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Space-time fluctuations in a quasi-static limit

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Abstract

We consider the macroscopic limit for the space-time density fluctuations in the open symmetric simple exclusion in the quasi-static scaling limit. We prove that the distribution of these fluctuations converge to a gaussian space-time field that is delta correlated in time but with long-range correlations in space.

Keywords: Interacting particle systems, Transport processes, Long-range correlations

MSC Classification: 82C22 , 82C70 , 60K35

1 Introduction

Consider a stochastic dynamics of interacting particles where the only conserved quantity is given by the number of particles, in equilibrium with a reversible stationary (equilibrium) Gibbs measure μ_ρ , ρ indicates here the average density of particles. The

fluctuation field of the density of particles at a given time t is defined by averaging in space on a macroscopic scale N . After a diffusive rescaling of space and time the fluctuation field in space of the density of particles converges to a distribution $\mathcal{Y}_t(u)$ expected to evolve in time following the infinite dimensional Langevin equation ([11], section II.2.9)

$$\partial_t \mathcal{Y}_t(u) = \mathcal{D}_\rho \partial_u^2 \mathcal{Y}_t(u) - \sqrt{2\chi_\rho \mathcal{D}_\rho} \partial_u w(t, u), \quad (1)$$

where \mathcal{D}_ρ is the diffusivity at density ρ , χ_ρ is the static compressibility and $w(t, u)$ is a standard space-time white noise. The equation (1) shall be considered in a weak sense and we wrote it in one dimension for simplicity. The only stationary distribution for (1) is given by the centered Gaussian distribution on with covariance

$$\mathbb{E}(\mathcal{Y}_t(u)\mathcal{Y}_t(v)) = \chi_\rho \delta(u - v), \quad (2)$$

that correspond to the static fluctuation field of the stationary Gibbs measure, that typically have exponential decay of space correlations on the microscopic scale. The time correlations of the solution of (1) have exponential decay.

In this article we consider the space-time fluctuations of the density of a open system on the space-time box: $\{1, \dots, N - 1\} \times [0, N^{2+\alpha}T]$. When $\alpha > 0$ as $N \rightarrow \infty$ this correspond to a quasi-static scaling limit, where the time scale is larger than the typical scale where the dynamics relax to equilibrium (i.e. the diffusive time scale with $\alpha = 0$ [2]).

We expect that the limit space-time field is a centered gaussian field $\mathbb{Y}(t, u)$ with covariance

$$\mathbb{E}(\mathbb{Y}(t, u)\mathbb{Y}(s, v)) = \chi_\rho \delta(t - s)(-\partial_u^2)^{-1}(u, v), \quad (3)$$

where $(-\partial_u^2)^{-1}(u, v)$ is the Green's function of the laplacian on $[0, 1]$ with Dirichlet boundary conditions. While the delta-correlation in time is a natural consequence of the quasi-static time scale, at first sight it may look surprising the long range correlation in space appearing in (3).

In the present article we prove such macroscopic behaviour for the open symmetric simple exclusion on the discrete interval $\{1, \dots, N - 1\}$ with boundary reservoirs, but we go beyond the equilibrium case (reservoirs at the same density ρ) and we consider reservoirs at different densities ($\rho_-(t)$ on the left and $\rho_+(t)$ on the right) that change in time on the macroscopic quasi-static scale, in the same situation as considered in [2]. When ρ_\pm do not change in time, a stationary state exists that has a density profile $\rho(u) = (\rho_+ - \rho_-)u + \rho_-$ and has long range correlations. The static fluctuation field in this stationary state in the limit $N \rightarrow \infty$ has covariance [10, 12]

$$\mathbb{E}(\mathcal{Y}(u)\mathcal{Y}(v)) = \chi_{\rho(u)} \delta(u - v) - (\rho_+ - \rho_-)^2 (-\partial_u^2)^{-1}(u, v). \quad (4)$$

where $\chi_\rho = \rho(1 - \rho)$. A similar limit is found for the density fluctuation at macroscopic time t in [2] when $\rho_\pm(t)$ are time dependent. Then we prove in this situation that the space-time fluctuation field converges to a centered gaussian field with covariance defined by (15) (16).

Even though long-range space correlations related to the Green function of the Dirichlet Laplacian appears in (3) (and its generalization (15)) as well as in (4), there

is not a direct connection between the two results. Notice that in the covariance of the space-time fluctuations (3) the long range correlations appear also in equilibrium (i.e. $\rho_- = \rho_+$), while in the space-fluctuations in the stationary state they appears only if $\rho_- \neq \rho_+$. Furthermore in (3) there is no term with a delta correlation in space and the long range terms appear with a positive sign (while they are negative in (4)).

In Section 2 we present the model under investigation, we recall from [2] the hydrodynamic limit but also estimates on correlation functions (which are crucial to achieve our results) and we state our main theorem. We carry out the complete proof of our main theorem for the open SSEP in Section 3, while in Section 4 we sketch the extension to other symmetric stochastic dynamics like the zero-range model.

We believe the results of the present article can be easily extended to multidimensional cases, for example on the open square of side length N . In the equilibrium case, with reservoirs at the boundaries with density ρ we expect that the macroscopic space-time fluctuations are distributed like the gaussian field with covariance

$$\mathbb{E}(\mathbb{Y}(t, u)\mathbb{Y}(s, v)) = \chi_\rho \delta(t - s)(-\Delta)^{-1}(u, v), \quad (5)$$

where $(-\delta^2)^{-1}(u, v)$ is the Green's function of the laplacian on $[0, 1]^d$ with Dirichlet boundary conditions. For time changing boundaries densities and in non-equilibrium, under reasonable conditions we expect the corresponding generalization of (15) (16) should hold.

Another interesting generalization would be the investigation of the space-time fluctuation for the asymmetric dynamics, like the ASEP where there exists results for the quasi-static limit [3].

