Harmonizable Fractional Stable Motion: simultaneous estimators for the both parameters

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Abstract

There are two classical very different extensions of the well-known Gaussian fractional Brownian motion to non-Gaussian frameworks of heavy-tailed stable distributions: the harmonizable fractional stable motion (HFSM) and the linear fractional stable motion (LFSM). As far as we know, while several articles in the literature, some of which appeared a long time ago, have proposed statistical estimators for the parameters of LFSM, no estimator has yet been proposed in the framework of HFSM. Among other things, what makes statistical estimation of parameters of HFSM to be a difficult problem is that, in contrast to LFSM, HFSM is not ergodic. The main goal of our work is to propose a new strategy for dealing with this problem and obtaining solutions of it. The keystone of our new strategy consists in the construction of new transforms of HFSM which allow to obtain, at any dyadic level, a sequence of *independent* stable random variables.

Running head: Estimation of Harmonizable Fractional Stable Motion

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1 Introduction and statement of the two main results

A real-valued harmonizable fractional stable motion[1] (HFSM), denoted by $\{X(t)\}_{t\in\mathbb{R}}$, is a paradigmatic example of a continuous symmetric stable self-similar stochastic process with stationary increments. It was introduced, about 35 years ago, by Cambanis and Maejima in [9]. Basically, it depends on two parameters: the Hurst parameter H belonging to the open interval $(0,1)$, and the stability parameter α belonging to the interval $(0,2]$. Among other

[1]Notice that the HFSM is somtimes called harmonizable fractional stable process.

things, the parameter H governs roughness of sample paths of $\{X(t)\}_{t\in\mathbb{R}}$ and its self-similarity property:

$$\big\{a^{-H}X(at)\big\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t)\}_{t\in\mathbb{R}}, \quad \text{for any fixed } a\in(0,+\infty),$$

where $\stackrel{d}{=}$ means that the equality holds in the sense of the finite-dimensional distributions. While the parameter α determines heaviness of tails of marginal distributions of $\{X(t)\}_{t\in\mathbb{R}}$: except in the very particular Gaussian case $\alpha=2$ in which the probability $\mathbb{P}\big(|X(t)|\geq z\big)$ vanishes exponentially fast when $z\to+\infty$, for any other value of α, one has, for each fixed $t\neq 0$ (notice that $X(0)\stackrel{a.s.}{=}0$),

$$c'(t)z^{-\alpha}\leq \mathbb{P}\big(|X(t)|\geq z\big)\leq c''(t)z^{-\alpha}, \quad \text{for all } z\in[1,+\infty),$$

where $0<c'(t)<c''(t)$ are two finite constants.

The HFSM $\{X(t)\}_{t\in\mathbb{R}}$ is defined, for all $t\in\mathbb{R}$, through the stable stochastic integral in the frequency domain:

$$X(t):=\mathrm{Re}\Big(\int_{\mathbb{R}}\frac{e^{it\xi}-1}{|\xi|^{H+1/\alpha}}\,d\widetilde{M}_\alpha(\xi)\Big), \tag{1.1}$$

where $\widetilde{M}_\alpha$ is a complex-valued rotationally invariant α-stable random measure with Lebesgue control measure. A detailed presentation of such a random measure and the corresponding stable stochastic integral and related topics can for instance be found in Chapter 6 of the well-known book [19]. The following remark, which provides two very important properties of this stochastic integral, will play a fundamental role in our work.

Remark 1.1

(i) The stable stochastic integral $\int_{\mathbb{R}}(\,\cdot\,)\,d\widetilde{M}_\alpha$ is a linear map on the Lebesgue space $L^\alpha(\mathbb{R})$ such that, for any deterministic function $g\in L^\alpha(\mathbb{R})$, the real part $\mathrm{Re}\big(\int_{\mathbb{R}^N}g(\xi)\,d\widetilde{M}_\alpha(\xi)\big)$ is a real-valued Symmetric α-Stable (SαS) random variable with a scale parameter satisfying

$$\sigma\Big(\mathrm{Re}\big\{\int_{\mathbb{R}}g(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}\Big)^\alpha=\int_{\mathbb{R}}|g(\xi)|^\alpha\,d\xi. \tag{1.2}$$

The equality (1.2) is reminiscent of the classical isometry property of Wiener integrals; in particular, it implies that $\mathrm{Re}\big\{\int_{\mathbb{R}}g_n(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ converges to $\mathrm{Re}\big\{\int_{\mathbb{R}}g(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ in probability, when a sequence $(g_n)_n$ converges to g in $L^\alpha(\mathbb{R})$.

(ii) Let $m\in\mathbb{N}$ be arbitrary and let $f_1,\ldots,f_m$ be arbitrary functions of $L^\alpha(\mathbb{R})$ whose supports are disjoint up to Lebesgue negligible sets, then the real-valued SαS random variables $\mathrm{Re}\big\{\int_{\mathbb{R}}f_1(\xi)\,d\widetilde{M}_\alpha(\xi)\big\},\ldots,\mathrm{Re}\big\{\int_{\mathbb{R}}f_m(\xi)\,d\widetilde{M}_\alpha(\xi)\big\}$ are independent.

In the very particular Gaussian case where the stability parameter $\alpha=2$, the HFSM represented by (1.1) reduces to the very classical Gaussian fractional Brownian motion (FBM) with Hurst parameter H, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$. It is well-known that, up to a deterministic multiplicative constant, the Gaussian process $\{B_H(t)\}_{t\in\mathbb{R}}$ can also be represented as a moving average stochastic Wiener integral in the time domain, whose integrand is no longer the complex-valued function $\xi\mapsto\frac{e^{it\xi}-1}{|\xi|^{H+1/\alpha}}$ but the real-valued function $s\mapsto\Big((t-s)_+^{H-1/2}-(-s)_+^{H-1/2}\Big)$.

One recalls, in passing, the usual convention that, for all $(x, \beta) \in \mathbb{R}^2$,

$$(x)_+^\beta := x^\beta \text{ when } x > 0 \quad \text{and} \quad (x)_+^\beta := 0 \text{ else.}$$

When the stability parameter $\alpha \neq 2$, the HFSM in (1.1) can no longer be represented as a moving average stable stochastic integral in the time domain. Actually, it is very different from the real-valued linear fractional stable motion (LFSM) $\{L(t)\}_{t\in\mathbb{R}}$ defined, for each $t \in \mathbb{R}$, by

$$L(t) := \int_{\mathbb{R}} \left((t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right) dM_\alpha(ds), \tag{1.3}$$

where M_α is a real-valued α-stable random measure. The large differences between the two processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{L(t)\}_{t\in\mathbb{R}}$ can be explained by several reasons. Let us mention two important ones of them: (1) in contrast to the process $\{L(t)\}_{t\in\mathbb{R}}$ the process $\{X(t)\}_{t\in\mathbb{R}}$ is not ergodic, (2) behavior of sample paths of $\{L(t)\}_{t\in\mathbb{R}}$ and $\{X(t)\}_{t\in\mathbb{R}}$ is far from being the same. Indeed, sample paths of $\{L(t)\}_{t\in\mathbb{R}}$ are multifractal functions (see [7]) which become discontinuous when $H \leq 1/\alpha$ and even unbounded on any interval when $H < 1/\alpha$ (see for example [19, 12]). While those of $\{X(t)\}_{t\in\mathbb{R}}$ are Hölder continuous of any order strictly less than H for any value of $H \in (0, 1)$ (see [13, 14, 19, 12]), namely, for each fixed $\delta > 0$ and $T > 0$, one has almost surely

$$\sup_{-T\leq t'<t''\leq T} \left\{ \frac{\left|X(t') - X(t'')\right|}{|t' - t''|^{H-\delta}} \right\} < +\infty\,. \tag{1.4}$$

Moreover, sample paths of $\{X(t)\}_{t\in\mathbb{R}}$ are monofractal functions; the latter fact results from their Hölderianity property combined with Corollary 4.4 in [6]. Also, for later purposes, one mentions that as regards their behavior at infinity, one can derive from Corollary 4.2 in [1], that, for all fixed $\delta > 0$, one has, almost surely,

$$\sup_{|t|\geq 1} \left\{ \frac{\left|X(t)\right|}{|t|^{H+\delta}} \right\} < +\infty\,. \tag{1.5}$$

