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We consider the Signorini problem with Coulomb friction in elasticity. Sufficient conditions of non-uniqueness are obtained for the continuous model. These conditions are linked to the existence of real eigenvalues of an operator in a Hilbert space. We prove that, under appropriate conditions, real eigenvalues exist for a non-local Coulomb friction model. Finite element approximation of the eigenvalue problem is considered and numerical experiments are performed.

KEY WORDS: Coulomb friction; elastostatics; non-uniqueness; eigenvalue problem; finite element approximation

1. INTRODUCTION

Many applications in solid mechanics involve contact problems between elastic structures. Very often, the Coulomb friction model is chosen in the modelling of the contact phenomena. From a mathematical point of view, the Coulomb frictional contact problem in (continuum) elastostatics causes considerable difficulties and is still open. From a mechanical point of view, there is special interest in the investigation of uniqueness of the solutions. The aim of this paper is to shed some light on this question.

The variational formulation of the continuous problem in elastostatics was given by Duvaut and Lions [1]. The first existence results were obtained by Nečas *et al*. in Reference [2] for an infinite elastic strip. Thereafter, existence results were obtained for an arbitrary domain [3–5]. In all these papers, the existence results hold for small friction coefficients and the uniqueness is not discussed. The so-called non-local Coulomb frictional models mollifying the normal stresses were introduced by Duvaut [6] and developed in References [7–9]. The

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smoothing map used in the non-local friction model allows to obtain existence results for any friction coefficient. Moreover, uniqueness results can also be established if the friction coefficient is small enough [6–9]. The same type of result (existence for any friction coefficient and uniqueness for small friction coefficients) was obtained by Klarbring *et al*. [10,11] in the case of the normal compliance model, introduced by Oden and Martins [12,13]. Finally, let us remark that the sufficient conditions for uniqueness (small friction) given in all the above-mentioned papers are not completed by neither sufficient conditions for non-uniqueness nor by examples of non-uniqueness.

The discrete (finite element) problem, associated with the continuous static Coulomb friction problem, always admits a solution [14–16] and it is unique if the friction coefficient is small enough. Moreover, a convergence result of the finite element model towards the continuous model was established by Haslinger [14]. In the finite dimensional context, numerous studies using truss elements have led to examples of non-uniqueness. The early work concerning non-uniqueness was done by Janovský [17] and was followed by Klarbring who constructed a concrete example of non-uniqueness involving a spring system in Reference [18]. Let us mention that Alart considered the general framework of finite dimensional systems. He obtained in Reference [19] abstract necessary and sufficient conditions for uniqueness. In elastostatics, all the uniqueness results in the finite dimensional context are valid for friction coefficients lower than a critical value (in the quasi-static case, this does not hold according to the counter-example of Ballard [20]). This critical value always depends on the number of degrees of freedom (on the mesh size when finite elements are used or on the dimension of the system in the case of truss elements). Since this critical value vanishes as the number of degrees of freedom increases, we cannot deduce any result for the continuous problem. Furthermore, the examples of non-uniqueness are specific to the finite dimensional system such that no continuous non-uniqueness example can be constructed from it.

The aim of this paper is to give simple sufficient conditions for non-uniqueness of the solution to the continuous Coulomb friction problem which are related to the analysis of an eigenvalue problem. The spectral approach developed here is different from the widespread fixed-point technique used in the search of solutions to the Coulomb friction problem. To our knowledge, this is the first preliminary result dealing with non-uniqueness conditions in the continuous context.

After the statement of the problem, we give in Section 3 sufficient conditions for non-uniqueness. They deal with a continuous branch of solutions and they do not cover the case of isolated multiple solutions. Only multiple solutions with the same distribution of slip, free and stick zones are considered. These conditions of non-uniqueness require that the friction coefficient is a real eigenvalue of a spectral problem. That means that if this spectral problem has a real eigenvalue then the Coulomb friction contact problem is open to non-uniqueness.

In Section 5 we prove the existence of a countable set of complex eigenvalues for the non-local friction model (recalled in Section 4). Moreover, we give there sufficient conditions for the existence of at least one real eigenvalue. The eigenvalue problem is approximated in Section 6, and convergence of the finite element method is discussed.

Finally in Section 7, we present some numerical results. First, we implement numerically the eigenvalue problem and we illustrate the convergence of the real eigenvalues. Second we show the non-uniqueness methodology using numerical computations, which unfortunately cannot prove an evidence of non-uniqueness since the convergence results of the finite element model are not established, but which explain quite well the spectral approach proposed in this paper.

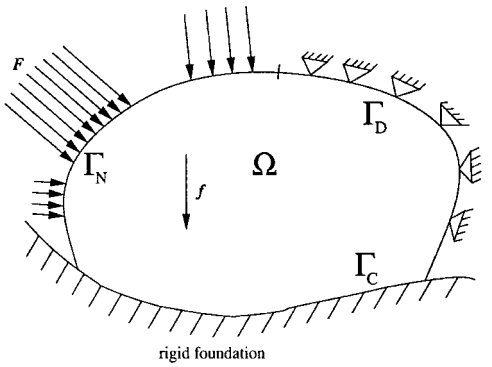

Figure 1. Setting of the problem. The domain Ω; its boundary is divided into three parts: Γ_D, Γ_N and Γ_C.

2. PROBLEM STATEMENT

Let an elastic body be given, occupying a domain $\Omega \times \mathbb{R}$ with Ω in $\mathbb{R}^2$. The generic point in $\mathbb{R}^3$ is denoted $\boldsymbol{x} = (x_1, x_2, x_3)$. We choose plane strain assumptions which means that the displacement field $\boldsymbol{u} = (u_1, u_2, u_3)$ is vanishing in the Ox_3 direction ($u_3 \equiv 0$) and u_1, u_2 depend only on (x_1, x_2). The boundary Γ of Ω is assumed to be Lipschitz and is divided as follows: $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_C$ where Γ_D, Γ_N and Γ_C are three open disjoint parts and $\text{meas}(\Gamma_D) > 0$. We assume that the displacement field is given on Γ_D (i.e. $\boldsymbol{u} = \mathbf{U}$) and that the boundary part Γ_N is acted on by a density of surface forces $\mathbf{F}$. The third part is Γ_C, which comprises all the points candidate to be in frictional contact with a rigid foundation (see Figure 1). The body Ω is acted upon by a given density of volume forces $\boldsymbol{f}$. Let $\boldsymbol{n} = (n_1, n_2)$ represent the unit outward normal vector on Γ and define the unit tangent vector $\boldsymbol{t} = (-n_2, n_1)$. We denote by $\mu > 0$ the friction coefficient on Γ_C.

The Coulomb frictional unilateral contact problem consists of finding the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ and the stress tensor field $\boldsymbol{\sigma}(\boldsymbol{u}) : \Omega \to \mathscr{S}_2$ satisfying (1)–(4):

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u}), \qquad \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega \tag{1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N, \qquad \boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D \tag{2}$$

where $\mathscr{S}_2$ stands for the space of second order symmetric tensors on $\mathbb{R}^2$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla^{\mathrm{T}} \boldsymbol{u})/2$ denotes the linearized strain tensor field, $\mathscr{C}$ is a fourth order symmetric and elliptic tensor of linear elasticity and **div** represents the divergence operator of tensor valued functions.

In order to introduce the equations on Γ_C, we adopt the following notation: $\boldsymbol{u} = u_n \boldsymbol{n} + u_t \boldsymbol{t}$ and $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}$. The equations modelling contact and friction are as follows on Γ_C:

$$u_n \leqslant 0, \quad \sigma_n(\boldsymbol{u}) \leqslant 0, \quad \sigma_n(\boldsymbol{u}) u_n = 0 \tag{3}$$

$$\begin{cases} u_t = 0 \Rightarrow |\sigma_t(\boldsymbol{u})| \leqslant \mu |\sigma_n(\boldsymbol{u})| \\ u_t \neq 0 \Rightarrow \sigma_t(\boldsymbol{u}) = -\mu |\sigma_n(\boldsymbol{u})| \dfrac{u_t}{|u_t|} \end{cases} \tag{4}$$

Remark 2.1
Let us mention that the true Coulomb friction law involves the tangential contact velocities and not the tangential displacements. However, a problem analogous to the one discussed here is obtained by time discretization of the quasistatic frictional contact evolution problem. In this case (see Reference [21]) $\boldsymbol{u}$, $\boldsymbol{f}$ and $\mathbf{F}$ stand for $\boldsymbol{u}((i+1)\Delta t)$, $\boldsymbol{f}((i+1)\Delta t)$ and $\mathbf{F}((i+1)\Delta t)$ respectively and u_t has to be replaced by $u_t((i+1)\Delta t) - u_t(i\Delta t)$, where Δt denotes the time step. For simplicity and without any loss of generality only the static case described above will be considered in the following.

