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Abstract

We consider a semilinear wave equation involving a time-dependent structural damping term of the form $\frac{1}{(1+t)^{\beta}}(-\Delta)^{\sigma/2}u_t$. Our results show the influence of the parameters β, σ on the nonexistence of global weak solutions under assumptions on the given system data.

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1. Introduction

The aim of the paper is to establish a nonexistence of global weak solutions to the Cauchy problem for the following semilinear structurally damped wave equation

$$\begin{cases} u_{tt} - \Delta u + \dfrac{b_0}{(1+t)^{\beta}}(-\Delta)^{\sigma/2}u_t = |u|^p & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $0 < \sigma < 2$, $n \geq 1$, $p > 1$, b_0 is a positive constant, and $\beta \in \mathbb{R}$. Without loss of generality, we assume that $b_0 = 1$.

Prior to our main blow-up results, we would like to emphasize that the special case of Eq. (1) for $b_0 = 0$ is governed by the classical semilinear wave equation, where the Strauss conjecture for this case states that: if $p \leq p_S$ then there is no global solution for (1) under suitable sign assumptions on the initial data, and for $p > p_S$ a global solution exists for small initial data; see e.g. [10, 14, 16, 18] among many other references, where p_S is the positive solution of the following quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$, and is given by

$$p_S = p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}.$$

In order to have more general initial data, but still with compact support, Kato [11] obtained with a slightly less sharp blow-up result an exponent of the form $(n+1)/(n-1)$ which is less than the Strauss exponent p_S, for $n \geq 2$.
We would like to emphasize here that the test function method, introduced by [17] and used by [8, 9, 12], plays a similar role as of Kato's method in the proof of blow-up results. In fact, the test function is effective in the case of parabolic equations as it provide exactly the critical exponent p_c, but in the case of hyperbolic equations (cf. [12]) we

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get the so-called Kato's exponent p^* which is less than p_c. This is one of the weakness of the test function method but in general it can be applied to a more general equation and system with no positive condition on solutions.

When $\beta = 0$, $\sigma \to 2$, and $b_0 = 1$, problem (1) is reduced to

$$u_{tt} - \Delta u - \Delta u_t = |u|^p, \quad x \in \mathbb{R}^n, t > 0, \tag{2}$$

which is called the viscoelastic damping case. D'Ambrosio and Lucente [3, Theorem 4.2] proved that the solution of (2) blows-up in finite time when $1 < p \le (n+1)/(n-1)_+$, where $(\cdot)_+ := \max\{0, \cdot\}$, by applying the test function method. Similar result has been obtained recently by Fino [6] in the case of an exterior domain. On the other hand, D'Abbicco-Reissig [2] proved that there exists a global solution for (2) when $p > 1 + \frac{3}{n-1}$ $(n \ge 2)$ for sufficiently small initial data. Therefore, the exact value of the critical exponent is still an open question.

Recently, problem (1) with $\sigma \to 2$,

$$u_{tt} - \Delta u - \frac{b_0}{(1+t)^\beta}\Delta u_t = |u|^p, \quad x \in \mathbb{R}^n, t > 0, \tag{3}$$

has been studied by Fino & Hamza [7]. They proved that if $\int_{\mathbb{R}^n} u_1(x)dx > 0$,

$$\begin{cases} p \in (1, \infty) & \text{for } n = 1, 2, \\ p \in (1, \frac{n}{n-2}] & \text{for } n \ge 3, \end{cases}$$

and

$$\begin{cases} 1 < p \le \frac{n+1}{(n-1)_+} & \text{if } \beta \ge -1, \\ 1 < p \le \frac{n(1-\beta)+2}{(n(1-\beta)-2)_+} & \text{if } \beta \le -1, \end{cases}$$

then the mild solution of (3) blows-up in finite time.

The goal of this paper is to prove the nonexistence of global weak solutions of (1) which is a generalization of the results of [7] to the case of fractional Laplacian damping term by using a recent version of the test function method that relies on Lemma 1 below. The novelty of this paper is the choice of this test function which is consistent with the suppression of positivity usually assumed by many authors.

In the case of $\beta \ne 0$ and $b_0 = 1$, we give an intuitive observation for understanding the influence of the damping term $(\frac{1}{(1+t)^\beta}(-\Delta)^{\sigma/2}u_t)$ by scaling argument. Let $u(t,x)$ be a solution of the linear strong damped wave equation

$$u_{tt}(t,x) - \Delta u(t,x) + \frac{1}{(1+t)^\beta}(-\Delta)^{\sigma/2}u_t(t,x) = 0. \tag{4}$$

When $\sigma \ge 1 - \beta$, we put

$$u(t,x) = v(\lambda(1+t), \lambda x), \qquad \lambda(1+t) = s, \ \lambda x = y, \tag{5}$$

with a parameter $\lambda > 0$, we have

$$v_{ss}(s,y) - \Delta v(s,y) + \frac{\lambda^{\sigma+\beta-1}}{s^\beta}(-\Delta)^{\sigma/2}v_s(s,y) = 0.$$

Thus, when $\sigma = 1 - \beta$ we notice that Eq. (4) is invariant, while when $\sigma > 1 - \beta$, letting $\lambda \to 0$, we obtain the wave equation without damping

$$v_{ss}(s,y) - \Delta v(s,y) = 0.$$

We note that $\lambda \to 0$ is corresponding to $t \to +\infty$.
On the other hand, when $\sigma < 1-\beta$, we put

$$u(t,x) = v(\lambda^{\frac{\sigma}{1-\beta}}(1+t), \lambda x), \qquad \lambda^{\frac{2}{1-\beta}}(1+t) = s, \ \lambda x = y,$$

with a parameter $\lambda > 0$, we have

$$v_{ss}(s,y) + \frac{1}{s^\beta}(-\Delta)^{\sigma/2} v_s(s,y) - \lambda^{\frac{2(1-\beta-\sigma)}{1-\beta}} \Delta v(s,y) = 0.$$

In this case, letting $\lambda \to 0$, we obtain the pseudo-parabolic equation

$$v_{ss}(s,y) + \frac{1}{s^\beta}(-\Delta)^{\sigma/2} v_s(s,y) = 0.$$

This means that the asymptotic behaviour of solutions depends on the behaviour of the coefficient of the damping term.