2 SSEP with boundary reservoirs

2.1 The model

Let $\alpha > 0$. We consider the symmetric simple exclusion process $\{\eta_t ; t \geq 0\}$ on the interval $\{1, \dots, N - 1\}$, with boundary creation/destruction of particles with rates $\rho_-(t)$ at $x = 1$ and $\rho_+(t)$ at $x = N - 1$. The generator of the time inhomogeneous pure jumps Markov process $\{\eta_{tN^{2+\alpha}} ; t \geq 0\}$ with state space $\Omega^N = \{0, 1\}^{\{1, \dots, N-1\}}$ is given by

$$\mathcal{L}_{N,t} = N^{2+\alpha} \left(\mathcal{L}_N^0 + \mathcal{L}_{N,t}^\pm \right)$$

where the generators above are acting on functions $f : \Omega^N \rightarrow \mathbb{R}$ according to

$$(\mathcal{L}_N^0 f)(\eta) = \sum_{x=1}^{N-2} [f(\eta^{x,x+1}) - f(\eta)]$$

and

$$(\mathcal{L}_{N,t}^- f)(\eta) = [1 - \rho_-(t)] \eta(1) [f(\eta^1) - f(\eta)] + \rho_-(t)(1 - \eta(1)) [f(\eta^1) - f(\eta)]$$

and

$$(\mathcal{L}_{N,t}^+ f)(\eta) = [1 - \rho_+(t)] \eta(N-1) [f(\eta^{N-1}) - f(\eta)] + \rho_+(t)(1 - \eta(N-1)) [f(\eta^{N-1}) - f(\eta)].$$

Above, $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation variables at sites x and y , while η^x is the configuration obtained from η by flipping the occupation variable at site x , that is:

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \notin \{x, y\}, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \end{cases}$$

and

$$\eta^x(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ 1 - \eta(x) & \text{if } y = x. \end{cases}$$

2.2 Hydrodynamic limit

It is proved in [2] the following Law of Large Numbers: if $\rho_0 : [0, 1] \rightarrow [0, 1]$ is a measurable profile and at initial time the (deterministic or random) initial probability measure μ_N on Ω^N is such that

$$\frac{1}{N} \sum_{x=1}^{N-1} G\left(\frac{x}{N}\right) \eta_0(x) \xrightarrow[N \rightarrow \infty]{\text{prob.}} \int_0^1 G(u) \rho_0(u) du, \quad (6)$$

for any continuous function $G : [0, 1] \rightarrow \mathbb{R}$, then at any time $t \geq 0$ it holds

$$\frac{1}{N} \sum_{x=1}^{N-1} G\left(\frac{x}{N}\right) \eta_{tN^{2+\alpha}}(x) \xrightarrow[N \rightarrow \infty]{\text{prob.}} \int_0^1 G(u) \rho_t(u) du, \quad (7)$$

with $\rho_t(u) = (\rho_+(t) - \rho_-(t))u + \rho_-(t)$.

Let μ_N be a probability measure on Ω^N and, for $t \geq 0$, we denote

$$\rho_t^N(x) := \mathbb{E}_{\mu_N} [\eta_{tN^{2+\alpha}}(x)], \quad x \in \{1, \dots, N-1\}, \quad (8)$$

with $\rho_t^N(0) := \rho_-(t)$, and $\rho_t^N(N) := \rho_+(t)$. Define for any $u, v \in [0, 1]$ and $s, t \geq 0$

$$\varphi^N(u, v; s, t) := \mathbb{E}_{\mu_N} [(\eta_{sN^{2+\alpha}}([Nu]) - \rho_s^N([Nu])) (\eta_{tN^{2+\alpha}}([Nv]) - \rho_t^N([Nv]))]. \quad (9)$$

We assume that

$$\sup_{N \in \mathbb{N}} \sup_{u, v \in [0, 1], u \neq v} N |\varphi^N(u, v; 0, 0)| \lesssim 1 \quad (10)$$

where here and in the following, \lesssim denotes an inequality that is correct up to a multiplicative constant independent of N .

It is proved in Theorem 2.2 of [2] that, for $u \neq v$ and $u, v \notin \{0, 1\}$,

$$\lim_{N \rightarrow \infty} N \varphi^N(u, v; t, t) = -(\rho_+(t) - \rho_-(t)) u(1 - v) \quad (11)$$

and for $t \neq s$

$$\lim_{N \rightarrow \infty} N \varphi^N(u, v; s, t) = 0. \quad (12)$$

We also note that from Lemma 4.3 in [10] and from (10) we get that

$$\sup_{N \geq 1} \max_{x, y \in \{1, \dots, N-1\}, x \neq y} N |\varphi^N(\frac{x}{N}, \frac{y}{N}; t, t)| \lesssim 1. \quad (13)$$

Above we used the proof of [10] that can be adapted to our case, since our correlation function $\varphi^N(u, v; t, t)$ (see (9) with $s = t$) also satisfies an equation similar to (4.5) in [10]. By following exactly the same steps as in the proof of Lemma 4.3 of [10] we conclude (13). We leave these details to the interested reader.

2.3 Main result

For a fixed $T > 0$, we denote the open set $\Omega_T = (0, T) \times (0, 1) \subset \mathbb{R}^2$. We define the (centered) space-time distribution valued random field \mathbb{Y}^N acting on $f \in \mathcal{C}^\infty(\Omega_T, \mathbb{C})$ as

$$\mathbb{Y}^N(f) = \int_0^T \frac{1}{\sqrt{N^{1-\alpha}}} \sum_{x=1}^{N-1} f(t, \frac{x}{N}) (\eta_{tN^{2+\alpha}}(x) - \rho_t^N(x)) dt. \quad (14)$$

Let \mathbb{Y} be the space-time distribution valued centered Gaussian field such that for any function f the variance $\mathbb{V}(f)$ of $\mathbb{Y}(f)$ is given by

$$\mathbb{V}(f) = 2 \int_0^T ds \int_0^1 du \rho_s(u)(1 - \rho_s(u)) |\partial_u F(s, u)|^2 \quad (15)$$

where F is the solution of

$$\begin{cases} (\partial_u^2 F)(t, u) = f(t, u), & u \in [0, 1], \\ F(t, 0) = F(t, 1) = 0, \end{cases} \quad (16)$$

for all $t \in [0, T]$.

Our main result is the following.

Theorem 1. *The sequence $\{\mathbb{Y}^N; N \geq 1\}$ converges in law, as $N \rightarrow +\infty$, to \mathbb{Y} .*

Remark 1. *We have defined \mathbb{Y}^N as an element of the dual of $\mathcal{C}^\infty(\Omega_T, \mathbb{C})$. In the theory of distributions is usually denoted $\mathcal{E}'(\Omega_T, \mathbb{C}) = \mathcal{C}^\infty(\Omega_T, \mathbb{C})$ and $\mathcal{D}'(\Omega_T, \mathbb{C}) = \mathcal{C}_0^\infty(\Omega_T, \mathbb{C})$. The restriction from $\mathcal{E}'(\Omega_T, \mathbb{C})$ to $\mathcal{D}'(\Omega_T, \mathbb{C})$ is injective (cf. [1], Chapter 8). In Subsection 3.2 we prove that $\{\mathbb{Y}^N; N \geq 1\}$ is tight in a Sobolev space $\mathcal{H}_{-m, -k} \subset \mathcal{E}'(\Omega_T, \mathbb{C})$.*

3 Proof of Theorem 1

The proof of Theorem 1 is a consequence of two ingredients: convergence of finite dimensional distributions and tightness. Therefore, in Proposition 1 we establish the convergence in law of the sequence $\{\mathbb{Y}^N(f) ; N \geq 1\}$ to $\mathbb{Y}(f)$ for any test function f and in Lemma 2 we prove that the sequence $\{\mathbb{Y}^N ; N \geq 1\}$ is tight in a suitable Sobolev space of distributions.