The variational formulation of problem (1)–(4) has been obtained by Duvaut and Lions [1]. It consists of finding $\boldsymbol{u}$ verifying:

$$\boldsymbol{u} \in \mathbf{K}_{\text{ad}}, \quad a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geqslant L(\boldsymbol{v} - \boldsymbol{u}), \quad \forall \boldsymbol{v} \in \mathbf{K}_{\text{ad}} \tag{5}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\Omega, \quad L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\Omega + \int_{\Gamma_{\text{N}}} \mathbf{F} \cdot \boldsymbol{v}\, \mathrm{d}\Gamma$$

are defined for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in the standard Sobolev space $(H^1(\Omega))^2$ (see Reference [22]) and the notations $\cdot$ and : stand for the canonical inner products in $\mathbb{R}^2$ and $\mathscr{S}_2$, respectively.

In (5), $\mathbf{K}_{\text{ad}}$ denotes the closed convex set of admissible displacement fields satisfying the non-penetration conditions:

$$\mathbf{K}_{\text{ad}} = \{\boldsymbol{v} \in (H^1(\Omega))^2 : \ \boldsymbol{v} = \mathbf{U} \text{ on } \Gamma_{\text{D}}, v_n \leqslant 0 \text{ on } \Gamma_{\text{C}}\}$$

The functional $j(\cdot, \cdot)$ given by

$$j(\boldsymbol{u}, \boldsymbol{v}) = -\int_{\Gamma_{\text{C}}} \mu \sigma_n(\boldsymbol{u}) |v_t|\, \mathrm{d}\Gamma \tag{6}$$

is defined for any $\boldsymbol{v}$ in $(H^1(\Omega))^2$ but more regularity is required for $\boldsymbol{u}$. Two different cases when $j(\boldsymbol{u}, \cdot)$ makes sense, are usually considered in the literature. The first one, which occurs in the continuous problem, involves the space

$$\tilde{\mathbf{V}} = \{\boldsymbol{v} \in (H^1(\Omega))^2 : \ \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{v}) \in (L^2(\Omega))^2\}$$

If $\boldsymbol{u} \in \tilde{\mathbf{V}}$ then $\boldsymbol{\sigma}(\boldsymbol{u})$ belong to $H(\mathbf{div}, \Omega)$ and $\sigma_n(\boldsymbol{u})$ is an element of $H^{-1/2}(\Gamma)$ (i.e. the dual of $H^{1/2}(\Gamma)$). Since $H^{-1/2}(\Gamma)|_{\Gamma_{\text{C}}}$ is different from $H^{-1/2}(\Gamma_{\text{C}})$ we have to suppose in addition that $\sigma_n(\boldsymbol{u}) \in H^{-1/2}(\Gamma_{\text{C}})$. With this assumption, (6) makes sense if we replace the integral term by the duality product. For a more precise formulation involving the convenient Sobolev spaces and the set of nonnegative Radon measures, a detailed study can be found in Reference [23]. In the second case, $\boldsymbol{u}$ belongs to a finite element set $\mathbf{V}_h \subset (H^1(\Omega))^2$, which implies that $\boldsymbol{\sigma}(\boldsymbol{u})$ is at least piecewise continuous so that $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ admits a trace on the boundary. In the latter case, the integral notation in (6) has to be understood in the classical sense.

The first existence result of (1)–(4) has been proven in Reference [2] when Ω is an infinitely long strip and the friction coefficient has compact support in Γ_{C} and is sufficiently

small. The extension of these results to domains with smooth boundaries as well as improvements can be found in References [3] and [4]. More recently in Reference [5], existence is stated when the friction coefficient μ is smaller than $\sqrt{3-4\nu}/(2-2\nu)$, ν denoting Poisson's ratio in Ω ($0 \leqslant \nu < \frac{1}{2}$). To our knowledge there exist neither uniqueness result nor non-uniqueness example of (5) (unless the loads $\mathbf{U}$, $\boldsymbol{f}$ and $\mathbf{F}$ are equal to zero).

3. SUFFICIENT CONDITIONS FOR NON-UNIQUENESS: A SPECTRAL APPROACH

Let us consider an equilibrium position $\boldsymbol{u}^0$ of the Coulomb frictional contact problem (i.e. a solution of (1)–(4)) supposed to be regular enough. The notation Γ_{f}^0 stands for the points of Γ_{C} which are currently free (separated from the rigid foundation). We denote by Γ_{s}^0 the points of Γ_{C} which are currently in contact but are stuck to the rigid foundation, and by Γ_{C}^0 the points of Γ_{C} which are currently in contact but are candidate to slip. That leads to the following definitions:

$$\Gamma_{\mathrm{f}}^0 = \{\boldsymbol{x} \in \Gamma_{\mathrm{C}}:\ u_n^0(\boldsymbol{x}) < 0\} \tag{7}$$

$$\Gamma_{\mathrm{s}}^0 = \{\boldsymbol{x} \in \Gamma_{\mathrm{C}}:\ u_n^0(\boldsymbol{x}) = 0,\ |\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})| < -\mu\sigma_n(\boldsymbol{u}^0)(\boldsymbol{x})\} \tag{8}$$

$$\Gamma_{\mathrm{C}}^0 = \{\boldsymbol{x} \in \Gamma_{\mathrm{C}}:\ u_n^0(\boldsymbol{x}) = 0,\ |\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})| = -\mu\sigma_n(\boldsymbol{u}^0)(\boldsymbol{x})\} \tag{9}$$

Let us adopt the following notation

$$\Gamma_{\mathrm{D}}^0 = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{s}}^0 \quad \text{and} \quad \Gamma_{\mathrm{N}}^0 = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{f}}^0$$

and consider now the following eigenvalue problem:
Eigenvalue problem. Find $\lambda \in \mathbb{C}$ and $\mathbf{0} \neq \boldsymbol{\Phi} \in (H^1(\Omega))^2$ such that

$$\boldsymbol{\sigma}(\boldsymbol{\Phi}) = \mathscr{C}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}), \quad \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{\Phi}) = \mathbf{0} \quad \text{in } \Omega \tag{10}$$

$$\boldsymbol{\Phi} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}^0, \quad \boldsymbol{\sigma}(\boldsymbol{\Phi})\boldsymbol{n} = 0 \quad \text{on } \Gamma_{\mathrm{N}}^0, \quad \boldsymbol{\Phi}_n = 0 \quad \text{on } \Gamma_{\mathrm{C}}^0 \tag{11}$$

$$\sigma_t(\boldsymbol{\Phi}) = \lambda\sigma_n(\boldsymbol{\Phi}) \quad \text{on } \Gamma_{\mathrm{C}}^0 \tag{12}$$

Remark 3.1
If we choose the commonly used Hooke's law, for homogeneous isotropic materials, given by

$$\sigma_{ij} = \frac{E\nu}{(1-2\nu)(1+\nu)}\delta_{ij}\varepsilon_{kk}(\boldsymbol{u}) + \frac{E}{1+\nu}\varepsilon_{ij}(\boldsymbol{u}) \quad \text{in } \Omega$$

where E denotes Young's modulus, ν represents Poisson's ratio and δ_{ij} is the Kronecker symbol, then the only constitutive constant involved in the eigenvalue problem (10)–(12) is the ratio $\eta = \nu/(1-2\nu)$. Indeed, the eigenvalues and eigenfunctions are independent of the Young modulus E.

The following theorem states sufficient conditions for the non-uniqueness of the equilibrium solution $\boldsymbol{u}^0$.

Theorem 3.2
Let $\boldsymbol{u}^0$ be a smooth solution of Coulomb's frictional contact problem (1)–(4) with $\mu>0$ as friction coefficient. Let $\boldsymbol{u}^1=\boldsymbol{u}^0+\delta\boldsymbol{\Phi}$ for some $\delta\in\mathbb{R}$ and $\boldsymbol{\Phi}$ a smooth eigenfunction of (10)–(12). Let us define the two following cases (i) and (ii):

(i) $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})\leqslant 0$ for all $\boldsymbol{x}\in\Gamma_C^0$ and μ is the corresponding eigenvalue for $\boldsymbol{\Phi}$. Assume that:

$$u_n^1(\boldsymbol{x})<0, \quad \text{for all } \boldsymbol{x}\in\Gamma_f^0 \tag{13}$$

$$\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x})\leqslant 0, \quad \text{for all } \boldsymbol{x}\in\Gamma_C^0 \tag{14}$$

$$|\sigma_t(\boldsymbol{u}^1)(\boldsymbol{x})|<-\mu\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}\in\Gamma_s^0 \tag{15}$$

$$u_t^1(\boldsymbol{x})\geqslant 0, \quad \text{for all } \boldsymbol{x}\in\Gamma_C^0 \tag{16}$$

(ii) $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})\geqslant 0$ for all $\boldsymbol{x}\in\Gamma_C^0$ and $-\mu$ is the corresponding eigenvalue for $\boldsymbol{\Phi}$. Assume that in addition to (13)–(15), one has:

$$u_t^1(\boldsymbol{x})\leqslant 0, \quad \text{for all } \boldsymbol{x}\in\Gamma_C^0 \tag{17}$$

If either (i) or (ii) holds then $\boldsymbol{u}^1$ is another (smooth) solution of (1)–(4).