This paper is organized as follows. We start in Sec. 2 by stating the main theorem of our work and we prove it in Sec. 4. Sec. 3 is to collect some preliminaries.

2. Main results

This section is aimed to state our main results.

Theorem 1 (**Blow-up**). *We assume that*

$$(u_0, u_1) \in (L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)) \times (L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$$

satisfying the following condition:

$$\int_{\mathbb{R}^n} u_1(x)dx > 0. \tag{6}$$

If

$$\begin{cases} 1 < p < \min\left\{ \dfrac{n+1}{(n-1)_+}; \dfrac{n}{(n-\sigma)_+} \right\} & \text{if } \sigma \geq 1-\beta, \\ 1 < p < \min\left\{ \dfrac{n(1-\beta)+\sigma}{(n(1-\beta)-\sigma)_+}; \dfrac{n}{(n-\sigma)_+} \right\} & \text{if } \sigma \leq 1-\beta, \end{cases} \tag{7}$$

or

$$\begin{cases} p = \dfrac{n+1}{n-1}, & \text{if } \sigma \geq 1-\beta, \ \sigma > \dfrac{2n}{n+1}, \ n \geq 2, \\ p = \dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}, & \text{if } n(1+\beta) < \sigma \leq 1-\beta, \ n \geq 1, \end{cases}$$

where $(\cdot)_+ := \max\{0, \cdot\}$*, then problem* (1) *has no global weak solutions.*

Remark 1. *We note that, by taking the limit case* $\sigma \to 2$ *we recover the same results of [7].*

Remark 2. *We stress that the exponent* $\dfrac{n+1}{(n-1)_+}$ *appearing in* (7) *was introduced first in [11] to prove the nonexistence of global solutions to the semilinear wave equation with the nonlinearity* $|u|^p$*, subject to small initial data with compact support.*

3. Preliminaries

Definition 1. *[13]*
Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$ and $s \in (0,1)$. The fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ is a non-local operator defined on $\mathcal{S}$ by

$$
\begin{aligned}
(-\Delta)^s v(x) \quad &:= \quad C_{n,s}\ p.v. \int_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}}\, dy \\
&= \quad \begin{cases} C_{n,s} \displaystyle\int_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}}\, dy, & \text{if } 0 < s < 1/2, \\ \\ C_{n,s} \displaystyle\int_{\mathbb{R}^n} \frac{v(x)-v(y)-\nabla v(x)\cdot(x-y)\mathcal{X}_{|x-y|<\delta}(y)}{|x-y|^{n+2s}}\, dy, \quad \forall\, \delta > 0, & \text{if } 1/2 \le s < 1, \end{cases}
\end{aligned}
$$

where p.v. stands for Cauchy's principal value, and $C_{n,s} := \dfrac{s\, 4^s \Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(1-s)}$.

In fact, we are rarely going to use the fractional Laplacian operator in the Schwartz space; it can be extended to less regular functions as follows: for $s \in (0,1)$, $\varepsilon > 0$, let

$$
\mathcal{L}_{s,\varepsilon}(\Omega) \quad := \quad \begin{cases} L_s(\mathbb{R}^n) \cap C^{0,2s+\varepsilon}(\Omega) & \text{if } 0 < s < 1/2, \\ \\ L_s(\mathbb{R}^n) \cap C^{1,2s+\varepsilon-1}(\Omega), & \text{if } 1/2 \le s < 1, \end{cases}
$$

where Ω be an open subset of $\mathbb{R}^n$, $C^{0,2s+\varepsilon}(\Omega)$ is the space of $2s+\varepsilon$- Hölder continuous functions on Ω, $C^{1,2s+\varepsilon-1}(\Omega)$ the space of functions of $C^1(\Omega)$ whose first partial derivatives are Hölder continuous with exponent $2s+\varepsilon-1$, and

$$
L_s(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} \quad \text{such that} \quad \int_{\mathbb{R}^n} \frac{u(x)}{1+|x|^{n+2s}}\, dx < \infty \right\}.
$$

Proposition 1. *[13, Proposition 2.4]*
Let Ω be an open subset of $\mathbb{R}^n$, $s \in (0,1)$, and $f \in \mathcal{L}_{s,\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then $(-\Delta)^s f$ is a continuous function in Ω and $(-\Delta)^s f(x)$ is given by the pointwise formulas of Definition 1 for every $x \in \Omega$.

Remark: A simple sufficient condition for function f to satisfy the conditions in Proposition 1 is that $f \in L^1_{loc}(\mathbb{R}^n) \cap C^2(\Omega)$.

Using [4, Lemma 2.11] and its proof, we have the following.

Lemma 1. *Let $\langle x \rangle := (1 + (|x|-1)^4)^{1/4}$ for all $x \in \mathbb{R}^n$, $n \ge 1$. Let $s \in (0,1]$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$
\phi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ \\ \langle x \rangle^{-n-2s} & \text{if } |x| \ge 1. \end{cases} \tag{8}
$$

Then

$$
\phi \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap H^2(\mathbb{R}^n), \qquad \partial_x^2 \phi \in L^\infty(\mathbb{R}^n),
$$

and the following estimate holds:

$$
\max\left\{ |\Delta\phi(x)|\,,\, |(-\Delta)^s \phi(x)| \right\} \le C\, \phi(x) \quad \text{for all } x \in \mathbb{R}^n.
$$

Lemma 2. *Let h be a smooth function satisfying $\partial_x^2 h \in L^\infty(\mathbb{R}^n)$. For any $R > 0$, let h_R be a function defined by*

$$
h_R(x) := h(R^{-1}x) \quad \text{for all } x \in \mathbb{R}^n.
$$

Then, $(-\Delta)^s(h_R)$, $s \in (0,1]$, satisfies the following scaling property

$$
(-\Delta)^s h_R(x) = R^{-2s}(-\Delta)^s h(R^{-1}x), \quad \text{for all } x \in \mathbb{R}^n.
$$

Lemma 3. *[4, Lemma 2.14]*
Let $s, \widetilde{s} \in (0,1]$, $R > 0$ and $p > 1$. Then, the following estimate holds

$$\int_{\mathbb{R}^n} (\phi_R(x))^{-\frac{1}{p-1}} \left|(-\Delta)^{\widetilde{s}} \phi_R(x)\right|^{\frac{p}{p-1}} dx \leq C\, R^{-\frac{2\widetilde{s}p}{p-1}+n},$$

where $\phi_R(x) := \phi(x/R)$ and ϕ as defined in (8).