3.1 Convergence of finite-dimensional distributions

Proposition 1. *For any $\theta \in \mathbb{R}$ and for any $f \in C^\infty(\Omega_T, \mathbb{C})$ it holds that the sequence $\{\mathbb{Y}^N(f) ; N \geq 1\}$ converges in law to the centred complex Gaussian random variable with variance $\mathbb{V}(f)$ where $\mathbb{V}(f)$ is defined in (15).*

Proof. We claim first that it is sufficient to prove that

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\exp \{i\theta \mathbb{Y}^N(f)\} \right) = \exp \left\{ -\frac{\theta^2}{2} \mathbb{V}(f) \right\} \quad (17)$$

for real-valued functions. Indeed, if this is the case, then let $f \in C^\infty(\Omega_T, \mathbb{C})$ be decomposed as $f = g + ih$ where $g, h \in C^\infty(\Omega_T, \mathbb{R})$. Therefore, for any real numbers λ, μ , (17) is valid for $\lambda g + \mu h$, i.e. the sequence of real random variables

$$\{\mathbb{Y}^N(\lambda g + \mu h) = \lambda \mathbb{Y}^N(g) + \mu \mathbb{Y}^N(h) ; N \geq 1\}$$

converges to a Gaussian random variable with variance $\mathbb{V}(\lambda g + \mu h)$. Let us define

$$\mathbb{V}(g, h) = 2 \int_0^T ds \int_0^1 du \rho_s(u) (1 - \rho_s(u)) \partial_u G(s, u) \partial_u H(s, u) \quad (18)$$

where G and H are the solutions, respectively, of

$$\begin{cases} (\partial_u^2 G)(t, u) = g(t, u), & u \in [0, 1], \\ G(t, 0) = G(t, 1) = 0, \end{cases} \quad (19)$$

for all $t \in [0, T]$, and

$$\begin{cases} (\partial_u^2 H)(t, u) = h(t, u), & u \in [0, 1], \\ H(t, 0) = H(t, 1) = 0, \end{cases} \quad (20)$$

for all $t \in [0, T]$. It is not difficult to check from the definition of \mathbb{V} given in (15) that $\mathbb{V}(\lambda g + \mu h) = \lambda^2 \mathbb{V}(g) + \mu^2 \mathbb{V}(h) + 2\lambda\mu \mathbb{V}(g, h)$. This implies that the sequence of random vectors

$$\{(\mathbb{Y}^N(g), \mathbb{Y}^N(h)) ; N \geq 1\}$$

converges in law to a centred Gaussian random vector with covariance matrix

$$\begin{pmatrix} \mathbb{V}(g) & \mathbb{V}(g, h) \\ \mathbb{V}(g, h) & \mathbb{V}(h) \end{pmatrix}.$$

Hence the sequence $\{\mathbb{Y}^N(f) = \mathbb{Y}^N(g) + i\mathbb{Y}^N(h) ; N \geq 1\}$ converges in law to a complex centered Gaussian variable with variance (observe that the solution of (16) is $F = G + iH$)

$$\mathbb{V}(g) + \mathbb{V}(h) = \mathbb{V}(f)$$

and this proves the claim. Now, it remains to prove (17) for real-valued functions.

Note that from (14) we have

$$\mathbb{Y}^N(f) = N^{\alpha/2} \int_0^T dt \mathcal{Y}_t^N(f)$$

where the space fluctuation field $\{\mathcal{Y}_t^N ; t \geq 0\}$ is defined on a space-time function $f : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ as

$$\mathcal{Y}_t^N(f) := \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} f(t, \frac{x}{N}) (\eta_{tN^{2+\alpha}}(x) - \rho_t^N(x)).$$

With the convention that $\eta_t(0) = \rho_-(t)$ and $\eta_t(N) = \rho_+(t)$ for all $t \in [0, T]$, we observe that

$$\mathcal{L}_{N,t}(\eta_t(x)) = N^\alpha (\Delta_N \eta_t)(x), \quad x \in \{1, \dots, N-1\},$$

where $(\Delta_N \eta_t)(x) = N^2 (\eta_t(x+1) + \eta_t(x-1) - 2\eta_t(x))$. In particular,

$$\partial_t [\rho_t^N(x)] = N^\alpha (\Delta_N \rho_t^N)(x), \quad x \in \{1, \dots, N-1\}.$$

Let $G : (t, u) \in [0, T] \times [0, 1] \rightarrow \mathbb{R}$ be a differentiable function in time with Dirichlet boundary conditions, i.e., for any $t \in [0, T]$

$$G(t, 0) = G(t, 1) = 0.$$

Since $\Delta_N \rho_s^N(x) = 0$, by Dynkin's formula it follows that the process $\{\mathcal{M}_t^N(G) ; t \geq 0\}$ defined by

$$\begin{aligned} \mathcal{M}_t^N(G) &= \mathcal{Y}_t^N(G) - \mathcal{Y}_0^N(G) \\ &\quad - \int_0^t \frac{1}{\sqrt{N}} \sum_{x=0}^N \partial_s G(s, \frac{x}{N}) (\eta_{sN^{2+\alpha}}(x) - \rho_s^N(x)) ds \\ &\quad - N^{\alpha-1/2} \int_0^t \sum_{x=1}^{N-1} G(s, \frac{x}{N}) [\Delta_N (\eta_{sN^{2+\alpha}}(x) - \rho_s^N(x))] ds \end{aligned} \tag{21}$$

is a martingale w.r.t. the natural filtration of $\{\eta_{tN^{2+\alpha}}; t \geq 0\}$. By performing a summation by parts and noting that G satisfies the Dirichlet boundary conditions, we get

$$\mathcal{M}_t^N(G) = \mathcal{Y}_t^N(G) - \mathcal{Y}_0^N(G) - \int_0^t \mathcal{Y}_s^N(\partial_s G + N^\alpha \tilde{\Delta}_N G) ds,$$

where $\tilde{\Delta}_N$ is defined by

$$(\tilde{\Delta}_N G)(u) = N^2 \left[G(u + \frac{1}{N}) + G(u - \frac{1}{N}) - 2G(u) \right], \quad u \in \{\frac{1}{N}, \dots, \frac{N-1}{N}\}.$$