Proof
Let us firstly remark that $\boldsymbol{u}^1=\boldsymbol{u}^0+\delta\boldsymbol{\Phi}$ satisfies Equations (1)–(2) for any $\delta\in\mathbb{R}$. Next, we have to check that $\boldsymbol{u}^1$ verifies the frictional contact conditions (3)–(4). We begin with the unilateral contact conditions (3).

If $\boldsymbol{x}\in\Gamma_f^0$ then from (13) we have $u_n^1(\boldsymbol{x})<0$. Since $\Gamma_f^0\subset\Gamma_C$, from (7) we get $\boldsymbol{\sigma}(\boldsymbol{u}^0)\boldsymbol{n}(\boldsymbol{x})=0$. Having in mind that $\Gamma_f^0\subset\Gamma_N^0$ and according to (11), we deduce $\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x})=0$. If $\boldsymbol{x}\in\Gamma_s^0\cup\Gamma_C^0$ then $\boldsymbol{u}_n^0(\boldsymbol{x})=0$ and $\boldsymbol{\Phi}_n(\boldsymbol{x})=0$, hence $u_n^1(\boldsymbol{x})=0$ and from (14) we deduce (3). Therefore $\boldsymbol{u}^1$ satisfies (3).

If $\boldsymbol{x}\in\Gamma_s^0$ the condition (4) implies $u_t^0(\boldsymbol{x})=0$. From assumption (15) we get $|\sigma_t(\boldsymbol{u}^1)(\boldsymbol{x})|<-\mu\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x})$ and since $\Gamma_s^0\subset\Gamma_D^0$ we obtain $u_t^1(\boldsymbol{x})=0$. If $\boldsymbol{x}\in\Gamma_f^0$, then owing to $\Gamma_f^0\subset\Gamma_N^0$, we have $\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x})=\sigma_t(\boldsymbol{u}^1)(\boldsymbol{x})=0$ and (4) is satisfied.

Let $\boldsymbol{x}\in\Gamma_C^0$. If case (i) holds, then, for all $\boldsymbol{x}\in\Gamma_C^0$ we have $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})\leqslant 0$ and $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})=\mu\sigma_n(\boldsymbol{u}^0)(\boldsymbol{x})$. Using (12) with $\lambda=\mu$, we obtain $\sigma_t(\boldsymbol{u}^1)(\boldsymbol{x})=\mu\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x})$ and (4) follows from (14) and (16). If we consider case (ii) then $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})=-\mu\sigma_n(\boldsymbol{u}^0)(\boldsymbol{x})$ and from (12) with $\lambda=-\mu$ we finally get $\sigma_t(\boldsymbol{u}^1)(\boldsymbol{x})=-\mu\sigma_n(\boldsymbol{u}^1)(\boldsymbol{x})$. Consequently, $\boldsymbol{u}^1$ satisfies (4). □

In order to apply Theorem 3.2, one has to check the conditions (13)–(17) dealing with the equilibrium $\boldsymbol{u}^0$, the eigenfunction $\boldsymbol{\Phi}$ and an appropriately chosen value of δ. The following corollary yields sufficient conditions concerning the solution $\boldsymbol{u}^0$ only. These conditions are more restrictive than those of the previous theorem but easier to handle. Indeed, we shall suppose that all the points of Γ_C are in a slipping contact, i.e. $\Gamma_s^0=\emptyset$ and $\Gamma_f^0=\emptyset$.

Corollary 3.3
Let $\boldsymbol{u}^0$ be a smooth solution of Coulomb's frictional contact problem (1)–(4) with $\mu>0$ as friction coefficient. Assume that $\Gamma_C^0=\Gamma_C$ and that there exist $\alpha,\beta>0$ such that

$$\sigma_n(\boldsymbol{u}^0)(\boldsymbol{x})\leqslant -\beta \quad \text{for all } \boldsymbol{x}\in\Gamma_C^0 \tag{18}$$

Moreover, suppose that one of the following two conditions (i) or (ii) holds:
(i) The pair $(\mu;\boldsymbol{\Phi})$ is a smooth solution of (10)–(12), and

$$u_t^0(\boldsymbol{x})\geqslant\alpha, \quad \text{for all } \boldsymbol{x}\in\Gamma_C^0 \tag{19}$$

(ii) The pair $(-\mu;\boldsymbol{\Phi})$ is a smooth solution of (10)–(12), and

$$u_t^0(\boldsymbol{x})\leqslant -\alpha, \quad \text{for all } \boldsymbol{x}\in\Gamma_C^0 \tag{20}$$

Then Coulomb's frictional contact problem (1)–(4) admits an infinity of solutions. In particular, there exists $\delta_0>0$ such that $\boldsymbol{u}^1=\boldsymbol{u}^0+\delta\boldsymbol{\Phi}$ is solution for any δ satisfying $|\delta|\leqslant\delta_0$.

Proof
Let $\beta_0=\sup_{x\in\Gamma_C}|\sigma_n(\boldsymbol{\Phi})(\boldsymbol{x})|$ and $\alpha_0=\sup_{x\in\Gamma_C}|\Phi_t(\boldsymbol{x})|$. Keeping in mind that $\sigma_n(\boldsymbol{u}^0+\delta\boldsymbol{\Phi})(\boldsymbol{x})\leqslant -\beta+|\delta|\beta_0$ we deduce that (14) holds when $|\delta|\leqslant\beta/\beta_0$.

If condition (i) holds, we can write $u_t^1(\boldsymbol{x})=u_t^0(\boldsymbol{x})+\delta\Phi_t(\boldsymbol{x})\geqslant\alpha-|\delta|_{\alpha_0}$ so that the bound $|\delta|\leqslant\alpha/\alpha_0$ leads to (16). Moreover, condition (19) implies that $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})\leqslant 0$ for all $\boldsymbol{x}\in\Gamma_C^0$. If we set $\delta_0=\min\{\beta/\beta_0,\alpha/\alpha_0\}$ then the first case in Theorem 3.2 proves the statement of the corollary.

If condition (ii) holds, then (17) is satisfied if $|\delta|\leqslant\alpha/\alpha_0$ and (20) implies that $\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})\geqslant 0$ for all $\boldsymbol{x}\in\Gamma_C^0$. Employing the second case in Theorem 3.2 completes the proof of the corollary. □

Remark 3.4
The above results are only sufficient conditions for non-uniqueness. They take into consideration only the possibility of existence of multiple solutions having the same distribution of the slip, free and stick zones. Moreover, the above corollary does not cover the case of isolated multiple solutions.

Indeed, as it follows from Corollary 3.3, if the problem is open to non-uniqueness then there exists an infinity of solutions located on a continuous branch.

The non-uniqueness conditions considered here imply that the friction coefficient μ (or $-\mu$) is an eigenvalue of (10)–(12). This eigenvalue problem depends exclusively on the geometry (the domain Ω and the distribution of the different types of boundaries) and on the elastic properties incorporated in the operator $\mathscr{C}$ (on the Poisson coefficient ν for an isotropic elastic material).

Remark 3.5
If (10)–(12) admits a real eigenvalue μ then the pair geometry-material is open to the non-uniqueness of the Coulomb frictional contact problem.

As a matter of fact, one can think of a distribution of loads $\mathbf{F}$, $\boldsymbol{f}$ and a displacement field $\mathbf{U}$ such that a solution $\boldsymbol{u}^0$ of (1)–(4) for this particular friction coefficient μ satisfies (18)–(19). We consider, for example, that Γ_C is a straight line segment located on the Ox_1-axis and that $\Gamma_N = \emptyset$. We choose

$$\mathbf{U}(\boldsymbol{x}) = \left(\alpha + 2\mu \frac{1-\nu}{1-2\nu} x_2, -x_2\right)$$

for all $\boldsymbol{x} = (x_1, x_2) \in \Gamma_D$, with $\alpha > 0$ and $\boldsymbol{f} = \mathbf{0}$. One can easily check that $\boldsymbol{u}^0(\boldsymbol{x}) = \mathbf{U}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \Omega$ is a solution of (1)–(4). Since $\Gamma_C^0 = \Gamma_C$, $\sigma_n(\boldsymbol{u}^0)(\boldsymbol{x}) = -E(1-\nu)/[(1-2\nu)(1+\nu)] < 0$ and $u_t^0(\boldsymbol{x}) = \alpha > 0$, we deduce that the sufficient conditions of the corollary hold.

4. THE NON-LOCAL FRICTION MODEL

There exist several laws 'mollifying' Coulomb's frictional contact model which lead generally to more existence and uniqueness properties. Among these regularization techniques, a special interest is devoted to the non-local procedure introduced in Reference [6] and developed in References [7–9]. Moreover, from a physical point of view, this law takes into account some interesting microscopic aspects: the normal pressure $\sigma_n(\boldsymbol{u})$ is distributed over a contact area of the deformed asperity (see Reference [7] for more arguments).