4. Proof of Theorem 1

The proof of Theorem 1 relies mainly on the concept of weak solution of the Cauchy problem (1) and the use of the test function method. Let

$$X_{\delta,T} = \{\varphi \in C([0,\infty), H^2(\mathbb{R}^n)) \cap C^1([0,\infty), H^\delta(\mathbb{R}^n)) \cap C^2([0,\infty), L^2(\mathbb{R}^n)), \text{such that supp}\varphi \subset Q_T\},$$

where $Q_T := [0,T] \times \mathbb{R}^n$, and the homogeneous fractional Sobolev space $H^\delta(\mathbb{R}^n)$, $\delta \in (0,2)$, is defined by

$$H^\delta(\mathbb{R}^n) = \begin{cases} \{u \in L^2(\mathbb{R}^n); (-\Delta)^{\delta/2} u \in L^2(\mathbb{R}^n)\}, & \text{if } \delta \in (0,1), \\ H^1(\mathbb{R}^n), & \text{if } \delta = 1, \\ \{u \in H^1(\mathbb{R}^n); (-\Delta)^{\delta/2} u \in L^2(\mathbb{R}^n)\}, & \text{if } \delta \in (1,2), \end{cases}$$

endowed with the norm

$$\|u\|_{H^\delta(\mathbb{R}^n)} = \begin{cases} \|u\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{\delta/2} u\right\|_{L^2(\mathbb{R}^n)}, & \text{if } \delta \in (0,1), \\ \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)}, & \text{if } \delta = 1, \\ \|u\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2(\mathbb{R}^n)} + \left\|(-\Delta)^{\delta/2} u\right\|_{L^2(\mathbb{R}^n)}, & \text{if } \delta \in (1,2). \end{cases}$$

The weak formulation associated with (1) reads as follows:

Definition 2. *(Weak solution)*
Let $T > 0$, and $u_0, u_1 \in L^2(\mathbb{R}^n)$. A function

$$u \in L^1((0,T), L^2(\mathbb{R}^n)) \cap L^p((0,T); L^{2p}(\mathbb{R}^n)),$$

is said to be a weak solution of (1) *on* $[0,T) \times \mathbb{R}^n$ *if*

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi \, dt\, dx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\ &+ \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{\sigma/2}\varphi(0,x)\, dx - \int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\, dx \\ &= \int_{Q_T} u\varphi_{tt}\, dt\, dx - \int_0^T \frac{1}{(1+t)^\beta} \int_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi_t\, dx\, dt \\ &- \int_{Q_T} u\Delta\varphi\, dt\, dx + \int_0^T \frac{\beta}{(1+t)^{\beta+1}} \int_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi\, dx\, dt, \end{aligned}$$

holds for all $\varphi \in X_{\sigma,T}$. We denote the lifespan for the weak solution by

$$T_w(u_0,u_1) := \sup\{T \in (0,\infty] \ \textit{for which there exists a unique weak solution } u \textit{ to (1)}\}.$$

Moreover, if $T > 0$ can be arbitrary chosen, i.e. $T_w(u_0,u_1) = \infty$, then u is called a global weak solution of (1).

Proof of Theorem 1. Let u be a global weak solution of (1), that is

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi \, dt\, dx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\
&+ \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{\sigma/2}\varphi(0,x)\, dx - \int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\, dx \\
&= \int_{Q_T} u\varphi_{tt}\, dt\, dx - \int_0^T \frac{1}{(1+t)^\beta} \int_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi_t\, dx\, dt \\
&- \int_{Q_T} u\Delta\varphi\, dt\, dx + \int_0^T \frac{\beta}{(1+t)^{\beta+1}} \int_{\mathbb{R}^n} u\,(-\Delta)^{\sigma/2}\varphi\, dx\, dt,
\end{aligned} \tag{9}$$

for all $T > 0$, and all $\varphi \in X_{\sigma,T}$.

Let $T > 0$. Now, we introduce the following test function

$$\varphi(t,x) = \psi^\eta(t)\phi_{T^d}(x) = \psi^\eta(t)\phi\left(\frac{x}{T^d}\right) \tag{10}$$

where ϕ is defined in (8), $\psi(t) = \Psi(\frac{t}{T})$, $\eta \gg 1$, $d > 0$ are constants that will be determined later, and $\Psi \in C^\infty(\mathbb{R}_+)$ is a cut-off non-increasing function such that

$$\Psi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1/2, \\ \searrow & \text{if } 1/2 \le r \le 1, \\ 0 & \text{if } r \ge 1. \end{cases}$$

From the formulation (9), we get the following inequality

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi \, dt\, dx + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\
&\le \int_{\frac{T}{2}}^T \int_{\mathbb{R}^n} |u|\,|\varphi_{tt}|\, dx\, dt + \int_{Q_T} |u|\,|\Delta\varphi|\, dt\, dx \\
&+ \int_{\frac{T}{2}}^T \frac{1}{(1+t)^\beta} \int_{\mathbb{R}^n} |u||(-\Delta)^{\sigma/2}\varphi_t|\, dx\, dt + \int_0^T \frac{\beta}{(1+t)^{\beta+1}} \int_{\mathbb{R}^n} |u|\,|(-\Delta)^{\sigma/2}\varphi|\, dx\, dt \\
&+ \int_{\mathbb{R}^n} |u_0|\left(|(-\Delta)^{\sigma/2}\varphi(0,x)| + |\varphi_t(0,x)|\right) dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{11}$$

