

Assume first $x \in]0, 1[$. Then, if $t > (q^\flat)^{-1}(x)$, by (ii) in Lemma 3 and (ii) in Lemma 5, $\Delta(t, x) = (0, p_o(t, x))$ for a suitable $p_o(t, x) \in]0, \sqrt{2}[$. Moreover, by (11) in Lemma 5 and (iii) in Lemma 3, $t < \mathcal{T}(p_o(t, x))/2$. Hence, as $t \to +\infty$, also $\mathcal{T}(p_o(t, x)) \to +\infty$. By (iii) in Lemma 3, $\lim_{t\to+\infty} p_o(t, x) = \sqrt{2}$.

Assume now that $x \geq 1$. If $t > (q^\sharp)^{-1}(x)$, by Lemma 1, $\Delta(t, x) = (0, p_o(t, x))$ with $p_o(t, x) \in [\sqrt{2}, 2[$. By (ii) in Lemma 4 and (i) in Lemma 3, we have $\lim_{t\to+\infty} p_o(t, x) = \sqrt{2}$.

Recall that along solutions to (**HS**)–(5), H is conserved, that is

$$\frac{u(t, x)^2}{2} + g(x) = \frac{p_o(t, x)^2}{2} + g(0)\,, \tag{14}$$

so that

$$\lim_{t\to+\infty} u(t, x)^2 = 2(1 - g(x)) \geq 0\,, \tag{15}$$

proving (7) for $x \geq 1$.

For $x \in [0, 1[$, we proved above that, for t large, $p_o(t, x) \in]0, \sqrt{2}[$. This, together with (14) ensures that

$$u(t, x) < \sqrt{2(1 - g(x))}\,. \tag{16}$$

By (**CL**), understood in its classical sense thanks to Proposition 1, and by (14)

$$\partial_t u(t,x) = -\partial_x \left(H\left(x, u(t,x)\right)\right) = -\partial_x \left(\frac{p_o(t,x)^2}{2}\right) = -p_o(t,x)\ \partial_x p_o(t,x)\,.$$

Indeed, note first that by (14), $p_o(t,x) > 0$ and the Implicit Function Theorem, p_o is sufficiently regular. Item (ii) in Lemma 5 implies that $\partial_x p_o(t,x) \geq 0$, so that $\partial_t u(t,x) \leq 0$. Hence, (15) and (16) allow to complete the proof of (7).

As a final remark, we note that as $t \to +\infty$, $u(t)$ converges uniformly on compact subsets of $\mathbb{R}\setminus\{0\}$, thanks to pointwise convergence and monotonicity, by Dini's Theorem [13, Theorem 7.13]. A further technical argument, based on (ii) of Theorem 2, allows to extend the local uniform convergence on all of $\mathbb{R}$.

Acknowledgements The first author was partly supported by the GNAMPA 2022 project *Evolution Equations, Well Posedness, Control and Applications*. This research was funded, in whole or in part, by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010. For the purpose of open access, the second author has applied a CC-BY public copyright license to any Author Accepted Manuscript (AAM) version arising from this submission.

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