Hence, we consider the non-local normal stress $\sigma_n^*(\boldsymbol{u})$ given by

$$\sigma_n^*(\boldsymbol{u})(\boldsymbol{x}) = \frac{\int_{\Gamma_C} w_\rho(|\boldsymbol{x}-\boldsymbol{y}|)\sigma_n(\boldsymbol{u})(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}{\int_{\Gamma_C} w_\rho(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}} \tag{21}$$

where $w_\rho, (\rho > 0)$ stands for a non-negative function with its non-empty support in $[-\rho, \rho]$ such that $\boldsymbol{x} \mapsto w_\rho(|\boldsymbol{x}|)$ is an infinitely differentiable function. As for the functional j, the above expression of the non-local normal stress is meaningful in two different cases. The first one concerns the continuous problem when $\sigma_n(\boldsymbol{u}) \in H^{-1/2}(\Gamma_C)$ and the above integral has to be replaced by the duality product between $H^{-1/2}(\Gamma_C)$ and $H^{1/2}(\Gamma_C)$. The second case happens when using a finite element approximation when $\sigma(\boldsymbol{u})$ is at least piecewise continuous.

Another type of smoothing procedure was introduced in Reference [24] for friction problems in viscoplasticity. In this case the second order stress tensor field is averaged in the interior of the domain and its normal trace on the contact boundary provides the non-local normal stress. The definition of the non-local normal stress $\sigma_n^*(\boldsymbol{u})$ becomes

$$\sigma_n^*(\boldsymbol{u})(\boldsymbol{x}) = \frac{\int_\Omega w_\rho(|\boldsymbol{x}-\boldsymbol{y}|)\boldsymbol{\sigma}(\boldsymbol{u})(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}{\int_{\Gamma_\Omega} w_\rho(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}} \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \tag{22}$$

Unlike the first non-local approach in (21), this second procedure avoids the handling of dual Sobolev spaces such as $H^{-1/2}(\Gamma)$. Indeed, the latter expression is well defined for any $\boldsymbol{u} \in (H^1(\Omega))^2$.

If we replace the above formulas in (4) we get the following 'regularized', non-local friction law on Γ_C:

$$\begin{cases} u_t = 0 \Rightarrow |\sigma_t(\boldsymbol{u})| \leqslant \mu|\sigma_n^*(\boldsymbol{u})| \\ u_t \neq 0 \Rightarrow \sigma_t(\boldsymbol{u}) = -\mu|\sigma_n^*(\boldsymbol{u})|\dfrac{u_t}{|u_t|} \end{cases} \tag{23}$$

The variational formulation of (1)–(3) and (23) is inequality (5), the same as in the local friction case in which j is replaced by (see Reference [6]):

$$j(\boldsymbol{u},\boldsymbol{v}) = -\int_{\Gamma_C} \mu\sigma_n^*(\boldsymbol{u})|v_t|\,\mathrm{d}\Gamma \tag{24}$$

From a mathematical point of view, the smoothing map used in the non-local friction model implies compactness properties of the operators involved in the variational approach (5). These properties permit using the Schauder and Tykhonov fixed point theorems in order to deduce the existence of at least one solution of the variational inequality [6–9]. In addition, some uniqueness results can also be obtained for the non-local friction model. As a matter of fact, it was proved in References [6–9] that there exists a critical friction coefficient μ_c such that if $\mu < \mu_c$ (i.e. the friction is small) then the solution of (5) is unique. As for the local friction case there exist, to our knowledge, no non-uniqueness examples.

The eigenvalue problem corresponding to the non-local friction case is (10)–(11) and

$$\sigma_t(\boldsymbol{\Phi}) = \lambda\sigma_n^*(\boldsymbol{\Phi}) \quad \text{on } \Gamma_C^0 \tag{25}$$

where Γ_s^0 and Γ_C^0 defined in (8) and (9) have to be replaced by

$$\Gamma_s^0 = \{\boldsymbol{x} \in \Gamma_C:\ u_n^0(\boldsymbol{x}) = 0, |\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})| < -\mu\sigma_n^*(\boldsymbol{u}^0)(\boldsymbol{x})\}$$
$$\Gamma_C^0 = \{\boldsymbol{x} \in \Gamma_C:\ u_n^0(\boldsymbol{x}) = 0, |\sigma_t(\boldsymbol{u}^0)(\boldsymbol{x})| = -\mu\sigma_n^*(\boldsymbol{u}^0)(\boldsymbol{x})\}$$

If the normal stress σ_n is replaced by σ_n^* then all the sufficient conditions for non-uniqueness given in Theorem 3.2 and Corollary 3.3 remain valid.

5. EXISTENCE OF EIGENVALUES AND EIGENFUNCTIONS

In order to derive the variational formulation of (10)–(11) and (25) we consider the subspaces $\mathbf{V}^0$ and $\tilde{\mathbf{V}}^0$ of $(H^1(\Omega))^2$ and $\tilde{\mathbf{V}}$, respectively:

$$\mathbf{V}^0 = \{\boldsymbol{v} \in (H^1(\Omega))^2:\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D^0,\ v_n = 0 \text{ on } \Gamma_C^0\}, \quad \text{and} \quad \tilde{\mathbf{V}}^0 = \mathbf{V}^0 \cap \tilde{\mathbf{V}}$$

Let us introduce the bilinear form $b(.,.)$ given by:

$$b(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_C} \sigma_n^*(\boldsymbol{u})v_t\,\mathrm{d}\Gamma$$

for any $\boldsymbol{v}$ in $(H^1(\Omega))^2$. Concerning the first variable, $b(\boldsymbol{u},\boldsymbol{v})$ makes sense if the non-local normal stress $\sigma_n^*(\boldsymbol{u})$ can be defined. Hence, $b(\boldsymbol{u},\boldsymbol{v})$ is well defined for any $\boldsymbol{u}\in\tilde{\mathbf{V}}$ such that $\sigma_n(\boldsymbol{u})\in H^{-1/2}(\Gamma_C)$ or for $\boldsymbol{u}\in\mathbf{V}_h$ (the notation $\mathbf{V}_h$ represents a finite element type space) if the non-local normal stress (21) is considered. When adopting the non-local normal stress (22), this restriction disappears, so that $b(\boldsymbol{u},\boldsymbol{v})$ is well defined for all $\boldsymbol{u},\boldsymbol{v}\in(H^1(\Omega))^2$.

The variational formulation of problem (10)–(11) and (25) consists of finding $\lambda\in\mathbb{C}$ and $\mathbf{0}\neq\boldsymbol{\Phi}\in\tilde{\mathbf{V}}^0$ such that:

$$a(\boldsymbol{\Phi},\boldsymbol{v})=\lambda b(\boldsymbol{\Phi},\boldsymbol{v}),\quad \forall\boldsymbol{v}\in\mathbf{V}^0 \tag{26}$$

and it can be easily checked that if $\lambda\in\mathbb{C}$ and a non-zero $\boldsymbol{\Phi}$ satisfy (10)–(11) and (25) then there are a solution of (26). Conversely, if $\lambda\in\mathbb{C}$ and a nonzero $\boldsymbol{\Phi}$ satisfy (26), then the pair $(\lambda;\boldsymbol{\Phi})$ is a weak solution of (10)–(11) and (25).

Theorem 5.1
The eigenvalues of problem (26) consist of a countable set of complex numbers $\{\lambda_n\}_{n\in I}$ with $\lambda_n\neq 0$. Each eigenvalue λ_n is of finite algebraic multiplicity. If I is infinite then $\lim_{n\to\infty}|\lambda_n|=+\infty$.

Proof
Let us first remark that $\lambda=0$ is not an eigenvalue of (26). Otherwise, $a(\boldsymbol{\Phi},\boldsymbol{\Phi})=0$ where $\boldsymbol{\Phi}$ is an eigenvector associated with $\lambda=0$, which contradicts the $\mathbf{V}^0$-ellipticity of $a(.,.)$. Let us denote by

$$H=L^2(\Gamma_C^0)\quad\text{and}\quad \mathbf{W}^0=\{\boldsymbol{v}\in\mathbf{V}^0:\ \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{v})=\mathbf{0}\ \text{in}\ \Omega,\ \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n}=\mathbf{0}\ \text{on}\ \Gamma_N^0\}$$

and define $\mathbf{P}: H\to\mathbf{V}^0$ as follows: for any $f\in H$, $\mathbf{P}(f)$ is the unique solution of the variational equality

$$a(\mathbf{P}(f),\boldsymbol{v})=\int_{\Gamma_C^0} f v_t\,\mathrm{d}\Gamma,\quad\forall\boldsymbol{v}\in\mathbf{V}^0.$$

If we put $\boldsymbol{v}\in(\mathscr{D}(\Omega))^2\subset\mathbf{V}^0$ in the previous equation (the notation $\mathscr{D}(\Omega)$ stands for the space of infinitely differentiable functions with compact support in Ω), we deduce that $\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{P}(f))=\mathbf{0}$ in Ω. In the same way, it can be formally checked that $\boldsymbol{\sigma}(\mathbf{P}(f))\boldsymbol{n}=\mathbf{0}$ on Γ_N^0 which implies that $\mathbf{P}(f)\in\mathbf{W}^0$. Hence, $\mathbf{P}$ is a linear continuous operator from H into $\mathbf{W}^0$. Next, we prove the theorem separately for the two regularization techniques in (21) and (22).