Let $\varepsilon > 0$. By applying ε-Young's inequality

$$AB \le \varepsilon A^p + C(\varepsilon,p)B^{p'},\ A \ge 0,\ B \ge 0,\ p + p' = pp',\ C(\varepsilon,p) = (p-1)(\varepsilon\, p^p)^{-1/(p-1)},$$

we obtain the estimation for the first integral

$$\begin{aligned}
I_1 &\le \int_{Q_T} |u|\varphi^{1/p}\varphi^{-1/p}\phi_{T^d}|(\psi^\eta)_{tt}|\, dt\, dx \\
&\le \varepsilon \int_{Q_T} |u|^p \varphi\, dt\, dx + C\int_{Q_T} \phi_{T^d}\psi^{-\eta/(p-1)}|(\psi^\eta)_{tt}|^{p'}\, dt\, dx.
\end{aligned} \tag{12}$$

As $(\psi^\eta)_{tt} = \eta\psi^{\eta-1}\psi_{tt} + \eta(\eta-1)\psi^{\eta-2}|\psi_t|^2$, the inequality (12) becomes

$$I_1 \le \varepsilon \int_{Q_T} |u|^p \psi\, dt\, dx + C\int_{Q_T} \phi_{T^d}\psi^{\eta-p'}|\psi_{tt}|^{p'}\, dt\, dx + C\int_{Q_T} \phi_{T^d}\psi^{\eta-2p'}|\psi_t|^{2p\prime}\, dt\, dx. \tag{13}$$

Proceeding similarly as for (13), we get

$$I_2 \leq \varepsilon \int_{Q_T} |u|^p \varphi \, dt \, dx + C \int_{Q_T} \psi^\eta \phi_{T^d}^{-1/(p-1)} |\Delta \phi_{T^d}|^{p'} \, dt \, dx. \tag{14}$$

In the same way, we write

$$\begin{aligned} I_3 &= \int_{\frac{T}{2}}^T \frac{1}{(1+t)^\beta} \int_{\mathbb{R}^n} |u| \varphi^{1/p} \varphi^{-1/p} \, |(-\Delta)^{\sigma/2} \phi_{T^d}| |(\psi^\eta)_t| \, dx \, dt \\ &\leq \varepsilon \int_{Q_T} |u|^p \varphi \, dt \, dx \\ &\quad + C \int_{\frac{T}{2}}^T \frac{1}{(1+t)^{\beta p'}} \int_{\mathbb{R}^n} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^{\eta - p'} \, |\psi_t|^{p'} \, dx \, dt. \end{aligned} \tag{15}$$

Clearly,

$$\frac{1}{(1+t)^{\beta p'}} \leq C \, T^{-\beta p'}, \qquad \forall \, t \in \left(\frac{T}{2}, T \right),$$

therefore,

$$I_3 \leq \varepsilon \int_{Q_T} |u|^p \varphi \, dt \, dx + C \, T^{-\beta p'} \int_{Q_T} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^{\eta - p'} \, |\psi_t|^{p'} \, dt \, dx. \tag{16}$$

In the same manner,

$$\begin{aligned} I_4 &\leq C \int_0^T \frac{1}{(1+t)^{\beta+1}} \int_{\mathbb{R}^n} |u| \varphi^{1/p} \varphi^{-1/p} \, \psi^\eta \, |(-\Delta)^{\sigma/2} \phi_{T^d}| \, dx \, dt \\ &\leq \varepsilon \int_{Q_T} |u|^p \varphi \, dt \, dx + C \int_0^T \frac{1}{(1+t)^{(\beta+1)p'}} \int_{\mathbb{R}^n} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^\eta \, dx \, dt. \end{aligned} \tag{17}$$

Finally, it remains only to control the term I_5. By exploiting the identity $\varphi_t(0,x) = \eta \psi_t(0) \phi_{T^d}(x)$, we infer

$$I_5 \leq C \int_{\mathbb{R}^n} |u_0| \left(|(-\Delta)^{\sigma/2} \phi_{T^d}| + |\psi_t(0)| |\phi_{T^d}| \right) dx. \tag{18}$$

Plugging (11) together with (13)-(18) and choosing ε small enough, we deduce that

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^n} u_1(x) \phi_{T^d}(x) \, dx \\ &\leq C \int_{Q_T} \phi_{T^d} \psi^{\eta - p'} |\psi_{tt}|^{p'} \, dt \, dx + C \int_{Q_T} \phi_{T^d} \psi^{\eta - 2p'} |\psi_t|^{2p\prime} \, dt \, dx \\ &\quad + C \int_{Q_T} \psi^\eta \phi_{T^d}^{-1/(p-1)} |\Delta \phi_{T^d}|^{p'} \, dt \, dx \\ &\quad + C \, T^{-\beta p'} \int_{Q_T} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^{\eta - p'} \, |\psi_t|^{p'} \, dt \, dx \\ &\quad + C \int_0^T \frac{1}{(1+t)^{(\beta+1)p'}} \int_{\mathbb{R}^n} \phi_{T^d}^{-1/(p-1)} |(-\Delta)^{\sigma/2} \phi_{T^d}|^{p'} \, \psi^\eta \, dx \, dt \\ &\quad + C \int_{\mathbb{R}^n} |u_0| \left(|(-\Delta)^{\sigma/2} \phi_{T^d}| + |\psi_t(0)| |\phi_{T^d}| \right) dx. \end{aligned}$$

Taking account of the expression of φ given by (10) and Lemmas 1-3, we infer that