Let us now present the main motivations behind our present work and its main contributions. Statistical estimators for the parameters H and α of the LFSM $\{L(t)\}_{t\in\mathbb{R}}$ and related moving average stable processes have been proposed in several articles in the literature (see for instance [21, 20, 3, 2, 10, 17, 16, 15]), some of which appeared a long time ago. However, as far as we know, in the framework of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ and related harmonizable stable processes and fields no statistical estimator for any one of these two parameters has yet been proposed in the literature. Also, according to what is mentioned in Remark 1.2 (D) and on page 2 of the very recent preprint [8], their statistical estimation in such a framework is far from being an obvious problem due to the fact that HFSM and related harmonizable stable processes and fields are not ergodic. The main idea behind our strategy for dealing with the latter problem is to construct new transforms of HFSM which allow to obtain, at any dyadic level $j \in \mathbb{N}$, a sequence $Y_{j,k}$, $k \in \mathbb{N}$, of *independent* real-valued SαS random variables whose scale parameters $\sigma(Y_{j,k})$, $k \in \mathbb{N}$, are closely connected to the unknown parameters H and α of HFSM through simple formulas which are rather easy to handle. Roughly speaking, these new transforms $Y_{j,k}$, $(j, k) \in \mathbb{N}^2$, of HFSM are at the same time inspired by discrete wavelet

transforms $\mathcal{W}_{j,k}$, $(j,k) \in \mathbb{N}^2$, of HFSM and significantly different from them. Indeed, while $\mathcal{W}_{j,k}$ is defined (see e.g. [11, 18]), up to normalizing factor, as

$$\mathcal{W}_{j,k} := \int_{\mathbb{R}} \psi_{j,k}(t) X(t)\, dt,$$

where $\psi_{j,k}(t) := \psi(2^j t - k)$, for all $t \in \mathbb{R}$, with ψ being a "nice" real-valued function; we define $Y_{j,k}$ as

$$Y_{j,k} := \frac{2}{2\pi} \int_{\mathbb{R}} \mathrm{Re}\left(\widehat{\psi}_{j,k}(t)\right) X(t)\, dt,$$

where $\widehat{\psi}_{j,k}$ is the Fourier transform of $\psi_{j,k}$. Also, we assume that the function ψ is given by the formula (2.4); we mention that many other choices for ψ are possible, and there is no need to impose to ψ to have any vanishing moment, while such a condition on moment(s) of ψ plays a crucial role in the case of the discrete wavelet transform $\mathcal{W}_{j,k}$. Notice that the definition of $Y_{j,k}$ is given in a more explicit way in (2.8).

Let us now state the main two theorems of our work.

Theorem 1.2 *One assumes that $\alpha \in [\underline{\alpha}, 2]$, where the lower bound $\underline{\alpha} \in (0,2]$ is known. Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence (that is $m_j \leq m_{j+1}$, for all $j \in \mathbb{N}$) of integers larger than 2 which satisfy the condition*

$$m_j \geq j\,, \quad \text{for all } j \in \mathbb{N}. \tag{1.6}$$

For each fixed $\gamma \in (0, 4^{-1}\underline{\alpha})$, and for every $j \in \mathbb{N}$, the statistics $V_{j,\gamma}^{m_j}$ is defined as

$$V_{j,\gamma}^{m_j} := \sum_{k=1}^{m_j} \left|Y_{j,k}\right|^{\gamma}, \tag{1.7}$$

where the $Y_{j,k}$'s are the real-valued SαS random variables defined in (2.8). Then, the following two results hold.

(i) Let

$$\widehat{H}_{j,\gamma} := \frac{\log_2(V_{j,\gamma}^{m_j})}{\gamma\, j}, \tag{1.8}$$

where $\log_2$ denotes the binary logarithm (that is $\log_2(2) = 1$). Under the condition

$$\lim_{j\to+\infty} \frac{\log_2(m_j)}{j} = 0, \tag{1.9}$$

$\widehat{H}_{j,\gamma}$ is a strongly consistent (almost surely convergent) estimator of the Hurst parameter H of HFSM.

(ii) Let

$$\widehat{\alpha}_{j,\gamma} := \frac{\gamma\, j}{j - \log_2(V_{j,\gamma}^{m_j})}. \tag{1.10}$$

Under the condition

$$\lim_{j\to+\infty} \frac{\log_2(m_j)}{j} = 1, \tag{1.11}$$

$\widehat{\alpha}_{j,\gamma}$ is a strongly consistent (almost surely convergent) estimator of the stability parameter α of HFSM.

Theorem 1.3 *Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence of integers larger than 2 which satisfies the condition (1.6). For every $j \in \mathbb{N}$, the statistics $V_{j,\log_2}^{m_j}$ is defined as*

$$V_{j,\log_2}^{m_j} := \sum_{k=1}^{m_j} \log_2 \left|Y_{j,k}\right|, \tag{1.12}$$

where the $Y_{j,k}$'s are the real-valued SαS random variables defined in (2.8). Then, the following two results hold.

(i) Let

$$\widehat{H}_{j,\log_2} := \frac{V_{j,\log_2}^{m_j}}{j\, m_j}. \tag{1.13}$$

Under the condition (1.9), $\widehat{H}_{j,\log_2}$ is a strongly consistent (almost surely convergent) estimator of the Hurst parameter H of HFSM.

(ii) Let

$$\widehat{\alpha}_{j,\log_2} := \frac{j\, m_j}{V_{j,\log_2}^{m_j}}. \tag{1.14}$$

Under the condition (1.11), $\widehat{\alpha}_{j,\log_2}$ is a strongly consistent (almost surely convergent) estimator of the stability parameter α of HFSM.

Remark 1.4

(i) We believe that the new strategy introduced in our present work would open the door to statistical estimation of parameters of harmonizable stable fields extending the HFSM, as for instance the harmonizable fractional stable field studied in e.g. [6], or the harmonizable fractional stable sheet studied in e.g. [5].

(ii) Since, for each for each fixed $j \in \mathbb{N}$, the random variables $Y_{j,k}$, $k \in \mathbb{N}$, have the very nice property to be independent, we believe that the centered and standardized versions of the two statistics $V_{j,\gamma}^{m_j}$ and $V_{j,\log_2}^{m_j}$ would be asymptotically normal. This would imply that the four estimators $\widehat{H}_{j,\gamma}$, $\widehat{\alpha}_{j,\gamma}$, $\widehat{H}_{j,\log_2}$ and $\widehat{\alpha}_{j,\log_2}$ would be asymptotically normal as well.

(iii) In our present work, the four estimators $\widehat{H}_{j,\gamma}$, $\widehat{\alpha}_{j,\gamma}$, $\widehat{H}_{j,\log_2}$ and $\widehat{\alpha}_{j,\log_2}$ are obtained from the observation of a sample path of the HFSM X in continuous time, we believe that it would be possible to extend our estimation procedures to a framework where only a discretized sample path of X is observed.

We intend to study these three issues (i), (ii) and (iii) in future works.

The remaining of our present work is organized in the following way. In Section 2, basically we show that the real-valued SαS random variables $Y_{j,k}$, $(j,k) \in \mathbb{N}^2$, defined in (2.8), can be represented in terms of the stable stochastic integral $\int_{\mathbb{R}} \left(\,\cdot\,\right) d\widetilde{M}_\alpha$ (see Lemma 2.2); a straightforward consequence of this representation is the independence property, for any fixed $j \in \mathbb{N}$, of the random variables $Y_{j,k}$, $k \in \mathbb{N}$ (see Remark 2.3). Section 3 is devoted to the proofs of Theorems 1.2 and 1.3.