It follows that

$$\mathbb{Y}^N \left(\tilde{\Delta}_N G + N^{-\alpha} \partial_t G \right) = \frac{1}{N^{\alpha/2}} \{ \mathcal{Y}_T^N(G) - \mathcal{Y}_0^N(G) \} - \frac{1}{N^{\alpha/2}} \mathcal{M}_T^N(G). \quad (22)$$

Let now $f \in \mathcal{C}^\infty(\Omega_T, \mathbb{R})$ and consider, for any $N \geq 1$, the function $F^N(t, \frac{x}{N}), x \in \{0, \dots, N\}$, satisfying the finite-dimensional linear differential equation

$$\begin{cases} N^{-\alpha} \partial_t F^N(t, \frac{x}{N}) + \tilde{\Delta}_N F^N(t, \frac{x}{N}) = f(t, \frac{x}{N}), & x \in \{1, \dots, N-1\}, \quad 0 < t < T, \\ F^N(t, 0) = F^N(t, N) = 0, \\ F^N(T, \cdot) = F(T, \cdot). \end{cases} \quad (23)$$

The quadratic variation of the martingale $N^{-\alpha/2} \mathcal{M}_T^N(F^N)$ satisfies

$$\begin{aligned} \mathbb{V}^N(f) &:= N \int_0^T \sum_{x=0}^{N-1} \left(\eta_{sN^{2+\alpha}}(x) - \eta_{sN^{2+\alpha}}(x+1) \right)^2 \left| F^N\left(s, \frac{x+1}{N}\right) - F^N\left(s, \frac{x}{N}\right) \right|^2 \\ &\xrightarrow[N \rightarrow \infty]{\mathbb{L}^2} 2 \int_0^T \int_0^1 ds \int_0^1 du \rho(s, u)(1 - \rho(s, u)) |\partial_u F(s, u)|^2 = \mathbb{V}(f). \end{aligned} \quad (24)$$

To prove the previous convergence, we use (35) proved below. It is then sufficient to prove that

$$\begin{aligned} &\int_0^T \frac{1}{N} \sum_{x=0}^{N-1} \left(\eta_{sN^{2+\alpha}}(x) - \eta_{sN^{2+\alpha}}(x+1) \right)^2 \left| (\partial_u F)(s, \frac{x}{N}) \right|^2 ds \\ &\xrightarrow[N \rightarrow \infty]{\mathbb{L}^2} 2 \int_0^T \int_0^1 \rho(s, u)(1 - \rho(s, u)) |(\partial_u F)(s, u)|^2 duds, \end{aligned} \quad (25)$$

but this is a simple consequence of Theorem 3.1 of [2].

Let us now prove (17). Recalling that $F^N(T, \cdot) = F(T, \cdot)$ we rewrite (22) with G replaced by F^N as

$$\mathbb{Y}^N(f) = -N^{-\alpha/2} [\mathcal{Y}_0^N(F^N) - \mathcal{Y}_T^N(F) + \mathcal{M}_T^N(F^N)].$$

We have then

$$\begin{aligned} & \mathbb{E} \left[\exp\{i\theta \mathbb{Y}^N(f)\} \right] \\ &= \mathbb{E} \left[\left(\exp \left\{ -i\theta N^{-\alpha/2} (\mathcal{Y}_0^N(F^N) - \mathcal{Y}_T^N(F)) \right\} - 1 \right) \exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) \right\} \right] \\ &+ \mathbb{E} \left[\exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) + \frac{\theta^2}{2} \mathbb{Y}^N(f) \right\} \exp \left\{ -\frac{\theta^2}{2} \mathbb{Y}^N(f) \right\} \right]. \end{aligned} \quad (26)$$

The modulus of the first term on the right-hand side of the previous display is bounded from above by

$$\begin{aligned} & \left| \mathbb{E} \left[\left(\exp \left\{ -i\theta N^{-\alpha/2} (\mathcal{Y}_0^N(F^N) - \mathcal{Y}_T^N(F)) \right\} - 1 \right) \exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) \right\} \right] \right| \\ &\leq \mathbb{E} \left[\left| \exp \left\{ -i\theta N^{-\alpha/2} (\mathcal{Y}_0^N(F^N) - \mathcal{Y}_T^N(F)) \right\} - 1 \right| \right] \\ &\lesssim |\theta| N^{-\alpha/2} \left(\mathbb{E} \left[\left| \mathcal{Y}_0^N(F^N) \right| \right] + \mathbb{E} \left[\left| \mathcal{Y}_T^N(F) \right| \right] \right) \\ &\leq |\theta| N^{-\alpha/2} \sqrt{\mathbb{E} \left[\left| \mathcal{Y}_0^N(F^N) \right|^2 \right]} + |\theta| N^{-\alpha/2} \sqrt{\mathbb{E} \left[\left| \mathcal{Y}_T^N(F) \right|^2 \right]} \\ &\lesssim |\theta| N^{-\alpha/2} \left[N^{-1} \sum_{x,y=1}^{N-1} F^N(0, \frac{x}{N}) F^N(0, \frac{y}{N}) \varphi^N(\frac{x}{N}, \frac{y}{N}; 0, 0) \right]^{1/2} \\ &+ |\theta| N^{-\alpha/2} \left[N^{-1} \sum_{x,y=1}^{N-1} F^N(T, \frac{x}{N}) F^N(T, \frac{y}{N}) \varphi^N(\frac{x}{N}, \frac{y}{N}; T, T) \right]^{1/2} \end{aligned} \quad (27)$$

where the space-time density correlation function φ^N is defined in (9). Then by assumption on the initial condition, see (10), and by Lemma 1, we have that

$$\begin{aligned} & N^{-1} \sum_{x,y=1}^{N-1} F^N(0, \frac{x}{N}) F^N(0, \frac{y}{N}) \varphi^N(\frac{x}{N}, \frac{y}{N}; 0, 0) \\ &\leq N^{-2} \sum_{\substack{x,y=1 \\ x \neq y}}^{N-1} |F^N(0, \frac{x}{N})| |F^N(0, \frac{y}{N})| + N^{-1} \sum_{x=1}^{N-1} F^N(0, \frac{x}{N})^2 \varphi^N(\frac{x}{N}, \frac{x}{N}; 0, 0) \quad (28) \\ &\leq \left(N^{-1} \sum_{x=1}^{N-1} |F^N(0, \frac{x}{N})| \right)^2 + CN^{-1} \sum_{x=1}^{N-1} F^N(0, \frac{x}{N})^2 = o(N^\alpha) \end{aligned}$$

we get that the first term on the right-hand side of (26) goes to 0 as $N \rightarrow +\infty$. The term involving $F(T, \cdot)$ can be treated analogously by noting that F is bounded and by using (13) with $t = T$.