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^n} u_1(x) \phi_{T^d}(x) \, dx \\ &\leq C \, T^{-2p'+1+nd} + C \, T^{-2dp'+1+nd} + C \, T^{-\beta p' - p' - \sigma d p' + 1 + nd} \\ &\quad + C \, T^{-\sigma d p' + nd} \int_0^T (1+t)^{-\frac{(\beta+1)p}{p-1}} \, dt + C \left(T^{-\sigma d} + T^{-1} \right) \int_{\mathbb{R}^n} |u_0(x)| \, dx. \end{aligned} \tag{19}$$

Since $0 < \sigma < 2$, we notice that the cases of $\sigma \geq 1-\beta$ and $\sigma \leq 1-\beta$ are equivalent to

$$\sigma \geq 1-\beta, \quad \text{for all } \beta > -1, \qquad \text{and} \qquad \sigma < 1-\beta, \quad \text{for all } \beta < 1,$$

therefore, we distinguish two cases:

I. Case: $\sigma \geq 1-\beta$, for all $\beta > -1$.
In this case, we choose $d = 1$.

Subcritical case $p < \min\{\dfrac{n+1}{(n-1)_+}; \dfrac{n}{(n-\sigma)_+}\}$.

Note that,

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \leq C \begin{cases} T^{1-\frac{(\beta+1)\,p}{p-1}} & \text{if } \beta\, p < -1, \\ \ln T & \text{if } \beta\, p = -1, \\ 1 & \text{if } \beta\, p > -1. \end{cases}$$

We have two cases to distinguish.

a) If $\beta \geq 0$. In this case, we have $\beta\, p > -1$ and so

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \leq C.$$

Therefore, (19) implies

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \\ &\leq C\, T^{-2p'+1+n} + C\, T^{-(\beta+1)p'-\sigma p'+1+n} + C\, T^{-\sigma p'+n} + C\left(T^{-\sigma} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx. \end{aligned}$$

Using the fact $\beta\, p > -1 \Longrightarrow -(\beta+1)p' - \sigma p' + 1 + n < -\sigma p' + n$, we conclude that

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \\ &\leq C\, T^{-2p'+1+n} + C\, T^{-\sigma p'+n} + C\left(T^{-\sigma} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx. \end{aligned} \tag{20}$$

Note that, we can easily see that

$$-2p' + 1 + n < 0 \Longleftrightarrow p < \frac{n+1}{(n-1)_+} \quad \text{and} \quad -\sigma p' + n < 0 \Longleftrightarrow p < \frac{n}{(n-\sigma)_+}.$$

Letting $T \to \infty$, and using the Lebesgue dominated convergence theorem together with $u_1 \in L^1(\mathbb{R}^n)$, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts our assumption (6).

b) If $-1 < \beta < 0$. We have three cases for n.

Note that when $n = 1$, we have $n = 1 < \frac{1-\beta}{1+\beta}$.

i) If $n \geq \dfrac{\sigma}{1+\beta}$, then $n \geq \dfrac{1-\beta}{1+\beta}$ and therefore

$$p < \min\left\{\frac{n+1}{n-1}; \frac{n}{n-\sigma}\right\} \leq -\frac{1}{\beta}, \quad \text{for all } n \geq 2,$$

i.e. $\beta\, p > -1$, which implies a contradiction by following the same calculations as in part a).

ii) If $\dfrac{1-\beta}{1+\beta} < n < \dfrac{\sigma}{1+\beta}$ when $\sigma > 1-\beta$, then we have

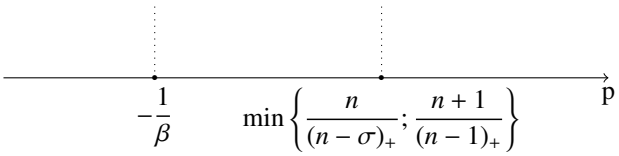
$$n < \frac{\sigma}{1+\beta} < \frac{\sigma}{2-\sigma},$$

and therefore

$$p < \min\left\{\frac{n+1}{n-1}; \frac{n}{n-\sigma}\right\} = \frac{n+1}{n-1} < -\frac{1}{\beta}, \quad \text{for all } n \geq 2,$$

i.e. $\beta\, p > -1$, which implies a contradiction by following the same calculations as in part a).

iii) If $n \leq \dfrac{1-\beta}{1+\beta}$, then $n \leq \dfrac{\sigma}{1+\beta}$, i.e.



• If $p \leq -1/\beta$, i.e. $p\beta \geq -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \leq C\, \ln T, \qquad \text{for all } T \gg 1.$$

Then, (19) implies

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \\
&\leq C\, T^{-2p'+1+n} + C\, T^{-(\beta+1)p'-\sigma p'+1+n} + C\, T^{-\sigma p'+n}\, \ln T + C\left(T^{-\sigma} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx,
\end{aligned}$$

for all $T \gg 1$. As $p\beta \geq -1 \Longrightarrow (\beta+1)p' \geq 1$, we conclude that

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \\
&\leq C\, T^{-2p'+1+n} + C\, T^{-\sigma p'+n}\, \ln T + C\left(T^{-\sigma} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx, \quad \forall\, T \gg 1. \qquad (21)
\end{aligned}$$

Note that, we can easily see that

$$-2p' + 1 + n < 0 \Longleftrightarrow p < \frac{n+1}{(n-1)_+} \quad \text{and} \quad -\sigma p' + n < 0 \Longleftrightarrow p < \frac{n}{(n-\sigma)_+}.$$

Letting $T \to \infty$, using the fact that $\ln T \leq T^{\frac{\sigma p'-n}{2}}$ (because $p < \frac{n}{(n-\sigma)_+}$) and the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts our assumption (6).