2 The keystone

Remark 2.1 Throughout our work, we use the very classical convention that, for any function $f \in L^1(\mathbb{R})$, the Fourier transform $\widehat{f}$, also denoted by $\mathcal{F}(f)$, is defined as

$$\mathcal{F}(f)(t) = \widehat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x)\, dx, \quad \text{for all } t \in \mathbb{R}. \tag{2.1}$$

While, the inverse Fourier transform of f, denoted by $\mathcal{F}^{-1}(f)$, is defined as

$$\mathcal{F}^{-1}(f)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} f(t)\, dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.2}$$

It is well-known that the two maps $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to $L^2(\mathbb{R})$, and satisfy

$$\mathcal{F}^{-1}\big(\mathcal{F}(g)\big) = \mathcal{F}\big(\mathcal{F}^{-1}(g)\big) = g, \quad \text{for every } g \in L^2(\mathbb{R}). \tag{2.3}$$

One denotes by ψ the even piecewise linear continuous function from $\mathbb{R}$ into $[0, 2]$ with defined as

$$\psi(\xi) := 4\big(\mathbb{1}_{[-4^{-1},4^{-1}]} * \mathbb{1}_{[-4^{-1},4^{-1}]}\big)(\xi) = 2\big(1 - |2\xi|\big)\mathbb{1}_{[-2^{-1},2^{-1}]}(\xi), \quad \text{for all } \xi \in \mathbb{R}, \tag{2.4}$$

where "$*$" denotes the convolution product. It is clear that ψ is compactly supported and that

$$\operatorname{supp} \psi = [-2^{-1}, 2^{-1}]. \tag{2.5}$$

Moreover, it follows from elementary calculations and a classical property of Fourier transform that

$$\widehat{\psi}(t) = \big(4^{-1}\, t\big)^{-2} \sin^2(4^{-1}\, t), \quad \text{for every } t \in \mathbb{R}, \tag{2.6}$$

which in particular implies that there is a finite deterministic constant $c > 0$ such that

$$\big|\widehat{\psi}(t)\big| \leq c\big(1 + |t|\big)^{-2}, \quad \text{for every } t \in \mathbb{R}. \tag{2.7}$$

For all $(j, k) \in \mathbb{N}^2$, the random variable $Y_{j,k}$ is defined through the pathwise Lebesgue integral

$$Y_{j,k} := \frac{2^{1-j}}{2\pi} \int_{\mathbb{R}} \cos(2^{-j}kt)\widehat{\psi}(2^{-j}t)X(t)\, dt\,. \tag{2.8}$$

Notice that the fact that the latter integral is well-defined and finite results from (2.7), (1.5) and continuity of sample paths of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$.

Lemma 2.2 *For all $(j, k) \in \mathbb{N}^2$, $Y_{j,k}$ (defined in (2.8)) is a real-valued SαS random variable which can almost surely be expressed as*

$$Y_{j,k} = \operatorname{Re}\Big(\int_{\mathbb{R}} \frac{\widetilde{\psi}_{j,k}(\xi)}{|\xi|^{H+1/\alpha}}\, d\widetilde{M}_\alpha(\xi)\Big), \tag{2.9}$$

where

$$\widetilde{\psi}_{j,k}(\xi) := \psi(2^j\xi + k) + \psi(2^j\xi - k), \quad \textit{for every } \xi \in \mathbb{R}. \tag{2.10}$$

The following remark is very fundamental.

Remark 2.3 One knows from (2.10) and (2.4) that, for all $(j,k) \in \mathbb{N}^2$, $\widetilde{\psi}_{j,k}$ is an even piecewise linear continuous function from $\mathbb{R}$ into $[0,2]$ with compact support such that

$$I_{j,k} := \operatorname{supp} \widetilde{\psi}_{j,k} = \left[\frac{-k-2^{-1}}{2^j}, \frac{-k+2^{-1}}{2^j}\right] \bigcup \left[\frac{k-2^{-1}}{2^j}, \frac{k+2^{-1}}{2^j}\right]. \tag{2.11}$$

Then, in view of the fact that $k \geq 1$, it turns out that the function $\xi \mapsto |\xi|^{-H-1/\alpha}\, \widetilde{\psi}_{j,k}(\xi)$ belongs to the Lebegue space $L^\alpha(\mathbb{R})$, with guarantees that the SαS stochastic integral in (2.9) is well-defined.

More importantly, observe that, for each fixed $j \in \mathbb{N}$, since (see (2.11)) the supports of the functions $\widetilde{\psi}_{j,k}$, $k \in \mathbb{N}$, are disjoint (up to Lebesgue negligible sets), the corresponding (via (2.9)) SαS random variables $Y_{j,k}$, $k \in \mathbb{N}$ are *independent*.

Proof of Lemma 2.2 Throughout the proof the positive integers j and k are arbitrary and fixed. First observe that, one can derive from (2.10) and standard calculations that the Fourier transform of $\widetilde{\psi}_{j,k}$ is given by

$$\widehat{\widetilde{\psi}}_{j,k}(t) = 2^{1-j} \cos(2^{-j}kt)\widehat{\psi}(2^{-j}t) \quad \text{for all } t \in \mathbb{R}. \tag{2.12}$$

Thus, (2.8) reduces to

$$Y_{j,k} = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\widetilde{\psi}}_{j,k}(t) X(t)\, dt\,. \tag{2.13}$$

Moreover, one can derive from (2.12) and (2.7), that, for some finite deterministic constant $c_1(j) > 0$, only depending on j, one has

$$\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big| \leq c_1(j)\big(1+|t|\big)^{-2}, \quad \text{for every } t \in \mathbb{R}. \tag{2.14}$$

For all $(n,m) \in \mathbb{N} \times \mathbb{Z}$, the dyadic number $d_{n,m} := 2^{-n}\, m$. Having introduced the latter notation, for each $n \in \mathbb{N}$, the random variable $Y^n_{j,k}$ is defined as the finite sum:

$$\begin{aligned} Y^n_{j,k} &:= \frac{1}{2\pi} \sum_{|m| \leq 4^n} X(d_{n,m}) \int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widetilde{\psi}}_{j,k}(t)\, dt \qquad (2.15) \\ &= \operatorname{Re}\left(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \left(\frac{1}{2\pi} \sum_{|m| \leq 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{i d_{n,m}\xi} - 1\big) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt \right) d\widetilde{M}_\alpha(\xi) \right), \end{aligned}$$

where the last equality follows from (1.1).

First step: we show that, when n goes to $+\infty$, the random variable $Y^n_{j,k}$ converges almost surely to the random random variable $Y_{j,k}$ defined through (2.13).

Let N be an arbitrary fixed positive integer. One sets

$$Y^N_{j,k} := \frac{1}{2\pi} \int_{-N}^{N} \widehat{\widetilde{\psi}}_{j,k}(t) X(t)\, dt = \frac{1}{2\pi} \sum_{m=-N2^n}^{N2^n-1} \int_{d_{n,m}}^{d_{n,m+1}} X(t) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt \tag{2.16}$$

and, for every integer $n \geq N$,

$$Y^{N,n}_{j,k} := \frac{1}{2\pi} \sum_{m=-N2^n}^{N2^n-1} X(d_{n,m}) \int_{d_{n,m}}^{d_{n,m+1}} \widehat{\widetilde{\psi}}_{j,k}(t)\, dt. \tag{2.17}$$

Then, using (1.4) with $T = N$, and the fact that $d_{n,m} \in [-N, N]$ when $-N2^n \leq m \leq N2^n - 1$, one gets, on an event of probability 1 depending only on N and denoted by Ω_N^*, that

$$\begin{aligned}
\left|Y_{j,k}^N - Y_{j,k}^{N,n}\right| &\leq \sum_{m=-N2^n}^{N2^n-1} \int_{d_{n,m}}^{d_{n,m+1}} \left|X(t) - X(d_{n,m})\right| \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt \\
&\leq C_{N,\delta}^* \sum_{m=-N2^n}^{N2^n-1} \int_{d_{n,m}}^{d_{n,m+1}} |t - d_{n,m}|^{H-\delta} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt \\
&\leq \left(C_{N,\delta}^* \int_{\mathbb{R}} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt\right) 2^{-n(H-\delta)}, \qquad (2.18)
\end{aligned}$$

where δ is an arbitrarily small fixed positive real number, and $C_{N,\delta}^*$ is a positive finite random variable, only depending on N and δ. Next, let Ω^* be the event of probability 1 defined as $\Omega^* := \bigcap_{N\in\mathbb{N}} \Omega_N^*$. One can derive from (2.18) that

$$\lim_{n\to+\infty} \left|Y_{j,k}^N(\omega) - Y_{j,k}^{N,n}(\omega)\right| = 0, \quad \text{for all } N \in \mathbb{N} \text{ and } \omega \in \Omega^*. \qquad (2.19)$$