(i) Case (21). The function $\sigma_n^*(\boldsymbol{v})\in H$ is well defined for any $\boldsymbol{v}\in\mathbf{W}^0$. Set $Q:\mathbf{W}^0\to H$ so that $Q(\boldsymbol{v})=\sigma_n^*(\boldsymbol{v})$, Since Q is a linear and completely continuous operator [9, Theorem 11.2, p. 338] we deduce that $T=\mathbf{P}Q:\mathbf{W}^0\to\mathbf{W}^0$ is also completely continuous. In order to prove the statement of the theorem, we only have to mention that λ is a solution in (26) if and only if $1/\lambda$ is a non-zero eigenvalue for T which is true since $\lambda T(\boldsymbol{\Phi})=\boldsymbol{\Phi}$ if and only if $(\lambda;\boldsymbol{\Phi})$ is a solution for (26).

(ii) Case (22). The operator $Q:\mathbf{V}^0\to H$, given by $Q(\boldsymbol{v})=\sigma_n^*(\boldsymbol{v})$ is well defined. In addition, Q is a linear and completely continuous operator [24, Lemma 1.2, p. 181] and we deduce that $T=\mathbf{P}Q:\mathbf{V}^0\to\mathbf{V}^0$ is also completely continuous. As a consequence, the proof follows as in case (i). □

Remark 5.2
The technique used in the proof above cannot be used if the non-local assumption is removed. The existence of a countable set of eigenvalues is linked to the compactness of operator T which is assured by the regularized trace operator Q.

The following result ensures, under specific conditions, the existence of at least one real positive or negative eigenvalue for problem (26) which also minimizes the moduli among all eigenvalues satisfying (26). First, we need to define the convex cone $\mathbf{K}^0$:

$$\mathbf{K}^0 = \{\boldsymbol{v} \in \mathbf{W}^0:\ \sigma_t(\boldsymbol{v}) \geqslant 0 \text{ on } \Gamma_C^0\}$$

It is easy to see that each displacement filed $\boldsymbol{v}$ of $\mathbf{W}^0$ (and of $\mathbf{K}^0$) is determined uniquely by the tangential component $\sigma_t(\boldsymbol{v})$ of the stress vector on Γ_C^0.

Theorem 5.3
Suppose that one of the two following conditions (i) or (ii) holds:

(i) any $\boldsymbol{v}$ in $\mathbf{K}^0$ satisfies $\sigma_n^*(\boldsymbol{v}) \geqslant 0$ on Γ_C^0,
(ii) any $\boldsymbol{v}$ in $\mathbf{K}^0$ satisfies $\sigma_n^*(\boldsymbol{v}) \leqslant 0$ on Γ_C^0.

Then the eigenvalue λ_0, minimizing the moduli of all eigenvalues in problem (26), is real and its associated eigenvector lies in $\mathbf{K}^0$. Moreover $\lambda_0 > 0$ in the case (i) and $\lambda_0 < 0$ in the case (ii).

Let us first recall a weak form of the Krein–Rutman theorem [25–27] which we use in the proof of Theorem 5.3.

Theorem (Krein and Rutman [25])
Let X be a Banach space and let $K \subset X$ be a convex cone containing 0 (i.e. $\lambda x + \mu y \in K,\ \forall \lambda \geqslant 0,\ \forall \mu \geqslant 0,\ x \in K,\ y \in K$). Suppose that K is closed, $X = K - K$ and $K \cap (-K) = \{0\}$. Let T be a linear operator satisfying $T(K) \subset K$.

If T is compact and its spectral radius $r(T) \neq 0$ then there exists $\varphi \in K - \{0\}$ such that

$$T(\varphi) = r(T)\varphi$$

Proof
Let us consider operator $T = \mathbf{P}Q : \mathbf{W}^0 \to \mathbf{W}^0$ introduced in the proof of Theorem 5.1. For both non-local frictional approaches, the operator T is compact in the Hilbert space $\mathbf{W}^0$. Moreover, the closed convex cone $\mathbf{K}^0$ satisfies $\mathbf{K}^0 \cap (-\mathbf{K}^0) = \mathbf{0}$ and $\mathbf{K}^0 - \mathbf{K}^0 = \mathbf{W}^0$ (it suffices to write $\sigma_t(\boldsymbol{v}) = (\sigma_t(\boldsymbol{v}))_+ - (\sigma_t(\boldsymbol{v}))_-$ where the notations $(.)_+$ and $(.)_-$ represent the positive and the negative parts, respectively). We next show that $T(\mathbf{K}^0) \subset \mathbf{K}^0$.

The assumptions of the theorem imply that the operator Q defined by $Q(\boldsymbol{v}) = \sigma_n^*(\boldsymbol{v})$ maps $\mathbf{K}^0$ into $(L^2(\Gamma_C^0))_+$. The operator $\mathbf{P}$ defined for all $f \in L^2(\Gamma_C^0)$ by

$$a(\mathbf{P}(f), \boldsymbol{v}) = \int_{\Gamma_C^0} f v_t \, d\Gamma, \quad \forall \boldsymbol{v} \in \mathbf{W}^0$$

satisfies $\sigma_t(\mathbf{P}(f)) = f$. Hence $\boldsymbol{v} \in \mathbf{K}^0$, which implies that $Q(\boldsymbol{v}) \in (L^2(\Gamma_C^0))_+$ and thus $T(\boldsymbol{v}) \in \mathbf{K}^0$.

It follows then from Krein–Rutman's theorem that if T admits a positive spectral radius, then there exists an eigenvalue which is equal to the spectral radius with an associated eigenvector in $\mathbf{K}^0$.

The case (ii) is handled similarly by using the operator $-T$. □

6. FINITE ELEMENT APPROXIMATION OF THE EIGENVALUE PROBLEM

The problem we intend to approximate is as follows: find $\lambda \in \mathbb{C}$ and $\mathbf{0} \neq \mathbf{\Phi} \in \mathbf{V}^0$ such that:

$$a(\mathbf{\Phi}, \boldsymbol{v}) = \lambda b(\mathbf{\Phi}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{V}^0 \tag{27}$$

which is exactly the eigenvalue problem corresponding to the non-local frictional approach (22). Notice that when the regularization procedure (21) is adopted, then the convergence analysis is more complicated. A remark at the end of this section explains and gives partial answers to the convergence study in that case.

We denote by $\|\cdot\|_1$ the standard norm on $(H^1(\Omega))^2$. Our aim is to approximate the eigenvalues of problem (27). Let be given a family of finite dimensional subspaces $\mathbf{V}_h^0 \subset \mathbf{V}^0$ where h denotes the discretization parameter [28]. The finite dimensional problem consists then of finding $\lambda_h \in \mathbb{C}$ and $\mathbf{0} \neq \mathbf{\Phi}_h \in \mathbf{V}_h^0$ such that [29,30]:

$$a(\mathbf{\Phi}_h, \boldsymbol{v}_h) = \lambda_h b(\mathbf{\Phi}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h^0 \tag{28}$$

We assume that the following approximation property holds:

$$\lim_{h \to 0} \inf_{\boldsymbol{u}_h \in \mathbf{V}_h^0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 = 0, \quad \forall \boldsymbol{u} \in \mathbf{V}^0$$

Let λ^{-1} be an eigenvalue of T defined in (ii) of the proof in Theorem 5.1. Denoting by I the identity map, there exists a least integer α such that $\mathrm{Ker}((\lambda^{-1}I - T)^{\alpha}) = \mathrm{Ker}((\lambda^{-1}I - T)^{\alpha+1}) = E$ with $\dim(E) = m < \infty$. The algebraic multiplicity of λ^{-1} is m and α stands for the ascent of $\lambda^{-1}I - T$. The set E contains the generalized eigenvectors of T corresponding to λ^{-1}. Let T^* be the adjoint operator of T defined on the dual space $\mathbf{V}^{0*}$. Then $\bar{\lambda}^{-1}$ is an eigenvalue of T^* with algebraic multiplicity m and α is also the ascent of $\bar{\lambda}^{-1}I - T^*$. The notation $E^* = \mathrm{Ker}((\bar{\lambda}^{-1}I - T^*)^{\alpha})$ stands for the space of generalized eigenvectors of T^* associated with $\bar{\lambda}^{-1}$. Given two closed subspaces A and B of $\mathbf{V}^0$, define the gap between A and B by

$$\delta(A, B) = \max\left(\sup_{\boldsymbol{u} \in A,\, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v} \in B} \|\boldsymbol{u} - \boldsymbol{v}\|_1, \sup_{\boldsymbol{u} \in B,\, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v} \in A} \|\boldsymbol{u} - \boldsymbol{v}\|_1 \right)$$

Let λ be an eigenvalue of (27) and denote by m its algebraic multiplicity. When h tends to zero, there exist exactly m eigenvalues of (28) denoted $\lambda_{1,h}, \lambda_{2,h}, \ldots, \lambda_{m,h}$ converging to λ. Let E_h be the direct sum of the generalized eigenspaces associated with $\lambda_{1,h}, \lambda_{2,h}, \ldots, \lambda_{m,h}$ and define

$$\varepsilon_h = \sup_{\boldsymbol{u} \in E,\, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v}_h \in \mathbf{V}_h^0} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 \quad \text{and} \quad \varepsilon_h^* = \sup_{\boldsymbol{u} \in E^*,\, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v}_h \in \mathbf{V}_h^0} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1$$

The following theorem, taken from Kolata [31], describes the convergence of the finite element approximation.