• If $p > -1/\beta$, i.e. $p\beta < -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)p}{p-1}}\,dt \le C\,T^{1-\frac{(\beta+1)p}{p-1}}, \qquad \text{for all } T>0.$$

Then, (19) implies

$$\begin{aligned}&\int_{Q_T} |u|^p\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx\\ &\le C\,T^{-2p'+1+n} + C\,T^{-(\beta+1)p'-\sigma p'+1+n} + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx,\end{aligned}$$

for all $T \gg 1$. As $\sigma \ge 1-\beta \Longrightarrow n \le \frac{1-\beta}{1+\beta} \le \frac{\sigma}{2-\sigma}$, we conclude that

$$p < \min\left\{\frac{n+1}{(n-1)_+}; \frac{n}{(n-\sigma)_+}\right\} = \frac{n+1}{(n-1)_+} \le \frac{n+1}{(n-\beta-\sigma)_+} \Longrightarrow -(\beta+1)p' - \sigma p' + 1 + n < 0,$$

and

$$p < \min\left\{\frac{n+1}{(n-1)_+}; \frac{n}{(n-\sigma)_+}\right\} = \frac{n+1}{(n-1)_+} \Longleftrightarrow -2p' + 1 + n < 0.$$

Note that $\sigma \ge 1-\beta > 1$, and $2 > \sigma > \sigma+\beta$, therefore

$$\begin{cases} (n-\beta-\sigma)_+ = (n-1)_+ = (n-\sigma)_+ = 0, & \text{when } n=1,\\ (n-\beta-\sigma)_+ = n-\beta-\sigma,\ (n-1)_+ = n-1,\ (n-\sigma)_+ = n-\sigma, & \text{when } n\ge 2,\end{cases}$$

Letting $T \to \infty$, using the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\,dx \le 0.$$

This contradicts our assumption (6).

Critical case: $p < \infty$ and $\sigma \ge 1$ when $n=1$, or $p = \dfrac{n+1}{n-1}$ and $\sigma > \dfrac{2n}{n+1}$ when $n \ge 2$.

Note that when $2 \le n < \dfrac{\sigma}{2-\sigma}$, we have

$$p = \frac{n+1}{n-1} < \frac{n}{n-\sigma} \quad \text{i.e.} \quad -\sigma p' + n < 0,$$

and when $\sigma \ge 1$ and $n=1$, we also have $-\sigma p' + n < 0$.
We have two cases to distinguish.

a) If $\beta \ge 0$. In this case, we have $\beta\,p > -1$ and so

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \le C.$$

From the subcritical case, we can see that we have

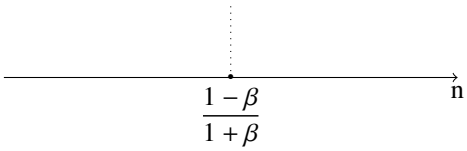
$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{22}$$

On the other hand, by applying Hölder's inequality instead of Young's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx \quad &\leq \quad C\left(\int_{\frac{T}{2}}^{T}\int_{\mathbb{R}^n} |u|^p\varphi\,dx\,dt\right)^{1/p} + C\left(\int_0^T\int_{|x|\geq T} |u|^p\varphi\,dx\,dt\right)^{1/p}\\ &\quad + C\,T^{-\sigma p'+n} + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx.\end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (22), we get a contradiction.

b) <u>If $-1 < \beta < 0$.</u> We have three cases for n.



Note that when $n = 1$, we have $n = 1 < \dfrac{1-\beta}{1+\beta}$.

i) If $n > \dfrac{1-\beta}{1+\beta}$, then

$$p = \frac{n+1}{n-1} < -\frac{1}{\beta},$$

i.e. $\beta\, p > -1$, which implies a contradiction by following the same calculations as in part a).

ii) If $n = \frac{1-\beta}{1+\beta}$, then

$$p = \frac{n+1}{n-1} = -\frac{1}{\beta},$$

i.e. $\beta\, p = -1$, and therefore

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\,dt \leq C\,\ln T, \qquad \text{for all } T > 0.$$

From the subcritical case (20), we can see easily that we have

$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{23}$$

On the other hand, by applying Hölder's inequality instead of Young's inequality and using $p = \dfrac{n+1}{n-1}$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\,dx \quad &\leq \quad C\left(\int_{\frac{T}{2}}^{T}\int_{\mathbb{R}^n} |u|^p\varphi\,dx\,dt\right)^{1/p} + C\left(\int_0^T\int_{|x|\geq T} |u|^p\varphi\,dx\,dt\right)^{1/p}\\ &\quad + C\,T^{-\sigma p'+n}\ln T + C\left(T^{-\sigma}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx.\end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (23) and the fact that $\ln T \leq T^{\frac{\sigma p'-n}{2}}$ (because $p < \dfrac{n}{n-\sigma}$), we get

$$\int_{\mathbb{R}^n} u_1(x)\,dx \leq 0.$$

This contradicts our assumption (6).

iii) If $n < \dfrac{1-\beta}{1+\beta}$, then

$$p = \frac{n+1}{(n-1)_+} > -\frac{1}{\beta},$$

i.e. $p\beta < -1$. In this case, we change the test function ψ by $\psi(t) = \Psi\left(\dfrac{t}{K^{-1}T}\right)$ where $K \geq 1$ is independent of T. Then

$$\int_0^{K^{-1}T} (1+t)^{-\frac{(\beta+1)p}{p-1}}\, dt \leq C\, K^{-1+(\beta+1)p'}\, T^{1-(\beta+1)p'}, \qquad \text{for all } T > 0.$$

From the subcritical case (20), we can see easily that we have

$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{24}$$

On the other hand, by applying Hölder's inequality in (19) instead of Young's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \quad \leq \quad & C\, K^{2p'-1}\left(\int_{\frac{K^{-1}T}{2}}^{K^{-1}T}\int_{\mathbb{R}^n} |u|^p\varphi\, dx\, dt\right)^{1/p} + C\, K^{-1}\left(\int_0^{K^{-1}T}\int_{|x|\geq T} |u|^p\varphi\, dx\, dt\right)^{1/p} \\ & + C\, K^{-1+(\beta+1)p'}\, T^{-(\beta+1)p'-\sigma p'+1+n} + C\left(K^\sigma T^{-\sigma} + KT^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\, dx.\end{aligned}$$