On another hand, it follows from (1.5) that there are Ω^{**} an event of probability 1 and C_δ^{**} a positive finite random variable only depending on δ, such that

$$\left|X(t,\omega)\right| \leq C_\delta^{**}(\omega)|t|^{H+\delta}, \quad \text{for all } t \notin (-1,1) \text{ and } \omega \in \Omega^{**}. \qquad (2.20)$$

Combining (2.15) and (2.17) with (2.20) and (2.14), one obtains, for every $\omega \in \Omega^{**}$ and positive integers N and $n \geq N$,

$$\begin{aligned}
&\left|Y_{j,k}^n(\omega) - Y_{j,k}^{N,n}(\omega)\right| \\
&\leq \sum_{m=N2^n}^{+\infty} \left|X(d_{n,m},\omega)\right| \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt + \sum_{m=-\infty}^{-N2^n-1} \left|X(d_{n,m},\omega)\right| \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt \\
&\leq C_\delta^{**}(\omega) \left(\sum_{m=N2^n}^{+\infty} \left|d_{n,m}\right|^{H+\delta} \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt + \sum_{m=-\infty}^{-N2^n-1} \left|d_{n,m}\right|^{H+\delta} \int_{d_{n,m}}^{d_{n,m+1}} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt \right) \\
&\leq C_\delta^{**}(\omega) \left(\sum_{m=N2^n}^{+\infty} \int_{d_{n,m}}^{d_{n,m+1}} |t|^{H+\delta} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt + \sum_{m=-\infty}^{-N2^n-1} \int_{d_{n,m}}^{d_{n,m+1}} \left(1+|t|\right)^{H+\delta} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| dt \right) \\
&\leq 2c_1(j) C_\delta^{**}(\omega) \int_N^{+\infty} \left(1+t\right)^{H+\delta-2} dt \leq 2(1-H-\delta)^{-1} c_1(j) C_\delta^{**}(\omega) N^{-(1-H-\delta)}. \qquad (2.21)
\end{aligned}$$

Putting together (2.13), (2.16), (2.20) and (2.14), it follows that, for all $\omega \in \Omega^{**}$ and positive integers N, one has

$$\begin{aligned}
\left|Y_{j,k}(\omega) - Y_{j,k}^N(\omega)\right| &\leq \int_{\{|t|\geq N\}} \left|\widehat{\widehat{\psi}}_{j,k}(t)\right| \left|X(t)\right| dt \\
&\leq 2c_1(j) C_\delta^{**}(\omega) \int_N^{+\infty} \left(1+t\right)^{H+\delta-2} dt \\
&\leq 2(1-H-\delta)^{-1} c_1(j) C_\delta^{**}(\omega) N^{-(1-H-\delta)}. \qquad (2.22)
\end{aligned}$$

Finally, observe that, for all positive integers N and $n \geq N$, and for each $\omega \in \Omega^* \cap \Omega^{**}$ (the event $\Omega^* \cap \Omega^{**}$ is clearly of probability 1), using the triangle inequality, (2.21) and (2.22), one gets that

$$\begin{aligned} \big|Y_{j,k}(\omega) - Y_{j,k}^n(\omega)\big| &\leq \big|Y_{j,k}(\omega) - Y_{j,k}^N(\omega)\big| + \big|Y_{j,k}^N(\omega) - Y_{j,k}^{N,n}(\omega)\big| + \big|Y_{j,k}^{N,n}(\omega) - Y_{j,k}^n(\omega)\big| \\ &\leq 4(1-H-\delta)^{-1}\, c_1(j) C_\delta^{**}(\omega) N^{-(1-H-\delta)} + \big|Y_{j,k}^N(\omega) - Y_{j,k}^{N,n}(\omega)\big|. \end{aligned}$$

Thus, one can derive from (2.19) that, for all positive integers N and for each $\omega \in \Omega^* \cap \Omega^{**}$,

$$\limsup_{n\to+\infty} \big|Y_{j,k}(\omega) - Y_{j,k}^n(\omega)\big| \leq 4(1-H-\delta)^{-1}\, c_1(j) C_\delta^{**}(\omega) N^{-(1-H-\delta)}. \tag{2.23}$$

Finally, when N goes to $+\infty$, (2.23) implies that

$$\limsup_{n\to+\infty} \big|Y_{j,k}(\omega) - Y_{j,k}^n(\omega)\big| = 0, \quad \text{for all } \omega \in \Omega^* \cap \Omega^{**},$$

which shows that $Y_{j,k}^n$ converges almost surely to $Y_{j,k}$, when n tends to $+\infty$.

Second step: we show that for each fixed $\gamma \in (0, \alpha)$, when n goes to $+\infty$, the SαS random variable $Y_{j,k}^n$ (see (2.15)) converges in $L^\gamma(\Omega)$ (Ω being the underlying probability space) to the SαS random variable $\check{Y}_{j,k}$ defined as

$$\check{Y}_{j,k} := \mathrm{Re}\Big(\int_{\mathbb{R}} \frac{\widetilde{\psi}_{j,k}(\xi)}{|\xi|^{H+1/\alpha}} d\widetilde{M}_\alpha(\xi) \Big). \tag{2.24}$$

Observe that, since $\widetilde{\psi}_{j,k}$ is a continuous function on $\mathbb{R}$ (see (2.10) and (2.4)) belonging to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ whose Fourier transform $\widehat{\widetilde{\psi}}_{j,k}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as well (see (2.6)), one knows from Remark 2.1, and more particularly from (2.3) in it, that the function $\widetilde{\psi}_{j,k}$ can be expressed as

$$\widetilde{\psi}_{j,k}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} \widehat{\widetilde{\psi}}_{j,k}(t)\, dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.25}$$

Thus combining (2.25) with (2.11) and the fact that $k \geq 1$, one obtains that

$$\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\widetilde{\psi}}_{j,k}(t)\, dt = \widetilde{\psi}_{j,k}(0) = 0. \tag{2.26}$$

Then (2.25) and (2.26) imply that one has

$$\widetilde{\psi}_{j,k}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \big(e^{i\xi t} - 1\big) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.27}$$

Next, putting together the second equality in (2.15), (2.24) and (2.27), it follows that

$$R_{j,k}^n := Y_{j,k}^n - \check{Y}_{j,k}$$

is a SαS random variable which can be expressed as

$$\begin{aligned} R_{j,k}^n = \frac{1}{2\pi} \mathrm{Re} \Bigg(\int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Bigg(\sum_{|m| \leq 4^n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{i d_{n,m}\xi} - e^{it\xi}\big) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt \\ + \int_{\{t \notin [-2^n, 2^n + 2^{-n})\}} \big(e^{it\xi} - 1\big) \widehat{\widetilde{\psi}}_{j,k}(t)\, dt \Bigg) d\widetilde{M}_\alpha(\xi) \Bigg). \end{aligned} \tag{2.28}$$

Thus, in view of (3.3) in Remark 3.3 in the next section, for proving that $\lim_{n\to+\infty}\mathbb{E}\big(|R^n_{j,k}|^\gamma\big)=0$, for each fixed $\gamma\in(0,\alpha)$, it is enough to show that the scale parameter of the SαS random variable $R^n_{j,k}$ satisfies

$$\lim_{n\to+\infty}\sigma(R^n_{j,k})=0. \tag{2.29}$$

It follows from Remark 1.1, (2.25) and the inequality $|a+b|^\alpha\leq 2^\alpha\big(|a|^\alpha+|b|^\alpha\big)$, for all comlex numbers a and b, that, for every $n\in\mathbb{N}$,

$$\sigma(R^n_{j,k})^\alpha\leq 2^\alpha\big(\mathcal{U}^n_{j,k}+\mathcal{V}^n_{j,k}\big), \tag{2.30}$$

where

$$\mathcal{U}^n_{j,k}:=\int_{\mathbb{R}}|\xi|^{-\alpha H-1}\bigg(\sum_{|m|\leq 4^n}\int_{d_{n,m}}^{d_{n,m+1}}\big|e^{i(d_{n,m}-t)\xi}-1\big|\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\bigg)^\alpha d\xi \tag{2.31}$$

and

$$\mathcal{V}^n_{j,k}:=\int_{\mathbb{R}}|\xi|^{-\alpha H-1}\bigg(\int_{\{|t|\geq 2^n\}}\big|e^{it\xi}-1\big|\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\bigg)^\alpha d\xi. \tag{2.32}$$