Now let us focus on the remaining quantity in (26):

$$\mathbb{E} \left[\exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) + \frac{\theta^2}{2} \mathbb{V}^N(f) \right\} \exp \left\{ -\frac{\theta^2}{2} \mathbb{V}^N(f) \right\} \right]. \quad (29)$$

Last display can be rewritten as

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) + \frac{\theta^2}{2} \mathbb{V}^N(f) \right\} \left(\exp \left\{ -\frac{\theta^2}{2} \mathbb{V}^N(f) \right\} - \exp \left\{ -\frac{\theta^2}{2} \mathbb{V}(f) \right\} \right) \right] \\ & + \mathbb{E} \left[\exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) + \frac{\theta^2}{2} \mathbb{V}^N(f) \right\} \exp \left\{ \frac{-\theta^2}{2} \mathbb{V}(f) \right\} \right]. \end{aligned} \quad (30)$$

Since on the second term of last display we have the exponential complex martingale

$$\exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) + \frac{\theta^2}{2} \mathbb{V}^N(f) \right\},$$

which has constant expectation equal to one, the second term of (30) is, in fact, equal to

$$\exp \left\{ -\frac{\theta^2}{2} \mathbb{V}(f) \right\}.$$

It remains to estimate the first term of (30) and show that it goes to zero as $N \rightarrow +\infty$. That term can be rewritten as

$$\mathbb{E} \left[\exp \left\{ -i\theta N^{-\alpha/2} \mathcal{M}_T^N(F^N) \right\} \left(1 - \exp \left\{ \frac{\theta^2}{2} [\mathbb{V}^N(f) - \mathbb{V}(f)] \right\} \right) \right]. \quad (31)$$

From the inequality $|1 - e^x| \leq |x|e^{|x|}$, plus the fact that $\mathbb{V}^N(f) \lesssim 1$, $\mathbb{V}(f) \lesssim 1$ and (24), the proof ends. \square

Lemma 1. *Let $f \in C^\infty(\Omega_T)$ be real-valued. Let $F(t, u)$ be the unique solution of the Laplace equation*

$$\begin{cases} (\partial_u^2 F)(t, u) = f(t, u), & u \in (0, 1), 0 < t < T, \\ F(t, 0) = F(t, 1) = 0. \end{cases} \quad (32)$$

For any $N \geq 1$, let $F^N(t, \frac{x}{N})$, $x \in \{0, \dots, N\}$, be the solution of the equation

$$\begin{cases} N^{-\alpha} \partial_t F^N(t, \frac{x}{N}) + \tilde{\Delta}_N F^N(t, \frac{x}{N}) = f(t, \frac{x}{N}), & x \in \{1, \dots, N-1\}, 0 < t < T, \\ F^N(t, 0) = F^N(t, 1) = 0, \\ F^N(T, \frac{x}{N}) = F(T, \frac{x}{N}). \end{cases} \quad (33)$$

Then

$$\frac{1}{N} \sum_{z=1}^{N-1} |F^N(0, \frac{z}{N})|^2 = o(N^\alpha) \quad (34)$$

and

$$\lim_{N \rightarrow +\infty} \int_0^T \frac{1}{N} \sum_{z=0}^{N-1} |N(F^N(t, \frac{z+1}{N}) - F^N(t, \frac{z}{N})) - \partial_u F(t, \frac{z}{N})|^2 dt = 0. \quad (35)$$

Proof. Notice that F can be explicitly computed and it is given by

$$F(t, u) = \int_0^u \left(\int_0^v f(t, w) dw \right) dv - u \left(\int_0^1 \left(\int_0^v f(t, w) dw \right) dv \right).$$

Let $G^N = F^N - F$ and $\varepsilon^N = N^{-\alpha} \partial_t F - (\partial_u^2 - \tilde{\Delta}_N)F$. Then G^N satisfies

$$\begin{cases} N^{-\alpha} \partial_t G^N(t, \frac{x}{N}) + \tilde{\Delta}_N G^N(t, \frac{x}{N}) = -\varepsilon^N(t, \frac{x}{N}), & x \in \{1, \dots, N-1\}, \quad 0 < t < T, \\ G^N(t, 0) = G^N(t, 1) = 0, \\ G^N(T, \cdot) = 0. \end{cases} \quad (36)$$

Multiply the first line of last display by G_N and sum over $x \in \{1, \dots, N-1\}$. Since G^N satisfies Dirichlet boundary conditions, we have that

$$\sum_{x=1}^{N-1} G^N(t, \frac{x}{N}) \tilde{\Delta}_N G^N(t, \frac{x}{N}) = -N^2 \sum_{x=0}^{N-1} \left(G^N(t, \frac{x}{N}) - G^N(t, \frac{x+1}{N}) \right)^2.$$

Performing a summation by parts and integrating in time between 0 and T , we get

$$\begin{aligned} \frac{1}{2N^\alpha} \frac{1}{N} \sum_{x=1}^{N-1} [G^N(0, \frac{x}{N})]^2 + \int_0^T \frac{1}{N} \sum_{x=0}^{N-1} [N(G^N(t, \frac{x+1}{N}) - G^N(t, \frac{x}{N}))]^2 dt \\ = \int_0^T \frac{1}{N} \sum_{x=1}^{N-1} \varepsilon^N(t, \frac{x}{N}) G^N(t, \frac{x}{N}) dt. \end{aligned} \quad (37)$$

Now let $E^N : [0, T] \times \{0, \frac{1}{N}, \dots, 1\} \rightarrow \mathbb{R}$ be the function defined by

$$E^N(t, \frac{x}{N}) = \frac{1}{N} \sum_{y=0}^x \varepsilon^N(t, \frac{y}{N}).$$

Performing a summation by parts, we have that

$$\frac{1}{N} \sum_{x=1}^{N-1} \varepsilon^N(t, \frac{x}{N}) G^N(t, \frac{x}{N}) = -\frac{1}{N} \sum_{x=0}^{N-1} E^N(t, \frac{x}{N}) [N(G^N(t, \frac{x+1}{N}) - G^N(t, \frac{x}{N}))]. \quad (38)$$