As $\sigma \geq 1-\beta$ and $2 > \sigma > \sigma+\beta$, we conclude that

$$p = \frac{n+1}{n-1} \leq \frac{n+1}{n-\beta-\sigma} \Longrightarrow -(\beta+1)p' - \sigma p' + 1 + n \leq 0, \quad \text{when } n \geq 2,$$

and

$$-(\beta+1)p' - \sigma p' + 1 + n < 0, \quad \text{when } n = 1,$$

so

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_T(x)\, dx \quad \leq \quad & C\, K^{2p'-1}\left(\int_{\frac{K^{-1}T}{2}}^{K^{-1}T}\int_{\mathbb{R}^n} |u|^p\varphi\, dx\, dt\right)^{1/p} + C\, K^{-1}\left(\int_0^{K^{-1}T}\int_{|x|\geq T} |u|^p\varphi\, dx\, dt\right)^{1/p} \\ & + C\, K^{-1+(\beta+1)p'} + C\left(K^\sigma T^{-\sigma} + KT^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\, dx.\end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (24), we get

$$\int_{\mathbb{R}^n} u_1(x)\, dx \quad \leq \quad C\, K^{-1+(\beta+1)p'}.$$

Letting $K \longrightarrow \infty$ and using $p\beta < -1 \Longrightarrow -1 + (\beta+1)p' < 0$, we infer that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts our assumption (6).

II. Case: $\sigma < 1-\beta$, for all $\beta < 1$.

In this case we take $d = \dfrac{1-\beta}{\sigma} > 1$.

Subcritical case $p < \min\left\{\dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}; \dfrac{n}{n-\sigma}\right\}$.

Note that,

$$\int_0^T (1+t)^{-\frac{(\beta+1)p}{p-1}}\, dt \leq C \begin{cases} 1 - \dfrac{(\beta+1)p}{p-1} & \\ T & \text{if } \beta p < -1, \\ \ln T & \text{if } \beta p = -1, \\ 1 & \text{if } \beta p > -1. \end{cases}$$

We have two cases to distinguish.

a) If $0 \le \beta < 1$. In this case, we have $\beta\, p > -1$ and so

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \le C.$$

Therefore, (19) implies

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \\
&\le C\, T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\, T^{-\frac{2(1-\beta)}{\sigma}p'+1+\frac{n(1-\beta)}{\sigma}} \\
&\quad + C\, T^{-(1-\beta)p'+\frac{n(1-\beta)}{\sigma}} + C\left(T^{-(1-\beta)} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx.
\end{aligned}$$

As

$$\sigma < 1-\beta \Longrightarrow -\frac{2(1-\beta)}{\sigma}p' + 1 + \frac{n(1-\beta)}{\sigma} < -2p' + 1 + \frac{n(1-\beta)}{\sigma},$$

we conclude that

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \\
&\le C\, T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\, T^{(1-\beta)\left[-p'+\frac{n}{\sigma}\right]} + C\left(T^{-(1-\beta)} + T^{-1}\right) \int_{\mathbb{R}^n} |u_0(x)|\, dx,
\end{aligned}$$

Note that, we can easily see that

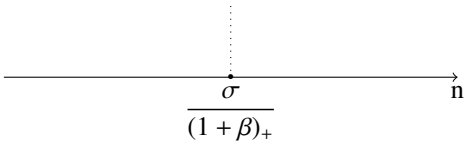
$$(1-\beta)\left[-p' + \frac{n}{\sigma}\right] < 0 \Longleftrightarrow p < \frac{n}{(n-\sigma)_+} \quad \text{and} \quad -2p' + 1 + \frac{n(1-\beta)}{\sigma} < 0 \Longleftrightarrow p < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}.$$

Note that $n(1-\beta) - \sigma \ge 1 - \beta - \sigma > 0$, for all $n \ge 1$. Letting $T \to \infty$, and using the Lebesgue dominated convergence theorem together with $u_1 \in L^1(\mathbb{R}^n)$, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \le 0.$$

This contradicts our assumption (6).

b) If $\beta < 0$. We have two cases for n.



Note that $n(1-\beta) - \sigma \ge 1 - \beta - \sigma > 0$ and $\sigma < \beta + 1 < 1 \le n \Longrightarrow n - \sigma > 0$, for all $n \ge 1$.

i) If $n \ge \dfrac{\sigma}{(1+\beta)_+}$, we have

$$p < \min\left\{\frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}; \frac{n}{n-\sigma}\right\} = \frac{n}{n-\sigma} \le -\frac{1}{\beta},$$

i.e. $\beta\, p > -1$, which implies a contradiction by following the same calculations as in part a).

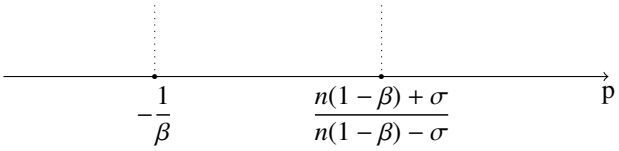
ii) If $n < \dfrac{\sigma}{(1+\beta)_+}$, we get

$$p < \min\left\{\frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}; \frac{n}{n-\sigma}\right\} = \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma},$$

and

$$-\frac{1}{\beta} < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma},$$

i.e.