For deriving appropriate upper bounds for $\mathcal{U}^n_{j,k}$ and $\mathcal{V}^n_{j,k}$, we will make use of the inequality

$$\big|e^{i\theta}-1\big|^\beta\leq\min\big\{|\theta|^\beta,2^\beta\big\},\quad\text{for all }\theta\in\mathbb{R}\text{ and }\beta\in(0,+\infty). \tag{2.33}$$

Let $B_0:=[-1,1]$ and $B_1:=\mathbb{R}\setminus B_0$. For $l\in\{0,1\}$, one sets

$$\mathcal{U}^{n,l}_{j,k}:=\int_{B_l}|\xi|^{-\alpha H-1}\bigg(\sum_{|m|\leq 4^n}\int_{d_{n,m}}^{d_{n,m+1}}\big|e^{i(d_{n,m}-t)\xi}-1\big|\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\bigg)^\alpha d\xi \tag{2.34}$$

and

$$\mathcal{V}^{n,l}_{j,k}:=\int_{B_l}|\xi|^{-\alpha H-1}\bigg(\int_{\{|t|\geq 2^n\}}\big|e^{it\xi}-1\big|\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\bigg)^\alpha d\xi. \tag{2.35}$$

In view of (2.31) and (2.32), one clearly has that

$$\mathcal{U}^n_{j,k}=\mathcal{U}^{n,0}_{j,k}+\mathcal{U}^{n,1}_{j,k}\quad\text{and}\quad\mathcal{V}^n_{j,k}=\mathcal{V}^{n,0}_{j,k}+\mathcal{V}^{n,1}_{j,k}. \tag{2.36}$$

Let us now bound $\mathcal{U}^{n,0}_{j,k}$ and $\mathcal{V}^{n,0}_{j,k}$. To this end, one introduces the positive finite constant c_2 defined as

$$c_2:=\int_{B_0}|\xi|^{\alpha 2^{-1}(1-H)-1}\,d\xi. \tag{2.37}$$

One can derive from (2.37), (2.33) with $\beta=2^{-1}(1+H)$, (2.34) and (2.35) with $l=0$, and the inequality

$$|t-d_{n,m}|\leq 2^{-n},\quad\text{for all }(n,m)\in\mathbb{N}\times\mathbb{Z}\text{ and }t\in[d_{n,m},d_{n,m+1}], \tag{2.38}$$

that one has, for every $n\in\mathbb{N}$,

$$\mathcal{U}^{n,0}_{j,k}\leq c_2\bigg(2^{2^{-1}(1-H)}\int_{\mathbb{R}}\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\bigg)^\alpha 2^{-n\alpha 2^{-1}(1+H)} \tag{2.39}$$

and

$$\mathcal{V}^{n,0}_{j,k}\leq c_2\bigg(2^{2^{-1}(1-H)}\int_{\{|t|\geq 2^n\}}\big|t\big|^{2^{-1}(1+H)}\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\bigg)^\alpha. \tag{2.40}$$

Moreover, one can derive from (2.14) that

$$c_3(j,k) := c_2\Big(2^{2^{-1}(1-H)}\int_{\mathbb{R}}\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\Big)^{\alpha} < +\infty \tag{2.41}$$

and that

$$\begin{aligned}&\int_{\{|t|\geq 2^n\}}|t|^{2^{-1}(1+H)}\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt \\ &\leq 2c_1(j)\int_{2^n}^{+\infty}\big(1+t\big)^{2^{-1}(H-3)}\,dt \leq 4(1-H)^{-1}c_1(j)2^{-n2^{-1}(1-H)}.\end{aligned} \tag{2.42}$$

Setting $c_4(j) := c_2\Big(2^{2^{-1}(1-H)}4(1-H)^{-1}c_1(j)\Big)^{\alpha}$, and putting together (2.39), (2.40), (2.41) and (2.42), one obtains, for all $n\in\mathbb{N}$, that

$$\mathcal{U}_{j,k}^{n,0} \leq c_3(j,k)2^{-n\alpha 2^{-1}(1+H)} \quad \text{and} \quad \mathcal{V}_{j,k}^{n,0} \leq c_4(j)2^{-n\alpha 2^{-1}(1-H)}. \tag{2.43}$$

Let us now bound $\mathcal{U}_{j,k}^{n,1}$ and $\mathcal{V}_{j,k}^{n,1}$. To this end, one introduces the two positive finite constants c_5 and c_6 defined as

$$c_5 := \int_{B_1}|\xi|^{-\alpha 2^{-1}H-1}\,d\xi \quad \text{and} \quad c_6 := \int_{B_1}|\xi|^{-\alpha H-1}\,d\xi. \tag{2.44}$$

One can derive from (2.44), (2.33) with $\beta = 2^{-1}H$ (for $\mathcal{U}_{j,k}^{n,1}$) and $\beta = 1$ (for $\mathcal{V}_{j,k}^{n,1}$), (2.34) and (2.35) with $l=1$, and the inequality (2.38), that one has, for every $n\in\mathbb{N}$,

$$\mathcal{U}_{j,k}^{n,1} \leq c_5\Big(2^{(1-2^{-1}H)}\int_{\mathbb{R}}\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\Big)^{\alpha}2^{-n\alpha 2^{-1}H} = c_5'(j,k)2^{-n\alpha 2^{-1}H} \tag{2.45}$$

and

$$\mathcal{V}_{j,k}^{n,1} \leq c_6\Big(2\int_{\{|t|\geq 2^n\}}\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\Big)^{\alpha} \leq c_6'(j)2^{-n\alpha}, \tag{2.46}$$

where $c_5'(j,k)$ and $c_6'(j)$ are two finite constants not depending on n. Notice that the last inequality in (2.46) is obtained similarly to (2.42),

Finally, putting together (2.30), (2.36), (2.43), (2.45) and (2.46), it follows that (2.29) is satisfied.

Conclusion: combining the main result obtained in the first step with the main one derived in the second step, it turns out that the almost sure equality (2.9) is valid. □

3 Proofs of the two main results

Lemma 3.1 *Under the sole condition (1.6) on the sequence $(m_j)_{j\in\mathbb{N}}$, one has*

$$\lim_{j\to+\infty}\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})} = 1, \quad \textit{for all fixed } \gamma\in(0,4^{-1}\underline{\alpha}), \tag{3.1}$$

where the convergences holds almost surely.

Lemma 3.2 *When the sequence $(m_j)_{j\in\mathbb{N}}$ satisfies the two conditions (1.6) and (1.9), or when it satisfies the two conditions (1.6) and (1.11), one has*

$$\lim_{j\to+\infty} \frac{V_{j,\log_2}^{m_j}}{\mathbb{E}(V_{j,\log_2}^{m_j})} = 1, \tag{3.2}$$

where the convergences holds almost surely.

For proving the two Lemmas 3.1 and 3.2, one needs several preliminary results.

Remark 3.3 For any fixed $\alpha \in (0,2]$, let Z be an arbitrary real-valued SαS random variable with scale parameter $\sigma(Z) \geq 0$. Then, for all $\gamma \in (0,\alpha)$, one has

$$\mathbb{E}\big(|Z|^\gamma\big) = c_0(\gamma,\alpha)\sigma(Z)^\gamma, \tag{3.3}$$

where the positive finite constant $c_0(\gamma,\alpha) := \mathbb{E}\big(|W|^\gamma\big)$, which only depends on γ and α, is the absolute moment of order γ of an arbitrary real-valued SαS random variable W with scale parameter equals to 1. Moreover, when $\sigma(Z) > 0$, letting $c_1(\alpha)$ be the finite constant only depending α defined as $c_1(\alpha) := \mathbb{E}\big(\log_2|W|\big)$, one has that

$$\mathbb{E}\big(\log_2(|Z|)\big) = \log_2(\sigma(Z)) + c_1(\alpha), \tag{3.4}$$

where $\log_2$ is the binary logarithm (that is $\log_2(2) = 1$).