Above we used again the fact that G^N satisfies Dirichlet boundary conditions. Since F is smooth, we have for any $x \in \{0, \dots, N-1\}$, that

$$|\varepsilon^N\left(t, \frac{x}{N}\right)| \lesssim N^{-\alpha} + N^{-2},$$

and consequently,

$$\lim_{N \rightarrow \infty} \int_0^T \frac{1}{N} \sum_{x=0}^{N-1} \left[E^N\left(t, \frac{x}{N}\right) \right]^2 dt = 0.$$

Hence by Cauchy-Schwarz inequality and (37) we conclude that

$$\lim_{N \rightarrow \infty} \int_0^T \frac{1}{N} \sum_{x=0}^{N-1} \left[N \left(G^N\left(t, \frac{x+1}{N}\right) - G^N\left(t, \frac{x}{N}\right) \right) \right]^2 dt = 0.$$

Since F is smooth this implies (35). Using this information in (37) we get that

$$\lim_{N \rightarrow \infty} \frac{1}{N^\alpha} \frac{1}{N} \sum_{x=1}^{N-1} \left[G^N\left(0, \frac{x}{N}\right) \right]^2 = 0,$$

and since F is bounded, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^\alpha} \frac{1}{N} \sum_{x=1}^{N-1} \left[F^N\left(0, \frac{x}{N}\right) \right]^2 = 0.$$

□

3.2 Tightness

We recall that $T > 0$ is some fixed finite time horizon. The sequence $\{\psi_{n,z} ; n \in \mathbb{Z}, z \in \mathbb{N}^*\}$ ¹ of functions on $[0, T] \times [0, 1]$, defined by

$$\psi_{n,z}(t, u) = \sqrt{2} \exp\left(2i\pi n \frac{t}{T}\right) \sin(\pi zu), \quad t \in [0, T], \quad u \in [0, 1], \quad (39)$$

forms an orthonormal basis of $L^2([0, T] \times [0, 1])$ and for any $f \in L^2([0, T] \times [0, 1])$ we denote $\hat{f}(n, z) = \langle f, \psi_{n,z} \rangle$ the Fourier coefficients of f in this basis. Above $\langle \cdot, \cdot \rangle$ is the inner product in $L^2([0, T] \times [0, 1])$. For $m, k \in \mathbb{R}$, let $\mathcal{H}_{m,k}$ be the Hilbert space obtained as the completion of $C^\infty(\Omega_T, \mathbb{C})$ endowed with the inner product²

$$\langle f, g \rangle_{m,k} = \sum_{\substack{n \in \mathbb{Z} \\ z \in \mathbb{N}^*}} (n^2 + 1)^m z^{2k} \hat{f}(n, z) \hat{g}^*(n, z).$$

¹ $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$

²Here, w^* denotes the complex conjugate of the complex number w .

The corresponding norm is denoted by $\|\cdot\|_{m,k}$. Then for $m, k \geq 0$ we have

$$\mathcal{H}_{-m,-k} \supset L^2([0, T] \times [0, 1]) \supset \mathcal{H}_{m,k}$$

and $\mathcal{H}_{-m,-k}$ can be identified with the dual of $\mathcal{H}_{m,k}$ with respect to the inner product of $L^2([0, T] \times [0, 1])$. The inner product in $\mathcal{H}_{-m,-k}$ is given by

$$\langle F, G \rangle_{-m,-k} = \sum_{\substack{n \in \mathbb{Z} \\ z \in \mathbb{N}^*}} (n^2 + 1)^{-m} z^{-2k} \hat{F}(n, z) \hat{G}^*(n, z). \quad (40)$$

and we denote the corresponding norm by $\|\cdot\|_{-m,-k}$.

Lemma 2. *For $m, k > 1/2$, the sequence $\{\mathbb{Y}^N; N \geq 1\}$ is tight in $\mathcal{H}_{-m,-k}$.*

Proof. We need to show that

$$\lim_{A \rightarrow +\infty} \limsup_{N \rightarrow +\infty} \mathbb{P}_{\mu_N} (\|\mathbb{Y}^N\|_{-m,-k} > A) = 0.$$

Recall from (40) the definition of the $\|\cdot\|_{-m,-k}$ -norm. From Chebychev's inequality the proof ends as long as we show that

$$\sup_{N \geq 1} \mathbb{E}_{\mu_N} (\|\mathbb{Y}^N\|_{-m,-k}^2) = \sup_{N \geq 1} \sum_{\substack{n \in \mathbb{Z} \\ z \in \mathbb{N}^*}} (n^2 + 1)^{-m} z^{-2k} \mathbb{E}_{\mu_N} (|\mathbb{Y}^N(\psi_{n,z})|^2) < \infty. \quad (41)$$

Observe that from (39) we have

$$\tilde{\Delta}_N \psi_{n,z}(t, \frac{x}{N}) + N^{-\alpha} \partial_t \psi_{n,z}(t, \frac{x}{N}) = [2i\pi \frac{n}{T} N^{-\alpha} - 4N^2 \sin^2(\frac{\pi z}{2N})] \psi_{n,z}(t, \frac{x}{N}).$$

Applying (22) with $G = \psi_{n,z}$ we get

$$\begin{aligned} \mathbb{Y}^N(\psi_{n,z}) &= \frac{1}{2i\pi \frac{n}{T} N^{-\alpha} - 4N^2 \sin^2(\frac{\pi z}{2N})} \mathbb{Y}^N \left(\tilde{\Delta}_N \psi_{n,z} + N^{-\alpha} \partial_t \psi_{n,z} \right) \\ &= \frac{1}{2i\pi \frac{n}{T} N^{-\alpha} - 4N^2 \sin^2(\frac{\pi z}{2N})} \left[\frac{1}{N^{\alpha/2}} \{ \mathcal{Y}_T^N(\psi_{n,z}) - \mathcal{Y}_0^N(\psi_{n,z}) \} - \frac{1}{N^{\alpha/2}} \mathcal{M}_T^N(\psi_{n,z}) \right]. \end{aligned} \quad (42)$$

From (24), we see that the quadratic variation of the martingale $\frac{1}{N^{\alpha/2}} \mathcal{M}_T^N(\psi_{n,z})$ is given by

$$8N \sin^2\left(\frac{\pi z}{N}\right) \cos^2\left(\frac{\pi z x}{N} + \frac{\pi z}{2N}\right) \int_0^T \sum_{x=0}^{N-1} \left(\eta_{tN^{2+\alpha}}(x) - \eta_{tN^{2+\alpha}}(x+1) \right)^2 dt \lesssim N^2 \sin^2\left(\frac{\pi z}{N}\right).$$