• If $p \le -1/\beta$, i.e. $p\beta \ge -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)p}{p-1}}\,dt \le C\,\ln T, \qquad \text{for all } T \gg 1.$$

Then, (19) implies

$$\begin{aligned}&\int_{Q_T} |u|^p\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx\\ &\le C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\,T^{-\frac{2(1-\beta)}{\sigma}p'+1+\frac{n(1-\beta)}{\sigma}}\\ &\quad + C\,T^{-(1-\beta)p'+\frac{n(1-\beta)}{\sigma}}\,\ln T + C\left(T^{-(1-\beta)}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx,\end{aligned}$$

for all $T \gg 1$. As

$$\sigma < 1-\beta \Longrightarrow -\frac{2(1-\beta)}{\sigma}p' + 1 + \frac{n(1-\beta)}{\sigma} < -2p'+1+\frac{n(1-\beta)}{\sigma},$$

we conclude that

$$\begin{aligned}&\int_{Q_T} |u|^p\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx\\ &\le C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\,T^{(1-\beta)\left[-p'+\frac{n}{\sigma}\right]}\,\ln T + C\left(T^{-(1-\beta)}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx,\end{aligned}$$

Note that, we can easily see that

$$(1-\beta)\left[-p'+\frac{n}{\sigma}\right] < 0 \Longleftrightarrow p < \frac{n}{n-\sigma} \quad \text{and} \quad -2p'+1+\frac{n(1-\beta)}{\sigma} < 0 \Longleftrightarrow p < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}.$$

Letting $T \to \infty$, using the fact that $\ln T \le T^{\frac{(1-\beta)(p'-\frac{n}{\sigma})}{2}}$ (because $p < \frac{n}{n-\sigma}$) and the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\,dx \le 0.$$

This contradicts our assumption (6).

• If $p > -1/\beta$, i.e. $p\beta < -1$, then

$$\int_0^T (1+t)^{-\frac{(\beta+1)p}{p-1}}\,dt \le C\,T^{1-\frac{(\beta+1)p}{p-1}}, \qquad \text{for all } T > 0.$$

Then, (19) implies

$$\begin{aligned}&\int_{Q_T} |u|^p\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\,dx\\ &\le C\,T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\,T^{-\frac{2(1-\beta)}{\sigma}p'+1+\frac{n(1-\beta)}{\sigma}} + C\left(T^{-(1-\beta)}+T^{-1}\right)\int_{\mathbb{R}^n}|u_0(x)|\,dx.\end{aligned}$$

As

$$\sigma < 1-\beta \Longrightarrow -\frac{2(1-\beta)}{\sigma}p' + 1 + \frac{n(1-\beta)}{\sigma} < -2p' + 1 + \frac{n(1-\beta)}{\sigma},$$

we conclude that

$$\begin{aligned}&\int_{Q_T} |u|^p \varphi \, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \\ &\leq C\, T^{-2p'+1+\frac{n(1-\beta)}{\sigma}} + C\left(T^{-(1-\beta)} + T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\, dx,\end{aligned}$$

Note that, we can easily see that

$$-2p' + 1 + \frac{n(1-\beta)}{\sigma} < 0 \Longleftrightarrow p < \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}.$$

Letting $T \to \infty$, using the Lebesgue dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts our assumption (6).

Critical case: $p = \dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}$ and $\sigma > n(1+\beta)$.

Note that when $n(1+\beta) < \sigma$, we have

$$p = \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma} < \frac{n}{n-\sigma} \quad \text{i.e.} \quad -\sigma p' + n < 0,$$

We have two cases to distinguish.

a) If $0 \leq \beta < 1$. In this case, we have $\beta\, p > -1$ and so

$$\int_0^T (1+t)^{-\frac{(\beta+1)\,p}{p-1}}\, dt \leq C.$$

From the subcritical case (20), we can see easily that we have

$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{25}$$

On the other hand, by applying Hölder's inequality instead of Young's inequality, (19) implies

$$\begin{aligned}\int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \quad \leq \quad & C\left(\int_{\frac{T}{2}}^T \int_{\mathbb{R}^n} |u|^p \varphi \, dx\, dt\right)^{1/p} + C\left(\int_0^T \int_{|x|\geq T^d} |u|^p \varphi \, dx\, dt\right)^{1/p} \\ & + C\, T^{(1-\beta)\left[-p'+\frac{n}{\sigma}\right]} + C\left(T^{-(1-\beta)} + T^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\, dx.\end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (25) we get

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts our assumption (6).

b) If $\beta < 0$. As $n(1+\beta) < \sigma \Longrightarrow n < \dfrac{\sigma}{(1+\beta)_+}$, so

$$p = \frac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma} > -\frac{1}{\beta},$$

i.e. $p\beta < -1$. In this case, we change the test function ψ by $\psi(t) = \Psi\Big(\dfrac{t}{K^{-1}T}\Big)$ where $K \geq 1$ is independent of T. Then

$$\int_0^{K^{-1}T} (1+t)^{-\frac{(\beta+1)p}{p-1}}\, dt \leq C\, K^{-1+(\beta+1)p'}\, T^{1-(\beta+1)p'}, \qquad \text{for all } T > 0.$$

From the subcritical case (20), we can see easily that we have

$$u \in L^p((0,\infty); L^p(\mathbb{R}^n)). \tag{26}$$

On the other hand, by applying Hölder's inequality in (19) instead of Young's inequality and using $p = \dfrac{n(1-\beta)+\sigma}{n(1-\beta)-\sigma}$, (19) implies

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\phi_{T^d}(x)\, dx \\
&\leq C\, K^{2p'-1}\left(\int_{\frac{K^{-1}T}{2}}^{K^{-1}T}\int_{\mathbb{R}^n} |u|^p \varphi\, dx\, dt\right)^{1/p} + C\, K^{-1}\left(\int_0^{K^{-1}T}\int_{|x|\geq T^d} |u|^p \varphi\, dx\, dt\right)^{1/p} \\
&+C\, K^{-1+(\beta+1)p'} + C\left(K^{1-\beta}T^{-(1-\beta)} + KT^{-1}\right)\int_{\mathbb{R}^n} |u_0(x)|\, dx.
\end{aligned}$$

Letting $T \longrightarrow \infty$ and taking into consideration (26), we get

$$\int_{\mathbb{R}^n} u_1(x)\, dx \quad \leq \quad C\, K^{-1+(\beta+1)p'}.$$

Letting $K \longrightarrow \infty$ and using $p\beta < -1 \Longrightarrow -1 + (\beta+1)p' < 0$, we infer that

$$\int_{\mathbb{R}^n} u_1(x)\, dx \leq 0.$$

This contradicts our assumption (6).

□

Acknowledgments

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