Proof Using the fact that Φ_Z and Φ_W, the characteristic functions of the random variables Z and W, satisfy (see e.g. [19]), for all $\lambda \in \mathbb{R}$,

$$\Phi_Z(\lambda) = \exp\big(-\sigma(Z)^\alpha|\lambda|^\alpha\big) \quad \text{and} \quad \Phi_W(\lambda) = \exp\big(-|\lambda|^\alpha\big),$$

it turns out that the random variable Z has the same distribution as the random variable $\sigma(Z)W$. The two equalities (3.3) and (3.4) are straightforward consequences of the latter fact. □

Lemma 3.4 *Let $\nu(\alpha)$ the positive finite constant defined as*

$$\nu(\alpha) := \left(2\int_{\mathbb{R}} \big|\psi(\eta)\big|^\alpha d\eta\right)^{1/\alpha} = \left(2\int_{-2^{-1}}^{2^{-1}} \big|\psi(\eta)\big|^\alpha d\eta\right)^{1/\alpha}, \tag{3.5}$$

where the last equality results from the fact that $[-2^{-1},2^{-1}]$ is the support of ψ (see (2.4)). The scale parameter of $Y_{j,k}$ satisfies, for all $(j,k) \in \mathbb{N}^2$,

$$\nu(\alpha)2^{jH}\big(k+2^{-1}\big)^{-(H+1/\alpha)} \leq \sigma(Y_{j,k}) \leq \nu(\alpha)2^{jH}\big(k-2^{-1}\big)^{-(H+1/\alpha)}. \tag{3.6}$$

Proof One knows from (2.9), (2.10), Remark 1.1 and (2.4) that, for all $(j,k)\in\mathbb{N}^2$, one has

$$\sigma(Y_{j,k})^\alpha=\int_{2^{-j}(-k+2^{-1})}^{2^{-j}(-k-2^{-1})}\frac{\big|\psi(2^j\xi+k)\big|^\alpha}{|\xi|^{\alpha H+1}}\,d\xi+\int_{2^{-j}(k-2^{-1})}^{2^{-j}(k+2^{-1})}\frac{\big|\psi(2^j\xi-k)\big|^\alpha}{|\xi|^{\alpha H+1}}\,d\xi$$

Then, using the fact that ψ is an even function and the change of variable $\eta=2^j\xi-k$ i.e $\xi=2^{-j}(\eta+k)$, one gets that

$$\sigma(Y_{j,k})^\alpha=2\int_{2^{-j}(k-2^{-1})}^{2^{-j}(k+2^{-1})}\frac{\big|\psi(2^j\xi-k)\big|^\alpha}{(\xi)^{\alpha H+1}}\,d\xi=2^{j\alpha H+1}\int_{-2^{-1}}^{2^{-1}}\frac{\big|\psi(\eta)\big|^\alpha}{(\eta+k)^{\alpha H+1}}\,d\eta,$$

which implies that

$$2^{j\alpha H+1}\big(k+2^{-1}\big)^{-(\alpha H+1)}\int_{-2^{-1}}^{2^{-1}}\big|\psi(\eta)\big|^\alpha\,d\eta\le\sigma(Y_{j,k})^\alpha\le 2^{j\alpha H+1}\big(k-2^{-1}\big)^{-(\alpha H+1)}\int_{-2^{-1}}^{2^{-1}}\big|\psi(\eta)\big|^\alpha\,d\eta.$$

Thus, in view of (3.5), it turns out that (3.6) is valid. □

Lemma 3.5 *For each fixed $\gamma\in(0,4^{-1}\underline{\alpha})$, the two positive finite constants $\mu'(\gamma,\alpha)$ and $\mu''(\gamma,\alpha)$ are defined as*

$$\mu'(\gamma,\alpha):=\big(1-(3/5)^{1/4}\big)\nu(\alpha)^\gamma c_0(\gamma,\alpha)\quad\textit{and}\quad\mu'(\gamma,\alpha):=8\nu(\alpha)^\gamma c_0(\gamma,\alpha),\tag{3.7}$$

where $\nu(\alpha)$ and $c_0(\gamma,\alpha)$ are the same positive constants as in (3.5) and (3.3). Then, one has, for all $j\in\mathbb{N}$,

$$\mu'(\gamma,\alpha)\,2^{j\gamma H}\,m_j^{1-\gamma(H+1/\alpha)}\le\mathbb{E}\big(V_{j,\gamma}^{m_j}\big)\le\mu''(\gamma,\alpha)\,2^{j\gamma H}\,m_j^{1-\gamma(H+1/\alpha)}.\tag{3.8}$$

Notice that, the condition that $\gamma\in(0,4^{-1}\underline{\alpha})$ implies that

$$1>1-\gamma(H+1/\alpha)>4^{-1}.\tag{3.9}$$

Remark 3.6 Let $\gamma\in(0,4^{-1}\underline{\alpha})$ be arbitrary and fixed. Two straightforward consequences of Lemma 3.5 are the following.

(i) When the conditions (1.6) and (1.9) hold, one has

$$\lim_{j\to+\infty}\frac{\log_2\big(\mathbb{E}(V_{j,\gamma}^{m_j})\big)}{\gamma\,j}=H.\tag{3.10}$$

(ii) When the conditions (1.6) and (1.11) hold, one has

$$\lim_{j\to+\infty}\frac{\log_2\big(\mathbb{E}(V_{j,\gamma}^{m_j})\big)}{\gamma\,j}=\frac{1}{\gamma}-\frac{1}{\alpha}\,.\tag{3.11}$$

Proof of Lemma 3.5 First observe that, one can derive from (1.7), (3.3) and (3.6) that, for all $j \in \mathbb{N}$,

$$\nu(\alpha)^\gamma c_0(\gamma,\alpha) 2^{j\gamma H} \sum_{k=1}^{m_j} \left(k+2^{-1}\right)^{-\gamma(H+1/\alpha)} \leq \mathbb{E}\big(V_{j,\gamma}^{m_j}\big) \leq \nu(\alpha)^\gamma c_0(\gamma,\alpha) 2^{j\gamma H} \sum_{k=1}^{m_j} \left(k-2^{-1}\right)^{-\gamma(H+1/\alpha)}. \tag{3.12}$$

Moreover, using (3.9) and the inequality $m_j \geq 2$, one has that

$$\begin{aligned}
&\sum_{k=1}^{m_j} \left(k+2^{-1}\right)^{-\gamma(H+1/\alpha)} \geq \sum_{k=1}^{m_j} \int_k^{k+1} \left(x+2^{-1}\right)^{-\gamma(H+1/\alpha)} dx = \int_1^{m_j+1} \left(x+2^{-1}\right)^{-\gamma(H+1/\alpha)} dx \\
&\geq \Big(m_j+\frac{3}{2}\Big)^{1-\gamma(H+1/\alpha)} - \Big(\frac{3}{2}\Big)^{1-\gamma(H+1/\alpha)} \\
&\geq \Big(m_j+\frac{3}{2}\Big)^{1-\gamma(H+1/\alpha)} - \Big(\frac{3}{5}\Big)^{1-\gamma(H+1/\alpha)} \Big(m_j+\frac{3}{2}\Big)^{1-\gamma(H+1/\alpha)} \\
&\geq \big(1-(3/5)^{1/4}\big) m_j^{1-\gamma(H+1/\alpha)}
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\sum_{k=1}^{m_j} \left(k-2^{-1}\right)^{-\gamma(H+1/\alpha)} = 2\sum_{k=1}^{m_j} \int_{k-2^{-1}}^{k} \left(k-2^{-1}\right)^{-\gamma(H+1/\alpha)} dx \\
&\leq 2\sum_{k=1}^{m_j} \int_{k-2^{-1}}^{k} \left(x-2^{-1}\right)^{-\gamma(H+1/\alpha)} dx = 2\sum_{k=1}^{m_j} \int_{k-1}^{k-2^{-1}} x^{-\gamma(H+1/\alpha)} dx \\
&\leq 2\sum_{k=1}^{m_j} \int_{k-1}^{k} x^{-\gamma(H+1/\alpha)} dx = 2\int_0^{m_j} x^{-\gamma(H+1/\alpha)} dx \leq 8\, m_j^{1-\gamma(H+1/\alpha)}.
\end{aligned} \tag{3.14}$$