Recalling (13) and following similar arguments as in (28), we conclude that

$$\mathbb{E} \left[\left\{ \mathcal{Y}_T^N(\psi_{n,z}) - \mathcal{Y}_0^N(\psi_{n,z}) \right\}^2 \right] \leq 2\mathbb{E} [|\mathcal{Y}_0^N(\psi_{n,z})|^2] + 2\mathbb{E} [|\mathcal{Y}_T^N(\psi_{n,z})|^2] \lesssim 1. \quad (43)$$

Collecting the previous bounds together we get that

$$\mathbb{E} (|\mathbb{Y}^N(\psi_{n,z})|^2) \lesssim \frac{N^2 \sin^2 \left(\frac{\pi z}{N} \right) + N^{-\alpha}}{n^2 N^{-2\alpha} + N^4 \sin^4 \left(\frac{\pi z}{2N} \right)}. \quad (44)$$

At this point we need to estimate

$$\sum_{\substack{n \in \mathbb{Z} \\ z \in \mathbb{N}^*}} \frac{1}{(n^2 + 1)^m} \frac{1}{z^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi z}{N} \right) + N^{-\alpha}}{n^2 N^{-2\alpha} + N^4 \sin^4 \left(\frac{\pi z}{2N} \right)}.$$

By symmetry, we split the sum in two cases depending on whether $n = 0$ or $n \neq 0$:

$$S_1 := \sum_{z \in \mathbb{N}^*} \frac{1}{z^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi z}{N} \right) + N^{-\alpha}}{N^4 \sin^4 \left(\frac{\pi z}{2N} \right)}.$$

and

$$S_2 := 2 \sum_{n, z \in \mathbb{N}^*} \frac{1}{(n^2 + 1)^m} \frac{1}{z^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi z}{N} \right) + N^{-\alpha}}{n^2 N^{-2\alpha} + N^4 \sin^4 \left(\frac{\pi z}{2N} \right)}.$$

First we observe that for $m > 1/2$ we have that

$$S_2 \lesssim \sum_{n \in \mathbb{N}^*} \frac{1}{(n^2 + 1)^m} \sum_{z \in \mathbb{N}^*} \frac{1}{z^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi z}{N} \right) + N^{-\alpha}}{N^4 \sin^4 \left(\frac{\pi z}{2N} \right)} \lesssim S_1,$$

therefore it is enough to estimate S_1 . Note that by writing $z = 2Np + q$ and using the periodicity of the function $\sin(\cdot)$ we have

$$S_1 := \sum_{p=0}^{+\infty} \sum_{q=1}^{2N-1} \frac{1}{(2Np + q)^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi q}{N} \right) + N^{-\alpha}}{N^4 \sin^4 \left(\frac{\pi q}{2N} \right)}.$$

Now we split last sum into

$$S_1^A := \sum_{p=0}^{+\infty} \sum_{q=1}^N \frac{1}{(2Np + q)^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi q}{N} \right) + N^{-\alpha}}{N^4 \sin^4 \left(\frac{\pi q}{2N} \right)},$$

$$S_1^B := \sum_{p=0}^{+\infty} \sum_{q=N+1}^{2N-1} \frac{1}{(2Np + q)^{2k}} \frac{N^2 \sin^2 \left(\frac{\pi q}{N} \right) + N^{-\alpha}}{N^4 \sin^4 \left(\frac{\pi q}{2N} \right)}.$$

We start with S_1^B . We do the change of variables $r = q - N$ and use a trigonometric identity, to get

$$S_1^B = \sum_{p=0}^{+\infty} \sum_{r=1}^{N-1} \frac{1}{(2Np + r + N)^{2k}} \frac{N^2 \sin^2\left(\frac{\pi r}{N}\right) + N^{-\alpha}}{N^4 \cos^4\left(\frac{\pi r}{2N}\right)}.$$

Now we use the fact that for $x \in [0, \frac{\pi}{2}]$ it holds that $\cos(x) \geq x - \frac{\pi}{2}$, to get the bound

$$S_1^B \lesssim \sum_{p=0}^{+\infty} \sum_{r=1}^{N-1} \frac{1}{(2Np + r + N)^{2k}} \frac{N^2 \sin^2\left(\frac{\pi r}{N}\right) + N^{-\alpha}}{(r - N)^4}.$$

From the change of variables $v = r - N$ and using again the periodicity of the function $\sin(\cdot)$, we can bound the last display from above by

$$S_1^B \lesssim \sum_{p=0}^{+\infty} \sum_{v=-N+1}^{-1} \frac{1}{(2N(p+1) + v)^{2k}} \frac{N^2 \sin^2\left(\frac{\pi v}{N}\right) + N^{-\alpha}}{v^4}$$

Now, we bound last display from above by a constant times

$$\sum_{p=0}^{+\infty} \sum_{v=1}^{N-1} \frac{1}{(2Np + 1)^{2k}} \frac{v^2 + N^{-\alpha}}{v^4} \lesssim 1 + \sum_{p=1}^{+\infty} \frac{1}{(2Np)^{2k}} < +\infty$$

where we used the fact that $k > 1/2$. Finally note that

$$S_1^A \lesssim \sum_{p=0}^{+\infty} \sum_{q=1}^N \frac{1}{(2Np + q)^{2k}} \frac{q^2 + N^{-\alpha}}{q^4} \lesssim 1 + \sum_{p=1}^{+\infty} \sum_{q=1}^N \frac{1}{(2Np)^{2k}} \frac{q^2 + N^{-\alpha}}{q^4} < +\infty.$$

This ends the proof. □

4 Generalization and discussion

Theorem 1 can be adapted without difficulties to the case of periodic boundary conditions at equilibrium, i.e. by considering the dynamics evolving on the discrete torus (without reservoirs) $\mathbb{T}_N = \{0, 1, \dots, N-1\}$ starting from its equilibrium state. Since the mass of the system is conserved we have to restrict the space of test functions f that satisfy $\int_{\mathbb{T}} f(t, u) du = 0$ for any $t \in [0, T]$. Here \mathbb{T} denotes the one-dimensional torus. In this case the sequence $\{\mathbb{Y}^N ; N \geq 1\}$ converges to a field \mathbb{Y} white in time and correlated in space according to the kernel $\rho(1 - \rho)(-\partial_u^2)^{-1}$, where ρ is the limit average density.