Theorem 6.1
If h is small enough, the following estimates hold:

$$\left|\lambda - \frac{1}{m}\sum_{i=1}^{m}\lambda_{i,h}\right| \leqslant C\varepsilon_h\varepsilon_h^*$$

$$|\lambda - \lambda_{i,h}| \leqslant C(\varepsilon_h\varepsilon_h^*)^{\frac{1}{\alpha}}, \quad 1 \leqslant i \leqslant m$$

$$\delta(E, E_h) \leqslant C\varepsilon_h$$

where the constant C does not depend on h.

If the first regularizing approach (21) is adopted then the eigenvalue problem becomes: find $\lambda \in \mathbb{C}$ and $\mathbf{0} \neq \mathbf{\Phi} \in \mathbf{W}^0$ such that $a(\mathbf{\Phi}, \boldsymbol{v}) = \lambda b(\mathbf{\Phi}, \boldsymbol{v})$, $\forall \boldsymbol{v} \in \mathbf{V}^0$. In such a case there are at least two alternatives for obtaining convergence results. The first one is to invoke again Kolata's studies in Reference [31] which are still valid. It suffices then to show two families of finite dimensional subspaces $\mathbf{V}_h^0$ and $\mathbf{W}_h^0$ of $\mathbf{V}^0$ and $\mathbf{W}^0$, respectively, where the dimensions of $\mathbf{V}_h^0$ and $\mathbf{W}_h^0$ are equal. This necessitates to introduce more specific finite element spaces which is out of the scope of this paper. The second possibility is to use a non-conforming finite element approach and approximating $\mathbf{V}^0$ and $\mathbf{W}^0$ with the same finite dimensional space $\mathbf{V}_h^0$ although $\mathbf{V}_h^0 \not\subset \mathbf{W}^0$. In that case the convergence result requires strong supplementary hypotheses as in Reference [32].

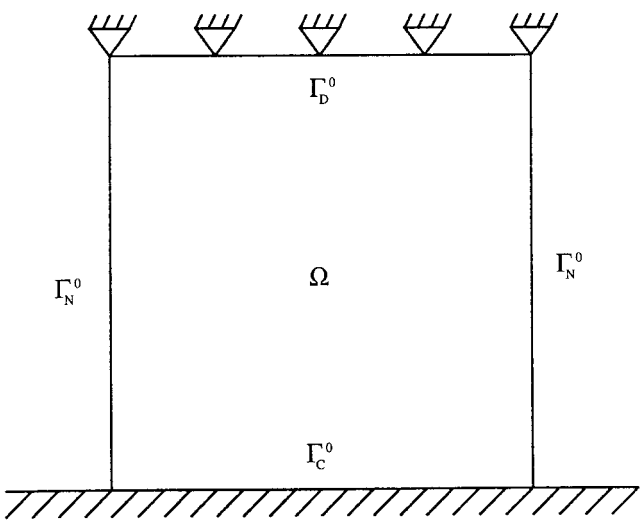

Figure 2. The geometry, the domain Ω and its boundary with its three open disjoint parts: Γ_D, Γ_N and Γ_C.

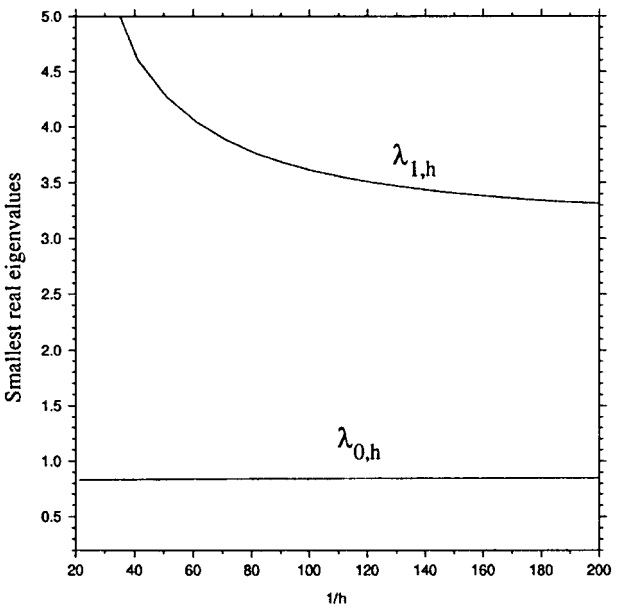

Figure 3. The two smallest real eigenvalues $\lambda_{0,h}$ and $\lambda_{1,h}$ vs the number of elements on a edge: $1/h$.

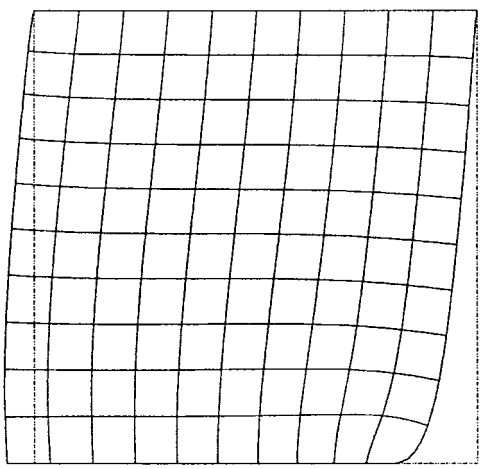

Figure 4. The eigenfunction $\mathbf{\Phi}_{0,h}$ computed for a mesh of 100×100 elements; all the nodes in Γ_C^0 are slipping.

7. NUMERICAL RESULTS

The sufficient conditions for non-uniqueness given in Section 3 concern the solution $\boldsymbol{u}^0$ of the continuous problem (1)–(4). Unfortunately, there are to our knowledge no available

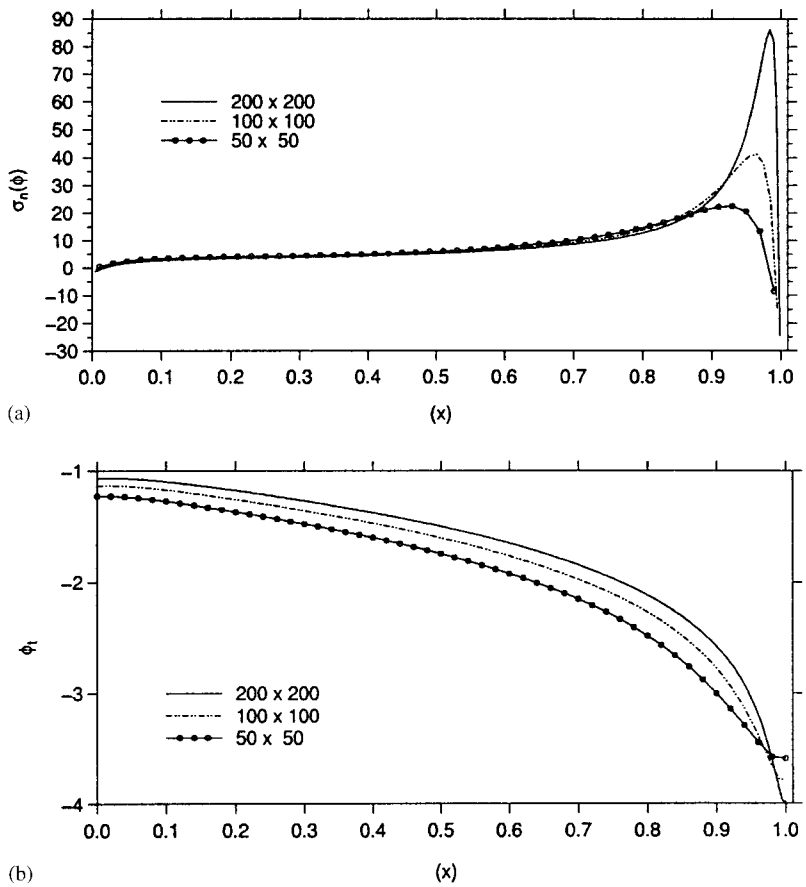

Figure 5. (a) The normal stress $\sigma_n(\mathbf{\Phi}_{0,h})$ and (b) the tangential slip $\mathbf{\Phi}_{0,ht}$ on Γ_C^0 for three different meshes: 50×50, 100×100 and 200×200 elements.

analytical examples of non-uniqueness in the continuous framework. That is why we cannot directly verify the sufficient conditions given in the Theorem 3.2 or Corollary 3.3. Next we try to illustrate the sufficient conditions from a numerical point of view. Since the convenient convergence results for the Coulomb friction model do not exist, the numerical computations cannot stand for a rigorous mathematical proof of the sufficient conditions for non-uniqueness. Our aim here is only to illustrate the methodology given in the continuous context.