Putting together (3.12), (3.13) and (3.14), one obtains (3.8). □

Lemma 3.7 *The following two results hold*

(i) Under the conditions (1.6) and (1.9), one has that

$$\lim_{j\to+\infty} \frac{1}{j\, m_j} \mathbb{E}(V_{j,\log_2}^{m_j}) = H. \tag{3.15}$$

(ii) Under the conditions (1.6) and (1.11), one has that

$$\lim_{j\to+\infty} \frac{1}{j\, m_j} \mathbb{E}(V_{j,\log_2}^{m_j}) = \frac{1}{\alpha}. \tag{3.16}$$

Proof First notice that, in view of (3.4) and (1.12), the expectation $\mathbb{E}(V_{j,\log_2}^{m_j})$ can be expressed, for all $j \in \mathbb{N}$, as

$$\mathbb{E}(V_{j,\log_2}^{m_j}) = c_1(\alpha) m_j + \sum_{k=1}^{m_j} \log_2\big(\sigma(Y_{j,k})\big). \tag{3.17}$$

Also notice that (3.6) implies that, for every $j \in \mathbb{N}$,

$$(H+1/\alpha)\sum_{k=1}^{m_j}\log_2(k-2^{-1}) \leq H(j\,m_j)+m_j\log_2(\nu(\alpha))-\sum_{k=1}^{m_j}\log_2\big(\sigma(Y_{j,k})\big) \leq (H+1/\alpha)\sum_{k=1}^{m_j}\log_2(k+2^{-1}). \tag{3.18}$$

Since

$$\lim_{j\to+\infty}\frac{m_j}{j\,m_j}=0,$$

combining (3.17) and (3.18), it turns out that for proving the lemma it is enough to show that, for $p \in \{-1,1\}$, one has

$$\lim_{j\to+\infty}\frac{S_{p,m_j}}{m_j\log_2(m_j)}=1, \quad \text{for } p\in\{-1,1\}, \tag{3.19}$$

where

$$S_{p,m_j}:=\sum_{k=1}^{m_j}\log_2(k+p2^{-1}). \tag{3.20}$$

Observe that when the conditions imposed to the sequence $(m_j)_{j\in\mathbb{N}}$ in Part (i) of the lemma hold, then (3.19) implies that

$$\lim_{j\to+\infty}\frac{S_{p,m_j}}{j\,m_j}=0.$$

Also, observe that when the conditions imposed to the sequence $(m_j)_{j\in\mathbb{N}}$ in Part (ii) of the lemma hold, then (3.19) entails that

$$\lim_{j\to+\infty}\frac{S_{p,2^j}}{j\,m_j}=1.$$

From now on, one focuses on the proof of (3.19). Using the inequality $\big|\log_2(1+x)\big|\leq 4|x|$, for all $x\in[-2^{-1},+\infty)$, one obtains that

$$\Big|S_{p,m_j}-\sum_{k=2}^{m_j}\log_2(k)\Big|\leq\sum_{k=1}^{m_j}\Big|\log_2\Big(1+\frac{p2^{-1}}{k}\Big)\Big|\leq 2\sum_{k=1}^{m_j}\frac{1}{k}\leq 2+2\log(m_j).$$

Thus, for proving (3.19), it turns out that it is enough to show that

$$\lim_{j\to+\infty}\frac{1}{m_j\log_2(m_j)}\sum_{k=2}^{m_j}\log_2(k)=1. \tag{3.21}$$

One clearly has, for all $j\in\mathbb{N}$,

$$\int_1^{m_j}\log_2(x)\,dx\leq\sum_{k=2}^{m_j}\int_{k-1}^{k}\log_2(k)dx=\sum_{k=2}^{m_j}\log_2(k)=\sum_{k=2}^{m_j}\int_k^{k+1}\log_2(k)dx\leq\int_2^{m_j+1}\log_2(x)\,dx. \tag{3.22}$$

Moreover, since $x \mapsto \big(x\log(x)-x\big)/\log(2)$ is a primitive of $x \mapsto \log_2(x)$, elementary calculations show that

$$\int_1^{m_j} \log_2(x)\,dx = m_j \log_2(m_j) + \frac{1-m_j}{\log(2)} \tag{3.23}$$

and

$$\int_2^{m_j+1} \log_2(x)\,dx = (m_j+1)\log_2(m_j+1) + \frac{1-m_j}{\log(2)} - 2. \tag{3.24}$$

Finally, combining (3.23) and (3.24) with (3.22), one obtains (3.21). □

We are now in position to prove the Lemmas 3.1 and 3.2 as well as Theorems 1.2 and 1.3 which are the two main results of our article.

Proof of Lemma 3.1 First notice that using Markov inequality, for each $j \in \mathbb{N}$, one has

$$\begin{aligned}\mathbb{P}\Big(\Big|\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})}-1\Big| \geq m_j^{-\rho}\Big) &= \mathbb{P}\Big(\Big|V_{j,\gamma}^{m_j}-\mathbb{E}(V_{j,\gamma}^{m_j})\Big| \geq m_j^{-\rho}\,\mathbb{E}(V_{j,\gamma}^{m_j})\Big)\\ &\leq m_j^{4\rho} \times \frac{\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j}-\mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)}{\big(\mathbb{E}(V_{j,\gamma}^{m_j})\big)^4},\end{aligned} \tag{3.25}$$

where ρ is a fixed positive constant small enough, which will be chosen more precisely later.

Let now provide an appropriate upper bound for the expectation $\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j}-\mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)$. Noticing that, for each $(j,k) \in \mathbb{N}^2$, the SαS random variable $Y_{j,k}$ is equal in distribution to $\sigma(Y_{j,k})W$, where W is an arbitrary real-valued SαS random variable with scale parameter equals to 1, one gets, for $q \in \{1,2\}$, that

$$\mathbb{E}\Big(\big||Y_{j,k}|^\gamma - \mathbb{E}(|Y_{j,k}|^\gamma)\big|^{2q}\Big) = c_q\,\sigma(Y_{j,k})^{2q\gamma}\,, \tag{3.26}$$

where the positive finite constant $c_q := \mathbb{E}\Big(\big||W|^\gamma - \mathbb{E}(|W|^\gamma)\big|^{2q}\Big)$ does not depend on (j,k). Then, one can derive from (1.7), (3.26), and the fact that, for any fixed $j \in \mathbb{N}$, the centered random variables $|Y_{j,k}|^\gamma - \mathbb{E}(|Y_{j,k}|^\gamma)$, $k \in \mathbb{N}$, are independent (see the very fundamental Remark

2.3), and from Lemma 3.4 that

$$\begin{aligned}
&\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j}-\mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)=\sum_{k_1,\dots,k_4=1}^{m_j}\mathbb{E}\Big(\prod_{l=1}^{4}\big||Y_{j,k}|^\gamma-\mathbb{E}(|Y_{j,k}|^\gamma)\big|\Big)\\
&\le\left(\sum_{k=1}^{m_j}\mathbb{E}\Big(\big||Y_{j,k}|^\gamma-\mathbb{E}(|Y_{j,k}|^\gamma)\big|^4\Big)\right)+6\left(\sum_{k=1}^{m_j}\mathbb{E}\Big(\big||Y_{j,k}|^\gamma-\mathbb{E}(|Y_{j,k}|^\gamma)\big|^2\Big)\right)^2\\
&=c_2\left(\sum_{k=1}^{m_j}\sigma(Y_{j,k})^{4\gamma}\right)+6c_1^2\left(\sum_{k=1}^{m_j}\sigma(Y_{j,k})^{2\gamma}\right)^2\le c_3\left(\sum_{k=1}^{m_j}\sigma(Y_{j,k})^{2\gamma}\right)^2\\
&\le c_4\,2^{j4\gamma H}\left(\sum_{k=1}^{m_j}\big(k-2^{-1}\big)^{-2\gamma(H+1/\alpha)}\right)^2\\
&\le c_4\,2^{j4\gamma H}\left(4^{\gamma(H+1/\alpha)}+\int_1^{m_j}\big(x-2^{-1}\big)^{-2\gamma(H+1/\alpha)}\right)^2 \qquad (3.27)\\
&\le c_5\,2^{j4\gamma H}\left(1+\mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big)\log\big(m_j\big)+m_j^{1-2\gamma(H+1/\alpha)}\right)^2,
\end{aligned}$$