In fact, in the case of periodic conditions, at equilibrium, Theorem 1 can be proved similarly for various interacting particle systems: the Kipnis-Marchioro-Presutti model

[9], the symmetric Ginzburg-Landau dynamics with harmonic potential [8, 13], the Symmetric Inclusion Process [5], independent random walks (i.e. the Symmetric Zero-Range Process with linear rate) and Generalized Exclusion Process [4].

In the sequel we see briefly how to extend our main theorem in the case of generic symmetric Zero-Range processes at equilibrium. Our result is however limited to a time scale $tN^{2+\alpha}$ with $\alpha < 4$. More precisely we consider a Zero-Range process $\{\eta_{tN^{2+\alpha}} ; t \geq 0\}$ with state space $\Omega^N = \mathbb{N}^{\mathbb{T}^N}$ with generator \mathcal{L} acting on a test function $f : \Omega^N \rightarrow \mathbb{R}$ as

$$(\mathcal{L}f)(\eta) = \sum_{\substack{x,y \in \mathbb{T}^N \\ |x-y|=1}} g(\eta(x)) [f(\eta^{x,y}) - f(\eta)].$$

Here $g : \mathbb{N} \rightarrow \mathbb{N}$ has to satisfy some technical assumptions listed in [7] in order to assure the validity of the second order Boltzmann-Gibbs bound (48). We refer the reader to [7] for details. We denote by ν_ρ the equilibrium probability measure with density ρ . Let us consider the dynamics at equilibrium under ν_ρ .

We define

$$\mathbb{Y}^N(f) = N^{\alpha/2} \int_0^T \mathcal{Y}_t^N(f) dt$$

where the random fluctuation field $\{\mathcal{Y}_t^N ; t \geq 0\}$ acts on a space-time function $f : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ satisfying $\int_{\mathbb{T}} f(t, u) du = 0$ for any $t \in [0, T]$, as

$$\mathcal{Y}_t^N(f) := \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}^N} f(t, \frac{x}{N}) (\eta_{tN^{2+\alpha}}(x) - \rho).$$

Let

$$V_g(\eta(x)) = g(\eta(x)) - \tilde{g}(\rho) - \tilde{g}'(\rho)(\eta(x) - \rho).$$

By Dynkin's formula it follows that the process $\{\mathcal{M}_t^N(F) ; t \geq 0\}$ defined by

$$\begin{aligned} \mathcal{M}_t^N(F) &= \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) \\ &- \int_0^t \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}^N} \partial_s F(s, \frac{x}{N}) (\eta_{sN^{2+\alpha}}(x) - \rho) ds \\ &- N^{\alpha-1/2} \int_0^t \sum_{x \in \mathbb{T}^N} \tilde{g}'(\rho) (\tilde{\Delta}_N F)(s, \frac{x}{N}) [\eta_{sN^{2+\alpha}}(x) - \rho] ds \\ &- N^{\alpha-1/2} \int_0^t \sum_{x \in \mathbb{T}^N} (\tilde{\Delta}_N F)(s, \frac{x}{N}) V_g(\eta_{sN^{2+\alpha}}(x)) ds \end{aligned} \tag{45}$$

is a martingale w.r.t. the natural filtration of $\{\eta_{tN^{2+\alpha}} ; t \geq 0\}$.

Let $\mathbb{T} = [0, 1)$ be the unit torus. Consider now $f : (t, u) \in [0, T] \times \mathbb{T} \rightarrow f(t, u) \in \mathbb{R}$ be a function satisfying $\int_{\mathbb{T}} f(t, u) du = 0$ for any $t \in [0, T]$ and consider, for any $N \geq 1$,

the function $F^N : (t, u) \in [0, T] \times \frac{1}{N}\mathbb{T}_N \rightarrow F^N(t, u) \in \mathbb{R}$ satisfying the equation

$$\begin{aligned} N^{-\alpha} \partial_t F^N(t, \frac{x}{N}) + \check{g}'(\rho) \tilde{\Delta}_N F^N(t, \frac{x}{N}) &= f(t, \frac{x}{N}), \quad x \in \{1, \dots, N-1\}, \\ F^N(T, \cdot) &= 0. \end{aligned} \quad (46)$$

We have then

$$\begin{aligned} \mathbb{Y}^N(f) &= -\frac{1}{N^{\alpha/2}} \mathcal{Y}_0^N(F^N) - \frac{1}{N^{\alpha/2}} \mathcal{M}_T^N(F^N) \\ &\quad + N^{(\alpha-1)/2} \int_0^T \sum_{x \in \mathbb{T}_N} (\tilde{\Delta}_N F^N)(s, \frac{x}{N}) V_g(\eta_{sN^{2+\alpha}}(x)) ds \end{aligned} \quad (47)$$

By adapting Remark 11 in [6] we have that for any $\ell \geq 1$

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T \sum_{x \in \mathbb{T}_N} (\tilde{\Delta}_N F^N)(s, \frac{x}{N}) V_g(\eta_{sN^{2+\alpha}}(x)) ds \right)^2 \right] \\ &\lesssim \left[\frac{\ell}{N^{2+\alpha}} + \frac{T}{\ell} \right] \int_0^T \sum_{x \in \mathbb{T}_N} [(\tilde{\Delta}_N F^N)(s, \frac{x}{N})]^2 ds. \end{aligned} \quad (48)$$

By taking $\ell = N^{1+\alpha/2}$ and using (46) we get

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^T \sum_{x \in \mathbb{T}_N} (\tilde{\Delta}_N F^N)(s, \frac{x}{N}) V_g(\eta_{sN^{2+\alpha}}(x)) ds \right)^2 \right] \\ &\lesssim N^{-(1+\alpha/2)} \int_0^T \sum_{x \in \mathbb{T}_N} [(\tilde{\Delta}_N F^N)(s, \frac{x}{N})]^2 ds \\ &\lesssim N^{-(1+\alpha/2)} \left\{ \int_0^T \sum_{x \in \mathbb{T}_N} [f(s, \frac{x}{N})]^2 ds + N^{-2\alpha} \int_0^T \sum_{x \in \mathbb{T}_N} [\partial_s F^N(s, \frac{x}{N})]^2 ds \right\}. \end{aligned} \quad (49)$$

Hence the square of the L^2 -norm of the third term on the right-hand side of (47) is bounded above by $N^{\alpha/2-2}$. It goes to 0 if, and only if, $\alpha < 4$. This ends the proof.

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