We consider the eigenvalue problem (10)–(12) where Ω represents a unit square whose partition of the boundary $(\Gamma_C^0, \Gamma_D^0, \Gamma_N^0)$ is depicted Figure 2. The elastic material is supposed to be isotropic and a Poisson ratio v equal to 0.4 is chosen. The finite element discretization is made of uniform quadrilateral meshes of edge size h. The ARPACK library is used to compute the eigenvalues and eigenfunctions of the discrete spectral problem (28).

Figure 3 depicts the two smallest real eigenvalues denoted $\lambda_{0,h}$ and $\lambda_{1,h}$ as a function of the number of elements on Γ_C^0 ($\lambda_{0,h}$ and $\lambda_{1,h}$ are single eigenvalues). As expected from

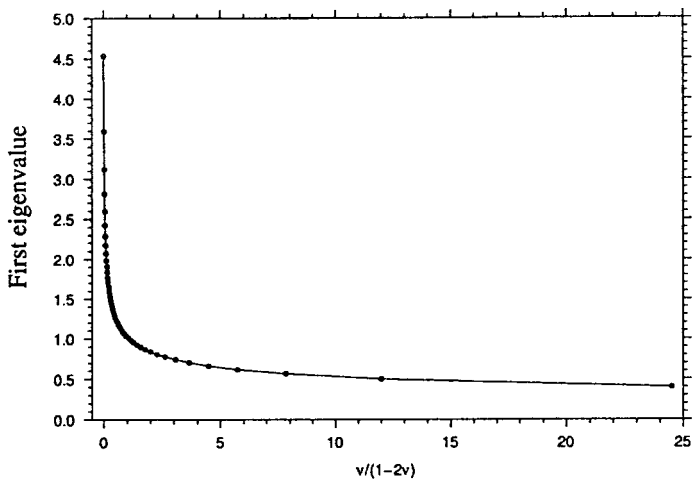

Figure 6. The dependence of the first eigenvalue on the ratio $\eta = v/(1-2v)$.

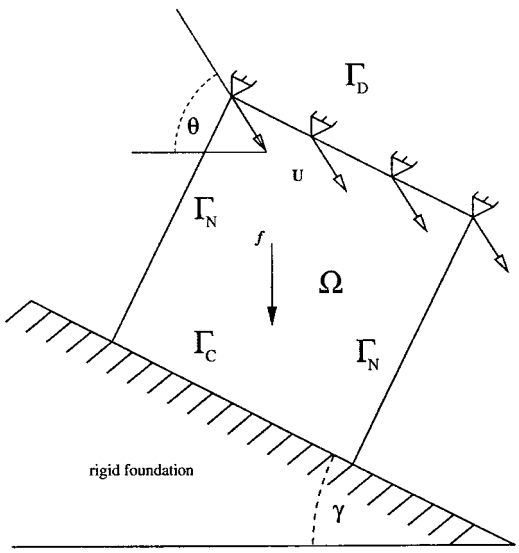

Figure 7. An example of mechanical problem (1)–(4) with non-uniqueness. An elastic square lying on a rigid inclined plane, loaded by the gravity $\boldsymbol{f}$ and by an imposed displacement $\mathbf{U}$.

Theorem 6.1, we note good convergence of these eigenvalues with respect to the mesh size. Note that $\lambda_{0,h}$, which admits the smallest modulus among all complex eigenvalues converges quite well.

In Figure 4, we show the eigenfunction $\boldsymbol{\Phi}_{0,h}$, corresponding to $\lambda_{0,h}$, computed for a mesh of 100×100 elements. Notice that all the nodes on Γ_C^0 are slipping. Figure 5 represents the

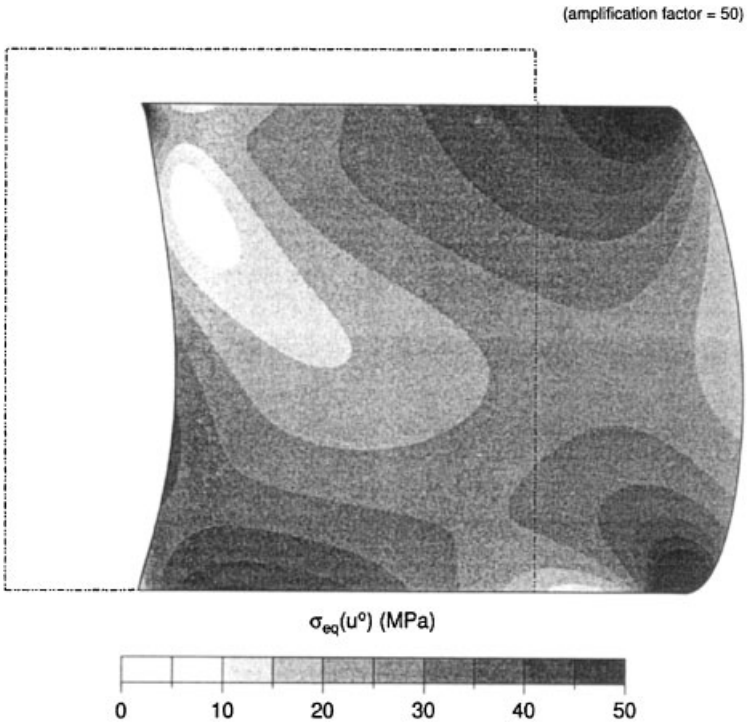

Figure 8. The deformed configuration and the Von–Mises stress field associated with the solution $\boldsymbol{u}_h^0$, computed with a 100×100 mesh.

normal stress $\sigma_n(\boldsymbol{\Phi}_{0,h})$ and the tangential slip displacement $\boldsymbol{\Phi}_{0,ht}$ on Γ_C^0 for three different sizes of meshes. We note the good agreement between these three computations, except maybe, in the neighborhood of the lower right corner of Ω.

The existence of a real eigenvalue for (10)–(12) is investigated in Section 5. As it was already pointed out in Remark 3.1, the eigenvalues depend only on the ratio $\eta = \nu/(1-2\nu)$ in the case of a homogeneous and isotropic elastic body. In Figure 6, we observe that for all ν the first eigenvalue is real and positive when the computations are performed on a 50×50 mesh. Note the sharp variation for small values of the ratio η. If for some ν the positivity of the first eigenvalue is preserved when h tends to 0 and it converges to an eigenvalue of the continuous problem, then the geometry is open to non-uniqueness according to Remark 3.5.

In order to discuss our non-uniqueness methodology, we choose the same geometry as in the previous test and a Poisson ratio ν of 0.4. So, we consider an elastic unit square lying on a rigid inclined plane (see Figure 7) as an example of the mechanical problem (1)–(4).

The elastic body is loaded by a density of gravity forces $\boldsymbol{f}$ and by an imposed displacement field $\mathbf{U}$. The following values are used: $\gamma = \mathrm{Arctan}\, 2 \simeq 63.43°$, $\theta = \mathrm{Arctan}\, 2.5 \simeq 68.2°$, $|\boldsymbol{f}| = 223.6\ \mathrm{N\ m^{-3}}$, $|\mathbf{U}| = 0.005385$, and a Young modulus $E = 10$ GPa. The computations of the numerical solution denoted $\boldsymbol{u}_h^0$ are achieved using the finite element code CASTEM. The