where c_3, c_4 and c_5 are three positive finite constants not depending on j. Next, combining (3.27) with the first inequality in (3.8), one gets, for all $j\in\mathbb{N}$,

$$\begin{aligned}
&m_j^{4\rho}\times\frac{\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j}-\mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)}{\big(\mathbb{E}(V_{j,\gamma}^{m_j})\big)^4}\\
&\le c_6\,m_j^{-4(1-\rho+\gamma(H+1/\alpha))}\left(1+\mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big)\log\big(m_j\big)+m_j^{1-2\gamma(H+1/\alpha)}\right)^2 \qquad (3.28)\\
&\le c_6\left(m_j^{-2(1-\rho+\gamma(H+1/\alpha))}+\mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big)m_j^{-2(1-\rho+\gamma(H+1/\alpha))}\log\big(m_j\big)+m_j^{-(1-2\rho)}\right)^2\\
&\le 3c_6\left(m_j^{-4(1-\rho+\gamma(H+1/\alpha))}+\mathbb{1}_{\{1\}}\big(2\gamma(H+1/\alpha)\big)m_j^{-4(1-\rho+\gamma(H+1/\alpha))}\log^2\big(m_j\big)+m_j^{-2(1-2\rho)}\right).
\end{aligned}$$

Next, notice that, in view of (3.9), the positive constant ρ can be chosen small enough so that one has

$$4\big(1-\rho-\gamma(H+1/\alpha)\big)>1\quad\text{and}\quad 2(1-2\rho)>1\,. \qquad (3.29)$$

Then, it follows from (3.28), (3.29), (1.6) and (3.25) that

$$\sum_{j=1}^{+\infty}\mathbb{P}\left(\Big|\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})}-1\Big|\ge m_j^{-\rho}\right)<+\infty.$$

Therefore, (3.1) results from Borel-Cantelli Lemma. □

Proof of Lemma 3.2 First notice that similarly (3.25), it can be shown that, for each $j\in\mathbb{N}$,

$$\mathbb{P}\left(\Big|\frac{V_{j,\log_2}^{m_j}}{\mathbb{E}(V_{j,\log_2}^{m_j})}-1\Big|\ge m_j^{-1/8}\right)\le m_j^{1/2}\times\frac{\mathbb{E}\Big(\big|V_{j,\log_2}^{m_j}-\mathbb{E}(V_{j,\log_2}^{m_j})\big|^4\Big)}{\big(\mathbb{E}(V_{j,\log_2}^{m_j})\big)^4}. \qquad (3.30)$$

Let now provide an appropriate upper bound for the expectation $\mathbb{E}\Big(\big|V_{j,\log_2}^{m_j} - \mathbb{E}(V_{j,\log_2}^{m_j})\big|^4\Big)$. Noticing that, for each $(j,k) \in \mathbb{N}^2$, the SαS random variable $Y_{j,k}$ is equal in distribution to $\sigma(Y_{j,k})W$, where W is an arbitrary real-valued SαS random variable with scale parameter equals to 1, one gets, for $q \in \{1,2\}$, that

$$\mathbb{E}\Big(\big|\log_2|Y_{j,k}| - \mathbb{E}(\log_2|Y_{j,k}|)\big|^{2q}\Big) = c_{0,q}\,, \tag{3.31}$$

where the positive finite constant $c_{0,q} := \mathbb{E}\Big(\big|\log_2|W| - \mathbb{E}(\log_2|W|)\big|^{2q}\Big)$ does not depend on (j,k). Then, one can derive from (1.12), (3.31), and the fact that, for any fixed $j \in \mathbb{N}$, the centered random variables $\log_2|Y_{j,k}| - \mathbb{E}(\log_2|Y_{j,k}|^\gamma)$, $k \in \mathbb{N}$, are independent (see the very fundamental Remark 2.3), that

$$\begin{aligned}
\mathbb{E}\Big(\big|V_{j,\log_2}^{m_j} - \mathbb{E}(V_{j,\log_2}^{m_j})\big|^4\Big) &= \sum_{k_1,\ldots,k_4=1}^{m_j} \mathbb{E}\Big(\prod_{l=1}^{4}\big|\log_2|Y_{j,k}| - \mathbb{E}(\log_2|Y_{j,k}|)\big|\Big) \\
&\leq \left(\sum_{k=1}^{m_j}\mathbb{E}\Big(\big|\log_2|Y_{j,k}| - \mathbb{E}(\log_2|Y_{j,k}|)\big|^4\Big)\right) + 6\left(\sum_{k=1}^{m_j}\mathbb{E}\Big(\big|\log_2|Y_{j,k}| - \mathbb{E}(\log_2|Y_{j,k}|)\big|^2\Big)\right)^2 \\
&= c_{0,2}\,m_j + 6c_{0,1}^2 m_j^2. \qquad (3.32)
\end{aligned}$$

Next, combining (3.32) with Lemma 3.7, and (3.30), under the assumptions of Lemma 3.2, it follows that

$$\sum_{j=1}^{+\infty}\mathbb{P}\left(\Big|\frac{V_{j,\log_2}^{m_j}}{\mathbb{E}(V_{j,\log_2}^{m_j})} - 1\Big| \geq m_j^{-1/8}\right) < +\infty.$$

Therefore, (3.2) results from Borel-Cantelli Lemma. □

Proof of Theorem 1.2 One clearly has, for all $\gamma \in (0, 4^{-1}\underline{\alpha})$ and for every $j \in \mathbb{N}$,

$$\frac{\log_2(V_{j,\gamma}^{m_j})}{\gamma\, j} = \frac{1}{\gamma\, j}\log_2\left(\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})}\right) + \frac{\log_2(\mathbb{E}(V_{j,\gamma}^{m_j}))}{\gamma\, j}. \tag{3.33}$$

On one hand, when the conditions (1.6) and (1.9) hold, one can derive from (3.33), (3.1) and (3.10) that

$$\lim_{j\to+\infty}\frac{\log_2(V_{j,\gamma}^{m_j})}{\gamma\, j} = H\,, \quad \text{(almost sure convergence)},$$

which means that $\widehat{H}_{j,\gamma}$, defined in (1.8), is a strongly consistent estimator of the Hurst parameter H. On another hand, when the conditions (1.6) and (1.11) are satisfied, it follows from (3.33), (3.1) and (3.11) that

$$\lim_{j\to+\infty}\frac{\log_2(V_{j,\gamma}^{m_j})}{\gamma\, j} = \frac{1}{\gamma} - \frac{1}{\alpha}\,, \quad \text{(almost sure convergence)},$$

which means that $\widehat{\alpha}_{j,\gamma}$, defined in (1.10), is a strongly consistent estimator of the stability parameter α. □

Proof of Theorem 1.3 One clearly has, for all j large enough,

$$\frac{V_{j,\log_2}^{m_j}}{j\, m_j} = \frac{\mathbb{E}(V_{j,\log_2}^{m_j})}{j\, m_j} \times \frac{V_{j,\log_2}^{m_j}}{\mathbb{E}(V_{j,\log_2}^{m_j})}\,. \tag{3.34}$$

On one hand, when the conditions (1.6) and (1.9) hold, one can derive from (3.34), (3.2) and (3.15) that

$$\lim_{j\to+\infty} \frac{V_{j,\log_2}^{m_j}}{j\, m_j} = H\,, \quad \text{(almost sure convergence)},$$

which means that $\widehat{H}_{j,\log_2}$, defined in (1.13), is a strongly consistent estimator of the Hurst parameter H. On another hand, when the conditions (1.6) and (1.11) are satisfied, it follows from (3.34), (3.2) and (3.16) that

$$\lim_{j\to+\infty} \frac{V_{j,\log_2}^{m_j}}{j\, m_j} = \frac{1}{\alpha}\,, \quad \text{(almost sure convergence)},$$

which means that $\widehat{\alpha}_{j,\log_2}$, defined in (1.14), is a strongly consistent estimator of the stability parameter α. □

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