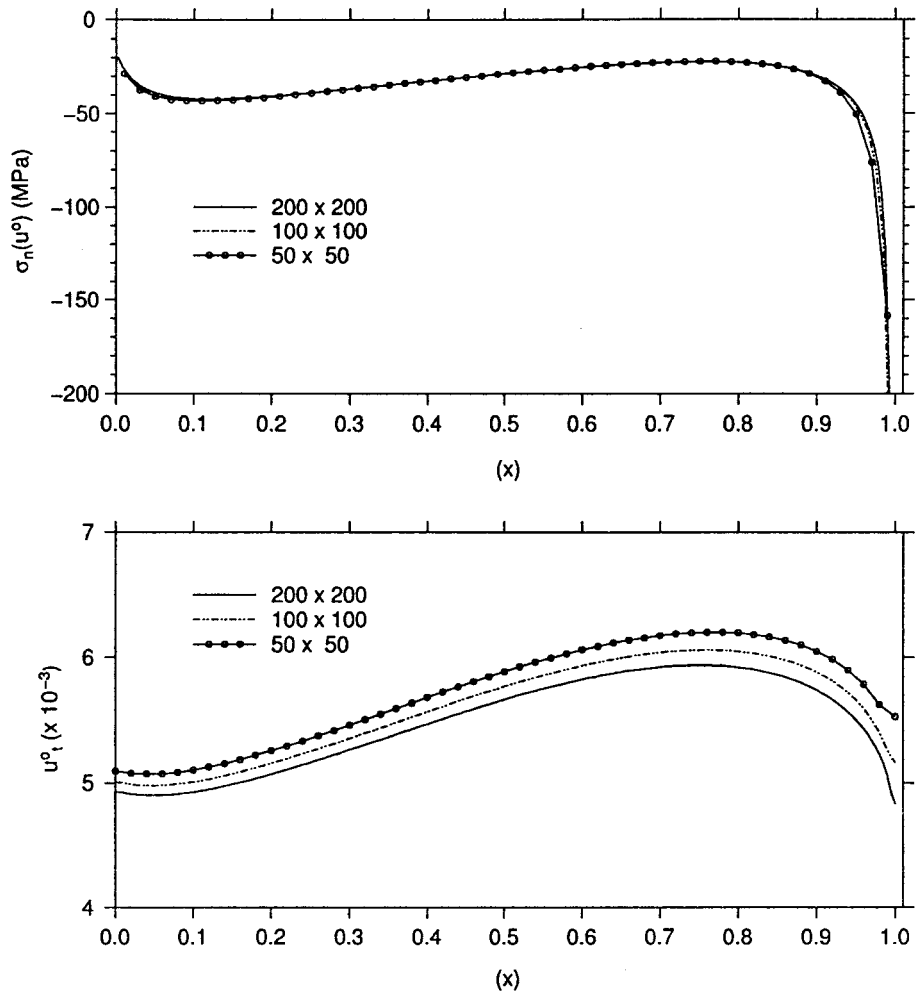

Figure 9. The normal stress and the tangential displacement of Γ_C, corresponding to the solution $\boldsymbol{u}_h^0$, computed with three different meshes.

numerical results plotted in Figure 8 are obtained using an uniform quadrilateral mesh of 100×100 elements.

The friction coefficient μ is chosen to be equal to $\lambda_{0,h} = 0.84232$, the first eigenvalue computed above for the same mesh. We remark that all the points of Γ_C are slipping and therefore we have $\Gamma_C^0 = \Gamma_C$, $\Gamma_D^0 = \Gamma_D$, and $\Gamma_N^0 = \Gamma_N$.

Figure 9 represents the normal stress and the tangential displacement on Γ_C for three different mesh sizes. We remark that the sufficient conditions (18)–(19) are satisfied for the three computations. From the computations we can reasonably expect that these conditions still hold as h tends to 0. If in addition we assume convergence of the finite element solution $\boldsymbol{u}_h^0$ to a solution $\boldsymbol{u}^0$ of the continuous model (1)–(4) such that L^∞ convergence of the quantities

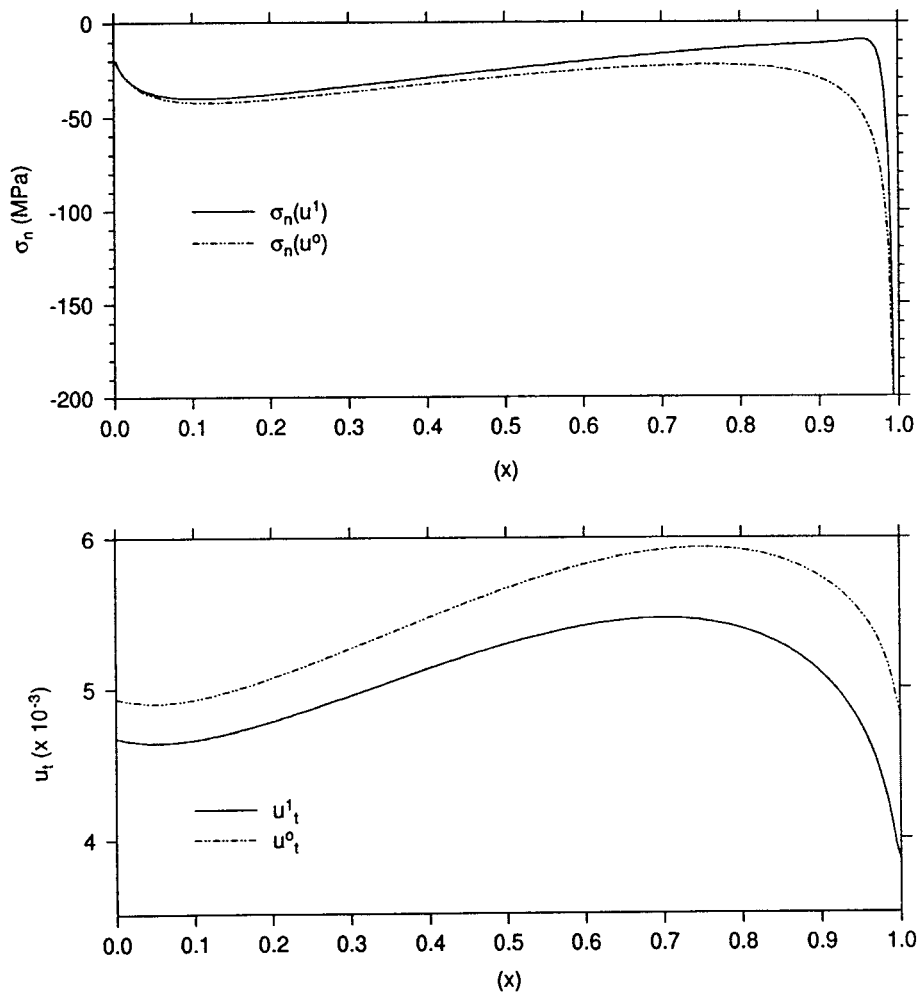

Figure 10. The normal stress and the tangential displacement corresponding to the solutions $\boldsymbol{u}_h^1$ and $\boldsymbol{u}_h^0$, computed with a 200×200 mesh.

depicted in Figure 9 holds, then the conditions (18)–(19) also hold for the solution $\boldsymbol{u}^0$. If the first eigenvalue $\lambda_{0,h}$ in Figure 3 converges to a positive eigenvalue λ_0 of (10)–(12), we choose $\mu = \lambda_0$ and from Corollary 3.3 we could deduce that $\boldsymbol{u}^0$ is not the unique solution of (1)–(4). Indeed, there exist an appropriate scalar δ such that $\boldsymbol{u}^1 = \boldsymbol{u}^0 + \delta \boldsymbol{\Phi}_0$ is another solution of (1)–(4). In Figure 10 we try to illustrate this with a very refined mesh (as fine as the computation allows). We have plotted the normal stress and the tangential displacement corresponding to the discrete solutions $\boldsymbol{u}_h^1$ and $\boldsymbol{u}_h^0$ computed with a 200×200 mesh. Although the solutions are not quite different, we nevertheless observe a local misfit (gap) of order of 10%. The Von-Mises stress corresponding to the difference $\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0$ is depicted in

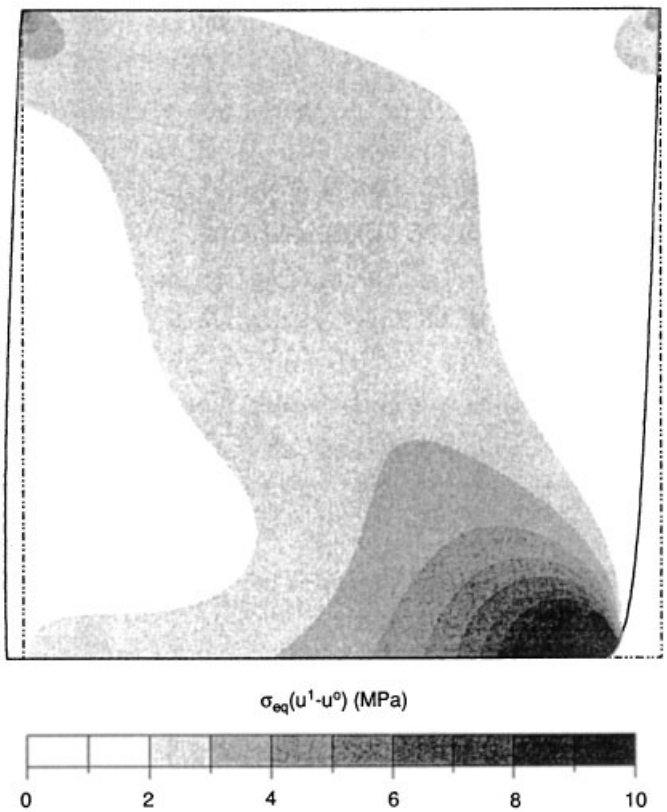

Figure 11. The Von-Mises stress corresponding to the difference $\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0$.

Figure 11. In this case the difference (of order of 20%) is concentrated on the lower right corner.

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